Endoscopy for GSp(4)

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1.Introduction

Let F be a local or global field of characteristic zero. Let $G = GSp(4)$ be the group of similitudes of an alternating nondegenerate form on a four dimensional F vector space, considered as a reductive algebraic group over the field F. If F is a local field, then it is generally expected, that some of the irreducible, admissible representations of the group $GSp(4,F)$ are obtained as lifts from endoscopic groups. In [Ch] we discussed this local character lift in detail. Endoscopic lifts naturally arise in the process of the stabilization of the Selberg trace formula [L],[K]. For the group $G = GSp(4)$ under consideration there is only one relevant elliptic endoscopic group M, that appears in the stabilization of the trace formula. As a reductive group over F it is isomorphic to

$$M = Gl(2) \times Gl(2)/\mathbb{G}_m \ .$$

Here the multiplicative group $\mathbb{G}_m$ is viewed to be diagonally embedded via $t \mapsto (t\cdot E, t^{-1}\cdot E)$ into the center of $Gl(2) \times Gl(2)$.

The instability of the trace formula gives rise to a lift both locally and globally, which maps irreducible, admissible representations σ of the group $M(F)$ to certain packets of representations π of $G(F)$.

In [Ch] we considered local and global properties of this lift. In loc. cit. the local theory was developed for p-adic fields. The local discussion can be easily extended to the case of an arbitrary local fields of characteristic zero. However the global properties of the endoscopic lift, i.e. the global multiplicity formula (Arthur's conjecture), were only partially obtained in [Ch] even in the case $F = \mathbb{Q}$. In fact the analysis of the topological or Selberg trace formula alone did not suffice to obtain a local-global principle for the endoscopic lift. With this missing ingredient the completion of the proof of the endoscopic multiplicity formula seemed to be available only by the use of a twisted trace formula. However, as explained in talks given 1996 at Tilouine's seminar (Paris 13) and also later 1998 at the Luminy conference, a more concrete and elementary approach is available. This approach is based on two related principles. One is the principle of exchange and the other is the key formula stated in section §4. We remark, that the key formula can be directly deduced from weak versions of the trace formula (see [Ch], cor. 3 or the appendix of this paper) and it deals with simultaneous changes at two places. The principle of exchange on the other hand deals with an exchange at one place exclusively, and its proof is based on explicit theta lifts. At this point unfortunately we have to make the assumption that F is a totally real number field, since we apply [ ]. However we apply results of [ ] in a very specific case, so it is very likely that this restriction can be easily removed. The information obtained from theta lifts provide the missing ingredient for the proof of the final version of the multiplicity formula. In fact, from the point of view above, the principle of exchange is essential, and its proof boils down to (a special case of) the Hasse-Brauer-Noether theorem (lemma 3).

The first draft of this manuscript dates from 1995, where globally only the case $F = \mathbb{Q}$ had been considered. In this revised version the global restriction $F = \mathbb{Q}$ has been removed. Nevertheless the author felt that it still makes sense to deal with the case $F = \mathbb{Q}$ separately and decided not to change the arguments used, since they are slightly different and this might be of some independent interest.

2.The local endoscopic lift

In the following we recall results of [Ch]. At the end we also include some improvements, obtained from corollary 1 of this paper. Let F_v be a local field of characteristic zero. Assume F_v be nonarchimedian. The archimedian case is similar, and well known. Let $M_v = M(F_v)$ be the local proper elliptic endoscopic group of $G_v = GSp(4, F_v)$. Let σ_v be an irreducible, admissible representation of $M_v = M(F_v)$. Then σ_v corresponds to a pair $\sigma_v = (\sigma_{1,v}, \sigma_{2,v})$ of irreducible, admissible representations $\sigma_{i,v}$ of the group $Gl(2, F_v)$ with the same central character $\omega_{\sigma_{1,v}} = \omega_{\sigma_{2,v}}$.

Let $R_{\mathbb{Z}}[G_v]$ resp. $R_{\mathbb{Z}}[M_v]$ denote the Grothendieck group of finitely generated, admissible representations of G_v resp. M_v. For $\alpha = \sum n_i \cdot \pi_{v,i} \in R_{\mathbb{Z}}(G_v)$, a finite sum with integer coefficients n_i and irreducible, admissible representations $\pi_{v,i}$, and a locally constant function f_v with compact support on G_v define $\alpha(f_v) = \sum n_i \cdot tr\ \pi_{v,i}(f_v)$. We use similar notation also for the group M_v.

In [Ch] $\S 10$ the local endoscopic character lift is described as a homomorphism

$$r = r_{M_v}^{G_v} : R_{\mathbb{Z}}[M_v] \to R_{\mathbb{Z}}[G_v]$$

on the level of Grothendieck groups, and it was shown to have the following properties:

i) The lift r is endoscopic in the sense, that for all $\alpha \in R_{\mathbb{Z}}[M_v]$

$$\alpha(f_v^{M_v}) \ = \ r(\alpha)(f_v)$$

holds whenever $f_v, f_v^{M_v}$ are locally constant functions with compact support on G_v resp. M_v with matching orbital integrals. Alternatively we could fix the central character ω of the representations, and consider test functions locally constant with compact support modulo the center Z_v, which transform with the inverse of this character under Z_v. The matching condition is understood in the following sense: $f_v, f_v^{M_v}$ have matching orbital integrals, if for all $G-M$ regular semisimple elements $\gamma \in G_v$ and $\gamma' \in M_v$, where both elements are related by an admissible isomorphism of there centralizing tori, the κ-orbital integral of f_v on G_v multiplied by a transfer factor (chosen as in [Ch],$\S 5$) is equal to the stable orbital integral of $f_v^{M_v}$ on M_v. For details we refer to [Ch] $\S 5$. The form of the transfer factor implies, that r preserves the central character.

ii) The lift r commutes with character twists

$$r(\sigma_v \otimes \chi_v) \ = \ r(\sigma_v) \otimes \chi_v \ .$$

Here χ_v is a quasicharacter, and the character twist for G_v is defined via the similitude homorphisms $G_v \to F_v^*$ ([Ch], lemma 1). Furthermore we use the abbreviation $\sigma_v \otimes \chi_v = (\sigma_{1,v} \otimes \chi_v, \sigma_{2,v} \otimes \chi_v)$.

iii) The lift r commutes with parabolic induction

$$r \circ r^{M_v}_{P_{M_v}}(\rho_v) \ = \ r^{G_v}_{P_v}(\rho_v) \ .$$

Here ρ_v is in $R_{\mathbb{Z}}[Gl(2,F_v)\times F_v^*]$, where P_{M_v} is a maximal proper parabolic of M_v and P_v is the Siegel parabolic of G_v. The similar statement also holds for Borel groups B_{M_v}, B_v of M_v and G_v

$$r \circ r^{M_v}_{B_{M_v}}(\rho_v) \ = \ r^{G_v}_{B_v}(\rho_v) \ .$$

iv) For the involution $\sigma \mapsto \sigma^*$ defined by $(\sigma_{1,v},\sigma_{2,v})^* := (\sigma_{2,v},\sigma_{1,v})$ we have

$$r(\sigma^*) \ = \ r(\sigma) \ .$$

v) The lift r commutes with Galois twists

$$r(\sigma_v^\tau) \ = \ r(\sigma_v)^\tau \quad , \quad \tau \in Aut(\mathbb{C}/\mathbb{Q}) \ .$$

For details see [Ch] $\S 10$. By the properties properties above r is known, once $r(\sigma_v)$ is known for the irreducible representations σ_v of discrete series of M_v. We do not repeat the more precise description given in [Ch], but summarize the qualitative results obtained from it.

<u>Description of r in the discrete series case</u>: Suppose σ_v is an irreducible, admissible representation of M_v, which belongs to the discrete series. Under this assumption there exists a set $\{\pi_+(\sigma_v),\pi_-(\sigma_v)\}$ of two isomorphism classes of irreducible, admissible representations of G_v attached to σ_v, such that

$$r(\sigma_v) \ = \ \pi_+(\sigma_v) - \pi_-(\sigma_v) \ .$$

The central character of $\pi_\pm(\sigma_v)$ and the central character of σ_v coincide. Furthermore the representations $\pi_+(\sigma_v)$ have Whittaker models, whereas the representation $\pi_-(\sigma_v)$ do not have Whittaker models. The first statement on Whittaker models was proved in [CH] whereas the latter statement concerning $\pi_-(\sigma_v)$ was only proved conditionally. See [Ch] $\S 11$, in particular cor.14. In fact it holds unconditionally. This will be a consequence of corollary 1 of this paper. The essential part is to describe $\pi_-(\sigma_v)$ in terms of a theta lift, for which this property can be verified.

Still suppose, that σ_v is irreducible and belongs to the discrete series. Then by [Ch] $\S 10$:

I) Either $\sigma_v^* \cong \sigma_v$. Then, although σ_v belongs to the discrete series, both $\pi_\pm(\sigma_v)$ do not belong to the discrete series. They are tempered elliptic representations of G_v (so called limits of discrete series).

II) Or $\sigma_v^* \not\cong \sigma_v$. Then for σ_v in the discrete series, the representation $\pi_-(\sigma_v)$ is always cuspidal and $\pi_+(\sigma_v)$ belongs to the discrete series. Furthermore $\pi_+(\sigma_v)$ is cuspidal if and only if σ_v is cuspidal.

The remaining cases: Suppose σ_v is irreducible, preunitary admissible but does not belong to the discrete series. Consider first the generic case. This is the case where neither of the two representations $\sigma_{i,v}$, which define σ_v, is one dimensional. In this case

$$r(\sigma_v) \;=\; \pi_+(\sigma_v)$$

for an irreducible, admissible representation $\pi_+(\sigma_v)$ of G_v, which has a Whittaker model. Furthermore $\pi_+(\sigma_v)$ is tempered, if σ_v is tempered. The central characters of σ_v and $\pi_+(\sigma_v)$ are the same.

Definition: Suppose σ_v is a (preunitary) irreducible generic admissible representation of $M(F_v)$. Then the support of $r(\sigma_v)$ – viewed as a subset of the equivalence classes of irreducible admissible representations of $GSp(4,F_v)$ – will be called the local L-packet attached to σ_v.

These local L-packets have cardinality either one or two.

If σ_v is preunitary but not generic, at least one of the two representations $\sigma_{i,v}$ of $Gl(2,F_v)$ defining σ_v is one dimensional. In these cases the image packets defined by the support of the lift r are again very interesting. Now they do not define 'L-packets' , but rather describe the 'Arthur-packets' . Their description is implicitly contained in [Ch].

Remark: The above mentioned properties of the endoscopic lift were stated and proved in [Ch] in the case of a p-adic field $\mathbb{Q}_p$. The arguments given there carry over to the case of a general nonarchimedian local field F_v of characteristic zero. One only has to modify the global argument used in [Ch]. Instead of the topological trace formula one has to use Arthur's version of the Selberg trace formula: For this choose a totally real number field F of degree at least two over $\mathbb{Q}$, whose completion at the nonarchimedian place v is the given local field F_v. One can find a global cuspidal automorphic representation $\sigma \not\cong \sigma^*$ extending the given σ_v. This σ can be chosen to be cuspidal at some additional nonarchimedian place $w \neq v$, and to be belong to the discrete series at all archimedian places. The fundamental lemma for $GSp(4)$ proven in [WW] allows to stabilize the Arthur trace formula for the group $GSp(4,A_F)$ under certain restrictions. See §6. Namely consider the simple version of this trace formula with test function of the form $f = \prod_w f_w$, where f_w is a difference of pseudo coefficients of the discrete series representations $\pi_+(\sigma_\infty), \pi_-(\sigma_\infty)$ at all infinite places of F. Furthermore for our purposes it is of no harm to assume, that f_w is a matrix coefficient of some cuspidal representation $\pi_-(\sigma_w)$ at one further chosen auxiliary place w, where $F_w = \mathbb{Q}_w$ splits. Here σ_w is chosen to be some irreducible cuspidal representation. Under this assumptions, one can stabilize the Arthur trace formula with arguments analogous to those in [Ch] starting from [A], corollary 7.2 and 7.4. For further details see also §6 below. The stabilized Arthur trace formula allows to deduce – with the same arguments as in [Ch] – the following statement: The endoscopic lift $r(\sigma_v)$ of the character of a discrete series representation σ_v of $M(F_v)$, a priori only defined as a distribution satisfying the properties i) and iii) from above, can be expressed as a finite integral linear combination $r(\sigma_v) \in R_{\mathbb{Z}}[G_v]$ of characters of irreducible, admissible representations of $G(F_v)$. Once this is known, $r(\sigma_v)$ can be expressed as an integral linear combination of irreducible characters of $G(F_v)$ for any irreducible character σ_v of $M(F_v)$ and the properties of the lift are deduced completely analogous to [Ch].

Local character transfer: For irreducible, admissible representations σ_v of M_v the endoscopic transfer of distributions applied to the character of σ_v defines distributions on G_v, which are called character lifts of the characters σ_v. It is expected, that generally these character lifts can be expressed as linear combinations of characters χ_{π_v} of irreducible, admissible representations π_v of G_v. So each character lift should be the character of a linear combination of representations

$$r(\sigma_v) \ = \ \sum_{\pi_v} \ n(\sigma_v, \pi_v) \cdot \pi_v \ .$$

The distribution defining the character lift uniquely determines the transfer coefficients $n(\sigma_v, \pi_v)$, once they are known to exist. This follows from the linear independence of characters (Rogawski [R], prop.13.8.1). So it remains to show the existence of such an expansion. In the archimedian case Shelstad [Sh], [Sh2] has established this in complete generality. Moreover it is shown in loc. cit. that the archimedian transfer coefficients $n(\sigma_\infty, \pi_\infty)$ belong to $\{\pm 1, 0\}$ with almost all of them being zero ([Sh], thm.4.1.1(i)). Shelstads proof builds on the Harish-Chandra theory, and uses the classification of irreducible, admissible representations and explicit character formulas. An analog of this is not available in the nonarchimedian case.

To obtain a corresponding local character lift in the case of the group $G_v = GSp(4, F_v)$ for nonarchimedian local fields of characteristic zero, the arguments used in [Ch] were to some extend of a global nature. Since these arguments involve several steps, it might be of some use to summarize the different steps involved.

(A) Global inputs: The stabilization of the global trace formula ([Ch],[W]) is the source for the desired weak character identities $r(\sigma_v) = \sum n(\sigma_v, \pi_v) \cdot \pi_v$, with certain coefficients $n(\sigma_v, \pi_v)$ in $\mathbb{C}$. These sums need not be finite a priori, but they are absolutely convergent in a suitable sense ([Ch], [R]). In fact the trace formula gives more than that, since it directly relates the coefficients $n(\sigma_v, \pi_v)$ to the global multiplicities $m(\pi)$ of irreducible, automorphic representations π of $G(\mathbb{A}_F)$.

The case $F = \mathbb{Q}$ studied in [Ch] is particularly simple. Since it is easier than the case of an arbitrary number field F, we therefore outline the proof in this case first by repeating the most important steps of the proof.

The first basic result is the following trace identity ([Ch], §3, cor.1)

$$(0) \qquad \prod_{v\neq\infty} n(\sigma_v,\pi_v) \;=\; m(\pi_\infty^W \pi_f) - m(\pi_\infty^H \pi_f),$$

where $\pi_f = \otimes_{v\neq\infty}\pi_v$ is any global weak endoscopic automorphic lift of a global irreducible cuspidal automorphic representation $\sigma_f = \otimes_{v\neq\infty}\sigma_v$ of $M(\mathbb{A}_F)$, for which $\sigma^* \not\cong \sigma$ holds. In the formula above $m(\pi)$ denotes the multiplicity of π in the discrete spectrum and $\pi_\infty^H, \pi_\infty^W$ are the irreducible archimedian representations defined by the archimedian character lift $r(\sigma_\infty) = \pi_\infty^W - \pi_\infty^H$ (Shelstad [Sh]).

This version (0) of the trace formula implies for the local components σ_v of the global representation σ

(i) $r(\sigma_v)$ is a absolutely convergent sum

$$r(\sigma_v) = n(\sigma_v)\cdot\sum_{\pi_v}\varepsilon(\sigma_v,\pi_v)\cdot\pi_v$$

for some $n(\sigma_v)\in\mathbb{C}^*$ such that

(ii) (integrality) $\varepsilon(\sigma_v,\pi_v)\in\mathbb{Z}$ holds and furthermore $\varepsilon(\sigma_v,\pi_v)=1$ for $\pi_v=\pi_+(\sigma_v)$.

(iii) (product formula) $\prod_v n(\sigma_v)=1$.

