Four dimensional Galois Representations

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Introduction

It is well known how to associate λ-adic representations to irreducible cuspidal automorphic representations of the group $Gl(2,\mathbb{A})$, if the infinite component belongs to the discrete series representations, $\mathbb{A}$ being the rational adele ring. In the case $Gl(2)$ this leads to the study of the classical holomorphic cuspforms of weight $k \geq 2$. In comparison, the related problem for irreducible cuspidal automorphic representations Π of the group $GSp(4,\mathbb{A})$ is not understood as well. Again assume, that the representation Π_∞ at the infinite place belongs to the discrete series. The discrete series representations of $GSp(4,\mathbb{R})$ are parametrized by L-packets, each L-packet contains two classes of irreducible representations. One of them is a member of the holomorphic discrete series and does not have a Whittaker model, whereas the other has a Whittaker model. The packets itself are parametrized by their weight up to a character twist. The weight is described by a pair of integers (k_1,k_2) such that $k_1 \geq k_2 \geq 3$. In fact, the lowest K_∞-type of the holomorphic discrete series is characterized by its highest weight vector, which is defined by the weight (k_1,k_2). In the Whittaker case the corresponding K_∞-type has highest weight $(k_1, 2-k_2)$.

In the following consider irreducible cuspidal automorphic representations $\Pi = \Pi_\infty \Pi_f$ of the group $GSp(4,\mathbb{A})$ with component Π_∞ belonging to the discrete series lying in a L-packet of weight (k_1,k_2). We abbreviate this by saying, that Π has weight (k_1,k_2). The main theorem is the following

Theorem I: Suppose Π is a unitary cuspidal irreducible automorphic representation of $GSp(4,\mathbb{A})$ with Π_∞ belonging to the discrete series of weight (k_1,k_2). Let S denote the set of ramified places of the representation Π. Put $w = k_1 + k_2 - 3$. Then there exists a number field E, such that for primes $p \notin S$ the local L-factor

$$L_p(p^{-s}) = L_p(\Pi_p, s - \frac{w}{2}) \quad , \quad L_p(X)^{-1} \in E[X]$$

of the degree 4 spinor L-series (suitably normalized) has coefficients in E, and such that for any prime number l and any extension λ of l to E there exists a four dimensional semisimple Galois representation

$$\rho_{\Pi,\lambda} : Gal(\overline{\mathbb{Q}} : \mathbb{Q}) \to Gl(4, \overline{E_\lambda}) \ ,$$

which is unramified outside $S \cup \{l\}$, and for $p \notin S \cup \{l\}$ the following holds

$$L_p(\Pi_p, s - \frac{w}{2}) = det(1 - \rho_{\Pi,\lambda}(Frob_p)p^{-s})^{-1} \ .$$

The eigenvalues of $\rho_{\Pi,\lambda}(Frob_p)$ for $p \neq l, p \notin S$ are algebraic integers. The representation $\rho_{\Pi,\lambda}$ arises from a λ-adic representation, if E is chosen suitably large. The so defined λ-adic representation $\rho_{\Pi,\lambda}$ is mixed. If Π is not a CAP representation (see [S]) the representation $\rho_{\Pi,\lambda}$ is pure of weight w, i.e. for all isomorphisms $\overline{\mathbb{Q}}_l \cong \mathbb{C}$ the image of the eigenvalues of $\rho_{\Pi,\lambda}(Frob_p)$ has absolute value $p^{\frac{w}{2}}$ for $p \neq l$ and $p \notin S$.

<u>1.Remark</u> $Frob_p$ is the geometric Frobenius. Our normalization of the λ-adic representation $\rho_{\Pi,\Lambda}$ is cohomological. It is the dual of the usual homological normalization, which is used in the literature on elliptic modular forms. In fact, the dual representation $\rho_{\Pi,\lambda}^\vee$ is

$$\rho_{\Pi,\lambda}^\vee \ \cong \ \rho_{\Pi,\lambda} \otimes \chi^{-1} \quad , \quad \chi = \omega_\Pi \cdot \mu_l^{-w},$$

where is the cyclotomic character $\mu_l(Frob_p) = p^{-1}$. This is a consequence of the Tchebotarev density theorem, since the formula $L_p(\Pi_p, s - \frac{w}{2}) = det(1 - \rho_{\Pi,\lambda}(Frob_p)p^{-s})^{-1}$ and $\Pi \cong \Pi^\vee \otimes \omega_\Pi$ implies that the two semisimple representations $\rho_{\Pi,\lambda}^\vee$ and $\rho_{\Pi,\lambda} \otimes \chi^{-1}$ with $\chi = \omega_\Pi \cdot \mu_l^{-w}$ have the same character. The representation $\rho_{\Pi,\lambda}$ only depends on the weak equivalence class of Π. Two irreducible automorphic representations Π_1, Π_2 are called weakly equivalent, if they are isomorphic $\Pi_{1,v} \cong \Pi_{2,v}$ at almost places v.

Theorem II: The representations $\rho_{\Pi,\lambda}$ constructed in theorem I are not reducible of the form $\rho_{\Pi,\lambda} \cong \rho_0 \oplus \rho_0$, for a two-dimensional λ-adic representations ρ_0 of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$. They contain a one dimensional invariant subspace iff Π is a CAP-representation (of Saito-Kurokawa type).

The representations $\rho_{\Pi,\lambda}$ can be viewed as representations of dimension $4 \cdot [E_\lambda : \mathbb{Q}_l]$ over $\mathbb{Q}_l$. In the way they are constructed, these $\mathbb{Q}_l$-vectorspaces are Hodge-Tate modules of $Gal(\overline{\mathbb{Q}}_l/\mathbb{Q}_l)$. This is a consequence of [CF] theorem 6.2. Moreover

Theorem III: Suppose Π is weakly equivalent to a generic (or multiplicity one) representation. Then the representations $\rho_{\Pi,\lambda}$ define Hodge-Tate modules over $\mathbb{Q}_l$ with four different Hodge types $(k_1+k_2-3, 0), (k_1-1, k_2-2), (k_2-2, k_1-1), (0, k_1+k_2-3)$, each of them occurs with $\mathbb{Q}_l$-dimension $[E_\lambda : \mathbb{Q}_l]$.

Theorem III follows from proposition 1.5. Note that a generic representation of $GSp(4, \mathbb{A})$ has multiplicity one. This follows from [V] p.506 and [Sha].

<u>2.Remark</u>: A weaker version of theorem I was obtained by Taylor [T]. As in [T], p.291ff we use the fact, that the representations Π contribute to the (interior) cohomology of a suitably defined projective limit M of Siegel modular threefolds with respect to a

coefficient system $\mathcal{V}_\mu(\overline{\mathbb{Q}}_l)$, which only depends on the weight (k_1, k_2) of Π. Our result is also deduced from the study of the etale cohomology groups. Results of [W] imply, that the etale cohomology of M defines mixed Galois representations of the absolute Galois group $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$. However, the representations $\rho_{\Pi,\lambda}$ does not completely arise in the cohomology of M in general. However this is true in most of the cases. There are only two situations, where this fails to be true: The case, where Π is a CAP-representation of Saito-Kurokawa type (see [P]) and the case where Π is a weak endoscopic lift (see [W]).

