A remark on the existence of Whittaker models for L-packets of automorphic representations of $GSp(4)$.

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Let Π be a cohomological irreducible cuspidal automorphic representation of $GSp(4, \mathbb{A})$, where $\mathbb{A}$ are the rational adeles. Here cohomological is an abbreviation for $\Pi \cong \Pi_\infty \Pi_f$, where Π_∞ belongs to the discrete series of weight w in the sense of [W].

Hypothesis $(*)$: *Suppose Π is weakly equivalent to a cohomological representation, which has a global Whittaker model.*

Hypothesis $(**)$: *Suppose Π has a weak lift to a representation Π' of $Gl(4, \mathbb{A})$.*

Hypothesis $(**)$ was deduced by U.Weselmann (unpublished) modulo the assumption, that the fundamental lemma holds. Concerning the fundamental lemma see Y.Flicker and [BWW].

Concerning hypothesis $(*)$ we remark, that the following hypothesis $(***)$ implies hypothesis $(*)$ by results of Piateskii-Shapiro, Soudry, Jaquet and Shalika which characterize the theta lift from $GO(6)$ to $GSp(4)$ in terms of nonvanishing of degree 5 L at $s = 1$, or in terms of poles for the exterior square degree 6 L-series of $Gl(4)$. In fact the image representations in $GSp(4, \mathbb{A})$ of this weak inverse lift do have global Whittaker models! The well known results of Shahidi on the nonvanishing of L-series of $Gl(n)$ on the line $Re(s) = 1$ guarantees the existence of the inverse lift provided we know

Hypothesis $(***)$:*Let the assumption be as in $(*)$. The restriction of Π to $Sp(4, \mathbb{A})$ can be weakly lifted to $PGl(5, \mathbb{A})$ or $Gl(5, \mathbb{A})$.*

Remark: The hypothesis $(***)$ should follow from the fundamental lemma and the trace formula similar to the case $(**)$. This will be considered in [BWW].

Some explanation are in order: Let F be a local field. The group $GSO(3,3)(F)$ can be realized acting on the six dimensional space $\Lambda^2(F^4)$. See [AG], p.39 ff and [Wa], p.44 f. It turns out, that

$$GSO(3,3)(F) \cong \Big(GL(4,F) \times F^*\Big)/F^* \ ,$$

where $t \in F^*$ acts on $(h,x) \in Gl(4,F) \times F^*$ via $(h,x) \mapsto (ht^{-1}, xt^2)$. The similitude character is $\lambda(h,x) = det(h)x^2$. The group $GO(3,3)(F)$ is generated by $GSO(3,3)(F)$ and a reflection θ of order two. Obviously irreducible admissible representations $\tilde{\Pi}$ of $GSO(3,3)(F)$ correspond to pairs (Π', ω), where Π' is an irreducible admissible representation of $Gl(4,F)$ and ω is a character of F^*, such that $\omega_{\Pi'} = \omega^2$. They can be extended to irreducible admissible representations of $GO(3,3)(F)$ at most in two ways. See [V], p. 482.

The local (or global) theta correspondence of the pair $(GSp(4), GO(3,3))$ preserves central characters. If Π is a global cuspidal representation of $GSp(4,\mathbb{A})$ with central character ω_Π, then its theta lift (if nonzero) to $GO(3,3)(\mathbb{A})$ satisfies $\omega = \omega_\Pi$. It is nonzero iff Π is globally generic [AG] p.40. In this case also its lift $\tilde{\Pi} = (\Pi', \omega_\Pi)$ is generic. Conversely, an irreducible cuspidal automorphic representation Π' of $Gl(4,\mathbb{A})$ lifts nontrivially to $GSp(4,\mathbb{A})$ iff $L(\Pi' \otimes \chi, s, \Lambda^2)$ (equivalently $L(\Pi' \otimes \chi, s, \Lambda^2)$ for a suitable finite set S of places including all bad primes) has a pole at $s = 1$ for some Grössencharacter. Then the lift is globally generic for $GSp(4,\mathbb{A})$. See [AG]. So if we start with some irreducible cohomological representation Π of $GSp(4,\mathbb{A})$, we weakly lift it to the representation Π' via $(**)$. Π' is uniquely determined by Π. But then $L^S(\Pi' \otimes \omega^{-1}, s, \Lambda^2) = L^S(\Lambda^2(\Pi), s)\zeta^S(s)$, where $\Lambda^2(\Pi)$ denotes the weak lift of $(***)$. By results of Shahidi $L^S(\Lambda^2(\Pi), s)$ does not vanish at $s = 1$. The pole of the zeta function therefore implies the existence of a nontrivial inverse theta lift $\theta(\Pi', \omega)$, which is generic on $GSp(4,\mathbb{A})$ and is locally isomorphic to Π at all unramified places (this implicitly uses the fundamental lemma). A local check at the archimedian place shows, that $\theta(\Pi', \omega)_\infty$ is in the discrete series. (The local theta lift is known on endoscopic representations of $GSp(4,\mathbb{R})$, so it is known for the archimedian discrete series representations). This explains, why $(*)$ follows from $(***)$ and $(**)$.