(iv) (Shelstad) $n(\sigma_\infty)=1$.

In fact these assertions follow from a result of Soudry, which provides the existence of a globally generic cuspidal representation $\pi_+ = \pi_+(\sigma)$ such that $(\pi_+)_v = \pi_+(\sigma_v)$ for all v and such that $m(\pi_+)=1$. See [Ch] §9 for further details. In fact $(\pi_+)_\infty = \pi_+(\sigma_\infty) = \pi_\infty^W(\sigma_\infty)$. Since $m(\pi_\infty^W\otimes(\pi_+)_f) = 1$ the principle of exchange (lemma 3 and 4 below) implies $m(\pi_\infty^H\otimes(\pi_+)_f)=0$. Hence by (0) we obtain the product formula $\prod_{v\neq\infty} n(\sigma_v,\pi_+(\sigma_v))=1$. Now define $n(\sigma_v) := n(\sigma_v,\pi_+(\sigma_v))$. Then by [Sh], thm.4.1.1(i)) we have in particular $n(\sigma_\infty)=1$. This gives (iv) and together with the product formula just obtained we now have shown the product formula (iii).

Assertion (i) on the other hand is a consequence of the global stabilized trace formula by defining $\varepsilon(\sigma_v,\pi_v) = n(\sigma_v,\pi_v)/n(\sigma_v,\pi_+(\sigma_v))$. Obviously $\varepsilon(\sigma_v,\pi_v)=1$ holds for $\pi_v=\pi_+(\sigma_v)$ by definition. To prove (ii) only $\varepsilon(\sigma_v,\pi_v)\in\mathbb{Z}$ remains to be shown. Again by [Sh] it is enough to prove this for the nonarchimedian places v. So let v be nonarchimedian.

Put $\tilde{\pi}_f = \pi_v \otimes_{w \neq v,\infty} \pi_+(\sigma_w)$ and $(\pi_+)_f = \otimes_{w\neq\infty} \pi_+(\sigma_w)$. Then by (iii) and (iv) we have $\prod_{w\neq\infty} n(\sigma_w, \pi_+(\sigma_w)) = 1$. Hence

$$\varepsilon(\sigma_v,\pi_v) = \varepsilon(\sigma_v,\pi_v) \prod_{w\neq\infty} n(\sigma_w,\pi_+(\sigma_w)) = \frac{n(\sigma_v,\pi_v)}{n(\sigma_v,\pi_+(\sigma_v))} \prod_{w\neq\infty} n(\sigma_w,\pi_+(\sigma_w)) = \prod_{w\neq\infty} n(\sigma_w,\tilde{\pi}_w) \ .$$

Together with the formula (0) applied to the representation $\tilde{\pi}_f$ instead of π_f this gives

$$\varepsilon(\sigma_v,\pi_v) \ = \ m(\pi_\infty^W \tilde{\pi}_f) - m(\pi_\infty^W \tilde{\pi}_f) \ .$$

Since the right side obviously is an integer, the integrality (ii) of the coefficients $\varepsilon(\sigma_v,\pi_v)$ follows.

<u>(B) Local inputs</u>: A purely local investigation of the character lift using compatibility with parabolic induction gives

(v) $n(\sigma_v) = 1$ for generic σ_v not in the discrete series.

See [Ch], $\S 10$.

Furthermore let $< \eta_v, \omega_v >_e$ denote the elliptic scalar product on the elliptic regular locus of G_v. Put $\|\eta_v\|_e^2 =< \eta_v,\eta_v >_e$. Then (see in particular [Ch] $\S 5$)

(vi) (Weyl integration formula) $\|r(\sigma_v)\|_e^2 = A_v \cdot \|\sigma_v\|_e^2 = A_v$

for σ_v in the discrete series of M_v. Here $A_v = 2$ or 4 depending on whether $\sigma_v^* \ncong \sigma_v$ or $\sigma_v^* \cong \sigma_v$.

Still for σ_v in the discrete series the classification of irreducible, admissible representations of G_v provides us with a second irreducible representation $\pi_-(\sigma_v)$, such that ([Ch] $\S 8$ and $\S 10$)

(vii) $\sum_{\pi_v} \varepsilon(\sigma_v,\pi_+(\sigma_v)) \cdot \pi_v = L(\sigma_v) + R(\sigma_v)$ with $L(\sigma_v) = \pi_+(\sigma_v) - \pi_-(\sigma_v)$

where furthermore

(viii) $\|L(\sigma_v)\|_e^2 = A_v$ and $< L(\sigma_v), R(\sigma_v) >= 0$

for the same coefficients $A_v = 2,4$ as above. See [Ch] $\S 5$, $\S 8$ cor.7 and $\S 10$.

If we combine the local and global information gathered above, we obtain

$$\prod_{v\in S}\|r(\sigma_v)\|_e^2 \;=\; \prod_{v\in S} n(\sigma_v)^2\cdot\prod_{v\in S}\|\sum_{\pi_v}\varepsilon(\sigma_v,\pi_+(\sigma_v))\cdot\pi_v\|_e^2 \;\leq 1\;\cdot\prod_{v\in S}\|L(\sigma_v)\|_e^2\;.$$

Here S is the finite set of places, where the suitable chosen global cuspidal irreducible representation $\sigma\not\cong\sigma^*$ belongs to the discrete series locally. Note $\prod_{v\in S} n(\sigma_v)^2=1$ follows from (iii), (iv) and (v).

But the inequality above is an equality, since both the left and the right side are equal to $\prod_{v\in S}A_v$ by (vi) and (viii). Hence equality holds, which forces $\|R(\sigma_v)\|_e^2=0$. Therefore

(ix) $\quad \sum_{\pi_v}\varepsilon(\sigma_v,\pi_v)\cdot\pi_v \;=\; \pi_+(\sigma_v)-\pi_-(\sigma_v)$

for σ_v in the discrete series (for σ). But then $\|r(\sigma_v)\|_e^2=|n(\sigma_v)|^2\cdot\|\sum\varepsilon(\sigma_v,\pi_v)\pi_v\|_e^2$ also implies $A_v=|n(\sigma)|^2\cdot A_v$, hence

(x) $\quad |n(\sigma_v)|=1.$

On the other hand the local nonarchimedian theory of Whittaker models implies

(xi) $\quad n(\sigma_v)$ must be a positive real number,

since $\pi_-(\sigma_v)$ does not have local Whittaker models ([Ch] $\S 11$, cor.14). This implies $n(\sigma_v)=1$, hence we obtain from (x) and (xi) the final formula

$$r(\sigma_v) \;=\; \pi_+(\sigma_v)-\pi_-(\sigma_v)$$

for all σ_v in the discrete series (of the global representation σ). Since up to a character twist every irreducible representation σ_v in the discrete series of M_v can be realized as the local component of a global cuspidal irreducible representation σ of $M(\mathbb{A}_F)$ (this follows from the existence of strong pseudo coefficients for $Gl(2,F_v)$), this completes the proof for σ_v in the discrete series. The general case is reduced to this case by purely local methods ([Ch] $\S 5-10$).

Let us see, how this carries over to the case of an arbitrary nonarchimedian local field of characteristic zero. All the arguments above were of general nature, only the trace identity (0) needs some equivalent. But this equivalent can be obtained from the strong multiplicity one theorem for $M(\mathbb{A}_F)$ (or $Gl(2,\mathbb{A}_F)$) and a version of the Arthur trace formula, which is stated in lemma 7 in the appendix. In fact one has to be more precise at this point. One has to use an avatar of lemma 7 in two respects. First, in the formulation of lemma 7 the properties of the local character lift has already been incorporated. Secondly, this was essential in order to be able to control the influence of twisting by multipliers as used in the proof of lemma 7. Going through the different steps in the proof of lemma 7, however allows to obtain formula (0) with two changes in the argument. First, one applies the formula for the local character lift only in those situations where it is already known, notably for the archimedian places ([Sh]). Next one has to avoid the use of multipliers. In fact they become unnecessary, since for the proof of (0) it suffices to use the simplified version of the trace formula of Deligne-Kazhdan type. We sketch the main steps of the argument to obtain formula (0) below, but leave the details as an exercise for the reader. In fact the reader is advised to first skip the next paragraph, and to first read §3 and §5 and §6, since the arguments below are adapted from arguments used there. We omit details to avoid the repeating of arguments given later anyway. Notice, although the situation later in §5 and §6 may be more complicated, the situation sketched below - although simpler - is in fact obscured by additional technical assumptions which are necessary for this kind of bootstrap argument with less information at hand to start with.

In fact, to obtain generalizations of formula (0) we choose v, v' as in lemma 7 to be archimedian for a suitable chosen auxiliary number field F, and a suitably chosen auxiliary global irreducible cuspidal automorphic representation $\sigma \not\cong \sigma^*$ of $M(\mathbb{A}_F)$, and we choose w to be some auxiliary 'harmless' place, where the global representation σ is unramified and for which the norm of w is sufficiently large; furthermore we choose F and σ so that for some additional auxiliary nonarchimedian place w' of where $F/\mathbb{Q}$ splits with residue characteristic different from two $\sigma^*_{w'} \not\cong \sigma_{w'}$ holds and is cuspidal. Then $\pi_\pm(\sigma_{w'})$ are cuspidal series and up to character twists the only representations in $\Theta_\pm(\sigma_{w''})$ (see proposition 1 below for the notation). In deed, the statement on the theta lift uses Waldspurgers proof of the Howe duality for the dual pair $Sp(4) \times O(4)$, whereas the cuspidality statement uses that we already considered the case of local fields which are completions of $\mathbb{Q}$. For the archimedian places we choose f_v to be some fixed auxiliary cuspidal function in the Hecke algebra satisfying $(*)_v$ such that $a_- = -a_+ = tr\pi_-(\sigma_v)(f_v) > 0$ (notation as in §5). At the archimedian place v' we choose two functions $f_{v'}$ such that $tr\pi_+(\sigma_{v'})(f_{v'}) = 0$ and $tr\pi_-(\sigma_{v'})(f_{v'}) = 1$ or vice versa. At the place w' we choose $f_{w'}$ functions in the Hecke algebra, whose traces separate the finitely many cuspidal representations of G_v which appear in the packets $\Theta_\pm(\sigma_{w'})$ of theta lift twisted by quadratic characters. For this we can assume $f_{w'}$ to be matrix coefficients of these finitely many cuspidal representations. Finally f_w is chosen with regular support in a maximal split torus of G_w. With these choices the global trace formula considerably simplifies as observed Henniart and Deligne-Kazhdan. In particular on the spectral side only the cuspidal automorphic spectrum contributes due to the cuspidal matrix coefficients $f_{w'}$ at the nonarchimedian place w'. This makes the use of multipliers (which is necessary in the more complicated situation of lemma 7) superfluous. Secondly the chosen $f_{w'}$ completely suffice to detect all local constituents of global endoscopic lifts at the place w', due to lemma 1 in §3. The next observation is, that 'moving' the unramified auxiliary place w to some w sufficiently large allows to get rid of the influence of f_w in the trace formula. This follows from well known finiteness results (i.e. we apply the same trick as in the proof of lemma 5 and of the key formula at the end of §5). Furthermore, similar as for the place w', the lemmas 1-4 of §3 completely control the archimedian constituents of the global endoscopic lift. So the chosen $f_{v'}$ again completely detect all local constituents of global endoscopic lifts at the place v' (lemma 1 and lemma 4 in §3). With these choices made we now follow the arguments in the proof of lemma 7 mutatis mutandis to obtain

$$\sum_{\pi'} m(\pi'_v \otimes (\pi')^v) \cdot tr\ \pi'(f_v) \ = \ \frac{1}{2}(a_+ - a_-) \prod_{w \neq v} n(\sigma_w, \pi'_w) \ ,$$

where $\pi' = \pi'_v \otimes (\pi')^v$ runs over all global cuspidal representations, which are weak endoscopic lifts (in the notation §3) of our fixed auxiliary global automorphic representation σ with fixed $(\pi')^v$ out side v. Since $\pi'_v \in \{\pi_\pm(\sigma_\infty)\}$ by lemma 4 below, we then get $tr\ \pi'(f_v) = a_\pm = \pm a_+$. Cancelling $a_+ = -a_- \neq 0$ from the formula leaves us with a formula, which is the precise analog of formula (0) with the unique archimedian place ∞ of $F = \mathbb{Q}$ now replaced by the fixed chosen archimedian place v of the number field F. In order to compare with the situation above we change notation and let v denote ∞ from now on. Then for any local nonarchimedian local field F_v of characteristic zero and any irreducible admissible representation σ_v of M_v in the discrete series we may have chosen F and σ such that they extend F_v and σ_v (may be up to a local twist by a character). With this additional choice made, the analog of formula (0) established above again implies (i)-(iv). This implies all the properties of the local endoscopic character lift for $r(\sigma_v)$ using (v)-(xi) as in the p-adic case.

This being said, we return to the representations $\pi_\pm(\sigma_v)$ defining the local L-packets introduced above. They can be further described in terms of the general classification of irreducible admissible representations of $GSp(4, F_v)$. See [Ch] $\S 6 - 10$. But they can also be described in terms of a theta lift (Weil representation). We now recall this second description. For details see [Ch] §11.

<u>Extended Jaquet-Langlands Correspondence</u>: Let D_v be the quaternion algebra over F_v. There is an inner form $\check{M}$ of the reductive F_v-group M, whose group of F_v-rational points is $\check{M}_v = (D_v^* \times D_v^*)/F_v^*$. The quotient group M_v is defined by the embedding $t \mapsto (t, t^{-1})$ of F_v^* in $D_v^* \times D_v^*$.

The well known Jaquet-Langlands lift describes the irreducible discrete series representations of the group $Gl(2, F_v)$ in terms of the irreducible representations of the multiplicative group D_v^* of the quaternion algebra.

We can use this to define an extended lift, denoted JL, which maps irreducible admissible representations of the group $\check{M}_v$ to irreducible, admissible, discrete series representations of the group M_v. Simply use the description of $\check{M}_v$ resp. M_v as quotient of two copies of D_v^* resp. $Gl(2,F_v)$. Extending the Jaquet-Langlands lift JL in the obvious way we see, that irreducible, admissible discrete series representations σ_v of M_v correspond uniquely to irreducible, admissible representations $\check{\sigma}_v$ of $\check{M}_v$ and vice versa via the correspondence $\sigma_v = JL(\check{\sigma}_v)$.

<u>Quaternary theta lifts</u>: M_v can be considered as the group of special orthogonal similitudes $GSO(V)$ of a split quaternary quadratic form over F_v. Similarly $\check{M}_v$ can be considered as the group of special orthogonal similitudes $GSO(V)$ of the quaternary anisotropic quadratic form over F_v.

With this point of view consider the theta correspondence for the groups $GO(V)$ and $GSp(4)$ as in [Ch]. It relates representations σ_v^{ext} and π_v of $GO(V)$ and $GSp(4)$, if $\sigma_v^{ext} \otimes \pi_v$ is a quotient of the big theta representation $\tilde{w}$ defined in [V] §4 or [Ch]. Although the big theta representation in [V] is defined in a slightly different way, we will use results of [V] whenever they carry over.

The groups $GSO(V)$ are subgroups of index 2 in $GO(V)$, so in general irreducible admissible representations σ_v of $GSO(V)$ cannot be extended uniquely to irreducible representations of the group $GO(V)$. If $\sigma_v \cong \sigma_v^*$, there exist possibly two nonisomorphic extensions σ_v^{ext} of the representation σ_v to $GO(V)$. One can consider the representations on $GSp(4,F_v)$ attached to σ_v^{ext} by the theta correspondence.

There is a subtlety always present in the definition of the theta correspondence: Either one deals with it on the level of representations of the groups $O(V), Sp(2n)$ or one the extended level of the groups $GO(V), GSp(2n)$. Both points of view are of relevance. The representations obtained from the theta lift for $GSp(4,F_v)$, say with a fixed central character of the center Z_v of $GSp(4,F_v)$, can also be obtained – via compact induction from $Sp(4,F_v)\times Z_v^*$ – from the $Sp(4,F_v)$-representations obtained from the $O(V), Sp(2n)$ correspondence. This follows as in [V], p.470. This comparison sometimes allows to decide, if σ_v^{ext} has some nontrivial theta lifts to $GSp(4,F_v)$. See [V], prop. 4.11. Notice also that the restriction from $G = GO(V)$ to $O(V)$ is multiplicity free by [HPS] lemma 7.2 and the remarks after it on p.94 for the split and the anisotropic quaternary spaces V.