Π is a weak endoscopic lift, if there exist two unitary irreducible cuspidal automorphic forms π_1, π_2 of $Gl(2, \mathbb{A})$ with the same central character $\omega_{\pi_1} = \omega_{\pi_2}$, such that

1) for almost all places $L_v(\Pi, s) = L_v(\pi_1, s)L_v(\pi_2, s)$
2) $\pi_{\infty,i}$ belong to the discrete series of weight r_i with $r_1 > r_2 \geq 2$.

The second assumption was added only for convenience. It is forced, if we consider representations Π, for which Π_∞ belongs to the discrete series.

Conversely let π_1, π_2 be as above. Then (π_1, π_2) lifts to a global L-packet consisting of a weak equivalence class of unitary cuspidal irreducible automorphic representations Π of $GSp(4, \mathbb{A})$, whose components at infinity belong to the discrete series of weight (k_1, k_2). The integers k_i and r_i are related by the formulas $r_1 = k_1 + k_2 - 2$ and $r_2 = k_1 - k_2 + 2$. Weak endoscopic lifts were discussed in [W]. It was shown there, that Arthur's conjectural multiplicity formula holds for these global L-packets.

If the representation Π of $GSp(4, \mathbb{A})$ considered in theorem 1.1 is a CAP-representation, then - again for primes outside some finite set S including the ramified places - $L^S(\Pi, s) = L^S(\pi_1, s)L^S(\pi_2, s)$ holds for a pair of irreducible automorphic representations π_1, π_2 of $Gl(2, \mathbb{A})$. The components of π_i at infinity still belong to the discrete series, however π_i need not be cuspidal any more.

Therefore in both these cases the statement of the main theorem immediately reduces to the corresponding statement for $Gl(2, \mathbb{A})$ proved in [D]. If $\rho_{\pi_i,\lambda}$ are the corresponding λ-adic Galois representations (in cohomological normalization), the representation $\rho_{\Pi,\lambda}$ of theorem 1.1 is formally defined as

$$\rho_{\Pi,\lambda} \;=\; \rho_{\pi_1,\lambda} \bigoplus \rho_{\pi_2,\lambda} \otimes_{\overline{\mathbb{Q}}_l} \mu_l^{\otimes(2-k_2)} .$$

For instance, if Π is a weak endoscopic lift, either $\rho_{\pi_1,\lambda}$ occurs in the Π_f-isotypic component of the third cohomology of the Siegel threefold, if Π_∞ is in the holomorphic discrete series, or the other summand $\rho_{\pi_2,\lambda} \otimes_{\overline{\mathbb{Q}}_l} \mu_l^{\otimes(2-k_2)}$ occurs, if Π_∞ has a Whittaker model. See [W].

<u>3.Remark</u>: Existence of a number field E, as stated in theorem I, is clear for the CAP-cases and the cases of weak endoscopic lifts. Otherwise, its existence follows from the finite dimensionality of the Betti-cohomology of Siegel modular threefolds for fixed levels.

This being said, we will assume for the proof of theorem I, that Π is not a lift of these types. Under this additional assumption - which will be maintained for simplicity starting with section 4 - the Galois representation constructed above are pure λ-adic representations of weight w, i.e. the Ramanujan conjecture holds for all unramified places of Π. This is shown in [W]. A short review is contained in the next section.

Let G denote the Zariski closure of the image $\rho_{\Pi,\Lambda}(Gal(\overline{\mathbb{Q}}/\mathbb{Q}))$ of the absolute Galois group in $Gl(4,k)$, an algebraic group defined over the algebraic closure k of E_λ. Let G^0 be its connected component, and $\pi_0(G)$ the group of components for the Zariski topology. Then

The possible cases are included in the following list:

(i) (later case 1a or 1b) $\pi_0(G)$ contains a finite subgroup of $PGl(2,k)$ as subgroup of index at most 2, and G^0 is a k-torus of dimension 2, or
(ii) (later a case 1a) $\pi_0(G)$ contains a cyclic normal subgroup, whose quotient Δ is a finite subgroup of $PGl(2,k)$, and G^0 is a k-torus of dimension 1, or
(iii) (case 2) G is a subgroup of the normalizer $N(T)$ of the maximal torus $T \subset GSp(4)$ and $\pi_0(G)$ is contained in the Weyl group D_8 of $GSp(4)$, or
(iv) (later case 3) $\pi_0(G)$ is a finite subgroup of $PGl(2,k)$ and $G^0 = Gl(2,k)$, or
(i) $G^0 \subset GSp(4,k)$ or $G^0 = GO(4,k)^0$ and $\pi_0(G) \subset \mathbb{Z}/2\mathbb{Z}$.

More precisely, in appendix D we show that in some cases G is contained in $GSp(4,k)$. For instance his rules out $G^0 = GO(4,k)^0$. Similar for instance case (iv) cannot occur, unless $\rho_{\Pi,\lambda}$ is reducible. In this situation we obtain, that either the representation $\rho_{\Pi,\lambda}$ is irreducible with $G = GSp(4,k)$ and $\rho_{\Pi,\lambda}$ inducing the standard representation, or with $G = Gl(2,k)$ and $\rho_{\Pi,\lambda}$ inducing the third symmetric power of the standard representation, or $\rho_{\Pi,\lambda}$ becomes reducible after restriction to a subgroup of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ of index two.

Theorem IV: Suppose Π is weakly equivalent to a generic (or multiplicity one) representation. Then the representation $\rho_{\Pi,\lambda}$ preserves a nondegenerate symplectic $\overline{\mathbb{Q}}_l$-bilinear form $\langle .,. \rangle$, such that the Galois group acts with the multiplier $\omega_\Pi \mu_l^{-w}$

$$\langle\ \rho_{\Pi,\lambda}(g)v, \lambda_{\Pi,\lambda}(g)w\ \rangle\ =\ \omega_\Pi(g)\mu_l^{-w}(g) \cdot\ \langle v,w \rangle \quad , \quad g \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})\ ,$$

where μ_l is the cyclotomic character.

The first sections 1 and 2 give some overview on facts, that follow from the study of the cohomology of Siegel threefolds. In section 1 we review results of [W], which were obtained by the stabilization of the topological trace formula. Section 2 gives on overview over results of Taylor. The results of these two sections allow to reduce the proof of theorem I to some critical cases. In fact these cases are defined group theoretically as cases 1 and 3. However, this can be reformulated in terms of the automorphic representation Π. It must be D-critical. This is discussed in section 3. The reformulation is contained in appendix B. It uses the classification of balanced representation, established in lemma A of appendix A. In section 4 (and in appendix C) it is shown, that those D-critical representations Π, which are relevant for the proof, contain a theta lift from a representation π of $Gl(2, \mathbb{A}_K)$ where $K/\mathbb{Q}$ is a quadratic algebra with involution σ. In sections 5-8 properties of this theta lift are analyzed. In section 9 a pole number $n_K(\Pi)$ is defined. Using this invariant $n_K(\Pi)$ it is shown in section 10 and appendix C, that $\sigma(\pi) \cong \pi \otimes \chi$ holds for some character χ. Using this the proof of theorem I is given in section 11, by analyzing this property at the archimedian place. Then the proof of theorem II is given in section 12, following similar lines. Appendix D is concerned with pairings and contains a proof of theorem IV. I am grateful to E.Urban for pointing out an error in the previous version of section D.

1. Multiplicity Results and Cohomology

1.1 Lemma: Any irreducible automorphic representation Π of $GSp(4,\mathbb{A})$ is isomorphic to its dual, twisted by the central character: $\Pi \cong \Pi^\vee \otimes \omega_\Pi$.

This answers a question of [T]. p.296.