Then by hypothesis $(*)$ and [W], theorem IV there exists a "continuous" semisimple Galois representation

$$\rho_\Pi : \Gamma = Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \to GSp(4, \overline{\mathbb{Q}}_l),$$

attached to the weak equivalence class of Π. It is uniquely determined by the characteristic polynomials of the Frobenius elements. Let λ be the similitude morphism and $\omega = \lambda \circ \rho_\Pi$. Then ω corresponds to the central character ω_Π of Π via class field theory up to a Tate twist. See [W]. Here – contrary to loc. cit. – we normalize Π by a character twist $\|.\|^{w/2}$ so that the central character match. Then $L_p(\Pi_p, s) = det(1 - \rho_\Pi(Frob_p)p^{-s})^{-1}$ holds for almost all places.

Of course Π' is uniquely determined by Π via the strong multiplicity one theorem for $Gl(n)$. By Chebotarev $\rho_\Pi = \rho_{\Pi'}$ is "the" four dimensional Galois representation uniquely attached to Π'. It admits, by construction, a Γ-equivariant symplectic pairing with similitude character $\omega = \omega_\Pi$. Furthermore $\omega^2 = \omega_{\Pi'}$ holds. Although the Galois representation ρ_Π may be reducible, it is multiplicity free. See [W]. Hence the sign $\varepsilon(\Gamma, \rho_\Pi, \omega) = 1$ of any such pairing is uniquely determined, if it exists. In other words ω is constrained by its companion, the weak lift Π', or equivalently by the associated Galois representation $\rho_{\Pi'} = \rho_\Pi$, as being a character ω of Γ with $\omega^2 = det(\rho_{\Pi'})$ for which there exists an isomorphism $\psi : \rho_{\Pi'}^\vee \cong \rho_{\Pi'} \otimes \omega^{-1}$ satisfying $\varepsilon(\Gamma, \rho_{\Pi'}, \omega, \psi) = +1$ (for some choice of ψ). For the notation see [W]. Note $(\psi^\vee)^\vee = \varepsilon(\Gamma, \rho_{\Pi'}, \omega, \psi) \cdot \psi$.

Langlands Philosophy: The weak lift studied in $(**)$ is constructed via the twisted trace formula on $Gl(4) \times Gl(1)$. In fact it is a weak lift of Π to a pair (Π', ω) of representations, such that $\Pi'^\vee \cong \Pi' \otimes \omega^{-1}$. The fundamental lemma implies, that the character ω is the central character ω_Π of Π. So for any pair (Π', ω) in the image of the trace lift $(**)$, Π' puts conditions on its companion ω in terms of the Langlands correspondence using the remarks of the last paragraph – even without further knowledge on Π.

Since conjecturally the Langlands correspondence is compatible with the local-global principle and with duality, the same should hold locally. In other words

Conjecture: *Suppose (Π', ω) is in the image of the lift $(**)$. Suppose v is a nonarchimedian place, such that Π'_v corresponds via the local Langlands correspondence to the Weil-Deligne group representation ρ'_v. Suppose ρ'_v is multiplicity free. Then the sign $\varepsilon(\Gamma_v, \rho'_v, \omega_v, \psi_v)$ is $+1$ independent of the choice of the isomorphism $\psi_v : \rho'_v \cong \rho'_v \otimes \omega_v^{-1}$.*

Corollary: *Suppose $(**)$ and $(***)$ and the conjecture holds for Π. Then multiplicity one holds for all cohomological representations $\tilde{\Pi}$ of $GSp(4, \mathbb{A})$, that are weakly equivalent to Π.*

