Double Cosets for Classical Groups in the Unramified Case

(R.Weissauer)

Let F be a nonarchimedian local field, let G be a connected simply connected semisimple group over F. Let T be a maximal F-torus in G, which is maximal split over F. Let $\mathcal{A}$ be an apartment of the Bruhat-Tits building of G belonging to T, and suppose $x \in \mathcal{A}$ is a hyperspecial point. Let K be the stabilizer of x in G. Let H be a maximal reductive F-subgroup of G, which contains T. We are interested in the double cosets $H(F) \setminus G(F)/K$. We expect that there always exist finitely many elements $r_i \in G(F)$ such that $\bigsqcup_{i=1}^h r_i T(F)$ contains a complete system of representatives for these double cosets. Since the maximal compact subgroup ${}^0T(F)$ of $T(F)$ is in K, a complete system of representatives then is already contained in $\bigsqcup_{i=1}^h r_i\Gamma$, where Γ is a set of representatives for the free abelian group $T(F)/{}^0T(F)$. Furthermore, to show that $\bigsqcup_{i=1}^h r_i T(F)$ contains a complete system of representatives, is equivalent to show that $\bigsqcup_{i=1}^h \{int_t(r_i) \mid t \in T(F)\}$ contains a complete set of representatives. In this note we verify this for cases, where the residue characteristic of F is different from two and where G is a classical group ([K], p.61).

As a first remark, we note that this problem can be immediately reduced to the case of a simply connected almost F-simple group. Then $G = Res_{F'/F}(G')$ for a simply connected absolutely almost simple group G' over F', for some finite unramified extension field F' of F. If $H = Res_{F'/F}(H')$, on can reduce to the absolute F-simple case replacing F by F'.

As a second remark, there is a surjective map $H(F) \setminus G(F)/K \twoheadrightarrow \tilde{H}(F) \setminus \tilde{G}(F)/\tilde{K}$ for every group $\tilde{G}$ in the isogeny class of G. Here $\tilde{H}$ denotes the image of H in $\tilde{G}$, and $\tilde{K}$ denotes the stabilizer of x in $\tilde{G}(F)$. If the isogeny $G \to \tilde{G}$ is separable, this follows immediately from the Kneser-Tits theorem $H^1(F,G) = 1$ and from $H^1(F,T) = 1$. Both statements together imply $\tilde{G}(F)/G(F) \cong \tilde{T}(F)/T(F)$. Furthermore $\tilde{H}(F) = H(F)\tilde{T}(F)$ and $\tilde{G}(F) = \tilde{T}(F)G(F)$. Therefore, if $\bigsqcup_{i=1}^h int_{T(F)}(r_i)$ contains a set of representatives for the double cosets in $G(F)$, $\bigsqcup_{i=1}^h int_{\tilde{T}(F)}(\tilde{r}_i)$ contains a set of representatives for the double cosets in $\tilde{G}(F)$. Here $\tilde{r}$ denotes the image of r in $\tilde{G}(F)$. The conclusion still holds for an arbitrary isogeny. One can argue similar since $H^1_{fppf}(F,T) = H^1_{et}(F,T) = 1$, using that T is a smooth group scheme over F.

Let us consider the fibers of this map. Suppose $H(F)rK$ and $H(F)sK$ for $r,s \in G(F)$ define the same double coset $\tilde{H}(F)\tilde{r}\tilde{K} = \tilde{H}(F)\tilde{s}\tilde{K}$. Then there exist $h \in H(F), \bar{t} \in \tilde{T}(F)$, $\bar{k} \in \tilde{K}$ such that $\tilde{s} = \tilde{h}\bar{t}\tilde{r}\bar{k} = \tilde{h} \cdot int_{\bar{t}}(\tilde{r}) \cdot \tilde{k}$ for some $k \in K$, since s, h and $int_{\bar{t}}(r)$ are in $G(F)$ and $H(F)$ contains the central subgroup, which is the kernel of the map $G(F) \to \tilde{G}(F)$. In other words, if the images of the elements $r_i \in G(F), i \in I$ contain a set of representatives for $\tilde{H}(F) \setminus \tilde{G}(F)/\tilde{K}$, then the elements $int_{\bar{t}}(r_i)$ contain a set of representatives of $H(F) \setminus G(F)/K$, for $i \in I$ and $\bar{t} \in \tilde{T}(F)$. If $\bigsqcup_{i=1}^{h} int_{\tilde{T}(F)}(\tilde{r}_i)$ contains a set of representatives for $\tilde{H}(F) \setminus \tilde{G}(F)/\tilde{K}$, then also $\bigsqcup_{i=1}^{h} int_{\tilde{T}(F)}(r_i)$ contains a set of representatives for $H(F) \setminus G(F)/K$. Note that $\bigsqcup_{i=1}^{h} int_{\tilde{T}(F)}(\tilde{r}_i) = \bigsqcup_{i=1}^{h'} int_{T(F)}(r'_i)$, since $\tilde{T}(F)/T(F)$ is a finite group.

So we may pass to the adjoint groups. Since the F-valued points of the normalizer of T_{ad} in the adjoint group act transitively on the hyperspecial points of $\mathcal{A}$, the assertion on the structure of double cosets is therefore essentially independent from the choice of the hyperspecial point x.

More generally one could ask in the case of arbitrary connected semisimple groups G for the structure of the double cosets $H(F)\backslash G(F)/K$ in those cases, where K is the stabilizer of an arbitrary special point of the apartment $\mathcal{A}$. We then expect, that again there exist finitely many elements $r_1, ..., r_h \in G(F)$, such that $\coprod_i r_i T(F)$ contains a complete set of representatives. To provide evidence for this, in the following we develop part of the arguments in greater generality.

Classical groups: Suppose the characteristic of F is different from 2. The simply connected absolutely almost simple classical groups over F are the following: The norm-1 groups of simple F-algebras, the special unitary groups of hermitian forms over quadratic extension fields, the symplectic groups and the unitary groups of hermitian forms over quaternion division algebras (inner forms of symplectic groups). The Spin groups of quadratic forms, and as inner forms the Spin groups of skew hermitian forms over quaternion algebras. We do not consider the norm-1 groups of simple F-algebras, since this case is easily dealt with. We therefore concentrate on the remaining cases. Existence of a hyperspecial point x in the apartment of T means, that we have to consider the following cases C_n (split symplectic group), B_n and D_n (split orthogonal groups), ${}^2A'_n$ (split unitary groups for unramified quadratic extension fields L/F) and ${}^2D_n$ (orthogonal groups for quadratic forms of even rank with two dimensional anisotropic kernel). See Tits [T], p. 67. We may also replace the Spin groups by the special orthogonal groups, since these are easier to handle.

Let D be a division field, whose center is a nonarchimedian local field of characteristic $\neq 2$. Let $\mathfrak{o}_D$ be a maximal order in D, π_D be a prime element and $*$ be an involution of D. Suppose F is the fixed field of $*$ in the center of D. For our purposes D may be either a quaternion division algebra with center F or a quadratic field extension of F, or F itself. Extend $*$ to an involution of the $n \times n$-matrices $M_n(D)$ over D by $(a_{ik})^* = (a_{ki}^*)$.

Witt normal form: Suppose $X \in M_n(D)$ is invertible and is ε-hermitian, i.e. satisfies $X^* = \varepsilon \cdot X$. Let $G \subset Gl(n,D)$ be the automorphism group of the sesquilinear form $(u,v) = u^*Xv$ defined on the D-right vector space D^n. The involution $*$ is supposed to act on D^n through its action on the coordinates and by transposition. Put $G = A(X)$. This algebraic group is in general not connected, its connected component is the subgroup $SA(X)$ of all matrices with determinant one. Similar define the group of similitudes $G = G(X)$ as the group of matrices $g \in Gl(n,D)$ such that $g^*Xg = \lambda \cdot X$ holds for a scalar $\lambda \in F^*$. D decomposes into D^+ and D^-, the even elements $x^* = x$ and the odd elements $x^* = -x$. The form $H(v) = (v,v)$ takes values in D^ε. H respectively X is called anisotropic if $H(v) = 0$ implies $v = 0$.