Proof: In fact this is a local statement. Consider the trace $\chi_{\Pi^\vee}(g) = \chi_\Pi(g^{-1})$. Since g is a symplectic similitude with similitude factor $\lambda(g)$, we have $g^{-1} = \lambda(g)^{-1} \cdot J^{-1} g' J$ with J as in [W]. Hence it is enough to show $\chi_\Pi(g) = \chi_\Pi(g')$ for the transposed matrix g' of g. We may assume $g \in GSp(4,\mathbb{Q}_v)$. It is then enough to prove the trace identity for regular semisimple elements g by a density argument, using well known properties of the trace distribution. Obviously, g and g' are stably conjugate, since g can be diagonalized $g = \gamma a \gamma^{-1}, a' = a$ for some similitudes a, γ defined over the algebraic closure of $\mathbb{Q}_v$. The analysis of instability for the two types of unstable tori in $GSp(4,\mathbb{Q}_v)$, as given in [W1], then implies, that g' and g are conjugate already over $\mathbb{Q}_v$. We leave this easy verification as an exercise for the reader. $\square$

In the following we fix an irreducible cuspidal admissible representation $\Pi = \Pi_\infty \Pi_f$ of $GSp(4,\mathbb{A})$, whose infinite component Π_∞ belongs to the discrete series of weight (k_1, k_2). We do not assume Π to be unitary in this section. Instead we use an "algebraic" normalization, where

$$\Pi \;\cong\; \Pi_0 \otimes \|.\|^{-\frac{c}{2}} \quad , \quad c = k_1 + k_2 - 6 = w - 3$$

as in [T] section 1. This algebraic normalization will be used in this section, the next section and in the appendices B and D. It arises naturally in the study of cohomology groups. In the remaining sections we only consider the unitary representation Π_0 (then often called Π again).

Let $\Pi = \Pi_0 \otimes \|.\|^{\frac{c}{2}}$ be as above with Π_0 unitary. Let $m(\Pi)$ denote its cuspidal multiplicity. In [T], section 1 Taylor defines a finite dimensional $\overline{\mathbb{Q}}_l$-representation $W^\bullet_{\Pi_f}$ of the absolute Galois group of $\mathbb{Q}$, such that $W^\bullet_{\Pi_f}$ arises from an λ-adic representation of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ by extension of scalars. It is obtained from the representation of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ on the Π_f-isotypic component in the interior cohomology of a Siegel modular threefold (image of the cohomology of compact support)

$$H^\bullet_P(M, \mathcal{V}_\mu(\overline{\mathbb{Q}}_l)) \;\cong\; \bigoplus_{\Pi_f} \Pi_f \otimes_{\overline{\mathbb{Q}}_l} W^\bullet_{\Pi_f}$$

of a coefficient system $\mathcal{V}_\mu$ depending on the weight (k_1, k_2). We tacitly assumed the choice of an isomorphism $\overline{\mathbb{Q}}_l \cong \mathbb{C}$, which allows to identify the complex representation Π_f with a representation over the field $\overline{\mathbb{Q}}_l$. See also [T] p.296.

The coefficient system $\mathcal{V}_\mu$ is essentially obtained from the decomposition of tensor products of the sheaf $R^1p_*\overline{\mathbb{Q}}_l$, where $p: A \to M$ is the universal principally polarized abelian variety of genus two over M. It is assigned to a rational finite dimension irreducible representation μ of $GSp(4,\mathbb{Q})$. The coefficient system admits a natural pairing $\lambda_\mu : \mathcal{V}_\mu \otimes_{\overline{\mathbb{Q}}_l} \mathcal{V}_\mu \ \to \ \overline{\mathbb{Q}}_l(-c)$ with $c = (k_1-3)+(k_2-3)$. We can define it by the Lieberman trick: It is enough to consider three cases.

a) If $k_1-3=1, k_2-3=0$, we have $\mathcal{V}_\mu = Rp_*(\overline{\mathbb{Q}}_l)$ of weight $c=1$, and the pairing λ_μ is the (odd) Weil pairing. Since we consider direct images in a cohomology theory, it turns out that μ is the dual of the four dimensional standard representation of $GSp(4,\mathbb{Q})$.

b) In the case $k_1-3=1, k_2-3=1$ we have $\mathcal{V}_\mu = R^2p_*(\overline{\mathbb{Q}}_l)^{prim}$ and $c=2$, and the pairing λ_μ is even. Hence λ_μ has parity $(k_1-3)+(k_2-3) \equiv k_1+k_2$ mod 2 in general.

c) Suppose μ is the dual of the similitude character ν of $GSp(4,\mathbb{Q})$. Then $\mathcal{V}_\mu$ is the line sheaf $\overline{\mathbb{Q}}_l(-1)$ of weight 2. The corresponding Galois action is given by $\mu_l^{-1}(Frob_p) = p$, where μ_l is the cyclotomic character.

All other irreducible representations μ of $GSp(4,\mathbb{Q})$ are contained as constituents $\mu \hookrightarrow (R^1p_*\overline{\mathbb{Q}}_l)^{\otimes i} \otimes (\nu^{-1})^{\otimes j}$. Then $c = c(\mu) = i+2j$. In particular, the trivial representation μ corresponds to the constant etale sheaf $\mathcal{V}_\mu = \overline{\mathbb{Q}}_l$.

Consider the cuspidal third cohomology group $H^3_P(M,\mathcal{V}_\mu)$. There is a cupproduct

$$H^3_P(M,\mathcal{V}_\mu) \times H^3_P(M,\mathcal{V}_\mu) \ \to \ H^6_P(M,\mathcal{V}_\mu \otimes_{\overline{\mathbb{Q}}_l} \mathcal{V}_\mu) \ .$$

The induced pairing

$$H^3_P(M,\mathcal{V}_\mu) \times H^3_P(M,\mathcal{V}_\mu) \ \to \ H^6_c(M,\overline{\mathbb{Q}}(-c)) \xrightarrow{tr} \overline{\mathbb{Q}}(-3-c) = \overline{\mathbb{Q}}_l(-w)$$

has parity $-(-1)^{k_1+k_2}$ (see Milne p.172, remark 1.18) with $c = (k_1-3)+(k_2-3)$. The Galois representation on the cuspidal cohomology group $H^3_P(M,\mathcal{V}_\mu)$ is pure of weight $w = k_1+k_2-3$.

<u>The cupproduct pairing</u> satisfies

$$tr((\sigma\times g)\cdot\eta \ \cup \ (\sigma\times g)\cdot\eta') \ = \ \mu_l(\sigma)^{-c-3}||g||^{-c}\cdot tr(\eta\cup\eta')$$

for all $\sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ and all $g \in GSp(4,\mathbb{A}_f)$. See [T], page 296. The cupproduct pairing, restricted to $(W_{\Pi_f}\otimes\Pi_f)\times(W_{\Pi'_f}\otimes\Pi'_f)$, is zero unless $(W_{\Pi_f}\mu_l^{c+3})\otimes(\Pi_f||.||^c) \cong (W_{\Pi'_f}\otimes\Pi'_f)^\vee$. Equivalently $W_{\Pi_f}^\vee \ \cong \ W_{\Pi'_f}\otimes\mu_l^{c+3}$ and $\Pi'_f \cong {\Pi_f}^\vee\otimes||.||^{-c}$, or $\Pi'_f \cong \Pi_f\otimes\omega_{\Pi_f}^{-1}||.||^{-c}$ by lemma 1.1. So $W_{\Pi'_f} \cong W_{\Pi_f\otimes\omega_{\Pi_f}^{-1}||.||^{-c}} \cong W_{\Pi_f}\otimes\omega_{\Pi_f}^{-1}\mu_l^{-c}$. Therefore $W_{\Pi_f}^\vee \cong W_{\Pi_f}\otimes\mu_l^3\omega_{\Pi_f}^{-1}$. In particular, since W_{Π_f} is pure of weight $w=c+3$, we obtain that $\mu_l^3\omega_{\Pi_f}$ has weight $-2w$. Hence $\omega_{\Pi_f}^{-1} = \mu_l^c\omega_0^{-1}$, where ω_0 is a Dirichlet character of finite order. Therefore $\Pi = \Pi_0\otimes||.||^{-c/2}$ and

$$\omega_\Pi = ||.||^{-c}\omega_0$$

for some unitary representation Π_0 with unitary central character ω_0. (One might prefer to normalize the coefficient systems $\mathcal{V}_\mu$ by a Tate twist, so that $c=0$ and $w=3$ always holds. However this is possible only for even c).