This corollary follows from the work of U.Weselmann, who proved all relevant local character identities, except for the cases where (Π'_v, ω_v) is exceptional. He furthermore conjectures, that these exceptional cases should not occur at all. (Π'_v, ω_v) is called exceptional, if it corresponds by Langlands duality to a representation $\rho'_v = \rho_{v1} \oplus \rho_{v2}$ where $\rho_{vi}, i = 1, 2$ are both irreducible and not isomorphic (so ρ'_v is multiplicity free), such that there exists a quadratic extension field $E_v/\mathbb{Q}_v$ and characters θ_{vi} of $Gal(\overline{\mathbb{Q}}_v/E_v)$ from which ρ_{vi} are obtained by induction: $\rho_{v1} = Ind(\theta_{v1})$ and $\rho_{v2} = Ind(\theta_{v2})$ such that $\omega_v = \det(\rho_{vi})\omega_{E_v/F_v}$ for $i = 1, 2$. Since Weselmann also shows that $(\rho'_v, \omega_v\omega_{E_v/F_v})$ appears in the lift for exceptional (ρ'_v, ω_v), we conclude from above or by direct calculation that $\varepsilon(\Gamma_v, \rho'_v, \omega_v\omega_{E_v/F_v}, \psi_v) = 1$. But then $\varepsilon(\Gamma_v, \rho'_v, \omega_v, \tilde{\psi}_v) = -1$ (independent from the choice of $\tilde{\psi}$). To see this we can replace ρ'_v by its components $\rho'_{vi}, i = 1, 2$. So assume ρ'_v is two dimensional and induced from $Gal(\overline{\mathbb{Q}}_v/E_v)$. Let $\theta_v, \overline{\theta}_v$ be the characters in the restriction to the absolute Galois group of E_v. Then $\theta_v \neq \overline{\theta}_v$. So ψ_v permutes these eigenspaces

$$\psi_v = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} ,$$

since $\theta_v^\vee \otimes (\theta_v\overline{\theta}_v) = \overline{\theta}_v$, and satisfies

$$\psi_v \circ \rho'_v(\tau^{-1})^t = \det(\rho'_v)(\tau)^{-1}\rho'_v(\tau) \circ \psi_v .$$

Here $\tau \in \Gamma_v$ is a lift of the generator $Gal(E_v/F_v)$. We can assume

$$\rho'_v(\tau) = \begin{pmatrix} 0 & \theta_v(\tau^2) \\ 1 & 0 \end{pmatrix} .$$

But then we can choose

$$\tilde{\psi}_v = \begin{pmatrix} 0 & b \\ -c & 0 \end{pmatrix} ,$$

since $\omega_{E_v/F_v}(\tau) = -1$. But $\psi_v^2 = +id$ then gives $\tilde{\psi}_v^2 = -id$. This being said, we can exclude that exceptional local representations appear in the lift (Π', ω) of Π. Hence Weselmann's result implies multiplicity one modulo hypothesis $(*)$.

In fact, one can deduce the last corollary without Langlands reciprocity – only using the theta correspondence.

Proposition: *Suppose* $(*)$ *and* $(**)$ *holds for* Π. *Then multiplicity one holds for all irreducible cohomological automorphic representations* $\tilde{\Pi}$ *of* $GSp(4,\mathbb{A})$, *that are weakly equivalent to* Π.

To prove the proposition, observe that (Π',ω) is the theta lift of a generic representation Π of $GSp(4,\mathbb{A})$ for the theta lift of the pair $(GO(3,3), GSp(4))$. One has to show, that locally exceptional representations (Π'_v,ω_v) do not occur in the theta lift (Π',ω). Since the theta lift satisfies the local-global principle, it is enough to show that locally exceptional representations do not occur in the local theta correspondence. We may suppose, that we are in a local nonarchimedian situation. We drop indices v in the following arguments.

We remark, that any pair (Π',ω) can be extended uniquely to an irreducible admissible representation $\tilde{\Pi} = (\Pi',\omega)^+$ of $GO(3,3)(F)$, which appears in the theta lift. This follows from [V], prop c) p.483 and $\pi = \tilde{\pi}$, [V] p.483. This extension corresponds to a unique Π by the Howe conjecture (theorem of Waldspurger) for odd residue characteristics. However this is not needed for the following.

We have to exclude the exceptional case, which is the following: (Π',ω) is a theta lift of a cuspidal irreducible representation Π of $GSp(4,F)$, such that Π' is induced from the standard parabolic $P = MN$ with Levi component $M = Gl(2,F)\times Gl(2,F)$ and a pair σ_1,σ_2 of irreducible non isomorphic representations $\sigma_1 \ncong \sigma_2$ with central characters $\omega_1 = \omega_2$ and common CM-character $\omega_0 \neq 1$ and $\omega = \omega_1\omega_0$. Note that we also know, that (Π',ω_1) is in the image of the local theta lift.