We may assume X to be of the form

$$X = \begin{pmatrix} 0 & 0 & J \\ 0 & X_0 & 0 \\ \varepsilon \cdot J & 0 & 0 \end{pmatrix} ,$$

where X_0 is diagonal and J is the matrix, whose entry is one in the antidiagonal and is zero else. We may also assume $X_0 = X_{an}$ to be anisotropic; in this case X is said to be in Witt normal form and $r = rank(J)$ is called the hyperbolic rank. If $X_{an} = 0$, we say that X is split hyperbolic. In the following we always assume X to be in Witt normal form. We do not assume X_0 to be anisotropic. In other words we will suppose $rank(J) = i \leq r$. Let the basis corresponding to the matrix coordinates be denoted $e_1, .., e_r, w_1, .., w_{n-2r}, e'_r, .., e'_1$. We also may assume $r > 0$, since otherwise the statements are trivial G being anisotropic and therefore compact.

Suppose

$$Y = \begin{pmatrix} A & B & C \\ 0 & Y_0 & M \\ G & F & E \end{pmatrix}$$

is an invertible matrix with in $A, E \in M_i(D)$, $i \leq r$. First suppose $G = 0$. Then $Y \in A(X)$ holds if and only if $Y_0 \in A(X_0)$ and $F = 0$, $A^*JE = J$, $B^*JE + Y_0^*X_0M = 0$ and $C^*JE + \varepsilon E^*JC + M^*X_0M = 0$ holds. The matrices Y with these properties define a maximal parabolic F-subgroup P of $A(X)$. Every maximal parabolic F-subgroup in $A(X)$ is conjugate to one of this form for some integer $i \leq r$. We define a corresponding group $H \subset A(X)$, by requiring $B = 0, M = 0, F = 0$ (and similar for the corresponding group of similitudes). Unless $H = A(X)$, this group H defines a proper maximal reductive F-subgroup of $A(X)$. H is the centralizer of the element $s = diag(E, -E, E) \in A(X)$. Obviously $H \cong A(X_{split}) \times A(X_0)$, where X_{split} is split hyperbolic of rank i. Furthermore $H \cap P$ is isomorphic to the semidirect product of $A(X_0) \times Gl(i, D)$ and the vector group $M_i(D)^{-\varepsilon} = \{C \in M_i(D) \mid C^*J + \varepsilon \cdot JC = 0\}$. Put $G = 0$, $F = 0$ and let A, Y_0, E be the identity, furthermore put $B = -\varepsilon \cdot JM^*X_0$ and $C = -\varepsilon \cdot JM^*X_0M/2$ for a matrix $M = M^{(n-2i,i)} \in Hom_D(D^i, D^{n-2i})$. The corresponding matrix $g(M)$ is in P.

The maximal lattice Λ: Let Λ in D^n be the lattice generated as an $\mathfrak{o}_D$-right module by $e_1, .., e_r, e_r', .., e_1'$ and Λ_0. Here Λ_0 is a sublattice of the vectorspace W generated by $w_1, .., w_{n-2r}$. Λ_0 is defined as follows; by definition $u \in \Lambda_0$ if and if there exists $x \in \mathfrak{o}_D$ such that $H(u) = \sigma(x) = x + \varepsilon \cdot x^*$. It is not difficult to show, that Λ_0 is a $\mathfrak{o}_D$-right module such that $(u, v) \in \mathfrak{o}_D$ holds for all $u, v \in \Lambda_0$; put $H(u) = \sigma(x)$ and $H(v) = \sigma(y)$. If $s = (u, v)$ for $u, v \in \Lambda_0$ were not in $\mathfrak{o}_D$, then $z \mapsto z + x + z^*(s^{-1})^*ys^{-1}z$ would be an isometry of $\mathfrak{o}_D$, hence surjective. Therefore $z \in \mathfrak{o}_D$ exists with $H(u + vs^{-1}z) = 0$. Then $u = -vs^{-1}z$ by anisotropy of X_{an}. This immediately gives the contradiction $s = -(s^{-1}z)^*H(v) \in \mathfrak{o}_D$. Obviously the proven statement implies, that Λ_0 is a lattice. Furthermore $H(u) \in \sigma(\mathfrak{o}_D)$ and $(u, v) \in \mathfrak{o}_D$ for all $u, v \in \Lambda$. Λ is a $\mathfrak{o}_D$ submodule of D^n, which is maximal with these properties. So if $u \in \Lambda$ is a primitive isotropic, there exists $v \in \Lambda$ such that $(u, v) \in \mathfrak{o}_D^*$, since otherwise $\langle \Lambda, u \cdot \pi_D^{-1} \rangle$ contradicts the maximality of Λ mentioned.

Let $\Lambda^* = \{v \in D^n \mid (u, \Lambda) \in \mathfrak{o}_D\}$ be its dual, then $\Lambda \subset \Lambda^*$. Let $A(X, \Lambda)$ be the stabilizer of Λ in $A(X)$ and let $A_0(X, \Lambda)$ be the normal subgroup, that acts trivially on Λ^*/Λ. These groups are compact open subgroups of $A(X)$. Λ is generated by Λ_0 and its primitive, isotropic vectors. Since we assumed $\Lambda_0 \neq \Lambda$, Λ is generated as $\mathfrak{o}_D$-module by its primitive isotropic vectors; the primitive basis vectors $e_1, .., e_r, e_1', .., e_r'$ are isotropic. Furthermore any $w \in \Lambda_0$ satisfies $H(w) + \sigma(t) = 0$ for some $t \in \mathfrak{o}_D$, hence $e_1 - e_1' \cdot t + w$ is primitive and isotropic. Any two primitive, isotropic vectors u, v in Λ are in the same $A(X,\Lambda)$-orbit. We can assume $u = e_1'$. First assume $v = e_1' + e_1 \cdot t + a$ for some $t \in \mathfrak{o}_D$ and some a in the $\mathfrak{o}_D$ span of $e_2, ..e_r, w_1, , e_r', .., e_2'$. Isotropy implies $\sigma(t) + H(a) = 0$. Hence $\psi(a,t)(x) = x + a(e_1, x) + e_1[t(e_1,x) - (a,x)]$ is in $A(X,\Lambda)$, and maps u to v. In general assume $v = \sum_{i=1}^r (e_i a_i + e_i' b_i) + w, w \in \Lambda_0$. If not all a_i, b_i are in $\pi_F \mathfrak{o}_D$, then permutations and a matrix in $P(\mathfrak{o}_D)$ reduce to the case above. In the other case there exists a $w = e_1\alpha + e_1'\beta + ..$ in Λ such that $(v, w) \in \mathfrak{o}_D^*$, as explained above. We can assume $(w, e_1) = (w, e_1') = 0$, since also $(v, w - e_1\alpha - e_1'\beta) \in \mathfrak{o}_D^*$. $H(w) + \sigma(s) = 0$ holds for some $s \in \mathfrak{o}_D$. Then $\psi(w,s)(v) = e_1 c_1 + ..$ with $c_1 \in \mathfrak{o}_D^*$. So we are again in the case already considered above. The proof is complete.