In other words, the cupproduct $\eta,\eta' \mapsto tr(\eta \cup \eta' \cup \omega_0^{-1})$ defines a nondegenerate pairing

$$(W_{\Pi_f} \otimes \Pi_f) \ \otimes \ (W_{\Pi_f} \otimes \Pi_f) \ \to \ \omega_0 \mu_l^{-3-c} \otimes \omega_0 ||.||^{-c} \ = \ \omega_\Pi \mu_l^{-3} \otimes \omega_\Pi \ .$$

Alternatively $\langle \eta, \eta' \rangle = tr(\eta \cup \eta' \cup \omega_\Pi^{-1})$ for $\eta, \eta' \in W_{\Pi_f} \otimes \Pi_f$ defines a pairing

$$(W_{\Pi_f} \otimes \Pi_f) \ \otimes \ (W_{\Pi_f} \otimes \Pi_f) \ \to \ \omega_0 \mu_l^{-3} \otimes \omega_0 \ .$$

It satisfies $\langle F_\infty \cdot \eta, F_\infty \cdot \eta' \rangle = -\langle \eta, \eta' \rangle$, since $F_\infty \cdot \omega_0 = \omega_0(-1) \cdot \omega_0$ and $tr(F_\infty \cdot \eta, F_\infty \cdot (\eta' \cup \omega_0^{-1})) = (-1)^{c+3}$.

Dimension formula: Assume Π is not CAP nor a weak endoscopic lift. This assumption implies, that $W^i_{\Pi_f} = 0$ vanishes except for the middle cohomology degree $i=3$. See [W]. We therefore write $W_{\Pi_f} = W^3_{\Pi_f}$. An irreducible cuspidal representation Π contributes $W_{\Pi_f} \neq 0$, iff Π_∞ belongs to the discrete series of weight (k_1,k_2). Let Π_∞^+ and Π_∞^- denote the discrete series representations of weight (k_1,k_2), of Whittaker type resp. of holomorphic type. We abbreviate $m^+(\Pi_f) = m(\Pi_\infty^+ \Pi_f)$ and $m^-(\Pi_f) = m(\Pi_\infty^- \Pi_f)$. Then the dimension of W_{Π_f} is

$$dim_{\overline{\mathbb{Q}}_l}(W_{\Pi_f}) \ = \ 2 \cdot m^+(\Pi_f) + 2 \cdot m^-(\Pi_f) \ .$$

In particular $dim_{\overline{\mathbb{Q}}_l}(W_{\Pi_f})$ is even. The factor 2 is due to the two Hodge types $(k_1+k_2-3, 0)$ and $(0, k_1+k_2-3)$ resp. (k_1-1, k_2-2) and (k_2-2, k_1-1), to which Π_∞ contributes.

Ramanujan's conjecture: Assume, that Π is neither CAP nor a weak endoscopic lift. Fix a nonarchimedian place v different from l, where Π_v is unramified. Then according to formula (*) in [W]

$$(*) \qquad \prod_{\Pi'} L_v(\Pi'_v, s - \frac{3}{2})^{2 \cdot m(\Pi')} = \prod_{\Pi'_f} \ det(1 - Frob_v | W_{\Pi'_f} \cdot p_v^{-s})^{-4} \ .$$

The products are over all isomorphism classes of irreducible cuspidal automorphic representations Π' of $GSp(4,\mathbb{A})$ resp. Π_f of $GSp(4,\mathbb{A}_f)$, which are unramified at v, such that Π' is isomorphic to Π at all places different from v and ∞. The right side of the formula above is pure of weight $3+c$ by Deligne's proof of the Weil conjectures. As a consequence, the unitary representation $\Pi_v \otimes ||.||_v^{\frac{c}{2}}$ satisfies Ramanujan's conjecture at all unramified places v (by varying l).

A refinement: Two irreducible automorphic representations are weakly isomorphic, if they are isomorphic for almost all places. It is expected, that weakly isomorphic cuspidal irreducible representations Π and Π' are locally isomorphic at all nonarchimedian places v, where Π and Π' are unramified. This would follow from the existence of a good theory of L-series for such representations. For the group $GSp(4)$ a good theory for the spinor L-series has been established by Piateskii-Shapiro and Soudry [PS].

1.2 Lemma: Suppose Π and Π' are weakly isomorphic irreducible cuspidal automorphic representations of $GSp(4,\mathbb{A})$. Assume $\Pi_\infty \cong \Pi'_\infty$ or assume that Π_∞, Π'_∞ are discrete series representations of the same weight. Then Π_v and Π'_v are isomorphic for all places v, where both Π and Π' are both unramified and Ramanujan's conjecture holds.

Proof: By saying, that Ramanujan's conjecture holds, we mean that it holds for the unitary representations $\Pi \otimes \|.\|^{\frac{c}{2}}$ and $\Pi' \otimes \|.\|^{\frac{c}{2}}$. For simplicity of notation assume for this proof, that $c = 0$. From the global functional equation of the L-series of Π and Π' in [P],[PS] we obtain $\prod_v \gamma(\Pi, s) = \prod_v \gamma(\Pi', s)$; a finite product over all places v, where the representations are not isomorphic. From our assumption at the archimedian place we obtain $\gamma_\infty(\Pi, s) = \gamma_\infty(\Pi', s)$; see remark below. Therefore, being rational functions in p_v^{-s}, we get $\gamma_v(\Pi, s) = c_v \cdot \gamma_v(\Pi', s)$ with constants c_v for all v. For unramified Π_v, Π'_v the ϵ-factors are known, therefore $L_v(\Pi^\vee, 1-s)/L_v(\Pi, s) = L_v(\Pi'^\vee, 1-s)/L_v(\Pi', s)$. Hence $L_v(\Pi, s) = L_v(\Pi', s)$ by Ramanujan's conjecture. This implies $\Pi_v \cong \Pi'_v$.