From the above remarks and [HPS] thm.9.1 and also [HPS] §10 it therefore follows, that the theta correspondence for $GSp(4,F_v)$ and $GO(V)$ is always nontrivial in the anisotropic case. In other words, to every irreducible representation $\check{\sigma}_v$ of $\check{M}_v$ there exists a nonempty set $\Theta(\check{\sigma}_v)$ of isomorphism classes of irreducible admissible representations of $GSp(4,F_v)$, which arise in the theta correspondence attached to an extension of $\check{\sigma}_v$ from $\check{M}_v = GSO(V)$ to $GO(V)$. It consists of the classes of irreducible admissible representations of G_v for which $\check{\sigma}_v \times \pi_v$ is a quotient of the big Weil representation of the pair $GO(V)\times G_v$. The next lemma says that all the representations in $\Theta(\check{\sigma}_v)$ are globally relevant.

Density lemma: (as in [Ch] cor.13, [Ch] preceding cor.14)

i) For $\check\sigma \cong \check\sigma_v^*$ the set $\Theta(\check\sigma_v)$ is the class of the single representation $\pi_-(\sigma_v)$.
ii) For $\check\sigma_v \not\cong \check\sigma_v^*$ all representations in $\Theta(\check\sigma_v)$ are cuspidal irreducible admissible representations π_v.
iii) Furthermore, given a finite set S of places w of an number field F, given unitary representations $\check\sigma_w$ of the discrete series of M_w and irreducible representations π_w of G_w in $\Theta(\check\sigma_w)$ for all $w \in S$, suppose the central characters ω_v of $\check\sigma_v$ are the components of a grossencharacter ω of F. Then there exists a corresponding automorphic irreducible cuspidal representation of $M(\mathbb{A}_F)$ and an nontrivial irreducible cuspidal automorphic theta lift π of σ, which realizes the given local representations π_w for all places in S.

Comment: One can enlarge S by a place w, choose some $\check\sigma_w \not\cong \check\sigma_w^*$ with central character $\tilde\omega_w$ to obtain the situation of [Ch]. Namely by the existence of very strong pseudo coefficients for $Gl(2)$ one can always find a global cuspidal σ with central character ω realizing σ_v for $v \in S \cup \{w\}$.

Remark: Furthermore, let w be the order of the group of roots of unity in F. If we choose the place w outside S of residue characteristic larger than w, we can always extend given unitary central characters $\omega_v, v \in S$ to a grossencharacter ω of F^*, which is unramified outside $\{w\} \cup S$ and induces the given local character up to an unramified character twist $\omega_v = \tilde\omega_v |.|_v^{it_v}$. For one place $v \in S$ we can choose $\omega_v = \tilde\omega_v$. See [T], p.342ff. Note that equation (A) of loc. cit can be solved by a suitable choice of a character $\tilde\omega_w : \mathfrak{o}_w^* \to \mathbb{C}^*$, which factorizes over the residue field. Then the equation (B) for the parameters t_v of loc. cit can be solved. This defines the character ω partially on the subgroup $(\prod_{v\in\{w\}\cup S} F_v^* \prod_{v\ else} \mathfrak{o}_v^*)/(F_S^*)$ of $(\mathbb{A}_F^1/F^*) \times \mathbb{R}^*$. Since $\mathbb{C}^*$ is divisible, hence injective, there is no obstruction to extend this partial ω from this subgroup to $\mathbb{A}_F^1/F^*$. The number of possible extensions to a character unramified outside w and S and tame at w is finite, since the S-ideal class group is finite.

If σ_v is an irreducible admissible representation of the group M_v and V is the split quaternary quadratic space, one can define a corresponding set $\Theta_+(\sigma_v)$. It consists of the classes of irreducible admissible representations of G_v for which $\sigma_v \times \pi_v$ is a quotient of the big Weil representation of the pair $GO(V) \times G_v$. By [S2], proof of thm.3.1 on p.366 there is at most one generic class of representations in $\Theta_+(\sigma_v)$. Otherwise one could easily show that that space of functionals considered in [S2], p.366 would have dimension ≥ 2, which gives a contradiction. A representative of this generic class we always be denoted $\theta_+(\sigma_v)$, whenever it exists. A generic class $\theta_+(\sigma_v)$ exists in $\Theta_+(\sigma_v)$, if for instance the representations σ_v is generic and in the discrete series of M_v. So in particular, $\Theta_+(\sigma_v)$ is nonempty in this case. The existence of $\theta_+(\sigma_v)$ in this case follows from global methods ([Ch] §9). This argument also gives $\theta_+(\sigma_v) \cong \pi_+(\sigma_v)$ for discrete series representations σ_v of M_v ([Ch] theorem 3).

For σ_v in the discrete series of M_v, there exists a unique $\check{\sigma}_v$, for which for $\sigma_v = JL(\check{\sigma}_v)$. Then we have $\theta_+(\sigma_v)$ (split case) and we define $\theta_-(\sigma_v)$ to be any member of the set $\Theta_-(\sigma_v) := \Theta(\check{\sigma}_v)$ (in the anisotropic theta lift). Then we have the next diagram, which is justified by the statemenets of proposition 1 below

$$\begin{tikzcd} \sigma_v \arrow[r, mapsto, "\Theta"] & \theta_+(\sigma_v) & \text{unique generic class in } \Theta_+(\sigma_v) \\ \check{\sigma}_v \arrow[u, mapsto, "JL"] \arrow[r, mapsto, "\Theta"] & \theta_-(\sigma_v) & \Theta_-(\sigma_v) = \{\theta_-(\sigma_v)\} \end{tikzcd}$$

<u>Proposition 1</u>: For irreducible, admissible representations σ_v in the discrete series of M_v the set $\Theta_-(\sigma_v)$ has cardinality one. Furthermore there is a unique generic representation in $\Theta_+(\sigma_v)$, and the endoscopic lift is given by these Weil representations as follows

$$r(\sigma_v) \ = \ \theta_+(\sigma_v) \ - \ \theta_-(\sigma_v) \ .$$

<u>Proof</u>: This description of r in terms of the theta lift was already obtained in [Ch], corollary 14 under some additional global hypothesis. As shown below in corollary 1 below, the statement holds unconditionally. Furthermore the corollary 4 below also shows, that the global hypothesis supposed in [Ch], corollary 14 is always satisfied.

<u>Remark</u>: Although the packets $\Theta_+(\sigma_v)$ presumably have cardinality one, we have not been able to show this. Since we want to include places of residue characteristic two, and also consider the theta lift for the groups of similitudes the result of Waldspurger on Howe duality can not be directly applied in our situation. To overcome this problem – for the global applications – we introduce the globally relevant subset

$$\Theta_\pm^{glr}(\sigma_v) \ \subset \ \Theta_\pm(\sigma_v) \ .$$

It consists of all classes of representations π_v in $\Theta_\pm(\sigma_v)$ for which there exists some global irreducible cuspidal automorphic representation $\sigma \not\cong \sigma^*$ (resp. $\check{\sigma} \not\cong \check{\sigma}^*$) for which π_v is the local component of a weak lift π attached to σ. The notion of weak lift will be explained in the next section. The density lemma, stated above, implies

$$\Theta_-^{glr}(\sigma_v) = \Theta_-(\sigma_v) \ .$$

Notice, that we will show later in corollary 1 resp. more generally in §5 that

$$\Theta_+^{glr}(\sigma_v) = \{\theta_+(\sigma_v)\}$$

consists only of the class of the representation $\theta_+(\sigma_v)$.

3.The global situation

General Assumption: Let F be a number field, $\mathbb{A}_F$ its ring of adeles. Since we apply [KRS] (in the argument preceding lemma 2) we have to assume F to be totally real. This restrictions would otherwise not be needed and most likely unnecessary. Let $G = GSp(4)$ and let $M = Gl(2) \times Gl(2)/\mathbb{G}_m^*$ be its proper elliptic endoscopic group. Let $\pi = \otimes' \pi_v$ be an irreducible, unitary cuspidal automorphic representation of the group $G(\mathbb{A}_F)$. Recall, π is a CAP representation (cuspidal associated to parabolic), if it is weakly equivalent to an automorphic representation associated to an Eisenstein series. Two irreducible automorphic representations are called weakly equivalent, if their local components are isomorphic at almost all places. In the following we assume π not to be a CAP representation. Then by the Langlands spectral theory π only contributes to the cuspidal part of the discrete spectrum

$$m_{cusp}(\pi) \;=\; m_{disc}(\pi) \ .$$

Definition: An irreducible, unitary cuspidal automorphic representation π of the group $G(\mathbb{A}_F)$, which is not a CAP representation, is called a weak endoscopic lift (from $M(\mathbb{A}_F)$), if there exist two automorphic representations

$$\sigma_1 \ , \ \sigma_2$$

of $Gl(2,\mathbb{A}_F)$ so that σ_i are either induced from a pair of grossencharacters or is irreducible cuspidal. Suppose the representations σ_i have the same central character $\omega_{\sigma_1} = \omega_{\sigma_2}$, such that

$$L_v(\pi_v, s) \;=\; L_v(\sigma_{1,v}, s) \cdot L_v(\sigma_{2,v}, s)$$

holds for almost all places v of F. We also call π a weak lift of the irreducible automorphic representation $\sigma = (\sigma_1, \sigma_2)$ of $M(\mathbb{A}_F)$ resp. $Gl(2,\mathbb{A}_F) \times Gl(2,\mathbb{A}_F)$ in this situation.

Suppose π is a weak endoscopic cuspidal lift as in the definition above, which is not CAP. Let σ be the corresponding automorphic representation of $Gl(2,\mathbb{A}_F) \times Gl(2,\mathbb{A}_F)$. Then $\sigma = (\sigma_1, \sigma_2)$ has to be a cuspidal representation. Otherwise either σ_1 or σ_2, say σ_1 has L-factors of the form $L_v(\sigma_1, s) = L_v(\lambda, s) L_v(\lambda^{-1}\omega_{\sigma_2}, s)$ for an idele class character λ for $v \notin S$. Put $\tau = \sigma_2 \otimes \lambda^{-1}$. Then $L^S(\pi, s) = L^S(\lambda, s) L(\lambda\omega_\tau, s) L^S(\tau \otimes \lambda, s)$. If $\tau^S \cong \nu^S \times \mu^S$ (locally induced from the unramified characters ν_v, μ_v), then this is the L-series with the local parameters $\lambda_v, \lambda_v\nu_v, \lambda_v\mu_v, \lambda_v\nu_v\mu_v$ or by [S], p.85 the L-series of the induced representation $\nu^S \times \mu^S \triangleleft \lambda^S$ of $GSp(4, \mathbb{A}^S)$ in the short notation of [W] and induced from a Borel subgroup. Since $\nu^S \times \mu^S \triangleleft \lambda^S = \tau^S \triangleleft \lambda^S$ (induction in steps) we get that $L^S(\pi, s)$ would be the partial L-series of the automorphic form $\tau \boxtimes \lambda$ of the Levi component of the

Siegel parabolic subgroup. Hence the weak lift would be associated to an automorphic Eisenstein representation, which is weakly equivalent to π. Hence π is CAP of Saito-Kurokawa type. This implies

$$\pi \text{ not CAP} \Rightarrow \sigma \text{ cuspidal.}$$

Notice, that σ is not uniquely defined by π. If π is a weak lift of $\sigma = (\sigma_1, \sigma_2)$, than π also is a weak lift of $\sigma^* = (\sigma_2, \sigma_1)$. These are the only possibilities.

Proposition 2: Suppose π is an irreducible cuspidal automorphic representation π of the group $G(\mathbb{A}_F)$, which is not a CAP representation. If π is a weak endoscopic lift of the representations σ and $\tilde{\sigma}$ of $M(\mathbb{A}_F)$, then $\tilde{\sigma}$ is either isomorphic to σ or σ^* and σ is cuspidal.

Proof: Suppose $\sigma = (\sigma_1, \sigma_2)$ and $\tilde{\sigma} = (\sigma_3, \sigma_4)$. $\sigma, \tilde{\sigma}$, hence $\sigma_1, \sigma_2, \sigma_3, \sigma_4$, are cuspidal as explained above. Then there exists a finite set S of places of F outside which we have an equality of partial L-series

$$L^S(\sigma_1 \times \rho, s) \cdot L^S(\sigma_2 \times \rho, s) \ = \ L^S(\sigma_3 \times \rho, s) \cdot L^S(\sigma_4 \times \rho, s).$$

Since all σ_i are cuspidal, there are complex numbers s_i for which the $|.|^{s_i} \otimes \sigma_i$ become unitary. By a suitable renaming suppose $Re(s_1) = max_i(Re(s_i))$. For the contragredient ρ of σ_1 the function $L^S(\sigma_1 \times \rho, s)$ has a simple pole at $s = 1$, and $L^S(\sigma_2 \times \rho, s)$ does not vanish at $s = 1$ using statement (2.2) and (2.3) of [AC], p.200. Furthermore, if σ_1 is neither isomorphic to σ_3 nor to σ_4, the right side is holomorphic at $s = 1$. This gives a contradiction. So, possibly by switching σ_3 and σ_4, we may assume $\sigma_1 \cong \sigma_3$. But then $L^S(\sigma_2, s) = L^S(\sigma_4, s)$, and the strong multiplicity one theorem for $Gl(2)$ implies $\sigma_2 \cong \sigma_4$.

Definition: Let π be irreducible (unitary) cuspidal automorphic representation π of the group $G(\mathbb{A}_F)$, which is not a CAP representation. Let π be a weak endoscopic lift. Suppose σ is a corresponding cuspidal automorphic representation of $M(\mathbb{A}_F)$. Then the set of equivalence classes of the irreducible, automorphic representations π' of $G(\mathbb{A}_F)$, which are weakly equivalent to π, is called the global L-packet of π.

Since it consists of all the weak endoscopic lifts attached to the given cuspidal representation σ it will be also called the global L-packet attached to σ. Since π was assumed not to be CAP, all representations in the global L-packet are cuspidal.

Main Theorem: Suppose π is an irreducible, cuspidal automorphic representation of $GSp(4, \mathbb{A}_F)$ and π is not CAP. Suppose the global L-packet attached to the cuspidal irreducible automorphic representation $\sigma = (\sigma_1, \sigma_2)$ of $Gl(2, \mathbb{A}_F)^2/\mathbb{A}_F^*$ is nonempty and contains π as a weak endoscopic lift. Then

1) σ_1 and σ_2 are not isomorphic as representations of $Gl(2,\mathbb{A}_F)$.

2) The restriction of π to $Sp(4,\mathbb{A}_F)$ is obtained as a Weil representation from the orthogonal group attached to a quaternary quadratic form of discrimant 1.

3) All local representations π_v of $\pi = \otimes'\pi_v$ belong to the local L-packets attached to σ_v. Hence $\pi_v \in \{\pi_+(\sigma_v), \pi_-(\sigma_v)\}$, if σ_v belongs to the discrete series, and $\pi_v = \pi_+(\sigma_v)$ otherwise.

4) The multiplicity of any irreducible representation $\pi' = \otimes\pi'_v$ weakly equivalent to π is

$$m(\pi') = \frac{1}{2}(1 + (-1)^{e(\pi')}) \ ,$$

where $e(\pi')$ denotes the (finite) number of representations π'_v, which do not have a local Whittaker model.

5) Let $d(\sigma)$ be the number of local components σ_v of σ in the discrete series. The global L-packet attached to σ contains a single representation if $d(\sigma) < 2$ and contains $2^{d(\sigma)-1}$ representations, each occurring with multiplicity one, otherwise.

<u>In addition</u>: For any cuspidal irreducible automorphic representation $\sigma = (\sigma_1, \sigma_2)$ with $\sigma_1 \not\cong \sigma_2$ the global L-packet attached to σ is nonempty.

<u>Remark</u>: For CAP representations a result analogous to statement 5) of the last theorem is [PS], thm. 2.6.

The proof of the main theorem has several steps: We deal separately with the case, where π_∞ belongs to the discrete series and F is the field of rational numbers. This situation is easier, and we can directly refer to [Ch]. The general case uses the Arthur trace formula and some of the arguments become more complicated. See §5 and §6.

Although the arguments are primarily global, they simultaneously provide us with the necessary local information, e.g. on local Whittaker models. The relevant local information is derived in the subsequent lemmas 3, 4 and 5. By a kind of bootstrap, one then improves to complete the proof for general π_∞ and for general F in §5 and §6.

We first prove statements 1) and 2) of the main theorem.