By induction it implies $A(X) = A(X,\Lambda) \cdot P = P \cdot A(X,\Lambda)$. In case $i = r$ this and the Iwasawa decomposition for $Gl(r, D)$ gives the Iwasawa decomposition for G. Note, $A(X,\Lambda)$ is a maximal compact subgroup of $A(X)$. Suppose K were a compact group containing $A(X,\Lambda)$. We claim $K = A(X,\Lambda)$, or equivalently $K(\Lambda) \subset \Lambda$. It is enough to show $k(v) \subset \Lambda$ for all primitive anisotropic vectors $v \in \Lambda$ and all $k \in K$. If this were not true, there would exist $k \in K$ and a primitive isotropic vector v such that the isotropic vector $k(v)$ is not in Λ. Let $\mu \neq 0$ be its height. By changing k from left and right by suitable elements in $A(X,\Lambda)$ we then find another $k \in K$ such that $k(e_1) = \pi_D^{-\mu}(e_1)$ and $\mu \neq 0$. But then – taking powers of k – this contradicts the fact, that the projection of the compact set $K(\Lambda)$ to $e_1 D$ has compact image.

The cosets: From the decomposition $A(X) = P \cdot A(X,\Lambda)$ we conclude, that the matrices $g(M)$ for $M \in Hom_D(D^i, D^{n-2i})$ contain a system of representatives for the double cosets $H \setminus A(X)/A(X,\Lambda)$. Since $g(M_1)g(M_2)$ and $g(M_1 + M_2)$ belong to the same double coset, it is immediately clear that $M \mapsto g(M)$ defines a surjective map α from

$$A(X_0,\Lambda) \setminus \Big(Hom_D(D^i, D^{n-2i})/Hom_{\mathfrak{o}_D}(\mathfrak{o}_D^i, \mathfrak{o}_D^{n-2i})\Big)/Gl(i, \mathfrak{o}_D)$$

to

$$H \setminus A(X)/A(X,\Lambda)\ .$$

Here $h \in A(X_0,\Lambda)$ and $U \in Gl(i,\mathfrak{o}_D)$ act on M via $M \mapsto h \cdot M \cdot U^{-1}$. Use conjugation with $diag(id,h,id)$ resp. $diag(J(U^*)^{-1}J,id,U)$.

Put

$$\Lambda_s = s \cdot X = \begin{pmatrix} 0 & 0 & J^{(i,i)} \\ 0 & -X_0 & 0 \\ \varepsilon \cdot J^{(i,i)} & 0 & 0 \end{pmatrix}\ .$$

So $\Lambda_s^* = \varepsilon \cdot \Lambda_s$ is ε-hermitian. For $g \in A(X)$ we have: $\lambda(g)^{-1}g^*\Lambda_s g = \Lambda_s$ is equivalent to $\Lambda_s g = XgX^{-1}\Lambda_s$ or equivalent to $sg = gs$, hence $g \in H$. This argument uses $\lambda(g)(g^*)^{-1} = XgX^{-1}$ and $X^{-1}\Lambda_s = s$. So we get an injection $\beta_H : H \setminus A(X) \hookrightarrow \{\varepsilon\text{-hermitian matrices}\}$ defined by $H \cdot g \mapsto \lambda(g)^{-1}g^*\Lambda_s g$; in fact $\lambda(g) = 1$ for $g \in A(X)$. From this we get a further injection

$$\beta : H \setminus A(X)/A(X,\Lambda) \quad \hookrightarrow \quad \{\varepsilon\text{-hermitian matrices}\}/A(X,\Lambda)\ .$$

The action of $A(X,\Lambda) \subset G(X)$ on the ε-hermitian matrices is given by $\Lambda \mapsto \lambda(g)^{-1}g^*\Lambda g$.

The composite $\beta \circ \alpha$ maps M to the $A(X,\Lambda)$-equivalence class of the ε-hermitian matrix

$$\Lambda_s(M) \ = \ \begin{pmatrix} 0 & 0 & J \\ 0 & -X_0 & -2X_0M \\ \varepsilon J & -2\varepsilon(X_0M)^* & -2M^*X_0M \end{pmatrix}\ .$$

Put $j = r - i$ and $r_{an} = n - 2(i+j)$ and write

$$M = \begin{pmatrix} B \\ M_{an} \\ A \end{pmatrix}$$

with matrices $B = B^{(j,i)}$, $M_{an} = M_{an}^{(r_{an},i)}$ and $A = A^{(j,i)}$. Then

$$\Lambda_s(M) = \begin{pmatrix} 0 & 0 & \mathcal{X} \\ 0 & -X_{an} & v \\ \varepsilon\mathcal{X}^* & \varepsilon v^* & \mathcal{A} \end{pmatrix}$$

with the following matrices: The $(r \times r)$-matrix

$$\mathcal{X} = \begin{pmatrix} 0^{(j,i)} & J^{(j,j)} \\ -J^{(i,i)} & -2J^{(j,i)} \cdot A \end{pmatrix} ,$$

the $(r_0 \times r)$-matrix

$$v = (\, 0^{(r_{an},j)} \quad , \; -2X_{an} \cdot M_{an} \,) \; ,$$

and the ε-hermitian $(r \times r)$-matrix

$$\mathcal{A} = \begin{pmatrix} 0^{(j,j)} & -2\varepsilon J^{(i,i)} B \\ -2B^* J^{(i,i)} & -2M^* X_0 M \end{pmatrix} .$$

Remark: If $M_{an} = 0$ – or equivalently $v = 0$ – then the "outer" $(2r \times 2r)$-block of the matrix $\Lambda_s(M)$, which involves $\mathcal{X}$ and $\mathcal{A}$, is contained in the subgroup $SA(X_{split})$ of $A(X_{split})$, consisting of all matrices g with multiplier $\lambda(g) = 1$. This is clear since in the "limit" $X_{an} \to E$ the matrix is in $SA(X)$. It does not change, since all entries involving X_{an} are multiplied by the entry M_{an}.

Assumptions: From now on make the following assumptions. Suppose

a) $1/2 \in \mathfrak{o}_D$

b) also suppose $X_{an} \in Gl(r_{an}, \mathfrak{o}_D)$.

The second assumptions holds in the case, where there exist hyperspecial points in the building. Both assumptions imply, that the matrix X_{an} is anisotropic modulo π_D. Note, $H(v) = \sigma(t\pi_D^\nu), \nu \geq 1$ implies $H(v + x\pi_D^\nu) \in \sigma(\mathfrak{o}_D \pi_F^{\nu+1})$ by solving the equation $t + v^* X_{an} x = 0$ modulo π_D.

Remark: More generally, now only suppose $1/2 \in \mathfrak{o}_D$. Let V be a D-right vectorspace with a nondegenerate ε-hermitian form X and let Λ be a right $\mathfrak{o}_D$-module in V containing a basis, such that $(x, y) \in \mathfrak{o}_D$ holds for all $x, y \in \Lambda$. Suppose $H(u) = H(v)$ for $u, v \in \Lambda$. Assume that either $H(u)$ or (u, v) is in $\mathfrak{o}_D^*$. Then u, v are in the same orbit under $A(X, \Lambda)$. Note $\phi(x) = x - s\sigma^{-1}(s, x) \in A(X, \Lambda)$ for $s = (u \pm v)/2$, provided $\sigma = (s, u)/2 \in \mathfrak{o}_D^*$. This is true at least for one choice of sign. Obviously $\phi(u) = \pm v$. Since $\pm id \in A(X, \Lambda)$, we are done.