Remark on the archimedian place: There exist weakly isomorphic representations in the weak endoscopic lift, whose infinite components are not isomorphic and belong to the discrete series representations $\Pi^+_\infty, \Pi^-_\infty$ of fixed weight (k_1, k_2) with the same central character. See [W]. A similar argument as above therefore implies $\gamma_\infty(\Pi^+_\infty, s) = \gamma_\infty(\Pi^-_\infty, s)$.
$\square$

This improves formula (*), since now summation only involves the representation $\Pi' = \Pi$ using lemma 1.2. We obtain

1.3 Corollary: Suppose that Π is an irreducible cuspidal automorphic representation of $GSp(4,\mathbb{A})$, which is neither CAP nor a weak endoscopic lift. If Π_∞ is one of the discrete series $\Pi^+_\infty, \Pi^-_\infty$ of weight (k_1, k_2), and if v is an unramified place of Π different from l, then

$$L_v(\Pi_v, s - \frac{3}{2})^{2\cdot m^+(\Pi_f) + 2\cdot m^-(\Pi_f)} = det(1 - Frob_v | W_{\Pi_f} \cdot p_v^{-s})^{-4} \ .$$

By the Tchebotarev density theorem the $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$-representations on the semisimplifications of W_{Π_f} for weakly isomorphic irreducible cuspidal representations are isomorphic. Hence the semisimplification $W^{ss}_{\Pi_f}$ of the Galois representation depends only the weak equivalence class of Π.

Comparing with theorem I, we see that $W^{ss}_{\Pi_f} = n\cdot\rho_{\Pi,\lambda}$ must be an isotypic multiple of the four dimensional representation $\rho_{\Pi,\lambda}$ defined in theorem 1.1, since $\rho_{\Pi,\lambda}$ is not reducible of type $\rho_{\Pi,\lambda} = 2\cdot\rho_0$ for some two dimensional representation ρ_0 by theorem II. Then the dimension formula implies

1.4 Lemma : Suppose Π is an irreducible automorphic representation, which is neither a CAP-representations nor a weak endoscopic lift, such that Π_∞ belongs to the discrete series. Then $m^+(\Pi_f) + m^-(\Pi_f)$ is an even integer.

This is false, if Π is CAP or a weak endoscopic lift. In this case $m^+(\Pi_f) + m^-(\Pi_f) = 1$.

1.5 Proposition: Suppose Π is an irreducible automorphic representation, such that Π_∞ belongs to the discrete series. Suppose Π is neither a CAP-representation nor a weak endoscopic lift. Then $m^+(\Pi_f) = 1$ and $m^+(\Pi'_f) = m^-(\Pi'_f)$ holds for all Π', which are weakly isomorphic to Π.

Proof: By multiplicity 1 Π either $m^+(\Pi_f)$ or $m^-(\Pi_f) = 1$. Therefore $m^{\pm 1}(\Pi_f) > 0$ by the last lemma. The Galois representation on the underlying $\mathbb{Q}_l$-vectorspace is Hodge-Tate ([CF] theorem 6.2). The theorem of Sen implies the existence of four different eigenvalues each with multiplicity $m^+(\Pi_f)$, $m^+(\Pi_f)$, $m^-(\Pi_f)$, $m^-(\Pi_f)$ respectively, in the sense of [T], p.296, [CF] theorem 6.2. By the formula of theorem I relating the Galois representation with the L-series, Galois substitutions with four different eigenvalues have eigenspaces of equal dimension. This forces $m^+(\Pi_f) = m^-(\Pi_f)$. Since the Galois representation depends only on the weak isomorphism class of Π, this also holds for Π' as stated. □

2. A review of Taylor's Results

We introduce some notations following [T], section 1 and 2. We fix and irreducible representation Π, such that Π_∞ belongs to the discrete series of weight (k_1, k_2) with algebraic (nonunitary) normalization of the central character as in the last section. We furthermore assume, that Π is neither CAP nor a weak endoscopic lift. In the following let S be a finite set of places of $\mathbb{Q}$, containing all archimedian places, and all places where the representations Π is ramified. Consider the semisimplification $W^{ss}_{\Pi_f}$ of the Galois representation W_{Π_f}. This is consistent with the assumption of [T] on top of p.297, since we have shown $\Pi^\vee \cong \Pi\otimes\omega_\Pi^{-1}$.

<u>The representation W attached to Π</u>:

Put $k = \overline{\mathbb{Q}}_l$. Then we define W to be the finite dimensional k-vector space $W = W^{ss}_\Pi \oplus W^{ss}_\Pi \oplus W^{ss}_\Pi \oplus W^{ss}_\Pi$, The absolute Galois group acts on W. Let G denote the Zariski closure of the image of the Galois group. This is a reductive group G defined over k. We have a homomorphism

$$r : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \to G \ .$$

The obvious embedding defines a faithful representation $s : G \hookrightarrow Gl(W)$ of the algebraic group G. Let denote G^0 the connected component of G in the Zariski topology. Let T be a maximal split torus in G^0. Let $R \in X^*(T)$ denote the roots of T of the representation s.

<u>Properties of Π</u>:

a) There exists an algebraic homomorphism

$$n : G \to k^* \quad , \quad n|G^0 \neq 1$$

nontrivial on G^0.

b) For all $g \in G$ there are at most 4 eigenvalues of $s(g)$, which come in at most two pairs $\alpha, n(g)/\alpha$.

c) For all roots $\lambda \in R$ we have $\lambda^2 \neq n$ on T.

This holds according to [T], Lemma 1 and Corollary 1. As remarked above, in addition to [T] we know the following further facts from the stabilization of the trace formula:

d) The unitary representation $\Pi_0 = \Pi \otimes \|.\|^{\frac{c}{2}}$ satisfies the Ramanujan conjecture for all unramified nonarchimedian places (Π is assumed not to be CAP).

e) The k-dimension of W is $4m$ and

$$L_v(\Pi, s - \frac{3}{2})^m = det(1 - s(Frob_v)p_v^{-s})^{-1}$$

for all unramified nonarchimedian places v. (Π is not CAP nor a weak endoscopic lift). We may furthermore assume $n \circ r = \omega_\Pi \mu_l^{-3}$.

<u>A list of possible cases</u>:

In [T] one finds a detailed discussion of the different possibilities for (G, s). This is listed on page 298 of loc. cit in a table. This list is derived under the hypotheses a), b), c) above, together with a weaker version of e). There are 11 possible cases for the group G^0 and the representations $s|G^0$. In the cases 4-11 of this list the index G/G^0 turns out to be at most 2, since according to [T], page 299 bottom the quotient group G/G^0 embeds into the group O in these cases 4-11. The group O is tabulated in the right column of the list, and is either trivial or cyclic of order two in these cases. (See also the discussion on the bottom of page 301 of [T] and the lemma below. The critical cases occur in Proposition 1, part 5 and 6 of [T]).

The cases 1 and 3 of Taylor's list will be called the critical cases.

First critical case (case 1): $G^0 = \mathbb{G}_m^r$ for $r = 1, 2$ and $s|G^0 = (\chi \oplus n\chi^{-1})^{2m}$ and $O = \mathbb{Z}/2\mathbb{Z}$. The center $Z(G^0) = G^0$ is connected.

Second critical case (case 3): $G^0 = Gl(2)$ and $s|G^0 = (st)^{2m}$ and $O = \{1\}$, where st denotes the 2-dimensional standard representation. The center $Z(G^0) = \mathbb{G}_m$ is connected.

The additional conditions d) and e) simplify Taylor's list. The proof of the main theorem reduces to the two critical cases

2.1 Lemma: In all the cases of Taylor's list except in the critical cases W is a isotypic multiple of a 4-dimensional representation ρ_Π and the main theorem holds.