<u>Lemma 1</u> (Global endoscopic = theta lift): Suppose π is an irreducible cuspidal automorphic representation of $GSp(4,\mathbb{A}_F)$, which is a weak lift in the global L-packet attached

to an irreducible automorphic representation $\sigma = (\sigma_1, \sigma_2)$ of $M(\mathbb{A}_F)$. (A) Suppose π is not CAP. Then

1) σ is cuspidal and

$$\sigma_1 \not\cong \sigma_2 \quad \text{for} \quad \sigma = (\sigma_1, \sigma_2).$$

2) Furthermore all irreducible constituents of the restriction of π to $Sp(4, \mathbb{A}_F)$ are contained in the image of theta lifts of the form $\Theta : \mathcal{A}_{cusp}(H(\mathbb{A}_F)) \to \mathcal{A}(Sp(4, \mathbb{A}_F))$, where H are orthogonal groups of similitudes over F, which are inner forms of $Gl(2,F)^2/F^*$. Hence $H(F) = D^* \times D^*/\{(x, x^{-1}) | x \in F^*\}$, where D^* is either the multiplicative group of a quaternion algebra over F or $Gl(2, F)$.

3) For each local component π_v of π there exists a quadratic character μ_v, such that $\pi_v \otimes \mu_v$ is contained in

$$\Theta_+^{glr}(\sigma_v) \cup \Theta_-^{glr}(\sigma_v) \ .$$

(B) Conversely suppose σ is cuspidal and $\sigma_1 \not\cong \sigma_2$. Then π is not CAP.

Concerning the notation: $\mathcal{A}$ denotes certain packets of irreducible automorphic representations.

Proof: For (A) it is enough to prove the global statements 1) and 2). The local statement 3) follows from this by Frobenius reciprocity [V], prop. 4.12(c) using compact induction.

So let us proof (A)1) and 2). Let $\omega = \omega_\pi$ denote the central character of the irreducible representation π of $GSp(4, \mathbb{A}_F)$. The standard L-function of π corresponds to the four dimensional standard representation st of the L-group ${}^LG$ of G. The alternating square of the standard representation is a six dimensional representation $\bigwedge^2(st)$ of the L-group ${}^LG$. The corresponding L-series of π outside a finite set S of ramified places is a product $\zeta^S(\pi, \bigwedge^2(st), s) = L^S(\omega, s) \cdot \zeta^S(\pi, \omega, s)$ of the Dirichlet L-series $L(\omega, s)$ for the character ω and a twist of the degree five standard L-series $\zeta(\pi, \omega, s)$ of the restriction of π to $Sp(4, \mathbb{A}_F)$. Here, by abuse of notation, we let again π denote any of the irreducible constituents of its restriction to $Sp(4, \mathbb{A}_F)$. Then in fact, the degree five L-series $\zeta(\pi, \omega, s)$ is the standard L-series attached to the representation $\pi \times \omega$ of $Sp(4, \mathbb{A}_F) \times Gl(1, \mathbb{A}_F)$, at least if we consider partial L-series at almost all unramified places. Since π is a weak lift, we get from this

The partial L-series: If we omit a suitable chosen finite set of places S depending on σ and π, then

$$\zeta^S(\pi, \bigwedge^2(st), s) \ = \ L^S(\omega_{\sigma_1}, s) \cdot L^S(\omega_{\sigma_2}, s) \cdot L^S(\sigma_1 \otimes \sigma_2, s) \ ,$$

hence by $\omega_{\sigma_1} = \omega_{\sigma_2}$

$$L^S(\omega_\pi \omega_{\sigma_2}^{-1}, s) \zeta^S(\pi, \omega_\pi \omega_{\sigma_2}^{-1}, s) \ = \ \zeta^S(s)^2 \cdot L^S(\sigma_1 \otimes \sigma_2 \otimes \omega_{\sigma_2}^{-1}, s) \ = \ \zeta^S(s)^2 \cdot L^S(\sigma_1 \otimes \check{\sigma_2}, s) \ .$$

For cuspidal σ_i the L-series $L^S(\sigma_1\otimes\check{\sigma_2},s)$ does not vanish at $s=1$. By a character twist the σ_i become unitary. A twist of the cuspidal representations σ_i becomes unitary iff their central character ω_{σ_i} becomes unitary. Since this is a common character, one can take the same character twist in both cases. So we can assume without restriction of generality that σ_1 and σ_2 are unitary. Notice, since both representations have the same central character they can both assumed to be unitary. If $\omega_\Pi\neq\omega_{\sigma_i}$, this implies that $\zeta^S(\pi,\omega_\pi\omega_{\sigma_2}^{-1},s)$ has a pole of order two at $s=2$. This is impossible as explained below. Therefore

$$\omega_\pi=\omega_{\sigma_2}$$

holds. Then the partial degree five standard L-function $\zeta^S(\pi,s)=\zeta(\pi,1,s)$ of π is given by the formula

$$\zeta^S(\pi,s) \;=\; \zeta^S(s)\cdot L^S(\sigma_1\otimes\sigma_2\otimes\omega_{\sigma_2}^{-1},s) \;=\; \zeta^S(s)\cdot L^S(\sigma_1\otimes\check{\sigma_2},s)\ .$$

Then $L^S(\sigma_1\otimes\check{\sigma_2},1)\neq 0$ holds at $s=1$ for cuspidal unitary representations σ_i. Hence

$$ord_{s=1}\zeta^S(\pi,s)\geq 1\ .$$

For $\sigma_1\cong\sigma_2$ the pole order would be ≥ 2. Hence the first assertion of lemma 1 comes from the next result of Soudry.

<u>Review of a result of Soudry</u>:

$\zeta^S(\pi,\chi,s)$ has at most a simple pole at $s=1$ for unitary cuspidal π.

In fact, this is proved in [S] for a special character $\chi=\chi_T$. Indeed this is the most difficult case. For other characters one can use the same approach. The pole order at $s=1$ is then again estimated by the pole order of an Eisenstein series. Up to a factor $L^S(\chi^2,2s)L^S(\chi,s+1)$ which is irrelevant at $s=1$, the L-series $\zeta^S(\pi,\chi_T\chi,s)$ is up to a function $A(s)$ with $A(1)\neq 0,\infty$ an integral $\int\varphi(g)\theta^{\phi}_{T,\psi}(g,1)E(f_{s,\chi};g)dg$ by [S] (2.2) and (2.5). The poles of $E(f_{s,\chi};g)$ therefore give an upper bound for the poles of $\zeta^S(\pi,\chi\chi_T,s)$. By [S] thm.2.4 the poles of these Eisenstein series are contained in the poles of $L(\chi^2,2s)L(\chi,s+1),L(\chi^2,2s)L(\chi,s),L(\chi^2,2s-1)L(\chi,s),L(\chi^2,2s-1)L(\chi,s-1)$ counted with multiplicity. There there is no pole at $s=1$ for $\chi^2\neq 1$, there is at most a simple pole for $\chi\neq 1$. The difficult case is $\chi=1$. In this case there is no pole by [S] thm.3.1 and thm.4.4.(b). Therefore the claim follows.

<u>Now apply Kudla, Rallis, Soudry [KRS] thm 7.1</u>: Here is now the point, where we have to restrict ourselves to totally real number fields, since this assumption is made [KRS]. Under this assumption the existence of a pole for the partial degree five L-series $\zeta^S(\pi,s)$ at

$s = 1$ shows that π – or more precisely any irreducible constituent of π as a representation of $Sp(4, \mathbb{A}_F)$ – is a constituent for a suitable theta lift

$$\Theta(\mathcal{A}(O(V', \mathbb{A}_F))$$

attached to a quadratic space V' of discriminant 1 over F:

$$\Delta(V') \ = \ \Delta(V_T) \cdot \Delta(V_{T'}) = 1 \ .$$

This follows from $\chi_T \chi_{T'} = 1 = (\ . \ , \Delta(V_T)) \cdot (\ . \ , \Delta(V_{T'}))$ using loc.cit. Theorem 7.1(ii), if $\chi_T \neq 1$, respectively Theorem 7.1(i), if $\chi_T = 1$.

We remark, that the assumption F totally real is very likely unnecessary since we need the case of split dicriminant. In fact, looking at the integral representation of [S] (2.2) it would suffice to show that the residues of the nonholomorphic Siegel Eisenstein series $E(f_s; g)$ at $s = 1$ ([S],2(b)) are binary theta series. A partial result in this direct – namely the local case of this statement – is [S], lemma 3.3. We hope to come back to this question on another occasion.

Structure of V': The quaternary quadratic F-spaces V' of discriminant $\Delta(V') = 1$ are classified by their local Hasse invariants, or in this case alternatively by the structure of their orthogonal groups of similitudes

$$GSO(V')(F) \cong D^* \times D^*/F^* \ .$$

Here D is the corresponding quaternion algebra or $Gl(2, F)$, having invariants determined by the signs of the local Hasse invariants of V'. This part (A) of the lemma and also proves statement 1) and 2) of the main theorem.

For the converse part (B) we may assume σ to have unitary central character. Notice $L^S(\pi, s) = L^S(\sigma_1, s)L^S(\sigma_2, s)$ and $\zeta^S(\pi \otimes \chi, s) = L^S(\chi, s)L^S(\sigma_1 \otimes \sigma_2\check{} \otimes \chi, s)$ outside a suitable finite set S. The known CAP-criteria for π in terms of poles of these L-series (Soudry, Piateskii-Shapiro) exclude that π is CAP for a cuspidal σ, since the known analytic behaviour of L-series of cuspidal forms on $Gl(2), Gl(2) \times Gl(2)$ rules out the existence of poles at $s = 3/2$ for the degree 4 L-series and poles at $s = 2$ for degree 5 L-series (for unitary central characters and unitary χ). This proves part (B).

Lemma 2: We have

$$\bigcup_{\sigma_v} \Theta_+(\sigma_v) \ \cap \ \bigcup_{\sigma_v} \Theta_-(\sigma_v) = \emptyset \ ,$$

where the union is over all generic irreducible admissible representation σ_v of M_v on the left, and over all generic irreducible admissible discrete series representation σ_v of M_v on the right. In other words, the two theta lifts attached to M_v (split case) and

$\check{M}_v$ (anisotropic case) have no irreducible representation of $GSp(4, F_v)$ in common. The same statement also holds globally or after restriction to the subgroup $Sp(4, F_v)$.

Proof: It is enough to consider the restrictions to $Sp(4, F_v)$. Here we refer to Howe and Piateskii-Shapiro [H-PS] thm. 9.4, where it is shown that there is at most one representation in common. In [Ch] §9 it was shown, that the common point corresponds to a one dimensional representation σ_v. However these are not generic and are excluded in the statement above.

Lemma 3 (Principle of exchange): Suppose a cuspidal irreducible automorphic representation $\pi = \otimes \pi_v$ of $GSp(4, \mathbb{A}_F)$ is a weak endoscopic lift, but not CAP. Let σ denote the corresponding cuspidal representation of $M(\mathbb{A}_F)$. 1) Fix some place v_0. Then π_{v_0} is in $\Theta^{glr}_{\varepsilon}(\sigma_{v_0} \otimes \mu_{v_0})$ for some quadratic character μ_v and some $\varepsilon \in \{\pm\}$. Suppose $\pi'_{v_0} \ncong \pi_{v_0}$ is an irreducible representation of $GSp(4, F_{v_0})$ and consider

$$\pi' \ = \ \pi'_{v_0} \ \otimes \ \bigotimes_{v \neq v_0} \pi_v \ .$$

If $\pi'_{v_0} \notin \Theta^{glr}_{\varepsilon}(\sigma_{v_0} \otimes \mu'_{v_0})$ for all quadratic characters μ'_v, then π' has global multiplicity

$$m_{disc}(\pi') = 0 \ .$$

2) If $d(\sigma) < 2$ (number of local discrete series representations), then $\pi_v = \theta_+(\sigma_v \otimes \mu_v)$ for all places v of F with certain quadratic characters μ_v. In particular π_v is generic.

Proof: Suppose to the contrary $m(\pi') \geq 1$. Then π' is in the global L-packet of π, hence it is a weak endoscopic lift attached to the same irreducible representation σ of $M(\mathbb{A})$ as π. According to lemma 1 all constituents of π' and π, after restriction to the group $Sp(4, \mathbb{A}_F)$, are in the image of some theta lifts Θ. These theta correspondences lift automorphic forms from orthogonal groups arising from global simple algebras of rank 4 to automorphic forms on $GSp(4)$. In our case let D and D' be the corresponding algebras respectively. From the known local properties of the theta lift it follows, that

$$D_v \cong D'_v \qquad (\forall v \neq v_0)$$

since from Lemma 2 we would otherwise get a contradiction to the fact

$$\pi_v \cong \pi'_v \qquad (\forall v \neq v_0) \ .$$

But then $D_{v_0} \cong D'_{v_0}$ follows from the theorem of Hasse-Brauer-Noether. $D_{v_0} \cong D'_{v_0}$ implies, that the constituents of both π_v, π'_v (after restriction to $Sp(4, F_v)$) either both do or both do not belong to the local theta lift $\Theta^{glr}_{\varepsilon}(\sigma_v)$. Since this contradicts the assumptions, the proof of the first part of lemma 3 is complete.

The second part is shown similarly. The assumptions imply, that D globally splits. Then [HPS] thm. 5.7b) or thm. 8.1b) implies that the restriction of π_v to $Sp(4,F_v)$ is generic. By compact induction also $\pi_v\otimes\mu_v$, hence π_v is generic. This implies $\pi_v\cong\theta_+(\sigma_v\otimes\mu_v)=\theta_+(\sigma_v)\otimes\mu_v$ for some quadratic character. This proves part 2) of the lemma.

The principle of exchange can be applied for an archimedian place in the following situation: Assume $F_{v_0}=R$ and let π_{v_o} and π'_{v_0} be representations in the local archimedian L-packet $\{\pi^W_\infty,\pi^W_\infty\}$ attached to a discrete series representation σ_∞ of M_∞. For the notation see [CH], except that in loc.cit the discrete series representations σ_∞ of $M(\mathbb{R})$ were parameterized by certain weights. In the situation above the principle of exchange can be applied by the next lemma 4. In fact, we have the following well known result

<u>Lemma 4</u>: Let $F_v=\mathbb{R}$ and $\pi'_v=\pi^W_\infty$ and $\pi_v=\pi^H_\infty$, then the constituents of π'_v restricted to $Sp(4,\mathbb{R})$ have Whittaker models, whereas the constituents of π_v restricted to $Sp(4,\mathbb{R})$ do not have a Whittaker model. Furthermore $\pi^W\otimes\mu_v\cong\pi^W,\pi^H\otimes\mu_v\cong\pi^H$ for all quadratic characters μ_∞, and also $\pi^W\in\Theta_+(\sigma_\infty\otimes\mu_\infty)$ and $\pi^H\in\Theta_-(\sigma_\infty\otimes\mu_\infty)$ for all quadratic characters μ_∞.

4.The Multiplicity formula

The main results of this section are the following: the local-global principle (lemma 5) and the key formula. They imply the corollaries 1-5 of this section. These corollaries include all statements of the main theorem of §3. Lemma 5 and the key formula are proven in this section only for $F = \mathbb{Q}$ and in cases, where at least one representation π is in the global L-packet for which the archimedean component π_∞ belongs to the discrete series. In this special case the proof follows from the weak multiplicity formula stated below. The proof of the local-global principle and the key formula in the general case is postponed to §5 and §6. However, for the proof of the implications of lemma 5 and the key formula proven no assumptions on F and π_∞ are made.

Let π be a cuspidal irreducible weak endocopic lift attached to some irreducible cuspidal representation σ of $M(\mathbb{A}_F)$, and assume π is not CAP. Then the representation σ must is cuspidal and $\sigma \not\cong \sigma^*$. For irreducible automorphic representations of the group $Gl(2, \mathbb{A}_F)$ cuspidal implies generic. So the same is true for the group $M(\mathbb{A}_F)$.

Put $\pi = \pi_\infty \otimes \pi_f$ where π_∞ denotes the tensor product over all archimedean components. If we make the assumptions, that π_v is in the discrete series for an archimedean place v, then $F_v = \mathbb{R}$ and π_v is contained in some local archimedean L-packet $\{\pi^W, \pi^H\}$ attached to a discrete series representation of M_v. If we furthermore make the assumption $F = \mathbb{Q}$, then the topological trace formula studied in [W] and [Ch] gives the following

<u>The Weak Multiplicity Formula</u>: ([Ch], §3 corollary 1) For $F = \mathbb{Q}$ and π_f as above

$$m(\pi_\infty^W \otimes \pi_f) - m(\pi_\infty^H \otimes \pi_f) \ = \ n(\pi_f) \ ,$$

where $n(\pi_f)$ is zero if there exists a nonarchimedean place v, for which π_v is not contained in the local L-packet of σ_v, and where $n(\pi_f)$ is equal to $(-1)^{e(\pi_f)}$ otherwise. Here $e(\pi_f)$ denotes the number of nonarchimedean places v, for which $\pi_v \cong \pi_-(\sigma_v)$ holds. Since $e(\pi_f) < d(\sigma)$, this number is finite.