The Genus: Two ε-hermitian $n \times n$ matrices X and $\tilde{X}$ are said to belong to the same genus, if there exists a matrix $g \in Gl(n, \mathfrak{o}_D)$ such that $\tilde{X} = g^* X g$. In the cases, where hyperspecial points exist, the genera of ε-hermitian forms are well understood. See [J], [OM]. Every genus is represented by a Jordan normal form $X \cong X_1 \pi^{a_1} \oplus X_2 \pi^{a_2} \oplus ... \oplus X_r \pi^{a_s}$ with scaling factors such that $a_1 < a_2 < ... < a_r$ and unimodular ε-hermitian Jordan constituents $X_1, ..., X_s$. If $1/2 \in \mathfrak{o}_D$ the classification simplifies considerably: Then two ε-hermitian forms in Jordan normal form are in the same genus iff all corresponding Jordan constituents are in the same genus. Since $1/2 \in \mathfrak{o}_D$, the signature and the rank are the only invariant of a Jordan block in the orthogonal case. In the symplectic case and the ${}^2A_n'$ case the rank is the only invariant.

A similar theory of genera also exist for the genera of ε-hermitian forms over quaternion divion algebras. Suppose, either the involution is nontrivial on $\mathfrak{o}_D/\pi_D\mathfrak{o}_D$ or $\varepsilon = 1$. Then $1/2 \in \mathfrak{o}_D$ implies, that any lattice Λ has an $\mathfrak{o}_D$-basis, for which a given ε-hermitian form $(.,.)$ is represented by a diagonal matrix $X = diag(a_1, ..., a_n)$ such that $|H(a_1)| \geq .... \geq |H(a_r)|$. These numbers will be called the elementary divisors and X is called to be in genus normal form: Choose $a \in \Lambda$ such that $|H(a)|$ is maximal. Under the assumptions $|H(a)| \geq |(x, a)|$ holds for all $x \in \Lambda$. Therefore $\Lambda = a \cdot \mathfrak{o}_D \oplus a^\perp$, and we can proceed by induction. If hyperspecial points exist, the assumptions above are satisfied unless G is a symplectic group (for this case see also [W]). Suppose hyperspecial points exist, suppose $1/2 \in \mathfrak{o}_D$ and suppose we are not in the symplectic case. Then $X = diag(a_1, ..., a_n)$ and $\tilde{X} = diag(\tilde{a}_1, ..., \tilde{a}_n)$ in genus normal form define the same genus iff the elementary divisors are the same and all submatrices $diag(a_\nu, ..., a_\mu)$, $diag(\tilde{a}_\nu, ..., \tilde{a}_\mu)$ for the entries, with common absolute value $|H(.)|$ of the hermitian norm, are in the same genus.

$A(X, \Lambda)$**-Orbits**: Let again Λ denote the $\mathfrak{o}_D$-right module generated by Λ_0 and $e_1, .., e_r, e_1', .., e_r'$. We assume $1/2 \in \mathfrak{o}_D$. Put $\Lambda_1 = e_1\mathfrak{o}_D + e_1'\mathfrak{o}_D$ (we assumed $r > 0$). Then $\Lambda = \Lambda_1 \oplus \Lambda_1^\perp$. Under the assumptions above any primitive vector $v \in \Lambda$ is in the orbit of a primitive vector in Λ_1 under $A(X, \Lambda)$. To see this consider $H(v) = \sigma(t)$ for $t = H(v)/2 \in \mathfrak{o}_D$. Put $v_1 = e_1 \cdot t + e_1'$. Then $H(v) = H(v_1)$. If $t \in \mathfrak{o}_D^*$ or $(v, v_1) \in \mathfrak{o}_D^*$ we are done by the remark above. So assume $t \in \mathfrak{o}_D\pi_D$. Since then $(v, v_1) = \alpha^* + \beta^* t\varepsilon \equiv \alpha^* \ (\pi_D)$ – if $v = e_1\alpha + e_1'\beta + ..$ – we are done using a permutation of the isotropic basis vectors e_ν, e'_ν unless $v \in \Lambda \cdot \pi_D + \Lambda_0$. But $v = \tilde{v} \cdot \pi_D + w, \tilde{v} \in \Lambda, w \in \Lambda_0$ can be excluded, since this gives $H(w) \in \pi_D \mathfrak{o}_D$. By the anisotropy of X_{an} modulo π_D this implies $w \in \Lambda_0 \cdot \pi_D$, contradicting the primitivity of v.

Standard forms: We can use this to bring $M \in Hom_D(D^i, D^{n-2i})$ in a standard form. This uses operations from $A(X_0, \Lambda)$ from the left, and from $Gl(i, \mathfrak{o}_D)$ from the right. We make successively B into a diagonal matrix and JA into a upper triangular matrix. Suppose $m \leq min(i,j)$. We say M is of m-standard form, if the m first columns of M have the form

$$\begin{pmatrix} B_m \\ 0 \\ A_m \end{pmatrix} \quad , \quad B_m = diag(\pi_D^{e_1}, .., \pi_D^{e_m}) \ ,$$

where $J^{(m,m)}A_m$ is an upper triangular matrix, where $e_1 \leq e_2 \leq ... \leq e_m \leq e$, e being the exponent greatest common divisor of all entries of the columns of M starting from the $m+1$-st column, such that the gcd of the ν-th column is $\pi_D^{e_\nu}$ for all $1 \leq \nu \leq m$. Suppose $m \leq min(i,j)$ and suppose M is in $(m-1)$-standard form. Then we claim, M can be modified under the action of $A(X_0, \Lambda) \times Gl(i, \mathfrak{o}_D)$ to become a matrix of m-standard form. For this, first make π_D^e to become the gcd of the $m-th$ column by a permutation of columns. Then we may achieve, that the $m-th$ column v is supported in its $m-th$ entries and beyond (by addition of multiples of the first m columns). Thext make the entries of v to be supported at its m-th entry and its entries at the $j+r_{an}+m$-th coordinates and beyond (using the left action of $A(X, \Lambda)$). We may also assume, that the m-th entry is of the form π_D^f for some $f \geq e$. If $f = e$ we are done. Otherwise $f < e$ and the $gcd(v_\mu)$ of the coordinates v_μ of v is attained for some index $2j + r_{an} - \mu$ for $\mu \leq m$. The substitution, which maps $e'_\mu \mapsto e'_\mu + e_m$, $e'_m \mapsto e'_m + \varepsilon e_\mu$ and fixes all other basis vectors, is in $A(X, \Lambda)$. It transforms the first m-th columns on M into

$$\begin{pmatrix} \tilde{B}_m \\ 0 \\ A_m \end{pmatrix}$$

with $\tilde{B}_m$ differing from B_m in the last row and in the μ-th entry of the last column. The last entry of this column is in $\mathfrak{o}_D^* \cdot \pi_D^e$. Left multiplication with a suitably chosen matrix $diag(U^{(m,m)}, id, J(U^*)^{-1}J) \in Aut(X, \lambda)$, where $U - id$ has only integral entries in the last row, leaves the lower block A_m unchanged and eliminates all entries in the last row and transform its last entry into π_D^e. All other entries of $\tilde{B}_m$ remain unchanged. Finally eliminate the μ-th entry of the m-th column by adding a multiple of the μ-th column. This brings the matrix in m-standard form.