Proof: It is enough to show, that except in the critical cases W is a isotypical copy of a 4-dimensional representation of the Galois group $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$. We prove slightly more: Property e) further simplifies Taylor's list, since each eigenvalue of $s(g), g \in G^0$ occurs with multiplicity divisible by m where $dim(W) = 4m$. In particular, the coefficient e, f which appear in the second column of this list of [T] must satisfy: $e = f = m$ in case 2,4, 5 and 9 and $f = 0$ in case 8. Here it is supposed that $e, f > 0$ in case 2 , $f > 0$ in case 4 and 5, $e, f > 0$ in case 9. This implies, that in all the cases 4-11 of this list, the representation s of G^0 on W is a m-fold direct sum of a four dimensional representation $\tilde{s} : G \to Gl(4, k)$. The image $\tilde{s}(G^0)$ is contained in $GSp(4, k)$ in the cases 2, 4, 5, 6, 7, 8, 9, 11 and in $GO(4, k)$ in case 10. Since the group O is independent from e, and $N \subset G^0$ in cases 4-11 and contained in the scalars for case 2 - see [T], p.299 - we conclude that also W decomposes as a representation of the full group G into a m-fold isotypic direct sum of a 4-dimensional representation ρ_Π with similar restrictions of the image. This completes the proof of the lemma. $\square$

For the proof of the main theorem it remains to understand, what happens in the critical cases 1 and 3 of the table in [T], p.298.

Notations: We have to get further information on the group $\pi_0(G)$ of connected components in these cases. Let us introduce further notations of [T] p.299. Let $\overline{G} \subset G$ be the kernel of the natural map $G \to O$, where

$$O = \{g \in Out(G^0) \mid s \circ g|G^0 \cong s|G^0\}$$

$$G/\overline{G} \hookrightarrow O \ .$$

Let N be the centralizer of G^0 in $\overline{G}$. Then $N \cap G^0 = Z(G^0)$ and $\overline{G} = (N \times G^0)/Z(G^0)$ and we have the two sequences

$$0 \to G^0 \to \overline{G} \to N/Z(G^0) \to 0 \ ,$$

$$0 \to N \to \overline{G} \to (G^0)_{ad} \to 0$$

for the normal subgroups G^0 and N respectively. Since $\overline{G}/G^0$ imbeds into the finite group $\pi_0(G)$ of connected components, the group $N/Z(G^0)$ is a finite group. In particular $Z(G^0) = N^0$, if the center of G^0 is connected.

From now on consider the cases one and three of Taylor's list. In the two critical cases the group $Z(G^0)$ is connected and therefore $Z(G^0) = N^0$, which is a torus over k of rank $r, r = 1, 2$. Hence

$$\pi_0(\overline{G}) \cong N/Z(G^0) \cong \pi_0(N) \ ,$$

which is a subgroup of $\pi_0(G)$ of index at most 2 in the two critical cases

$$0 \to \pi_0(N) \to \pi_0(G) \to G/\overline{G} \to 0 \ .$$

The group $G/\overline{G}$ may be nontrivial only in the first critical case. We distinguish the two subcases 1a and 1b, where $G/\overline{G}$ is trivial or not.

<u>The finite group $\tilde{N}$</u>: N and $\overline{G}$ can be obtained as pushout: Choose an integer n, which annihilates the group $H^2(\pi_0(N), k^*)$, e.g. which annihilates the order of $\pi_0(N)$. (Hopefully n will not be confused with the character n introduced earlier.) There exists a <u>finite</u> group $\tilde{N}$ and a central extension

$$0 \to (\mu_n)^r \to \tilde{N} \to \pi_0(N) \to 0 \ ,$$

such that $N = (\tilde{N} \times N^0)/(\mu_n)^r$ and $\overline{G} = (\tilde{N} \times G^0)/(\mu_n)^r$, where $(\mu_n)^r$ is the group of n-torsion points in the torus $N^0 \cong Z(G^0)$. The restriction of the representation s of G to the finite subgroup $\tilde{N}$ defines a faithful representation

$$\rho : \tilde{N} \to GL(W) \ .$$

A detailed analysis of the representation ρ in the critical cases in contained in appendix B. Either ρ or its restriction to a subgroup of index two is a balanced representation. Balanced representations are classified in appendix A. This leads to the proposition 3.1 of the next section.

3. D-critical automorphic representations

From now on we study L-series attached to the representation Π. For the study of the analytical properties of L-series attached to the cuspidal irreducible automorphic representation Π of $GSp(4, \mathbb{A})$, it is useful to assume that Π is a <u>unitary</u> representation. Then in particular, Π has a unitary central character ω_Π. This unitary normalization differs from the algebraic normalization used in the last sections. One gets from one to the other by a twist $\Pi_{alg} = \Pi_{unit}| \ . \ |^{-c/2}$, where $c = w - 3$. From now on assume, that Π is unitary (except for appendix B and D where the unitary representation will be called Π_0 again).

Let χ denote a Dirichlet character of finite order. Then the L-series $\zeta(\Pi, \chi, s)$ attached to Π below, depends on Π only up to a character twist. Therefore we often switch between unitary representations Π and Π', which only differ by a character twist. Π' need not even be assumed to be automorphic!

Outside a finite set S of bad places including the archimedian place, the representation $\Pi = \otimes_v \Pi_v$ has unramified local representations Π_v, which are completely characterized by their Satake parameters $(\nu_v, \tilde{\nu}_v, \mu_v, \tilde{\mu}_v)$. These are four complex parameters, such that $\nu_v \mu_v = \tilde{\nu}_v \tilde{\mu}_v$. They are uniquely determined by Π_v up to a reparametrization under the Weyl group.

There are two L-series attached to Π. One is the degree four spinor L-series

$$L^S(\Pi \otimes \chi, s) = \prod_{v \notin S} L_v(\Pi, s) \ ,$$

mentioned in the main theorem. The other one is the degree 5 standard L-series

$$\zeta^S(\Pi, \chi, s) = \prod_{v \notin S} \zeta_v(\Pi, \chi, s)$$

mention above. The local L-factors $\prod_j (1 - c_{v,j}\chi_v(p_v)p_v^{-s})^{-1}$ are determined in the first case by the fact, that the constants $c_{v,j}, j = 1, .., 4$ are the Satake parameters $\nu_v, \tilde{\nu}_v, \mu_v, \tilde{\mu}_v$ of Π_v. They are determined by the constants $c_{v,j}$ in $\{1, \frac{\nu_v}{\tilde{\mu}_v}, \frac{\tilde{\mu}_v}{\nu_v}, \frac{\nu_v}{\tilde{\nu}_v}, \frac{\tilde{\nu}_v}{\nu_v}\}$ for the degree 5 L-series.