<u>Comments</u>: The multiplicity $m(\pi)$ is the multiplicity of π in the discrete spectrum of $G(\mathbb{A}_\mathbb{Q})$ or equivalently the multiplicity in the cuspidal spectrum, since π is not CAP. We remark, that the multiplicity formula above is a reformulation of [Ch] cor. 1. In fact we used [Ch] cor.5 to identify the multiplicities $m_1(\pi_f), m_2(\pi_f)$ used in [Ch], cor.1 in terms of the representation π_∞. We also used, that the normalizing constant $\varepsilon = \pm 1$ from [Ch], cor.1 is always $\varepsilon = -1$. See [Ch] §3. Also recall, that in [Ch], cor 1. the value $n(\pi_f) = \prod_{v \neq \infty} n(\pi_v)$ is defined in terms of the multiplicities $n(\pi_v)$ of π_v in the local character expansion $r(\sigma(\sigma_v)) = \sum n(\pi_v) \cdot \pi_v$ for the local endoscopic lift $r : R_\mathbb{Z}(M_v) \to R_\mathbb{Z}(G_v)$. The expression for $n(\pi_f)$ in the form stated above follows from the structure of

the local L-packets of the local endoscopic lift $r(\sigma_v)$ of a generic representation σ_v of M_v. This was described in §2 in the nonarchimedian case and in lemma 4 in the archimedian case. The L-packets consist of two representations $\pi_+(\sigma_v), \pi_-(\sigma_v)$, if σ_v belongs to the discrete series, and of one representation $\pi_+(\sigma_v)$, if σ_v does not belong to the discrete series. Among these only the representations $\pi_-(\sigma_v)$ do not have Whittaker models. Hence $n(\pi_v) = 0$ unless π_v is in the local L-packet of σ_v. Furthermore $n(\pi_v)$ is $+1$ or -1 otherwise, and the sign depends on the existence of a Whittaker model.

The next lemma proves statement 3) of the main theorem.

Lemma 5 (Local-Global Principle): Let π be a weak endoscopic lift contained in the global L-packet attached to some generic, irreducible cuspidal automorphic representation σ of $M(\mathbb{A})$. Suppose π is cuspidal but not CAP. Then the local components π_v of π belong to the local L-packets $\{\pi_\pm(\sigma_v)\}$ of σ_v, for all places v of F.

Proof of lemma 5 (For $F = \mathbb{Q}$ and π_∞ in the discrete series. The general case is shown in $\S 5, 6$). If π_v is not in the local L-packet of σ_v, then v is nonarchimedian and the weak multiplicity formula implies $m(\pi_\infty^W \otimes \pi_f) = m(\pi_\infty^H \otimes \pi_f)$. But then $m(\pi) > 0$ and $m(\pi') > 0$ for $\pi = \pi_\infty^W \otimes \pi_f$ and $\pi' = \pi_\infty^H \otimes \pi_f$. But this contradicts the principle of exchange. By lemma 4 the assumptions of lemma 3) are satisfied for the place $v_0 = \infty$. Hence lemma 5 follows.

Now we drop again the assumptions $F = \mathbb{Q}$ and on π_∞, but instead we temporarily assume lemma 5 holds. From lemma 5 we deduce the next two corollaries.

In [Ch] §11 we considered the global $GSp(4), GO(V)$ theta correspondence. For σ_v in the discrete series we constructed a global irreducible automorphic cuspidal representation σ of $M(\mathbb{A}_F)$ and a nonvanishing global theta lift π of σ, such that $\pi_v \in \Theta_-(\sigma_v)$ and does not have local Whittaker models. Hence $\pi_v \not\cong \pi_+(\sigma_v)$. Hence $\pi_v = \pi_-(\sigma_v)$ by lemma 5. In particular $\pi_-(\sigma_v)$ is in $\Theta_-(\sigma_v)$ and does not have local Whittaker models. We also know from [Ch] thm.3 that $\pi_+(\sigma_v) = \theta_+(\sigma_v)$ is generic. Therefore as a consequence of lemma 5 we obtain

Corollary 1 (Local endoscopic lift = Theta lift): Suppose that lemma 5 holds and that its assumptions are satisfied. Then the representations $\pi_\pm(\sigma_v)$ of the local L-packet of a discrete series representation σ_v of M_v are two Weil representations $\theta_\pm(\sigma_v)$ of the group $GSp(4, F_v)$

a) $\Theta_+^{glr}(\sigma_v) \;=\; \{\pi_+(\sigma_v)\} \;=\; \{\theta_+(\sigma_v)\}$ for split D

b) $\Theta_-(\sigma_v) \;=\; \{\pi_-(\sigma_v)\} \;= \{\theta_-(\sigma_v)\}$ for non split D .

In particular, $\pi_+(\sigma_v)$ has a local Whittaker model, whereas $\pi_-(\sigma_v)$ does not have local Whittaker models.

Notice, that lemma 5 and its corollary 1 imply proposition 1.

Corollary 2 (Whittaker models): Suppose that lemma 5 holds and that its assumptions are satisfied. Let v be a place where σ_v belongs to the discrete series. Then $\pi_+(\sigma_v) \in \Theta_+(\sigma_v)$ has local Whittaker models, whereas $\pi_-(\sigma_v) \otimes \mu_v \notin \Theta_+(\sigma)$ for all (quadratic) characters μ_v, and these representations are cuspidal and do not have local Whittaker models.

Proof: Clear.

Corollary 3 (Refined principle of exchange): Suppose that lemma 5 holds and that its assumptions are satisfied. For the weak automorphic lift $\pi = \otimes_v \pi_v$ of σ put $\pi' = \pi'_v \otimes_{v \neq v_0} \pi_v$ for an irreducible admissible representation π'_v of G_v. Then

$$m(\pi') = 0$$

unless $\pi'_v \cong \pi_v$.

Proof: Suppose $m(\pi') > 0$ and $\pi'_v \ncong \pi_v$. Then both π and π' are weak automorphic endoscopic lifts of σ. Hence $\pi_v, \pi'_v \in \{\pi_+(\sigma_v), \pi_-(\sigma_v)\}$ by lemma 5. Since they are not isomorphic, we can apply the principle of exchange (lemma 3). By corollary 1 and 2 all the assumptions of lemma 3 are satisfied. Therefore $m(\pi') = 0$ follows from lemma 3. This completes the proof of corollary 3.

The proof of the remaining assertions 4) and 5) of the main theorem.

Suppose π is an automorphic represention of $G(\mathbb{A}_F)$ as in the main theorem. π is irreducible, cuspidal automorphic but not CAP, and π is a weak lift attached to some irreducible automorphic representation σ of $M(\mathbb{A}_F)$. Then, as already shown, σ is cuspidal and its local components σ_v are generic.

Conversely for an irreducible cuspidal automorphic representation $\sigma = (\sigma_1, \sigma_2)$ of $M(\mathbb{A}_F)$ consider the global L-packet of all weak endoscopic lifts attached to it. Since we are mainly interested in cuspidal automorphic representations, assume

$$\sigma_1 \ncong \sigma_2$$

because otherwise the global L-packet attached to σ does not contain a cuspidal representation (lemma 1). See also corollary 5 below.

The base point of the L-packet: By a result of Soudry this global L-packet is nonempty and contains the representation $\pi_+(\sigma) = \prod_v \pi_+(\sigma_v)$ as a cuspidal automorphic representation with multiplicity $m(\pi_+(\sigma)) = 1$.

Sketch of proof: See [Ch] §9. In fact the situation in loc. cit is more restricted. So for the convenience of the reader we sketch the argument. $\pi_+(\sigma)$ is the theta lift $\theta_D(\sigma)$ for the split case $D^* = Gl(2)$. By the assumptions on σ this theta lift does not vanish, and it is cuspidal by $\sigma_1 \not\cong \sigma_2$. Hence $m(\pi_+(\sigma) > 0$. Then by a result of Soudry

$$m(\pi_+(\sigma)) = 1 \ ,$$

since every cuspidal representation which is isomorphic to $\pi_+(\sigma)$ has a global Whittaker model. This follows again from a result of Soudry: Namely by construction $\pi_+(\sigma)$ is locally generic at all places v, it is enough to show the nonvanishing of $\zeta^S(\pi_+(\sigma), s)$ at $Re(s) = 1$ (if ω is normalized to be unitary). But this nonvanishing follows immediately from the computations made in the proof of lemma 1, since under our assumptions the L-series $L^S(\sigma_1 \times \hat{\sigma}_2, s)$ does not vanish on the line $Re(s) = 1$. This completes the proof.

From corollary 1 we know, that for every representation π in the global L-packet there exists a finite set of places S of F where π_v does not have a Whittaker model and such that

$$\pi \ = \ \bigotimes_{v\in S} \pi_-(\sigma_v) \ \otimes \ \bigotimes_{v\notin S} \pi_+(\sigma_v) \ .$$

Let $e(\pi) = \#S$ be cardinality of this uniquely defined set

$$S = S(\pi)$$

of places. Notice, that S is a subset of the finite set of places where σ_v belongs to the discrete series.

Corollary 4: Suppose lemma 5 and its assumptions hold. Also suppose the key formula (stated below) holds. Then for any π in the global L-packet of a cuspidal irreducible representation σ with $\sigma \not\cong \sigma^*$ the multiplicity $m_{cusp}(\pi) = m_{disc}(\pi)$ is

$$m(\pi) \ = \ \frac{1}{2}(1 + (-1)^{e(\pi)}) \quad , \quad e(\pi) = \#S(\pi) \ .$$

In other words: $m(\pi) = 0$ for $e(\pi)$ even and $m(\pi) = 0$ otherwise.

Proof: By lemma 3b) this is true for $e(\pi) < 2$. So we may assume $e(\pi) \geq 2$. Pick two places $v_1 \neq v_2$, which contribute to $e(\pi)$.

If some archimedian π_v belongs to the discrete series we always assume v_1 to be archimedian. More generally, let S be the set of places v where π_v is in the discrete series. Then we may assume v_1 to be some fixed place of S.

Now consider the representation π' of $G(\mathbb{A}_F)$, which is obtained from π by replacing the two local representations π_{v_1} and π_{v_2} within their local L-packets. Since π_v is in the discrete series for $v = v_1, v_2$ the local L-packets are both of cardinality two at v_1 and at v_2. With these notation we formulate the

Key Formula: Let π be a weak endoscopic lift of a cuspidal automorphic representation σ with $\sigma^* \not\cong \sigma$. Suppose π is not a CAP representation. Also suppose $e(\pi) \geq 2$. Let π' be defined as above by replacements at two places v_1, v_2, where π belongs to the discrete series. Then $m(\pi) + m(\pi') > 0$ implies, that $e(\pi)$ is even and that furthermore

$$m(\pi) + m(\pi') \;=\; 2 \;.$$

Continuation of the proof: It is clear that with this key formula we can proof corollary 4 by induction on $e(\pi)$, by reduction to the known cases $e(\pi) = 0, 1$. This is an immediate consequence of the fact, that every set S' of even cardinality of the set of places, where σ_v belongs to the discrete series, can be obtained by a finite number of exchanges at two places of this set. For this it is of no harm to assume that in addition one of the two places, say v_1 (for $F = \mathbb{Q}$ the archimedian place if π_∞ belongs to the discrete series), always is fixed in the set of places where σ_v is in the discrete series with only the place $v_2 \neq v_1$ varying. Using this successively one shows by induction $m(\pi') = 1$ or $m(\pi') = 0$ depending on whether $e(\pi')$ is even or odd. This completes the proof of corollary 4.

Proof of the key formula: (for $F = \mathbb{Q}$ and π_∞ in the discrete series. The general case is done in $\S 5$).

Put $\pi = \pi_{\varepsilon_2}(\sigma_{v_2}) \otimes \pi_{\varepsilon_1}(\sigma_{v_1}) \otimes \pi^{v_1,v_2}$. Then by the weak multiplicity formula, applied twice

$$\Big(m(\pi_{\varepsilon_2}(\sigma_{v_2}) \otimes \pi_{\varepsilon_1}(\sigma_{v_1}) \otimes \pi^{v_1,v_2}) - m(\pi_{\varepsilon_2}(\sigma_{v_2}) \otimes \pi_{-\varepsilon_1}(\sigma_{v_1}) \otimes \pi^{v_1,v_2}) \Big)$$

$$- \Big(m(\pi_{-\varepsilon_2}(\sigma_{v_2}) \otimes \pi_{\varepsilon_1}(\sigma_{v_1}) \otimes \pi^{v_1,v_2}) - m(\pi_{-\varepsilon_2}(\sigma_{v_2}) \otimes \pi_{-\varepsilon_1}(\sigma_{v_1}) \otimes \pi^{v_1,v_2}) \Big)$$

$$= \; (-1)^{e(\pi)} - (-1)^{e(\pi) \pm 1} \;=\; 2 \cdot (-1)^{e(\pi)} = \; 2 \cdot (-1)^{e(\pi)} \;.$$

By the principle of exchange two of these multiplicities must always vanish. If $m(\pi_{v_1 v_2}) > 0$ or $m(\pi) > 0$, then $m(\pi_{-\varepsilon_2}(\sigma_{v_2}) \otimes \pi_{\varepsilon_1}(\sigma_{v_1}) \otimes \pi^{v_1,v_2}) = m(\pi_{\varepsilon_2}(\sigma_{v_2}) \otimes \pi_{-\varepsilon_1}(\sigma_{v_1}) \otimes \pi^{v_1,v_2}) = 0$ vanish. Hence

$$m\Big(\pi_{\varepsilon_2}(\sigma_{v_2}) \otimes \pi_{\varepsilon_1}(\sigma_{v_1}) \otimes \pi^{v_1,v_2}\Big) + m\Big(\pi_{-\varepsilon_2}(\sigma_{v_2}) \otimes \pi_{-\varepsilon_1}(\sigma_{v_1}) \otimes \pi^{v_1,v_2}\Big) \;=\; 2 \cdot (-1)^{e(\pi)} \;.$$

The left side is nonnegative by assumption, so $e(\pi)$ is even and $m(\pi) + m(\pi_{\infty,v}) = 2$. so the statement of the key formula follows.

This being said we also have completed the proof of corollary 4 (in the case where $F = \mathbb{Q}$ and π_∞ belongs to the discrete series). So the main theorem is now proven in this (special) case.

Corollary 5: Suppose that σ is a (generic) irreducible cuspidal representation of $M(\mathbb{A}_F) = Gl(2,\mathbb{A}_F)\times Gl(2,\mathbb{A}_F)/A_F^*$. Suppose $\sigma = JL(\check{\sigma})$ is the Jaquet-Langlands lift of an irreducible representation $\check{\sigma}$ of some inner form $D^*(\mathbb{A})\times D^*(\mathbb{A})/\mathbb{A}^*$ of $M(\mathbb{A})$, where D is a quaternion algebra. Then the corresponding theta lift $\Theta_D(\check{\sigma})$ does not vanish.

Remark: The corresponding statement for $D^* = Gl(2)$ is easier and follows from the existence of Whittaker models. See [HPS] or [Ch]. In fact this fact was already used in the discussion preceding corollary 4.

Proof (of corollary 5): Suppose $\sigma_1 \cong \sigma_2$. Consider the theta lift $\Theta_D(\sigma)$ and its zero Fourier coefficient with respect to the maximal parabolic Q, which is not the Siegel parabolic. In the classical theory of Siegel modular forms this corresponds to consider the Siegel ϕ-operator. As representation of the $Gl(2,\mathbb{A}_F)$-factor of the Levi component this Fourier coefficient essentially defines Jaquet-Langlands lift attached to the automorphic representation $\check{\sigma}_1 \cong \check{\sigma}_2$ of $D^*(\mathbb{A}_F)$ (for details we refer to [HPS] and [Ch] $\S 11$). Since the Jaquet-Langlands lift is always non nontrivial, this implies that the theta lift $\Theta_D(\sigma)$ is not trivial. In particular it is not cuspidal under the assumption $\sigma_1 \cong \sigma_2$.