We may proceed with this until $m = min(i,j)$. There are two cases. In the first case $i \leq j$. Then M becomes

$$\begin{pmatrix} B_m \\ 0 \\ A_m \end{pmatrix} = \begin{pmatrix} diag(\pi_D^{e_1},..,\pi_D^{e_m}) \\ 0 \\ A_m \end{pmatrix} \quad , \quad m = i$$

being in i-standard form. If on the other hand $i > j$, then M will take the form

$$\begin{pmatrix} B_m & 0 \\ 0 & C_m \\ A_m & D_m \end{pmatrix} = \begin{pmatrix} diag(\pi_D^{e_1},..,\pi_D^{e_m}) & 0 \\ 0 & * \\ A_m & * \end{pmatrix} \quad , \quad m = j$$

with the first j columns in standard form and the gcd of the entries $*$ being divisible by $\pi_D^{e_j}$. We may also assume, that $e_m \leq 0$ for $m = min(i,j)$. Otherwise we could modify M by adding a suitable integral matrix, such that this holds. Then, in particular, $B_m = diag(\pi_D^{e_1},..,\pi_D^{e_m})$ is an invertible diagonal matrix such that both B_m^{-1} and $A_m B_m^{-1}$ are integral. In all cases

$$M_{an} = (0^{(r_{an},j)}, *) \ .$$

Using a right transform by a unimodular matrix, we can also assume that $C_{an}^* X_{an} C_{an}$ is in Jordan normal form, in particular by our assumption $1/2 \in \mathfrak{o}_D$ we can assume that it is block-diagonal with an invertible first block of rank $\leq r_{an}$ and zero blocks else. But then using anisotropy of X_{an}, C_{an} is also a block matrix with invertible first block of the same rank, and zero blocks else.

Elementary divisors: We only consider the case where $i \leq j$. In this case we have

$$\Lambda_s(M) = \begin{pmatrix} 0 & 0 & 0 & 0 & \mathfrak{X} \\ 0 & 0 & 0 & -J & 0 \\ 0 & 0 & -X_{an} & 0 & 0 \\ 0 & \varepsilon J & 0 & 0 & 0 \\ \varepsilon \mathfrak{X}^* & 0 & 0 & 0 & \mathfrak{A} \end{pmatrix}$$

with $\mathfrak{X}$ and $\mathfrak{A}$ as in the special case $i = j$. This allows to reduce all arguments for $i \leq j$ to the special case $i = j$, $r_{an} = 0$. We therefore will assume this for simplicity of notation. In particular we then write again $\mathfrak{X} = \mathcal{X}$ and $\mathfrak{A} = \mathcal{A}$ and

$$\Lambda_s(M) = \Lambda_s(fixed) \oplus \begin{pmatrix} 0 & \mathcal{X} \\ \varepsilon\mathcal{X}^* & \mathcal{A} \end{pmatrix} \quad , \quad \Lambda_s(fixed) = \begin{pmatrix} 0 & 0 & -J \\ 0 & -X_{an} & 0 \\ \varepsilon J & 0 & 0 \end{pmatrix} .$$

Now consider

$$g(Z,\alpha,\beta) = \begin{pmatrix} E & \alpha & Z \\ 0 & E & \beta \\ 0 & 0 & E \end{pmatrix} \in A(X),$$

i.e satisfying $\alpha^* J + X_{an}\beta = 0$ and $Z^*J + \varepsilon JZ + \beta^* X_{an}\beta = 0$ for a $(r \times r)$-matrix Z. If $\alpha = 0$, $\beta = 0$, then $Z^* = -\varepsilon JZJ$. Then for $g(Z) = g(Z,0,0)$

$$g(-Z)\Lambda_s(M)g(-Z)^* = \begin{pmatrix} \tilde{\mathcal{A}} & 0 & 0 \\ 0 & \Lambda_s(fixed) & 0 \\ 0 & 0 & \mathcal{A} \end{pmatrix}$$

with $\tilde{\mathcal{A}} = -Z\mathcal{A}Z^*$, if $\mathcal{X} = Z \cdot \mathcal{A}$.

Since M was chosen in standard form and since we are now in the case $i = j$, the matrix $\mathcal{A}$ is invertible; equivalently B_m is invertible. Since we are in the case $i = j$ (or $i \leq j$ to be precise) the matrix M_{an} vanishes.

This being said, the matrix $Z = \mathcal{X}\mathcal{A}^{-1}$ defined above is uniquely determined and satisfies $Z^* = -\varepsilon JZJ$. To see this, note that this is equivalent to JZ being $-\varepsilon$-hermitian, or $\mathcal{A}^*JZ\mathcal{A} = \varepsilon\mathcal{A}J\mathcal{X}$ being $-\varepsilon$-hermitian. Now

$$\mathcal{A}J\mathcal{X} = \begin{pmatrix} 0 & -2\varepsilon JB \\ 2B^*J & -2M^*X_0M + 4B^*JA \end{pmatrix} .$$

Since $4B^*JA - 2M^*X_0M = 2(B^*JA - \varepsilon A^*JB) - 2M_{an}^*X_{an}M_{an}$, this matrix is $-\varepsilon$-hermitian since $M_{an} = 0$. But then $g(Z) \in SA(X)$, hence

$$\begin{pmatrix} \tilde{\mathcal{A}} & 0 \\ 0 & \mathcal{A} \end{pmatrix} \in SA(X_{split}) .$$

As a consequence $\mathcal{A}^* J\tilde{\mathcal{A}} = J$ or

$$\tilde{\mathcal{A}} = \varepsilon J\mathcal{A}^{-1}J \ .$$

Note

$$\mathcal{A} = \begin{pmatrix} E & A \\ 0 & E \end{pmatrix}^* \begin{pmatrix} 0 & -2\varepsilon JB \\ -2B^*J & 0 \end{pmatrix} \begin{pmatrix} E & A \\ 0 & E \end{pmatrix} \ .$$

Hence $-2Z = -2\mathcal{X}\mathcal{A}^{-1}$ is equal to

$$-2Z = \begin{pmatrix} 0 & J \\ -J & -2JA \end{pmatrix} \begin{pmatrix} E & -A \\ 0 & E \end{pmatrix} \begin{pmatrix} 0 & \varepsilon JB \\ B^*J & 0 \end{pmatrix}^{-1} \begin{pmatrix} E & 0 \\ -A^* & E \end{pmatrix} \ .$$

This becomes

$$\begin{pmatrix} 0 & J \\ -J & -JA \end{pmatrix} \begin{pmatrix} 0 & J(B^*)^{-1} \\ \varepsilon B^{-1}J & 0 \end{pmatrix} \begin{pmatrix} E & 0 \\ -A^* & E \end{pmatrix}$$

$$= \begin{pmatrix} \varepsilon JB^{-1}J & 0 \\ (B^*)^{-1}A^* - JAB^{-1}J\varepsilon & -(B^*)^{-1} \end{pmatrix} \ .$$

In other words JZ is $-\varepsilon$-hermitian

$$-2JZ = -2\begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix} Z$$

$$= \begin{pmatrix} J(B^*)^{-1}A^* - \varepsilon AB^{-1}J & -J(B^*)^{-1} \\ \varepsilon B^{-1}J & 0 \end{pmatrix} \ .$$

Furthermore

$$-2\mathcal{A}^{-1} = \begin{pmatrix} E & -A \\ 0 & E \end{pmatrix} \begin{pmatrix} 0 & J(B^*)^{-1} \\ \varepsilon B^{-1}J & 0 \end{pmatrix} \begin{pmatrix} E & 0 \\ -A^* & E \end{pmatrix}$$

$$= \begin{pmatrix} -J(B^*)^{-1}A^* - \varepsilon AB^{-1}J & J(B^*)^{-1} \\ \varepsilon B^{-1}J & 0 \end{pmatrix} \ .$$

We conclude, that $g(Z) \in A(X,\Lambda)$ and $\tilde{\mathcal{A}}$ is integral ε-hermitian, since B is invertible and $B^{-1}J$ and $AB^{-1}J$ are integral matrices.