In the last section the proof of the main theorem was reduced to the two critical cases. In the critical cases, the properties a)-e) of the automorphic representation Π listed in the last section, allow to reformulate the information on the Galois representation in terms of the Satake parameters. This is done in the appendices A and B using elementary group theory. The result can be formulated as follows:

In appendix B a normal subgroup $\pi_L(G)$ of $\pi_0(G)$ is defined, which is trivial in cases 3 and 1b and cyclic and contained in $\pi_0(\overline{G})$ in case 1a. Let Δ be the quotient group

$$0 \to \pi_L(G) \to \pi_0(G) \to \Delta \to 0 \ .$$

Let L be the finite Galois extension of $\mathbb{Q}$ with Galois group Δ defined by the surjective homomorphism $r : Gal(\overline{\mathbb{Q}} : \mathbb{Q}) \to \pi_0(G) = G/G^0 \to \Delta$. Let $\overline{L}$ the field attached to the quadratic character with values in $G/\overline{G}$. Hence either $\overline{L} = \mathbb{Q}$, or it is a quadratic extension of $\mathbb{Q}$ contained in L. Let $\overline{\Delta}$ denote the Galois group $Gal(L/\overline{L})$.

$$\begin{tikzcd}
 & L^0 \arrow[d, dash, "\pi_L(G)"'] \\
 & L \arrow[dl, dash, "\overline{\Delta}"'] \arrow[ddl, dash, "\Delta"] \\
\overline{L} \arrow[d, dash, "G/\overline{G}"'] & \\
\mathbb{Q} &
\end{tikzcd}$$

Then there exists a finite set of roots of unities ζ_v and a set T of places of Dirichlet density zero, containing S, such that the Satake parameters of Π_v for $v \notin T$ have the shape $(\nu_v, \nu_v\zeta_v^{-1}, \mu_v, \mu_v\zeta_v)$, with $\omega_{\Pi,v}(p_v) = \nu_v\mu_v$. Furthermore $\zeta_v = 1$ if and only if v splits in L. The logarithmic local zeta factor for $v \notin T$ has the following asymptotic expansion

$$log\ \zeta_v(\Pi_v, \chi_v, s) = w_v \cdot \chi(p_v)p_v^{-s} + O(p_v^{-2s})$$

with real <u>weights</u> w_v

$$w_v = \zeta_v + \zeta_v^{-1} + {}^*Ad_v$$

and the real numbers $-1 \leq {}^*Ad_v \leq 3$ and ${}^*Ad_v = 1 + \frac{\nu_v}{\mu_v\zeta_v} + \frac{\mu_v\zeta_v}{\nu_v}$.

Under the additional hypothesis, that the main theorem would not hold for Π, lemma B2 and lemma B3 of the appendix B further imply: $\zeta_v = \pm 1$ and the Galois group $Gal(L/\mathbb{Q})$ is either elementary two-abelian of order $D \geq 4$ or is dihedral with normal elementary abelian subgroup $Gal(L/\overline{L})$ of order $\overline{D} \geq 4$. More precisely:

3.1 Proposition: If the main theorem does not hold for Π, then Π is D-critical either of CM type with $D \geq 8$ or nondegenerate of two-abelian type with $D \geq 4$, in the following sense

3.2 Definition: A unitary cuspidal representation Π of $GSp(4, \mathbb{A})$ is <u>D-critical</u>, if it is neither CAP nor a weak endoscopic lift, and if the following holds:

i) There exists a Galois extension $L/\mathbb{Q}$ with Galois group of degree $D = [L : \mathbb{Q}]$, a finite set S of $\mathbb{Q}$-places containing the ramified or archimedian places of Π and L, a set T of $\mathbb{Q}$-places containing S of Dirichlet density zero, such that the following holds: For all $v \notin T$ one has Π_v has Satake parameters

$$(\nu_v, \varepsilon_v\nu_v, \mu_v, \varepsilon_v\mu_v)$$

with $\varepsilon_v = \pm 1$ and $\epsilon_v = 1$, if and only if v splits in $L/\mathbb{Q}$.

ii) For $v \notin T$ we have $\omega_\Pi(p_v) = \nu_v\mu_v$ for central character ω_Π.

iii) The Ramanujan conjecture holds for all $v \notin S$: In particular $|\nu_v| = |\mu_v| = 1$ for $v \notin T$.

A D-critical representation Π is called nondegenerate, if $(\nu_v/\mu_v)^{2D} \neq 1$ holds for all $v \notin T$. It is abelian resp. two-abelian type, if the Galois group $\Delta = Gal(L/\mathbb{Q})$ is abelian resp. elementary two-abelian. It is called to be of CM type, if Δ contains an elementary two-abelian normal subgroup $\overline{\Delta}$ of index 2 and order $\overline{D} \geq 4$, such that for $v \notin T$ the Satake parameters are determined by a pair of Grossencharacters as in Lemma B5 and B6 of appendix B.

3.3 Remark: If Π is D-critical in the sense above, put $Ad_v = 1 + \frac{\mu_v}{\nu_v} + \frac{\nu_v}{\mu_v}$. Then for $v \notin T$ we have $|\nu_v| = |\mu_v| = 1$ by (iii). Therefore

iv) The numbers $Ad_v, v \notin T$ are real and satisfy $-1 \leq Ad_v \leq 3$ and the weights of the asymptotic expansion of the logarithmic zeta function $log\ \zeta_v(\Pi_v, \chi_v, s)$ are

$$w_v = 1 + \varepsilon_v \cdot (Ad_v + 1)\ .$$

Therefore Ramanujan's conjecture - property iii) - implies for $s \to 1^+$ the asymptotic behaviour

$$log\ \zeta^S(\Pi, \chi, s)\ \sim\ log\ L^S(\chi, s) + \sum_v \varepsilon_v \cdot (Ad_v + 1) \cdot \chi(p_v) p_v^{-s}\ ,$$

since χ is a unitary character of $\mathbb{A}^*/\mathbb{Q}^*$.

4. The theta lift

From now on assume, that the main theorem does not hold for Π or slightly less, that Π is D-critical as in the situation of the proposition of the last section. This implies

4.1 Proposition: Suppose Π is D-critical of two-abelian or CM type. Then the restriction of Π to $Sp(4, \mathbb{A})$ contains a theta lift.

This proposition is an immediate consequence of the following theorem 4.2 and the next three lemmas. □

Let $T \in Symm^2(\mathbb{Q}^2)$, $det(T) \neq 0$ belong to a nondegenerate non vanishing Fourier coefficient of Π as in [KRS], p.531. Such a T always exists, and T defines a binary quadratic space V_T over $\mathbb{Q}$ with character $\chi_T = (\ .\ , \Delta(V_T))$. Here $\Delta(V) = (-1)^{dim(V)/2} det(V)$ denotes the discriminant of a quadratic space V of even dimension over $\mathbb{Q}$, well defined in $\mathbb{Q}^*/(\mathbb{Q}^*)^2$, if we consider isomorphism class of quadratic spaces. Note $\Delta(V \oplus V') = \Delta(V)\Delta(V')$.

4.2 Theorem([KRS], thm. 7.1): If $\zeta^S(\Pi, \chi_0, s)$ has a pole at $s = 1$ - for sufficantly large finite set S and some quadratic character χ_0 - then

1) $s = 1$ is a simple pole.
2) The restriction of Π to $Sp(4, \mathbb{A})$ is a theta lift

from some automorphic representation on the orthogonal group $O(V, \mathbb{A})$, where V is a four dimensional quadratic space $V_T \oplus V_{T'}$. V_T is as above, and $V_{T'}$ is the binary quadratic space with quadratic character $\chi_0 \chi_T$.

4.3 Remark: From [S], thm. 2.4 or [KRS] introduction a character χ_0, such that $\zeta^S(\Pi, \chi_0, s)$ has a pole at $s = 1$, is necessarily quadratic: $\chi_0^2 = 1$.

Let K be the rank two commutative algebra over $\mathbb{Q}$, which is attached to the quadratic character $\chi_K = \chi_0$ by class field theory. Note, that χ_K is the quadratic character attached to the discriminant $\Delta(V)$ of $V = V_T \oplus V_{T'}$. K is a quadratic field extension of $\mathbb{Q}$ unless χ_K is the trivial character. Then $K = k^2$.