To prove the first assertion we can now assume $\sigma_1 \not\cong \sigma_2$. We will then show, that the theta lift $\Theta_D(\sigma)$ does not vanish. Let S be the set of non split places of the quaternion algebra D. As the cardinality of S is even, the multiplicity of the representation $\pi = \prod_{v\in S}\pi_-(\sigma_v)\otimes\prod_{v\notin S}\pi_+(\sigma_v)$ is one by corollary 4. Hence in particular it is not zero, hence π is a cuspidal automorphic representation. As π is a weak endoscopic lift by definition, proposition 1 implies that π is a theta lift. The set S of places, where π_v does not have a Whittaker model, is the set of non split places of the corresponding algebra by corollary 1. Hence this algebra is D. But then by proposition 2 π is a theta lift of type $\Theta_D(\sigma)$ or $\Theta_D(\sigma^*)$. Since $\Theta_D(\sigma) = \Theta_D(\sigma^*)$, recall the theta lift originally is defined by passing from $M = GSO(4)$ to $GO(4)$ (see the remarks at the end of $\S 1$), the claim follows. This proves corollary 5.

5.Local and Global Trace Identities

We say a locally constant functions f_v on G_v with compact support on G_v satisfies (RS), if its support is contained in the locus of regular points. A semisimple element of G_v is called elliptic, if it is an element of an elliptic Cartan subgroup. The set of regular elliptic point is open in G_v. For locally constant functions f_v on G_v with compact support we say <u>condition</u> (ES) holds, if the support is contained in the regular elliptic locus. Condition (ES) implies, that all the orbital integrals $O_{\gamma_v}^{G_v}(f_v)$ of f_v vanish for G_v-regular nonelliptic elements $\gamma_v \in G_v = GSp(4, F_v)$

$$O_{\gamma_v}^{G_v}(f_v) = 0 \quad \text{for } \gamma_v \text{ regular and not elliptic .}$$

Hence $f_v \in A(G_v)$ in the sense of [KA]. And $f_v \in A(G_v)$ implies, that f_v is <u>cuspidal</u> in the sense of [A], p.538. This, by definition, means $(f_v)_{M_v} = 0$ for all Levi subgroups $M_v \neq G_v$ of G_v, or equivalently by the adjunction formula for induced representations

$$tr\ \pi_v(f_v) = 0 \quad , \quad \text{for all } \pi_v = Ind_{P_v}^{G_v}(\sigma_v) \ .$$

Here π_v runs over all induced representations of tempered representations σ_v of M_v for proper parabolic subgroups $P_v = M_v N_v$. See [A2], p.328.

We say <u>condition</u> $(*)_v$ holds for f_v, if the stable orbital $SO_{\gamma_v}^{G_v}(f_v)$ of f_v vanishes for all regular semisimple $\gamma_v \in G_v$

$$(*)_v \qquad SO_{\gamma_v}^{G_v}(f_v) = 0 \quad , \quad (\text{for all } \gamma_v \text{ regular semisimple}) \ .$$

Since unstable tori of the group $G_v = GSp(4, F_v)$ are elliptic, the orbital integrals and stable orbital coincide for regular points outside the elliptic locus. Hence condition $(*)_v$ implies $O_{\gamma_v}^{G_v}(f_v) = 0$ for all regular non-elliptic γ_v.

Let $\tilde{G}_v \subset G_v$ denote the subgroup of elements, whose value under the similitude character are squares in $(F_v)^*$. Then $tr\ (\pi_v \otimes \mu_v)(f_v) = tr\ \pi_v(f_v)$ for all quadratic characters μ_v, if $supp(f_v) \subset \tilde{G}_v$. Note $\tilde{G}_v = Z_v \cdot Sp(4, F_v)$, and $\tilde{G}_v$ is open in G_v. Every element in G_v, which is stably conjugate to an element of $\tilde{G}_v$, is contained in $\tilde{G}_v$.

<u>Remark</u>: We later use the following auxiliary result, which is related to proposition 3 below. For $F_v = \mathbb{R}$ there exists some σ_v in the discrete series and some K-finite infinitely differentiable function f_v with compact support satisfying $(*)_v$ and $tr\ \pi_-(\sigma_v)(f_v) > 0$. This is constructed by smooth truncation of a function $f_{\pi_+(\sigma_v)} - f_{\pi_+(\sigma_v)}$, where $f_{\pi_\pm(\sigma_v)}$ are pseudo coefficients. We leave this as an exercise. See also [Sh](4.7.1).

Lemma 6 (Stability): For σ_v in the discrete series of M_v the character T of $\pi_+(\sigma_v)\oplus\pi_-(\sigma_v)$ is a stable distribution. In other words for any locally constant function f_v on G_v condition $(*)_v$ implies $T(f_v)=0$.

Proof of the stability lemma 6: In the archimedean case this follows from [Sh2], lemma 5.2. In the nonarchimedian case one can argue as in [R], prop. 12.5.3 and cor. 12.5. to show, that the stability of a distribution T is equivalent to the statement $< T, r(\rho_v) >_e = 0$ for all discrete series representations ρ_v of M_v. Since for $T = tr\ \pi_+(\sigma_v) + tr\ \pi_-(\sigma_v)$ the latter means $< \pi_+(\sigma_v)+\pi_-(\sigma_v), \pi_+(\rho_v)-\pi_-(\rho_v) >_e = 0$, this follows from the scalar product formulas proved in [Ch]. There are two cases to be distinguished. Either $\sigma_v \not\cong \sigma_v^*$. Then we can apply the orthogonality relations, since both $\pi_\pm(\sigma_v)$ belong to the discrete series, and since furthermore $\pi_\varepsilon(\sigma_v) \cong \pi_{\varepsilon'}(\rho_v)$ implies $\sigma_v \cong \rho_v$ and $\varepsilon = \varepsilon'$. Or $\sigma_v \cong \sigma_v^*$. Then $< \pi_\varepsilon(\sigma_v), \pi_{\varepsilon'}(\rho_v) >_e = 1$ for $\sigma_v \cong \rho_v$ and is zero else as shown in [Ch]. So this gives the proof.

Remark: The same argument proves stability for the traces $T = tr\ \pi_v$ of discrete series representations π_v of G_v, for which π_v is not isomorphic to one of the representations $\pi_\pm(\sigma_v)$, σ_v is in the discrete series.

Lemma 6 implies the existence of auxiliary functions $f_v \in A(G_v)$ as follows

Proposition 3 (Instability): For σ_v in the discrete series of M_v, there exists a locally constant function f_v with condition (ES) and support in $\tilde{G}_v$, such that $(*)_v$ holds and such that $tr\ \pi_-(\sigma_v) > 0$.

Proof of Proposition 3: We can assume v to be nonarchimedian. Recall $f_v \in A(G_v)$, and this implies $T(f_v) = 0$ for $T = tr\ \pi_+(\sigma_v) + tr\ \pi_-(\sigma_v)$ by lemma 6. Therefore $tr\ \pi_-(\sigma_v)(f_v) > 0$ and $tr\ r(\sigma_v)(f_v) < 0$ are equivalent statements. Hence it is enough to show $tr\ \sigma_v(f_v^{M_v}) < 0$, for a matching function $f_v^{M_v}$. The character of σ_v does not vanish in any neighborhood of the identity element for at least for one elliptic torus T_v in M_v. Such a T_v defines two conjugacy classes of tori in G_v, which are stable conjugate, together with admissible isomorphisms between T_v and these tori. Since regular points are smooth points of the conjugation map, one can easily construct matching functions f_v and $f_v^{M_v}$ with support in these tori sufficiently near to the identity such that $(*)_v$ holds for f_v, but such that $tr\ \sigma_v(f_v^{M_v}) < 0$. This is done by the implicit function theorem. Simply consider bump functions f_v with small support near $\gamma_v \in T_v$ such that $SO_{\gamma_v}^{G_v}(f_v) = 0$ and such that $O_{\gamma_v}^{\kappa}(f_v) = SO_{\gamma_v}^{M_v}(f_v^{M_v}) \neq 0$ for a corresponding bump function with sufficiently small support near γ_v. Then the support of f_v is in $\tilde{G}_v$, and f_v satisfies (ES). Furthermore $tr\ r(f_v) < 0$ and $f_v \in A(G_v)$. This proves the claim.

We now prove the following results

Proposition 4: For irreducible σ_v in the discrete series $\Theta_-(\sigma_v) = \{\pi_-(\sigma_v)\}$.

Proposition 5: Suppose σ_v is irreducible and generic. Then $\pi_v \in \Theta_+^{glr}(\sigma_v)$ implies $\pi_v \cong \theta_+(\sigma_v)$.

In some cases these statements are known already, for instance in the archimedian case. For p-adic fields (completions of $\mathbb{Q}$) proposition 3 follows from the density lemma ([Ch], formulated in §2) and lemma 5 in §4. Still another case: If σ_v is cuspidal nonarchimedian such that $\sigma_v \cong \sigma_v^*$, then $\Theta_+(\sigma_v) = \pi_+(\sigma_v)$ ([Ch],[Wa] p.64, l.10-15 and p.55) is slightly stronger than proposition 5 above. The distribution T of lemma 6 is stable in this case for trivial reason. Namely $\pi_+(\sigma_v)$ and $\pi_-(\sigma_v)$ are the two constituents of an induced representation called $1 \times \sigma_v$ in [Ch], §6 p.31 and p.35. Since the orbital integral of any $f_v \in A(G_v)$ has elliptic support this implies $T(f_v) = 0$, since the trace of the induced representation $1 \times \sigma_v$ vanishes on the regular elliptic locus. In particular we obtain

Auxiliary functions: For nonarchimedian p-adic local fields F_v and given irreducible cuspidal representation σ_v with $\sigma_v \cong \sigma_v^*$, there exist functions $f_v \in A(G_v)$ with support in the regular elliptic point of $\tilde{G}_v$, such that $tr\ \pi_v(f_v) > 0$ holds for all $\pi_v \in \Theta_-(\sigma_v) \otimes \mu_v$, μ_v quadratic.

Proof of Proposition 4: The archimedian case is well known, so assume v to be nonarchimedian.

Suppose for some σ_v in the discrete series of M_v the assertion of prop. 4 were false. Then there is a $\pi_v \in \Theta_-(\sigma_v)$ not isomorphic to $\pi_-(\sigma_v)$. Choose a global field F for the given local field F_v, such that the central character of σ_v is induced from a grossencharacter ω of F. For reasons to become clear soon, we better write v'' for the place v from now on.

We choose additional auxiliary nonarchimedian places v, v', for which $F/\mathbb{Q}$ is split. Choose $\sigma_v, \sigma_{v'}$ at these auxiliary places in the discrete series such that $\sigma_v \cong \sigma_v^*$ and $\sigma_{v'} \cong \sigma_{v'}^*$ with central characters ω_v resp. $\omega_{v'}$. Then we fix auxiliary functions $f_v \in A(G_v)$ and $f_{v'} \in A(G_{v'})$ with (ES) and support in $\tilde{G}_v$ resp. $\tilde{G}_{v'}$ as constructed above (proposition 3).

By the density lemma formulated in §2 and the appropriate choice of central characters, we find a global cuspidal automorphic representation σ of $M(\mathbb{A}_F)$ such that $\sigma \not\cong \sigma^*$ realizes the given discrete series representations $\sigma_v, \sigma_{v'}$ and $\sigma_{v''}$ at the places v, v' and v''. Consider the set C of classes of irreducible cuspidal automorphic representation π' in in the global L-packet of σ, which specialize to the given representations $\pi_-(\sigma_v), \pi_-(\sigma_{v'})$ and $\pi_{v''}$ at the places v, v', v''. Such π' is never CAP, and by the density lemma we can assume C to be nonempty.

We now apply the lemma 7 of the appendix for the auxiliary functions $f_v, f_{v'}$ chosen above. Hence the condition (ii),(iv),(v) and (vi) of lemma 7 hold for $w = v$. For our choice of π also (i) and (iii) are satisfied. Furthermore $f_{v''}$ may be arbitrary. Then all assumption for lemma 7 are satisfied. It gives - since $f_{v''}$ is now arbitrary -

$$\sum_{\pi',(\pi')^{v,v'}=\pi^{v,v'}} m(\pi')\cdot tr\ \pi'_v(f_v)\cdot tr\ \pi'_{v'}(f_{v'})\ =\ 0\ .$$

The right side vanishes, since $\pi_{v''}\not\cong\pi_\pm(\sigma_{v''})$ at the local place v'', where we originally started from. Furthermore, as we know, the summation is over theta lifts. Hence every π'_v is in $\Theta_\pm(\sigma_v)$ up to a quadratic character twist (lemma 1, part 1) and similar for $\pi'_{v'}$.

Since the places v,v' are split in F, they are p-adic. Hence $\Theta_-(\sigma_v)=\{\pi_-(\sigma_v)\}$ by §4, and similarly $\Theta_-(\sigma_{v'})=\{\pi_-(\sigma_{v'})\}$. By our choice of $f_v,f_{v'}$ (prop 4) therefore

$$tr\ \pi'_v(f_v)\cdot tr\ \pi'_{v'}(f_{v'})>0$$

holds for all relevant π' in the sum, unless either $\pi_v\in\Theta_+(\sigma_v)$ or $\pi_{v'}\in\Theta_+(\sigma_{v'})$ holds up to quadratic character twists. But then the principle of exchange at v resp v' (lemma 3) implies that π_v and $\pi_{v'}$ must be both in the plus-space! By the assumption $\sigma_v\cong\sigma_v^*$ and $\sigma_{v'}\cong\sigma_{v'}^*$ we know from [Wa] loc. cit. that $\Theta_+(\sigma_v)=\{\theta_+(\sigma_v)\}$ and $\Theta_+(\sigma_{v'})=\{\theta_+(\sigma_{v'})\}$. Hence $tr\ \pi'_v(f_v)=tr\ \pi'_v\otimes\mu'_v(f_v)=tr\ \pi_+(\sigma_v)(f_v)=-tr\ \pi_-(\sigma_v)(f_v)$, and similar at the place v'. Since the signs from v and v' cancel, again

$$tr\ \pi'_v(f_v)\cdot tr\ \pi'_{v'}(f_{v'})>0\ .$$

In other words, this term is positive for all relevant π' appearing in the trace formula stated above. Furthermore $m(\pi')\geq 0$ and $m(\pi)\geq 1$. Since the total sum of the terms $tr\ \pi'_v(f_v)\cdot tr\ \pi'_{v'}(f_{v'})$ is zero, this gives a contradiction and completes the proof of proposition 4.

<u>Proof of proposition 5</u>: Fix a global cuspidal representation $\sigma\not\cong\sigma^*$ of $M(\mathbb{A}_F)$ for some number field F. An irreducible cuspidal automorphic representation π in the global L-packet of σ is called strange, if for some place it is strange locally, i.e. if $\pi_v\in\Theta_+^{glr}(\sigma_v)$ for $\pi_v\not\cong\pi_+(\sigma_v)$. Notice, such v is never archimedian, since in this case the theta lift is understood well enough. We therefore have to show, that strange representation π can not arise from the nonarchimedian places.

For any π' in the global L-packet attached to σ let $S=S(\pi')$ be the set of places v, where $\pi'_v\in\Theta_-(\sigma_v)$ holds up to some quadratic character twist. If strange π exist in this L-packet, we now choose π to be minimal with respect to the cardinality of $S(\pi)$ among all the strange π in this L-packet. For this π there exists a nonarchimedian place v'', where $\pi_{v''}$ is strange locally.

We claim the cardinality of $S(\pi)$ is ≥ 2. Otherwise the underlying quaternion algebra $D=D(\pi)$ were split and $S=\emptyset$ (second assertion of lemma 3). In this situation the corresponding theta lift is locally and also globally generic by results of Soudry and

Howe-Piateskii-Shapiro [H-PS]. But this implies $\pi_v \cong \theta_+(\sigma_v)$ for all v, which contradicts the strangeness assumption. For this recall from [S2] and §2, that there is at most one class of irreducible generic representations in $\Theta_+(\sigma_v)$. It follows that $\#S(\pi) \geq 2$.

Fix two different places v, v' in $S = S(\pi)$. Choose $f_v, f_{v'}$ cuspidal with $supp(f_v) \subset \tilde{G}_v$ and $supp(f_{v'}) \subset \tilde{G}_{v'}$, such that $f_v, f_{v'}$ satisfies $(*)_v$ resp. $(*)_{v'}$. Furthermore suppose $tr\ \pi_v(f_v) > 0$ and $tr\ \pi_{v'}(f_{v'}) > 0$ (proposition 3).

Furthermore choose an auxiliary nonarchimedian place $w \neq v, v', v''$ of residue characteristic different from 2, where σ_w and $\pi_w = \pi_+(\sigma_w)$ are unramified. In particular $w \notin S(\pi)$. Choose f_w with regular support contained in $\tilde{G}_w$, such that $tr\ \pi_+(\sigma_w)(f_w) > 0$. E.g. take a bump function with support in the maximal split torus concentrated at a regular point near the origin, where the character of the unramified representation $\pi_+(\sigma_w)$ is nontrivial. Then in particular (RS) holds.