By assumption the representation (Π',ω) of $\tilde{G} = Gl(4,F)\times F^*$ survives in the (unnormalized) Jaquet module

$$\theta(\Pi) \twoheadrightarrow J_{\tilde{P}}(\theta(\Pi)) = (\theta(\Pi))[N] \ .$$

Here – slightly changing notation from [Wa], p.44 ff – we put $\tilde{P} = P\times F^*$. However this Jaquet module under consideration is decribed in [Wa] p.63. As a representation of the group $GSp(4,F)\times\tilde{P}$ it is isomorphic to a representation $\mathcal{A}$, which contains a subrepresentation $\mathcal{J}$ such that

$$0 \to \mathcal{J} \to \mathcal{A} \to \mathcal{S} \to 0 \ ,$$

where $\mathcal{J}$ and $\mathcal{S}$ are the following representations of $GSp(4,F) \times \tilde{P}$: $\mathcal{S}$ is the big Weil representation $\omega_{\Gamma'}$ of the pair $GSp(4), GO(2,2)$ on the space of Schwartz-Bruhat functions $\mathcal{S} = \mathcal{S}(Y' \times F^*)$, up to a certain explicit character twist. See [Wa] p.63 for the details and for the notations, in particular p. 63 line -5 and line -1. Furthermore the map $\mathcal{A} \to \mathcal{S}$ is the defined by $(\varphi, \xi) \mapsto \xi$ in the notations of loc. cit. On the other hand

$$\mathcal{J} \subset \{\varphi : GSp(4,F) \to \mathcal{S}(V_0 \times F^*)\}$$

is the space of all functions φ satisfying

(i) φ is right invariant under an open subgroup.

(ii) $\varphi(diag(1,\sigma,det(\sigma))u \cdot s) = |det\sigma|^{-3}\omega_{\Gamma''}(\sigma)\varphi(s)$ for $diag(1,\sigma,det(\sigma)) \in R$, $u \in U$ for the Klingen parabolic subgroup $Q = R \cdot U$. Here $\sigma \in Gl(2,F)$ and $\omega_{\Gamma''}$ denotes the Weil representation for the pair $(GO(2,2), Gl(2,F))$ on the Schwartz-space $\mathcal{S}(V_0 \times F^*)$. In fact, using the notation of [Wa], p. 45, we get $GO(V_0) = GO(2,2)$, since V_0 is a four dimensional split quadratic space over F.

(iii) For a compact set $C \subset GSp(4,F)$ there exists $\varepsilon(C) > 0$ and $N(C)$, such that $\varphi(diag(x,E,x^{-1})s) = 0$ for all $s \in C$ and all x with $|x| < \varepsilon(C)$ or $|x| > N(C)$. Note $diag(x,E,x^{-1}) \in R$.

We remark, that line 14, p. 63 of [Wa] should read $\varphi(r_2(x)s) = \mu \circ \omega_{\Gamma'}(r_2(x)s) \circ \xi$. The map $\mathcal{A} \to \mathcal{S}$ is surjective: Namely for given $\xi \in \mathcal{S} = \mathcal{S}(Y' \times F^*)$, there exists $f \in \mathcal{S}(Y \times F^*)$ such that $\xi = \xi_f = \lambda(f)$. This is obvious from the definition of the map λ, see [Wa], p.63, line 2. Put $\varphi(s) = \varphi_f(s)$ as in [Wa], p.63. Then $(\varphi_f, \xi) \in \mathcal{A}$ by [Wa], lemma 8.1.

As a representation of $GSp(4,F)$ the module $\mathcal{J}$ is isomorphic to the representation $Ind_Q^{GSp(4,F)}(V)$ induced from a smooth representation V of the Levi component $R \cong Gl(2,F) \times F^*$. Induction is from the parabolic subgroup $Q = R \cdot U$. See [Wa] p.63 line 6 and -6. In fact, if we do not normalize the induced representation, V is the external tensor product of the Weil representation $|det(\sigma)|^{-3}\omega_{\Gamma''}(\sigma)$ for $\sigma \in Gl(2,F)$ and the regular representation of F^* on $\mathcal{S}(F^*)$. Hence V is a smooth representation of R.