Finally there is the obvious map to the genera of ε-hermitian forms

$$\gamma : \{\varepsilon\text{-hermitian matrices}\}/A(X,\Lambda) \longrightarrow \{\varepsilon\text{-hermitian matrices}\}/Gl(n,\mathfrak{o}_D)\ .$$

We consider the image of $\mathfrak{Y} = g(-Z)\Lambda_s(M)g(-Z)^*$

$$\mathfrak{Y} \ \cong\ \mathcal{A} \oplus \begin{pmatrix} 0 & 0 & -J \\ 0 & -X_{an} & 0 \\ -\varepsilon J & 0 & 0 \end{pmatrix} \oplus \tilde{\mathcal{A}}\ .$$

Proposition: *Suppose hyperspecial points exist. Furthermore assume* $1/2 \in \mathfrak{o}_D$ *and suppose* $s = diag(E_i, -E, E_i)$ *for* $i \leq j = r - i$. *Let* H *be the centralizer of* s *in* $A(X)$. *Then the map* γ, *hence also the composite map*

$$H \setminus A(X)/A(X,\Lambda) \ \to\ \{\textit{genera of } \varepsilon\textit{-hermitian } n \times n\textit{-forms}\}\ ,$$

which maps the representative $g(M)$ *with* M *in standard form to the genus of the* ε*-hermitian matrix* $\mathfrak{Y} = \mathfrak{Y}(M)$, *has finite fibers. The map* δ

$$\delta :\ H\backslash SA(X)/SA(X,\Lambda) \ \to\ \{\textit{genera of integral } \varepsilon\textit{-hermitian } 2i \times 2i\textit{-forms}\}\ ,$$

which maps the representative $g(M)$ *with* M *in standard form to the genus of the integral* ε*-hermitian matrix* $\tilde{\mathcal{A}} = \tilde{\mathcal{A}}(M)$, *is well defined and injective. Here* H *now denotes the centralizer of* s *in* $SA(X)$.

To prove the proposition, consider the Jordan decomposition of $\mathfrak{Y}$. All three summands of $\mathfrak{Y}$ contribute to essentially different Jordan constituents: the integral ε-hermitian matrix

$$-2\varepsilon \cdot \tilde{\mathcal{A}} = -2 \cdot J\mathcal{A}^{-1}J \ =\ \begin{pmatrix} 0 & \varepsilon JB^{-1} \\ (B^*)^{-1}J & \Omega \end{pmatrix}\ ,$$

with $\Omega = -\varepsilon(JAB^{-1}) - (JAB^{-1})^*$, has only integral Jordan constituents with positive scaling exponents, hence $\mathcal{A} = \varepsilon J\tilde{\mathcal{A}}^{-1}J$ has only Jordan constituents with negative scaling exponents. The third summand is a unimodular Jordan constituent. So the only possible common Jordan block is the unimodular one. If two representatives M and M' in standard form correspond to matrices $\mathfrak{Y}$ and $\mathfrak{Y}'$ in the same genus, then by our assumption $1/2 \in \mathfrak{o}_D$ the submatrices $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{A}}'$ necessarily are in the same genus in the C_n and unitary cases ${}^2A_n$. There may be one small exception in the $D_n, B_n, {}^2D_n$-cases the unimodular block may be replaced by one with different signature; in these cases the fibers may therefore have cardinality two.

Suppose there exists $U \in Gl(i, \mathfrak{o}_D)$ such that $\tilde{\mathcal{A}}' = U^*\tilde{\mathcal{A}}U$. Then $g = diag(U, E, J(U^*)^{-1}J) \in SA(X, \Lambda)$. Furthermore $\lambda(g)^{-1}g^*\mathfrak{Y}'g$ is equal to $\mathfrak{Y}$. Hence the map δ is injective, if it is well defined. To show that δ is well defined, it is enough to consider the orthogonal cases $D_n, B_n, {}^2D_n$. So for the remaining part of the proof of the proposition we assume in addition, that $G = SA(X)$ is a special orthogonal group over F.

For a invertible symmetric matrix X over F, the smallest integer c such that $\pi_F^c X^{-1}$ is integral, is called the conductor of X. Let X_0, Y_0 and X_1, Y_1 be integral symmetric invertible matrices of equal rank. Suppose $\pi_F^c X_0^{-1}$ and $\pi_F^c Y_0^{-1}$ and $\pi_F^d X_1^{-1}$ and $\pi_F^d Y_1^{-1}$ are integral for some integers c, d. Put $X = diag(X_0, \pi_F^{c+1}X_1)$ and $Y = diag(Y_0, \pi_F^{c+1}Y_1)$ and $f = c + 1 + d$, and let m denote the rank of X and Y. Suppose $g \in Gl(m, \mathfrak{o}_F)$ such that g^*Xg is congruent to Y modulo π^{f+1}. For

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

this is equivalent to the following congruences: $\alpha^*X_0\alpha + \pi_F^{c+1}\gamma^*X_1\gamma \equiv Y_0(\pi_F^{f+1})$, $\alpha^*X_0\beta + \pi_F^{c+1}\gamma^*X_1\delta \equiv 0(\pi_F^{f+1})$, $\beta^*X_0\beta + \pi_F^{c+1}\delta^*X_1\delta \equiv \pi_F^{c+1}Y_1(\pi_F^{f+1})$.

The first congruence implies $Y_0\alpha^*X_0\alpha \equiv E - \pi_F(\pi_F^c Y_0^{-1})\gamma^*X_1\gamma(\pi_F^{f-c+1})$, since $f \geq c$. Hence $Y_0\alpha^*X_0\alpha \equiv E(\pi_F)$. Now assume in addition that $det(X_0)/det(Y_0) \in \mathfrak{o}_F^*$. Then we get $det(\alpha) \in \mathfrak{o}_F^*$. Thus α is unimodular, $det(X_1)/det(Y_1) \in \mathfrak{o}_F^*$ and $\alpha^*X_0\alpha \equiv Y_0(\pi^{c+1})$. Then the second of the congruences above implies $\beta \equiv -\pi_F^{c+1}X_0^{-1}(\alpha^*)^{-1}\gamma^*X_1\delta(\pi_F^{f-c+1})$, hence $\beta \equiv 0(\pi_F)$ since $\pi_F^c X_0^{-1}$ is integral. α being unimodular and β being congruent to zero, implies that δ is unimodular. Substituting $X_0\beta$ from the second congruence into the third congruence $\pi_F^{-c-1}\beta^*X_0\beta + \delta^*X_1\delta \equiv Y_1(\pi_F^{f-c})$ gives $\pi_F^{-c-1}\beta^*X_0\beta \equiv -\beta^*(\alpha^*)^{-1}\gamma^*X_1\delta(\pi_F^{f-c})$. Inserting the congruence for β, this term is congruent to $\pi_F^{c+1}X_0^{-1}[(\alpha^*)^{-1}\gamma^*X_1\delta](\pi_F^{f-c})$. Hence $\delta^*X_1\delta \equiv Y_2(\pi_F^{f-c})$, where $Y_2 = Y_1 - \pi_F \cdot (\pi_F^c X_0^{-1})[(\alpha^*)^{-1}\gamma^*X_1\delta]$. In particular $Y_2 \equiv Y_1$ modulo π_F. In fact, the last congruence even implies $E \equiv X_1^{-1}(\delta^{-1})^*Y_2\delta^{-1}(\pi_F)$. Hence $det(Y_2)/det(X_1) \in \mathfrak{o}_F^*$. Therefore X_1 and Y_2 have the same elementary divisors.