With these data fixed we apply the lemma 7 from the appendix §6. It gives

$$\sum_{\pi', (\pi')^{v,v',w} = \pi^{v,v',w}} m(\pi') \cdot tr\ \pi'_v(f_v) \cdot tr\ \pi'_{v'}(f_{v'}) \cdot tr\ \pi'_w(f_w) \ = \ 0 \ .$$

The right side vanishes, since $\pi_{v''} \ncong \pi_\pm(\sigma_{v''})$ at the place v'' were $\pi_{v''}$ is locally strange.

The minimality of π and proposition 3 and 4 imply $\pi'_v \in \Theta_-(\sigma_v)$ (up to quadratic character twists) for all π', which contribute to the trace formula above, i.e. those π' which are isomorphic to π outside v, v'. The same assertion is true at the place v'. Hence $tr\ \pi'_v(f_v) = tr\ \pi_-(f_v) > 0$ and also for the place v', by our choice of test functions. It remains to consider the auxiliary place w. Here $\pi'_w \otimes \mu_w \in \Theta^+(\sigma_w)$ holds up to some quadratic character μ_w. The unramified Sp-O Howe correspondence matches unramified representations with unramified representations. Hence the restriction of π'_w to $Sp(4, F_w)$ contains an unramified representation, since σ_w was unramified by assumption. Hence up to a quadratic character twist both π_w and π'_w are constituents of $Ind_{Z_w \cdot Sp(4,F_w)}^{G_w}(\tilde{\pi}'_w)$ for the same irreducible unramified representation $\tilde{\pi}'_w$ of $\tilde{G}_w$. This implies $\pi'_w = \chi_w \otimes \pi_w$ for some quadratic character χ_w ([V],p.480 l.5). Therefore $tr\ \pi'_w(f_w) = tr\ \pi_w(\sigma_w) > 0$ holds independently from π', because the support f_w is contained in $\tilde{G}_w$. This gives a contradiction, since in the trace formula above all these nonnegative terms should sum up to zero with at least one of them being positive (for π itself). Hence there are no strange representations π in the global L-packet of σ. Proposition 5 is proven.

To complete the proof of the main theorem stated in §3 and also to obtain all the results stated in §3 and §4 in full generality, it remains to prove the key formula and the local-global principle (lemma 5) stated in §4. The essential part of the local-global principle is stated in proposition 4 and 5. What remains to obtain lemma 5, will be shown below together with the key formula (as stated in §4). So once more we explore the trace formula.

Proof: Let $\sigma \not\cong \sigma^*$ be a cuspidal irreducible automorphic representation of $M(\mathbb{A}_F)$. Then any π in the global L-packet attached to σ is not CAP. Fix π and suppose $e(\pi) \geq 2$. Choose two places $\{v, v'\}$ in $S(\pi)$ and consider the lemma in the appendix in the special case, where the the assumptions $(*)_v$ and $(*)_{v'}$ hold simultaneously. In particular the functions $f_v, f_{v'}$ are cuspidal. We choose f_v and $f_{v'}$ so that $a_+ \neq 0$ resp. $a'_+ \neq 0$ and (ES) holds in addition with supports in $\tilde{G}_v$ resp. $\tilde{G}_{v'}$. This is possible by proposition 3. Lemma 6 implies $a_- = -a_+ \neq 0$ and $a'_- = a'_+ \neq 0$. For the notation see the appendix. To make assumption (vi)' of the appendix satisfied, if both v, v' are archimedian, we also choose some additional nonarchimedian place w and some function f_w as in the proof of proposition 5.

The semilocal trace identity of lemma 7 in this case considerably simplifies. This means, that in the sum $\sum' \ a_{disc}(\pi'_v \pi'_{v'} \pi'_w \pi^{vv'w}) \cdot tr \ \pi'_v(f_v) \cdot tr \ \pi'_{v'}(f_{v'}) \cdot tr \pi'_w(f_w)$ only those representations π' contribute, whose local components are in the local L-packet of σ, at least up to a quadratic character twist. In particular π_w is unramified up to a quadratic character twist. In fact all this follows from the last two propositions, and the arguments used for their proof. By the support conditions of $f_v, f_{v'}$ and f_w quadratic character twists do not have an effect on the trace. Therefore the contribution $tr \ \pi'_w(f_w) = tr \ \pi_+(f_w) \neq 0$ is independent from π' and can be cancelled from both sides of the semilocal identity. Similarly the nonvanishing terms $a_+ = -a_- \neq 0$ and $a'_+ = -a'_- \neq 0$ are independent from π'. Cancelling these terms therefore gives $M_{++} + M_{+-} + M_{-+} + M_{--} = 2(-1)^{e(\pi)}$. Here $M_{\varepsilon,\varepsilon'}$ is the sum over all multiplicities $m(\pi')$ over all π' such that $\pi' \otimes \mu_v \mu_{v'} \mu_w \cong \pi_\varepsilon(\sigma_v) \pi_{\varepsilon'}(\sigma_v) \pi^{vv'}$ for some local quadratic characters $\mu_v, \mu_{v'}$ and μ_w. Since the right side $2(-1)^{e(\pi)}$ equals the left side, which is ≥ 0, we get $2 | e(\pi)$. Furthermore by the principle of exchange the condition $M_{++} > 0$ (note $m_{++} > 0$ by assumption) implies $M_{-+} = M_{+-} = 0$. Hence $M_{++} + M_{--} = 2$ and $e(\pi)$ is even. This almost completes the proof of the key formula. In fact we know $M(\pi') = 1$ from lemma 3 part 2), if the cardinality of $S(\pi')$ is $0, 1$. So by induction on this cardinality, the argument surrounding the statement of the key formula in $\S 4$ proves $M_{++} = M_{--} = 1$. It only remains to show $m(\pi') = 0$ unless π_v and $\pi_{v'}$ are in the local L-packet of σ_v resp. $\sigma_{v'}$. For $\pi_v \in \mu_v \otimes \Theta_+(\sigma_v)$ this follows with the same argument, that was used in the proof of prop. 5. In particular we always get $\pi'_w \cong \pi_+(\sigma_w)$ at the auxiliary place w. (This could also be seen by moving this place around, i.e. change to some other place where σ_w is unramified and where $\pi' = \pi_w(\sigma_w)$ holds). So by the symmetry of v, v' it only remains to show $m(\pi') = 0$, whenever $\pi'_v \cong \mu_v \otimes \pi_-(\sigma_v)$ holds for some quadratic character μ_v but $\pi'_v \not\cong \pi_-(\sigma_v)$. Assume this were not true. To obtain a contradiction again apply the trace formula with $f_{v'}, f_w$ as above, but now with f_v satisfying only (ES) but arbitrary otherwise. Still the assumptions of the lemma in the appendix are satisfied. From this we obtain a contradiction, provided the characters χ_1 and χ_2 of $\mu_v \otimes \pi_-(\sigma_v)$ and $\pi_-(\sigma_v)$ are linear independent on the regular elliptic locus of G_v (the support of f_v). Notice, the first character χ_1 appears on the left side of the trace identity whereas the other character χ_2 appears on the endoscopic left side of the assertion in lemma 7. So it remain for us to show that linear independence holds on the elliptic locus. For $\sigma_v \ncong \sigma_v^*$ both representations are cuspidal and the claimed linear independence follows from the orthogonality relations for cuspidal characters with respect to the elliptic scalar product. If $\sigma_v \cong \sigma_v^*$ both representations do not belong to the discrete series. But in this case the distribution T of lemma 6 vanishes on the regular elliptic locus. Hence $-2\chi_1 = r(\mu_v \otimes \sigma_v)$ and $-2\chi_2 = r(\sigma_v)$ holds on the regular elliptic locus, so the linear independence of χ_1, χ_2 on the elliptic locus follows from the corresponding linear independence of $\mu_v \otimes \sigma_v$ and σ_v. Since these representations are in the discrete series, and since they are not isomorphic by our assumptions, they are linear independent on the regular elliptic locus of M_v. This contradiction implies, that any π' in the global L-packet with $m(\pi') > 0$ has its local components π'_v in the local L-packet attached to σ_v for all places v of F. In other words the local-global principle holds. This, together with the multiplicity result obtained above, implies the key formula.

<u>Remark</u>: If we relax the assumptions on $\sigma = (\sigma_1, \sigma_2)$ in 4)(i), we can similarly consider the case where σ belongs to the discrete series with σ_1 one dimensional (see [Art]). With the results on L-series shown in [PS] (instead of the results of Soudry used in the non CAP case) the coefficients a_{disc} can be worked out directly, and the arguments from above should extend to give a characterization of the CAP-representation in terms of the local Arthur packets. This indeed would give a refined description of the Saito-Kurokawa lift in term of local Arthur packets.

6.Appendix on the Arthur Trace Formula

In this appendix we apply the Arthur trace formula. We deduce from it local character identities. Its coefficients contain global information. We therefore refer to them as semilocal character relations. This semilocal character identities are useful for several reasons. First they are the starting point for the proof of the local character identities of the endoscopic lift r. Next we used them in the last section to show local statements like stability lemma. Finally they provide global information, once the local concepts are well understood. For this reason, and in order to be flexible enough for all these applications, we formulate a number of technical condition. Under these conditions we prove the semilocal character identities in the lemma below.

Make the following assumptions:

(i) Let σ be a irreducible automorphic of $M(\mathbb{A}_F)$ with σ_v, $\sigma_{v'}$ in the discrete series for at least two different places v, v'.

(ii) Suppose that σ is <u>cuspidal</u> and $\sigma \not\cong \sigma^*$. Then the global L-packet of σ contains cuspidal representations. These are not CAP-representations, since otherwise there exists poles for partial degree 4 or 5 L-functions of π, which contradict the cuspidality of σ. See $\S 3$.

Choose signs $\varepsilon, \varepsilon' \in \{\pm\}$. Consider the representations $\pi = \pi_\varepsilon(\sigma_v) \otimes \pi_{\varepsilon'}(\sigma_{v'}) \otimes \pi^{v,v'}$ and $\pi' = \pi_{-\varepsilon}(\sigma_v) \otimes \pi_{-\varepsilon'}(\sigma_{v'}) \otimes \pi^{v,v'}$ in the global L-packet of the weak lift of σ. We write $\pi = \pi_{++}$, $\pi' = \pi_{--}$ in the following. Let $m = m_{++} = m(\pi)$ and $m' = m_{--} = m(\pi')$ denote their multiplicity in the discrete spectrum of $G(\mathbb{A}_F)$. Assume

(iii) $m_{++} + m_{--} > 0.$

From (iii) follows, that either π or π' is an automorphic representation. By switching $\varepsilon, \varepsilon'$ into their negatives, we can assume without loss of generality $m(\pi) > 0$. Then the restriction of π to $Sp(4, \mathbb{A}_F)$ is a theta lift and is cuspidal (lemma 1). It is not CAP by the assumption (i). So π resp. π' only contribute to the cuspidal spectrum. Every automorphic representation in the discrete spectrum isomorphic to π or π' belongs to the cuspidal spectrum. This follows from the Langlands theory of spectral decomposition and the fact, that π is not CAP. Therefore Put $m = m_{++} = m_{cusp}(\pi)$ resp. $m' = m_{--} = m_{cusp}(\pi')$.

We now come to the essential conditions:

(iv) Let $f_v, f_{v'}$ be locally compact function with compact support on G_v resp. $G_{v'}$, which are in the Hecke algebra in the sense of [A] . Assume that both functions have vanishing orbital integrals at regular nonelliptic senisimple elements. In particular they are cuspidal.

(v) Assume that either $(*)_v$ or $(*)_{v'}$ holds.

Furthermore suppose

(vi) (RS) holds for the test function f_w at least at one auxiliary nonarchimedian place w, the case $w \in \{v, v'\}$ is not excluded.

Lemma 7: Suppose the assumptions (i) - (vi) hold. Then

$$\sum_{\pi'}{}' a_{disc}(\pi'_v \pi'_{v'} \pi'_w \pi^{vv'}) \cdot tr\ \pi'_v(f_v) \cdot tr\ \pi'_{v'}(f_{v'}) \cdot tr\ \pi'(f_w)$$

is either zero if for a place $v'' \neq v, v', w$ the class of $\pi_{v''}$ satisfies $\pi_{v''} \notin \{\pi_\pm(\sigma_{v''})\}$, or is equal to

$$\frac{1}{2}(-1)^{e(\pi)}(a_+ - a_-)(a'_+ - a'_-) \cdot \tilde{T}(f_w)$$

otherwise. Here $\tilde{T} = tr\ \pi_+(\sigma_w) - tr\ \pi_-(\sigma_w)$ if σ_w is in the discrete series, and $\tilde{T} = tr\ \pi_+(\sigma_w)$ otherwise.

Concerning the notations: In these formulas the summation $\sum'$ is over all classes of global representations π' in the global L-packet of σ for which $(\pi')^{v,v',w} \cong \pi^{v,v',w}$ holds. Here $a_\pm$ and $a'_\pm$ is an abbreviation for $a_\pm = tr\ \pi'_{\pm\varepsilon}(\sigma_v)(f_v)$ resp. $a'_\pm = tr\ \pi'_{\pm\varepsilon'}(\sigma_{v'})(f_{v'})$. The coefficients $a_{\varepsilon\varepsilon'}$ are zero unless the corresponding multiplicities $m_{\varepsilon\varepsilon'}$ of the weak lifts $\pi_\varepsilon(\sigma_v)\pi_{\varepsilon'}(\sigma_{v'})\pi^{v,v'}$ in the discrete spectrum are nonzero. In fact they are equal to the multiplicities of these representation. This follows from [A] (see the references given below) and the Langlands theory of spectral decomposition, since by assumption (ii) all these representation are not CAP as explained above. In fact $a_{++} := a^G_{disc}(\pi) = m_{cusp}(\pi)$ for any π in the weak lift of σ, which is in the discrete spectrum and similar for all π' in the global L-packet of σ.

Proof: For simplicity we suppose for the proof $m(\pi) > 0$, as above.

We apply the results of Arthur [A] on the trace formula for $G(\mathbb{A}_F)$. Choose any test functions f_λ for $\lambda \neq v, v', w$ and put $f = \prod_\lambda f_\lambda$. By assumption f_v and f'_v are cuspidal. So the orbital integrals of these two functions vanishes outside the regular elliptic locus and these functions are cuspidal in the sense of [A] p.538. Since f is cuspidal at the two different places v, v' the Arthur trace formula simplifies. The spectral side is a sum over the traces of the discrete spectrum suitably ordered using the archimedian infinitesimal characters ([A] , thm. 7.1 and cor. 7.2)

$$\sum_{\pi'} a^G_{disc}(\pi') \cdot I_G(f) \ .$$

Notice, that $I_G(\pi', f) = tr\ \pi'(f)$ (notation of [A2], p.325). The coefficients $a^G_{disc}(\pi')$ are complex numbers, by grouping together the linear combinations of weighted characters defined by [A], formula (4.3)

$$\sum_{M_0 \subset L_0 \subset G_0} |W_0^{L_0}||W_0^G|^{-1} \sum_s |det(s-1)|^{-1}_{\mathfrak{a}^G_{L_0}} \cdot tr\ \Big(M_{Q_0|sQ_0}(0) \circ \rho_{Q_0,t}(s,0,f^1) \Big) \ .$$

Here $\rho_{Q_0,t}$ is an induced representation, induced from the part $L^2_{disc,t}(L_0(F)A_{L_0,\infty} \backslash L_0(\mathbb{A}_F))$ of the discrete spectrum of the Levi subgroup L_0. If π' is cuspidal but not CAP, only $L_0 = G_0$ contributes to the coefficient $a_{disc}(\pi)$ and the sum becomes the trace of f on $L^2_{disc,t}(L_0(F)A_{L_0,\infty} \setminus L_0(\mathbb{A}_F))$. Hence $a_{disc}(\pi) = m_{cusp}(\pi)$ holds in this case. In general of course $a_{disc}(\pi) = 0$ unless $m_{disc}(\pi') \neq 0$.