Now consider the cuspidal exceptional representation (Π, V_Π) of $GSp(4,F)$, and its corresponding theta lift (Π', ω) as a representation of $\tilde{G}$. It has nontrivial image in the Jaquet module $\mathcal{A}$ described above. We need to show, that $\omega = \omega_\Pi$ is equal to the common central character $\omega_1 = \omega_2$ of the pair σ_1, σ_2 of representations of $Gl(2,F)$, as defined above.

Being related by the theta lift, there exists a surjection from the big Weil representation on $\mathcal{S}(Y \times F^*)$ to the representation $\Pi \times (\Pi', \omega)$. From this we get an induced surjective $GSp(4,F)$ intertwining operator $\nu : \mathcal{A} \twoheadrightarrow V_\Pi$ from the Jaquet module $\mathcal{A} = \mathcal{S}(Y \times F^*)[N]$ to V_Π. For the notation see [Wa] p. 49 and 63. The restriction of ν to $\mathcal{J}$ must be zero; if not we would get a nontrivial morphism

$$\nu^\vee : \Pi^\vee \hookrightarrow Ind_Q^{GSp(4,F)}(V)^\vee \cong Ind_Q^{GSp(4,F)}(V^\vee \otimes \Delta_Q^{-1})$$

by duality; the modulus Δ_Q appears due to the fact, that we used normalized induction. Existence of an injection $\nu^\vee$ however is impossible by the well known adjointness of Jaquet functors, if we recall that the Jaquet module $J_Q(\Pi^\vee) = \Pi^\vee[U]$ of the cuspidal representation $\Pi^\vee$ vanishes. In other words ν induces a surjection

$$\mathcal{S} \twoheadrightarrow V_\Pi \ .$$

Hence the maximal Π-quotient of the Jaquet module is – up to some explicit character twist by a character χ – the theta lift of the pair $(GSp(4), GO(2,2))$

(i) $s \in GSp(4,F)$ acts by $\xi \mapsto \xi' = |\lambda(s)|^{-1}\omega_{\Gamma'}(s)\xi$, where $\lambda(s)$ is the similitude factor of s

(ii) $(A, D, z) \in Gl(2,F) \times Gl(2,F) \times F^*$ acts by

$$\xi \mapsto \xi' = |det(D)|^{-2}|z|^{-2}\omega_{\Gamma'}(m(A,D),z)\xi \ .$$

See [Wa] (8.3)(i),(ii). The character $\chi(s,A,D,z) = |\lambda(s)|^{-1}|det(D)|^{-2}|z|^{-2}$ has positive real values. (Warning: [Wa], p.46, line 11 should read $m(A,D)\ m = A \cdot m \cdot D^*$). In fact, $\omega_{\Gamma'}$ denotes the Weil representation of $GSp(4,F) \times GO(2,2)$ on the Schwartz-space $\mathcal{S}(Y' \times F^*)$, where $Y' = L \otimes_F V_0$. Here L is a two dimensional maximal isotropic subspace of a four dimensional symplectic space, and $V_0 \cong M_{2,2}(F)$ is a split four dimensional quadratic space over F, with form

$$q(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = 2(ad - bc) \ .$$

However the local correspondence of the pair $(GO(2,2), GSp(4))$ defined by the Weil representation $\omega_{\Gamma'}$ is well understood; recall $GSO(2,2)(F) \cong Gl(2,F) \times Gl(2,F)/F^*$. It is shown in [We], that this theta lift coincides with the generic part of the endoscopic lift for the group $GSp(4,F)$. Hence there is no room for an additional twist by a quadratic character $\omega_0 \neq 1$. In fact for the endoscopic lift the central character $\omega = \omega_\Pi$ is uniquely determined by the central characters $\omega_1 = \omega_2$ of the representation (σ_1, σ_2) of $Gl(2,F) \times Gl(2,F)$, namely $\omega = \omega_1 = \omega_2$. Hence $\omega_0 = 1$. This conclusion remains unaffected, if the theta correspondence is twisted by a positive character χ, as above. It is then still impossible that $\omega_1 = \omega\omega_0$, since ω_0 takes negative values. So this contradicts the occurrence of exceptional representations and completes the proof of the proposition.

Remark: It should be possible to verify the above conjecture concerning the local Galois representations directly from Weselmann's description of the lift from $GSp(4)$ to $Gl(4) \times G(1)$ (modulo the hypotheses $(**)$ and $(***)$.

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