Let now $X = diag(X_0, \pi_F^{c+1} X_1)$ and $Y = diag(Y_0, \pi_F^{c+1} Y_1)$ two integral symmetric matrices with conditions on the conductors as above, such that $g^* X g = Y$ for some $g \in Gl(m, \mathfrak{o}_F)$. Then, by the theory of genera or inductively by similar arguments as above, the condition $det(X_0)/det(Y_0) \in \mathfrak{o}_F^*$ automatically holds. As a consequence

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

has unimodular submatrices α and δ, β being congruent to zero modulo π_F. Furthermore $\alpha^* X_0 \alpha \equiv Y_0(\pi_F)$ and $\delta^* X_1 \delta \equiv Y_1(\pi_F)$. As a further consequence any isomorphism $g^* X g = Y$ for $g \in GL(m, \mathfrak{o}_F)$ between two integral symmetric invertible matrices $X = diag(X_0, \pi_F X_1, ..)$ and $Y = (Y_0, \pi_F Y_1, ..)$ in Jordan normal form is given by a matrix g congruent to a lower tringular block matrix

$$g \equiv \begin{pmatrix} \delta_0 & 0 & ... \\ * & \delta_1 & ... \\ * & * & *.. \end{pmatrix}$$

modulo π_F, such that all δ_ν are unimodular and satisfy $\delta_\nu^* X_\nu \delta_\nu \equiv Y_\nu$ for all the unimodular Jordan blocks X_ν, Y_ν.

This being said, the injectivity of the map δ

$\delta:\ H \backslash SA(X)/SA(X,\Lambda) \ \to \ \{$genera of integral ε-hermitian $2i \times 2i$-forms$\}$

follows , if there is no unimodular matrix $h \in SA(X, \Lambda)$

$$h^* \begin{pmatrix} 0 & 0 & J \\ 0 & X_{an} & 0 \\ J & 0 & 0 \end{pmatrix} h \ = \ \begin{pmatrix} 0 & 0 & J \\ 0 & X_{an} & 0 \\ J & 0 & 0 \end{pmatrix} \ ,$$

such that $h^* \mathfrak{Y} h = \mathfrak{Y}'$ holds for unimodular matrices $\mathfrak{Y}$, $\mathfrak{Y}'$ with unimodular $\tilde{\mathcal{A}}' = diag(1, ..., 1, \kappa)$ and $\mathcal{A}' = J(\tilde{\mathcal{A}}')^{-1} J$ and similar for $\mathfrak{Y}$ with $\tilde{\mathcal{A}} = E$. Here $\mu \in \mathfrak{o}_F^*$ denotes any element $\mu \notin (\mathfrak{o}_F^*)^2$. In other words this means

$$h^* h = diag(1,,,1,\mu,E,\mu^{-1},1..,1) \ .$$

To show this we may pass to the residue class field κ of F, provided the following matrices (viewed as matrices in κ by abuse of notation)

$$S = \begin{pmatrix} 0 & 0 & J \\ 0 & X_{an} & 0 \\ J & 0 & 0 \end{pmatrix} \quad , \quad S' = \begin{pmatrix} 0 & 0 & J(\mu) \\ 0 & X_{an} & 0 \\ J(\mu)^* & 0 & 0 \end{pmatrix}$$

are not conjugate $Sh = hS(\mu)$ under a κ-valued matrix h in $SA(X)$. Here $J(\mu)$ denotes $diag(1,..1,\mu^{-1})J$. Since X_{an} can be assumed diagonal one can easily reduce to the case, where the entry X_{an} in the matrices $S, S(\mu)$ is replaced by the identity matrix. We can also embed into a larger orthogonal group, such that the entry J has even rank $2k$. By this we can achieve that $S, S(\mu)$ are matrices in $G(\kappa)$, where G is the special orthogonal group $G = SA(X)$ over κ. Both matrices

$$S = \begin{pmatrix} 0 & 0 & J \\ 0 & E & 0 \\ J & 0 & 0 \end{pmatrix} \quad , \quad S' = \begin{pmatrix} 0 & 0 & J(\mu) \\ 0 & E & 0 \\ J(\mu)^* & 0 & 0 \end{pmatrix}$$

are conjugate in $G(\overline{\kappa})$, where $\overline{\kappa}$ is the algebraic closure of the residue class field κ. In fact, by computing the characteristic polynomials of the matrices S and $S(\mu)$, both matrices are conjugate to the matrix $s_k = diag(E_k, -E_{2k+r_{an}}, E_k)$ over $\overline{\kappa}$. Hence the obstruction for conjugacy over the field κ is a cohomology class in the Galois cohomology set $H^1(\kappa, G_S)$, where G_S is the centralizer of the element $S \in G(\kappa)$. Since $zSz^{-1} = S(\mu)$ holds for $z = diag(1,..,1,\rho,E,\rho^{-1},1,..,1)$, where $\rho \in \overline{\kappa}$ is a square root of μ, the obstruction class is represented by the following 1-cocycle $\phi(\sigma) = z^{-1}\sigma(z) \in G_S(\overline{\kappa})$:

$$\phi(\sigma) = diag(1,..,1,-1,E,-1,1,..,1)$$

if σ induces a nontrivial automorphism of $\kappa(\rho)/\kappa$ and $\phi(\sigma) = 1$ otherwise.

Over $\overline{\kappa}$ the centralizer G_S is conjugate to the maximal reductive subgroup $H = G_s \subset G$ introduced above, hence isomorphic to a subgroup of index two (equal determinants) of a product of two orthogonal groups. This is also true over the field κ itself. In fact $S^2 = E$, hence $P_\pm = (E \pm S)/2$ define matrix projectors commuting with G_S, which provide the isomorphism. For simplicity we restrict ourselves to the case, where $X_{an} = 0$. Then

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in G_S satisfies $c = JbJ, d = JaJ$. Furthermore under these conditions $g \in G_S$ is equivalent to $a^*a + Jb^*bJ = E$, $a^*b + Jb^*aJ = 0$, or equivalent to $(a \pm bJ)^*(a \pm bJ) = E$. Therefore the map $g \mapsto (g_1, g_2) = (a+bJ, a-bJ)$ maps the centralizer G_S isomorphically to the group $\{(g_1, g_2) \in O(2k,\kappa) \mid det(g_1) =$

$det(g_2)\}$. Therefore by Lang's theorem, since $SO(2k)$ is connected, we get a bijection

$$H^1(\kappa, G_J) \cong H^1(\kappa, \{\pm 1\})$$

induced by the homomorphism $g \mapsto det(g_1)$. Applied to the cocycle $\phi(\sigma)$ from above, we get as the image cocycle $\psi(\sigma) = det(\phi(\sigma)_1$. Since $\psi(\sigma) = -1$, if σ induces the nontrivial automorphism of $\kappa(\rho)/\kappa$, the cohomology class of ψ is nontrivial. Hence S and $S(\mu)$ are not conjugate over κ. This completes the proof of the proposition.

Corollary: *Under the hypotheses of the last proposition there exist finitely many elements $r_i, i = 1, ..., h$ in $G(F) = SA(X)$, such that $\bigsqcup_{i=1}^h r_i T(F)$ contains a set of representatives for the double cosets $H(F)\backslash G(F)/K$, where K is the maximal compact subgroup $K = SA(X, \Lambda)$.*

To prove the statement consider the injective map

$$\delta: \ H\backslash SA(X)/SA(X,\Lambda) \ \to \ \{\text{genera of integral } \varepsilon\text{-hermitian } 2i \times 2i\text{-forms}\} .$$

The elements on the right side will be identified a complete set of inequivalent integral matrices in Jordan normal form. All Jordan blocks are of the form $\pi_F^\nu X_\nu$, where X_ν is a unimodular matrix. Let Σ denote the image of the map δ. We have already seen, that for every integral matrix $X \in \Sigma$ in Jordan normal form, there exists an element $r = r(\tilde{A}) \in SA(X)$, such that $r^*\Lambda_s r = diag(X, \Lambda_s(fixed), J(X^{-1})J)$. In fact we can choose $r = g(M) \cdot g(-Z) \cdot diag(U, E, J(U^*)^{-1}J)$, which is contained in the special orthogonal group $SA(X)$. Under the map δ the double coset $H(F)rK$ maps to the genus of $\tilde{\mathcal{A}}$. The matrices $t = diag(t_1, .., t_{2i}, E, t_{2i}^{-1}, .., t_1^{-1})$ are in $T(F)$. We have

$$\delta\Big(H(F)rtK\Big) \ = \ genus(T^*XT) \ ,$$

where $T = diag(t_1, .., t_{2i})$. Hence the set Σ is stable under the action of such diagonal matrices T. There exist finitely many matrices X_j in Jordan normal form, such that every integral matrix is equivalent to a matrix of the form T^*X_jT. So there are finitely many representatives $X_j, j = 1, .., h$ for the action of these matrices T on Σ. The corresponding matrices $r_j, j = 1, .., h$ have the required property, the corollary follows.