Concerning the geometric side of this trace formula we obtain from [A], cor. 7.2

$$\sum_{\gamma \in (G(F))_{G,S}} a^G(S,\gamma) I_G(\gamma, f) \ .$$

Here $I_G(\gamma, f)$ is the global orbital integral $O^G_\gamma(f) = \prod_v O^{G_v}_{\gamma_v}(f_v)$ of f (see [A], p.325). Moreover, by our assumption (vi) on f_w the orbital at w vanishes unless γ_w is regular semisimple. This implies γ regular semisimple. Moreover by our assumption (iv) on $f_v, f_{v'}$ the geometric side only involves regular elliptic terms (as in [A] corollary 7.4). But in this case one can express $a^G(S,\gamma)$ explicitly. As in [A], cor. 7.4 one obtains the simpler expression for the geometric side

$$\sum_{\gamma \in (G(F)_{ell})} vol(G(F,\gamma)A_{G,\infty} \setminus G(\mathbb{A}_F, \gamma)) \int_{G(\mathbb{A}_F,\gamma) \backslash G(\mathbb{A}_F)} f(x^{-1}\gamma x) dx \ .$$

Here $G(F,\gamma) = Zent(\gamma, G^0)(F)$. In our case $G = G^0$. Furthermore in our case G_{der} is simply connected.

Stabilization of the elliptic terms of the geometric side as in [K], using the fundamental lemma [WW], gives for the geometric side of the Arthur trace formula a rearrangement in terms of stable orbital integrals for G and stable orbital integrals for the endoscopic group M

$$\sum_{\pi'} a_{disc}(\pi') \cdot tr\ \pi'(f) \ = \ ST^{G,**}(f) + \frac{1}{4} ST^{M,**}(f^M) \ .$$

The $**$-condition on central terms of [K] can be ignored by the regular support condition (vi) at the places w. Moreover the global stable orbital $ST^{G,**}(f) = ST^G(f) = 0$ vanishes by the local assumption $(*)_v$ or $(*)_{v'}$ of assumption (v)

$$ST^{G,**}(f) \ + \ \frac{1}{4}ST^{M,**}(f^M) \ = \ \frac{1}{4}ST^{M,**}(f^M) \ .$$

Since M is quasisplit the terms omitted in $ST^{M,**}(f^M)$ are again the central terms of the stable (semisimple) trace $ST^M(f^M)$ or preferable of a suitable stable trace on the z-extension $\tilde{M} = Gl(2) \times Gl(2)$. One of the functions $f_v, f_{v'}, f_w$ satisfies (RS). Without restriction of generality suppose it is $f_{v'}$. Then there is a matching function $f_{v'}^M$ with regular support. This uses smoothness of the regular locus of an elliptic torus argument and the implicit function theorem. Furthermore we can assume that for the two places v, v' the corresponding functions $f_v^M, f_{v'}^M$ have vanishing orbital integrals for regular non-elliptic elements. Then

$$ST^{M,**}(f^M) \ = \ ST^M(f^M) \ .$$

The z-extension $\tilde{M}$ of M does not have nontrivial endoscopy. The strong cuspidal condition (iv) is inherited by f^M, as well as condition (vi). So again the geometric side of Arthur's trace formula for $f^{\tilde{M}}$ is simple, and in particular only involves elliptic regular terms. Hence stabilization gives

$$\frac{1}{4}ST^{M,**}(f^M) \ = \ \frac{1}{4}T^{\tilde{M}}(f^{\tilde{M}}) \ .$$

If we compare the geometric terms $T^{\tilde{M}}(f^{\tilde{M}})$ with the spectral side, the simple form of the Arthur trace formula now applied for $\tilde{M}$, yields the character expansion

$$\frac{1}{4}T^{\tilde{M}}(f^{\tilde{M}}) \ = \ \frac{1}{4}\sum_{\sigma} \ a_{disc}(\otimes_v' \sigma_v) \cdot \prod_v tr \ r(\sigma_v)(f_v)$$

for the geometric side. The right side is a sum with σ running over the discrete spectrum of $\tilde{M}(\mathbb{A}_F)$, suitably ordered. To obtain this formula we used the local character identities $tr \ \sigma_v(f_v^{\tilde{M}_v}) = r(\sigma_v)(f_v)$. By the multiplicity one theorem for $Gl(2)$ resp. $\tilde{M}$ and the spectral theory for $Gl(2)$ we get $a_{disc}(\sigma) = 1$ for all cuspidal representations $\sigma = \otimes_v' \sigma_v$. See [JL], [GJ].

Notice, the expansion for $\frac{1}{4}T^{\tilde{M}}(f^{\tilde{M}})$ above is a character expansion in terms of representations of $G(\mathbb{A}_F)$. It involves only representations π which are weak lifts coming from M. All local components π_v appearing are in the local L-packet of an underlying global representation σ in the discrete spectrum of $M(\mathbb{A}_F)$. Comparing this character expansion with the one obtained from the trace formula for $G(\mathbb{A}_F)$ gives the formula (CharIdent)

$$\sum_{\pi'} \ a_{disc}(\pi') \cdot tr \ \pi'(f) \ = \ \frac{1}{4} \cdot \sum_{\sigma} \ a_{disc}(\sigma) \cdot \prod_v \ tr \ r(\sigma_v)(f_v)$$

A fixed representation $\pi^{v,v',w}$ of $G(\mathbb{A}_F^{v,v',w})$ belongs to the global L-packet of a fixed pair σ,σ^* of representations of $M(\mathbb{A}_F)$. This follows from proposition 2 and the strong multiplicity one theorem for $\tilde{M}$. So the last character identity should imply – so to say by the linear independence of characters of $G(\mathbb{A}_F^{v,v',w})$ and by separating the component $\pi^{v,v',w}$ – the following <u>semilocal identity</u>

$$\sum{}' a_{disc}(\pi'_v\pi'_{v'}\pi'_w\pi^{vv'w})\cdot tr\ \pi'_v(f_v)\cdot tr\ \pi'_{v'}(f_{v'})\cdot tr\ \pi'_w(f_w)$$

is equal to

$$= \frac{1}{2}(-1)^{e(\pi)}a_{disc}(\sigma)\cdot(a_+ - a_-)\cdot(a'_+ - a'_-)\cdot\tilde{T}(f_w)\ ,$$

if all local components of $\pi^{v,v',w}$ are in the local L-packets, and which is zero otherwise. Since σ and σ^* were supposed to be not isomorphic, and since both σ and σ^* contribute (prop. 2) we got the factor $\frac{1}{2}$ instead of the factor $\frac{1}{4}$ from the sum over the σ on the right side. In the sum the representations vary over all π' with $\pi^{v,v',w}$ fixed up to isomorphism.

Since the Arthur trace formula is not known to converge absolutely, a naive argument which implies linear independence of characters in the sense above, is not known at present. However, the assumptions (vi) or (vi) put us into a situation where the above semilocal identity can nevertheless be extracted from the global trace formula.

To extract the semilocal identity stated above from the global Arthur trace formula in this case, one uses multipliers at the archimedian places.

<u>Multipliers</u>: Let $f_\infty \in C_c^\infty(G_\infty, K_\infty)$ be a K_∞ finite test function at the archimedian places. A W-invariant distribution α with compact support on the Lie algebra of the standard Cartan subgroup h^1 is called a multiplier. In our cases h^1 is the Lie algebra of a maximal split torus; for the general case see [A],§6). Typical examples are elements in the center of the universal enveloping algebra or W-invariant smooth functions with compact support. Multipliers act on $C_c^\infty(G_\infty, K_\infty)$ in a natural way as shown by Arthur. Let $f_\infty \mapsto (f_\infty)_\alpha$ denote this action. For an irreducible admissible unitary representation π_∞ of G_∞ let ν_π denote its infinitesimal character viewed as a W-orbit in h^1. Let t_π denote the length of its imaginary part with respect to a suitable norm on h^1 (see [A]). Then $\pi_\infty((f_\infty)_\alpha)) = \hat{\alpha}(\nu_\pi)\pi_\infty(f_\infty)$, for the Fourier transform $\hat{\alpha}$ of the distribution α. Indeed this formula uniquely characterizes the action.

Notice, that $(f_\infty)_\alpha$ is cuspidal, if f_∞ is cuspidal. This following, since cuspidality is characterized by the vanishing of the traces $tr\ \pi_\infty(f_\infty$ for all representations π_∞, which are properly induced from irreducible tempered representations. Since $tr\ \pi_\infty((f_\infty)_\alpha)) = \hat{\alpha}(\nu_\pi)tr\ \pi_\infty(f_\infty)$ this property is preserved.

Furthermore the condition $(*)_\infty$ is preserved by the action of multipliers. See [S] and [A], def. 1.2.1. Without restriction of generality $F_v = \mathbb{R}$. Then $(*)_\infty$ is equivalent to $T(f) = 0$ for all characters $T = tr\ \pi_\infty$ of discrete series representations of G_v not isomorphic to

$\pi_\pm(\sigma_v)$, plus for all $T = tr\ \pi_+(\sigma_v) + tr\ \pi_-(\sigma_v)$ attached to discrete series representations σ_v of M_v and all characters T of representations properly induced from tempered representations. This follows from [Sh2], lemma 5.3. Furthermore, for an irreducible unitary representation σ_∞ the infinitesimal character ν_{σ_∞} determines the infinitesimal character $\nu_\pm$ of the representations $\pi_\pm(\sigma_\infty)$. Indeed

$$\nu_+ = \nu_- = \xi(\nu_{\sigma_\infty})$$

for a suitable 'linear' map ξ, up to some shift. This follows from the description of the endoscopic lift in terms of the theta correspondence or from [Sh], lemma 4.2.1. The precise nature of the map ξ is of no importance here. For $\nu_v = \xi(\nu_{\sigma_v})$ this implies for the archimedian places v

$$T((f_v)_\alpha) = \hat{\alpha}(\xi(\nu_v)) \cdot T(f_v) = \widehat{\tilde{\alpha}}(\nu_v) \cdot T(f_v)$$

for $\tilde{\alpha} = \alpha \circ \xi$.

Multipliers act on global test K-finite test function $f = \prod_v f_v$ in $C_c^\infty(G_\infty \times G((\mathbb{A}_F)_f))$ via their action on the archimedian component f_∞. Hence the nonarchimedian condition (vi) is preserved by the action of multipliers by trivial reasons.

The infinitesimal character $\nu = \nu_{\pi_\infty}$ of an unitary irreducible admissible representation π_∞ defines W-orbit. Given ν, a smooth multiplier α is constructed on p.182 ff of [AC], such that for $\alpha_m = \alpha * .. * \alpha$ (m-fold convolution) the following holds

$$lim_{m\to\infty} \sum_{\pi'} \ a_{disc}(\pi') \cdot tr\ \pi'(f_{\alpha_m}) \ = \sum_{\pi',\ \nu_{\pi'}=\nu} a_{disc}(\pi') \cdot tr\ \pi'(f)\ .$$

Recall, the left side is the spectral side of the Arthur trace formula in a simple form (for our purposes f is supposed to satisfy the assumption of the trace hypothesis; under the second assumption of condition (vi) these assumption are stable under the action of multipliers!). This sum is not necessarily absolute convergent, so summation is with respect to a suitably ordering using the parameter $t_{\pi'}$. The sum on the right is absolute convergent due to the admissibility statement [A], lemma 4.1. Hence linear independence of the characters involved holds in the sense of [JL], lemma 16.1.1. We recall, that for the above limit formula it suffices to know that $\hat{\alpha}(\nu) = 1$ and $|\hat{\alpha}(\nu_{\pi'})| < 1$ unless the W-orbits of $\nu_{\pi'}$ and ν coincide.

A similar separation of the infinitesimal character can be obtained for the spectral side of the simple trace formula for the endoscopic group. In fact, one can do better: For an irreducible unitary representation σ_∞ the infinitesimal character ν_{σ_∞} determines the infinitesimal character $\nu_\pm$ of the representations $\pi_\pm(\sigma_\infty)$ as explained above $\nu_+ = \nu_- = \xi(\nu_{\sigma_\infty})$ for a suitable 'linear' map ξ. For $\nu_v = \xi(\nu_{\sigma_v})$ this implies for the archimedian places v

$$tr\ r(\sigma_v((f_v)_\alpha) = \hat{\alpha}(\xi(\nu_v)) \cdot tr\ r(\sigma_v)(f_v) = \hat{\alpha}(\xi(\nu_v)) \cdot tr\ \sigma_v((f_v)^{M_v}) = tr\ \sigma_v((f_v)_{\tilde{\alpha}})$$

for $\tilde{\alpha} = \alpha \circ \xi$. If we identify the domains of the functions α and $\tilde{\alpha}$, as we may do, we can symbolically write

$$\big((f_v)_\alpha\big)^{M_v} \;=\; \big(f_v^{M_v}\big)_\alpha \;.$$

In other words, the action of multipliers commutes with the endoscopic matching condition. Since the Weyl group of M_v is a subgroup of the Weyl group of G_v, the smooth multiplier α for G_v can therefore be considered as a smooth ($*$-invariant) multiplier for M_v. This being said we can consider the formulas (CharIdent) from above for the various test functions f_{α_m}. From the spectral limit formulas both for G and for M we obtain for $m \to \infty$

$$\sum_{\pi', \nu_{\pi'} = \xi(\nu)} a_{disc}(\pi') \cdot tr\ \pi'(f) \;=\; \frac{1}{4} \cdot \sum_{\sigma,\ \nu_\sigma = \nu} a_{disc}(\sigma) \cdot \prod_v\ tr\ \sigma_v((f_v)^{M_v})$$

$$= \frac{1}{4} \cdot \sum_{\sigma,\ \nu_\sigma = \nu} a_{disc}(\sigma) \cdot \prod_v\ tr\ r(\sigma_v)(f_v) \;.$$

From this formula, [A] lemma 4.1 and [JL], lemma 16.1.1 the semilocal identity follows. This completes the proof of lemma 7.

<u>References</u>

[A] Arthur J., The invariant trace formula II, global theory, J. of AMS 1, (1988), 501-554

[A2] Arthur J., The invariant trace formula I, local theory, J. of AMS 1, (1988), 323-383

[AC] Arthur J.-Clozel L., Simple Algebras, Base Change, and the Advanced Theory of the Trace Formula, Annals of Princeton Studies, n.120, (1989)

[Art] Arthur J., On some problems suggested by the trace formula, in Lie Group Representations II, SLN 1041 (1984)

[GJ] Gelbart S.-Jaquet H., Forms of Gl(2) from the analytic point of view, in: Automorphic Forms, Representations, and L-functions, Proc. Symp. Pure Math. vol. 23, part 1, AMS (1979), 213-251

[H-PS] Howe R.-Piatetski-Shapiro I, Some examples of automorphic forms on Sp_4, Duke Math. J. vol 50, No 1 (1983)

[KA] Kazhdan D., Cuspidal Geometry of p-adic Groups, Journal d'Analyse Mathematique vol 47 (1986)

[K] Kottwitz R., Stable trace formula: Elliptic terms, Math. Ann. 275 (1986), 629-646

[KRS] Kudla S.S.-Rallis S.- Soudry D., On the degree 5 L-function for $Sp(2)$, Invent. math. 107 (1992)

[L] Langlands R., Sur les debuts des une formule des traces stables, Pub. Math. de l'Univ. Paris VII, 13 (1982)

[PS] Piateskii-Shapiro I.I, On the Saito-Kurokawa Lifting, Invent. Math. 71, 309-338 (1983)

[R] Rogawski J., Automorphic representations of unitary groups in three variables, Annals of Mathematics Studies 123, Princeton University Press (1990)

[Sh] Shelstad D., L-indistinguishability for real groups, Math. Ann. 259 (1982), 385-430

[Sh2] Shelstad D., Characters and inner forms of a quasi-split group over $\mathbb{R}$, Compositio Math. 39 (1979), 11-45

[S] Soudry D., The CAP representations of GSp(4,A), J. reine angew. Math. 383 (1988), 97- 108

[S2] Soudry D., The L and γ factors for generic representations of $GSp(4,k) \times Gl(2,k)$ over a local nonarchimedian field k, Duke math. journal 51, n.2 (1984), 335-394

[T] Tate J.T., Fourier analysis in number fileds and Hecke's zeta-functions, in Algebraic Number Theory, ed. Cassels-Fröhlich, Academic Press (1967)

[Ch] Weissauer R., Character identities and Galois representations related to the group GSp(4), preprint Mannheim 1994, revised version 1995

[V] Vigneras, M.F., Correspondances entre representations automorphes de $Gl(2)$ sur une extension quadratique de $GSp(4)$ sur $\mathbb{Q}$, conjecture locale de Langlands pour $GSp(4)$, Contemporary Math. vol. 53, 1986

[WW] Weissauer R., A special case of the fundamental lemma I-IV

[W] Weissauer R., The Ramanujan conjecture for genus two Siegel modular forms (An application of the trace formula), preprint 1993

[Wa] Waldspurger J.-L., Un exercice sur GSp(4,F) et les representations de Weil, Bull. Soc. math. France 115, 1987, p.35-69