The image of δ: We finally want to describe a complete set of inequivalent representatives for the double cosets. For this we have to determine the image

Σ of the map δ. This is particularly simple in the case of the symplectic group. In this case $h = 1$. See [W]. The unitary case ${}^2A_n$ is similar to the symplectic case. By the theory of hermitian genera [J] every Jordan block can be chosen to be of the form $\pi_F^\nu X_\nu$, where X_ν is the unit matrix. Hence the genera of matrices

$$\tilde{\mathcal{A}} = \begin{pmatrix} 0 & \varepsilon J B^{-1} \\ (B^*)^{-1} J & \Omega \end{pmatrix} ,$$

where B^{-1} and Ω are integral diagonal matrices whose entries are powers of π_F, exhaust all genera of matrices of the more general form with arbitrary integral symmetric matrices Ω, as they arise in the image Σ of the map δ. In fact an integral $2i \times 2i$ matrix X in Jordan normal form is in the image set Σ if and only if $\sum_\nu rank(X_\nu) \cdot \nu$ is even. We can therefore find a complete set of representatives among the unipotent matrices $g(M)$ with M in standard form, such that both JA and B are diagonal matrices, whose entries are powers of π_F. Furthermore h can be chosen to be $h = 2^{2i-1}$.

The description of the image set Σ in the orthogonal cases has to take care of the fact, that each Jordan block $\pi_F^\nu X_\nu$ now has two invariants, the rank of X_ν and the class of the determinant $det(X_\nu) \in \mathfrak{o}_F^*/(\mathfrak{o}_F^*)^2 \cong \mathbb{Z}/2\mathbb{Z}$. The genera X of matrices in the image Σ

$$\tilde{\mathcal{A}} = \begin{pmatrix} 0 & \varepsilon J B^{-1} \\ (B^*)^{-1} J & \Omega \end{pmatrix}$$

are characterized by the following property:

X splits over F.

In fact, every matrix $\tilde{\mathcal{A}}$ splits over F. Its first $i \times i$-block defines a maximal isotropic subspace, which can be complemented over F. Conversely, if X splits over F let W be a maximal isotropic subspace of dimension F^i. The intersection $\Lambda_0 = W \cap \mathfrak{o}_F^{2i}$ defines a sublattice in $\mathfrak{o}_F^{2i}$, which has a torsion free quotient $\mathfrak{o}_F^{2i}/\Lambda_0$. Hence an $\mathfrak{o}_F$-basis of Λ_0 can be complemented to a basis of $\mathfrak{o}_F^{2i}$. For such a basis, the quadratic form X is represented by a symmetric integral matrix

$$(-2\varepsilon)^{-1} \cdot \begin{pmatrix} 0 & Y \\ Y^* & \Omega \end{pmatrix} .$$

This symmetric matrix being nondegenerate, the matrix Y has nonvanishing determinant. Using unimodular transformations and the elementary divisor theorem, the matrix Y can assumed to be antidiagonal of the form εJB^{-1} with $B = diag(\pi_F^{e_1}, ..., \pi_F^{e_i})$ with increasing exponents $e_1 \leq e_2 \leq ... \leq e_i \leq 0$. The matrix Ω can be written as the symmetrization of an integral upper triangular matrix. This can be written in the form $-(JA)B^{-1}$ for some integral upper triangular matrix JA.

The condition, that X splits over F, imposes two conditions on the components of the Jordan splitting of $X \cong X_0 \oplus \pi_F X_1 \oplus ....$ Note that $rank(X) = 2i$ is even. So the first condition is, that the discriminant $(-1)^i det(X)$ is a square in F^* . The second condition is $\prod_\nu \gamma(\pi_F^\nu X_\nu) = 1$ for the product of the γ-invariants (an eight root of unity). Equivalently this can be stated as a certain condition on the Hasse invariant of X. This completely describes the image Σ of the map δ in the orthogonal cases $B_n, D_n, {}^2D_n$.

In case, where -1 is a square in F^*, we remark that again a complete set of representatives for the double cosets $H(F) \setminus SA(X)/SA(X, \Lambda)$ in the orthogonal groups can be chosen among the unipotent matrices $g(M)$, where M is in standard form with a diagonal matrix JA. In fact, an integral F-split 2×2 matrix of the form

$$\begin{pmatrix} 0 & \pi_F^e \\ \pi_F^e & \omega \end{pmatrix}$$

has Jordan normal form either $\pi_F^e \cdot diag(1, -1)$, or $\omega_0 \pi_F^a \oplus -\omega_0^{-1} \pi_F^{2e-a}$ if $\omega = \omega_0 \pi_F^a$ with $\omega_0 \in \mathfrak{o}_F^*$ where $a < e$. So in case $-1 \in \mathfrak{o}_F^*$ such a two dimensional integral F-split block can be split of from a diagonal Jordan matrix $X = diag(x_1, ..., x_{2i})$ as a direct summand, whenever $x_j x_{j'} \in (F^*)^2$ holds for a pair $j \neq j'$ of indices. Let $1, \mu, \pi_F, \mu\pi_F$ with $\mu \in \mathfrak{o}_F^*$ be a set of cosets for $F^*/(F^*)^2$. Then $det(X) \in (F^*)^2$ always implies existence of such a pair of indices unless $rank(X) = 4$ and X has Jordan splitting isomorphic to $\tilde{X} \cong t_1 \oplus \mu t_2 \oplus \pi_F t_3 \oplus \mu\pi_F t_4$ for certain squares $t_1, t_2, t_3, t_4 \in (F^*)^2$. In this case X is equivalent over F to the norm form $1 \oplus -\mu \oplus -\pi_F \oplus \mu\pi_F$ of a quaternion algebra. Inductively this implies, that X is a direct sum of two dimensional integral F-split forms and at most a four dimensional remaining summand $\tilde{X}$ as above. However comparing γ-invariants, one finds that this four dimensional summand $\tilde{X}$ cannot occur, since its γ-invariant is -1. Namely $1 = \gamma(X) = \gamma(\tilde{X})$, since all two dimensional F-split summands have a trivial γ-invariant. Any direct sum of two dimensional integral F-split forms is obtained from a pair (B, JA), where JA is a diagonal matrix.

If $\mu = -1$ is not a square, then a two dimensional integral F-split summand can be split off if and only if there exist a pair of indices $j \neq j'$ such that $x_j x_{j'} \in \mu \cdot (F^*)^2$ holds. Now for instance the forms $X = diag(1, \pi_F^{2a_1}, \mu\pi_F, \mu\pi_F^{1+2a_2}, \pi_F^2, \pi_F^{2+2a_3}, \mu\pi_F^3, \mu\pi_F^{3+2a_4})$ are split over F, but do not contain two dimensional F-split direct summands over $\mathfrak{o}_F$ if $2 \leq a_1 \leq a_2 \leq a_3 \leq a_4$.

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