4.4 Lemma: Suppose Π is D-critical of abelian type. Then for $s \to 1^+$ we have the asymptotic behaviour

$$log\ (\prod_{\chi \in \hat{\Delta}} \zeta(\Pi, \chi, s)) \ \sim \ D \cdot \sum_{v\ \ L-split} (Ad_v + 2) \cdot p_v^{-s}$$

with product over all D characters χ of the Galois group Δ of $L/\mathbb{Q}$. Furthermore, $\zeta(\Pi, \chi_0, s)$ has a pole at $s = 1$ for at least one character $\chi_0 \in \hat{\Delta}$.

Proof: The first statement is an immediate consequence of the asymptotic formula for the logarithmic zeta function, stated at the end of the last section. Here primes $p_v, v \in T$, which are completely split in L, were called L-split. They have Dirichlet density $1/D$. Since $Ad_v + 2 \geq 1$ for these primes, in the limit $s \to 1^+$ we get $log\ (\prod_\chi \zeta(\Pi, \chi, s)) \succ log\ \zeta(s)$, where $\succ$ means $\geq$ up to some function growing like $o(log\ \zeta(s))$. Hence $\zeta(\Pi, \chi, s)$ has a pole for at least one character χ of Δ. $\square$

4.5 Lemma: Let Π be D-critical of CM type. Then $D = 8$ and $Gal(L/\mathbb{Q})$ is either the dihedral group D_8 or elementary abelian of order eight.

Proof: Consider the quadratic extension field $\overline{L}$ defined by the nontrivial quadratic character $\chi_Q = \chi_{\overline{L}/\mathbb{Q}}$, related to the nontrivial quadratic character $Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \to G/\overline{G}$. Then $Gal(L/\overline{L})$ has order $\overline{D} \geq 4$ with $D = 2\overline{D} \geq 8$.

Similar to lemma 4.4, in the limit $s \to 1^+$

$$log\ (\zeta(\Pi, \chi_Q, s)\zeta(\Pi, 1, s)) \ \sim \ 2 \cdot \sum_{v\ \ \overline{L}-\text{split}} (1 + \varepsilon_v \cdot (Ad_v + 1)) \cdot p_v^{-s} \ .$$

By lemma B5 and B6 in the appendix B we can replace in the sum $(1 + \varepsilon_v \cdot (Ad_v + 1))$ by the weight $w_v = 3$, in the case A where v is L-split, and by $w_v = -1$, in the case B where v is $\overline{L}$- but not L-split. These cases have Dirichlet density $\frac{1}{2\overline{D}}$ and Dirichlet density $\frac{\overline{D}-1}{2\overline{D}}$ respectively. This gives

$$2 \cdot (-\frac{\overline{D}-1}{2\overline{D}} + 3 \cdot \frac{1}{2\overline{D}}) \cdot log\ \zeta(s) \ = \ (\frac{4}{\overline{D}} - 1) \cdot log\ \zeta(s)$$

on the right side. However, arising from the logarithm of a meromorphic function, the number $\frac{4}{\overline{D}} - 1$ has to be an integer. Since $\overline{D} \geq 4$ we get get $\overline{D} = 4$. This implies $D = 8$ and $Gal(L/\mathbb{Q}) \ \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ or $(\mathbb{Z}/2\mathbb{Z})^3$ or D_8. See the example in appendix B after lemma B3. Finally the first group is $\langle N, S | N^4 = S^2 = 1, NS = SN \rangle$ and is easily excluded. The character χ of this group defined by $\chi(S) = 1$ and $\chi(N) = i$ take imaginary values except on the subgroup with elements $1, N^2, S, SN^2$. This is the elementary two-abelian subgroup $Gal(L/\overline{L})$ of $Gal(L/\mathbb{Q})$. These elements belong to the cases A, B, B, B with statistical weights $w_1 = 3, w_{N^2} = w_S = w_{SN^2} = -1$ and Dirichlet densities $1/8$ in the sense of lemma B5 in appendix B. Therefore the real part of $log\ \zeta(\Pi, \chi, s)$ behaves for $s \to 1^+$ as

$$(3/8 - 1/8\chi_N(N^2) - 1/8\chi_N(S) - 1/8\chi_N(SN^2)) \cdot log\ \zeta(s) = \frac{1}{2} \cdot log\ \zeta(s)\ .$$

This is impossible. $\square$

4.6 Lemma: Suppose Π is D-critical of CM type with group $Gal(L/\mathbb{Q}) = D_8$ or $(\mathbb{Z}/2\mathbb{Z})^3$. Then there exists an abelian character χ of $Gal(L/\mathbb{Q})$, such that $\zeta(\Pi, \chi, s)$ has a pole at $s = 1$.

Proof: The case, where the Galois group is abelian, is covered by lemma 4.4. We can therefore assume $Gal(L/\mathbb{Q}) = D_8$. The group $\Delta = D_8$ has four different abelian characters $1, \chi_Q, \chi_P, \chi_R$. Their common kernel in D_8 is the group commutator group $\{1, N^2\}$ of order two. For the notation see the remark after lemma B3 in the appendix B. The nontrivial element N^2 in the commutator group belongs to case B, whereas the element 1 belongs to case A, in the sense of lemma B5. The corresponding statistical weights are $w_{N^2} = -1$ and $w_1 = 3$. They occur with Dirichlet density $1/8$, hence

$$log\ (\prod_{\chi \in \hat{\Delta}} \zeta(\Pi, \chi, s)) \ \sim \ 4 \cdot (3/8 - 1/8) \cdot log\ \zeta(s)\ .$$

Therefore the function $\zeta(\Pi, \chi, s)$ has a pole at $s = 1$ at least for one character $\chi \in \hat{\Delta}$. $\square$

More information on the analytic behaviour of these zeta functions is needed later. This information can not entirely be obtained by the method used above. One also has to use the result proved above, that Π contains a theta lift. What can be obtained by the method above together with this additional information, is shown in appendix C. There, the reader finds a complete analysis of the behaviour of $\zeta(\Pi, \chi, s)$ at $s = 1$ in the D-critical cases of CM type.

5. The orthogonal similitude group $GSO(V)$

Let V be a nondegenerate quadratic space of dimension four over a local or global field k of characteristic zero. Consider the orthogonal group $GO(V)$ of similitudes and its subgroup $GSO(V)$ of proper similitudes. $GSO(V)$ is geometrically connected in the Zariski topology and of index two in $GO(V)$. The kernel of the similitude character is the special orthogonal group $SO(V)$

$$0 \to SO(V)(k) \to GSO(V)(k) \to M(k) \to 0 \quad , \quad (k^*)^2 \subset M(k) \subset k^* \ ,$$

which is of index two in the orthogonal group $O(V)$. Obviously $GO(V)$ is generated by $GSO(V)$ and $O(V)$. Similar quadratic spaces have isomorphic groups $GSO(V)$.

Let K be a commutative k-algebra of dimension two. Hence K is a field extension of degree two, or is split $K = k \oplus k$. Let σ be the canonical nontrivial involution of the algebra K/k. The norm form defines a nondegenerate binary quadratic form over k with discriminant Δ_K (minus the determinant), such that $K = k[T]/(T^2 - \Delta_K)$. This algebra depends only on the class $\Delta_K \in k^*/(k^*)^2$.

Consider a central simple algebra D_K of rank four over K. Assume $D_K = D \otimes_k K$ for some central simple algebra D of rank four over k. Under this assumption, the involution σ of $K \hookrightarrow D_K$ extends to a σ-linear involution $d \mapsto d^*$ of D_K, such that $(d_1 d_2)^* = d_2^* d_1^*$ and $(d_1 + d_2)^* = d_1^* + d_2^*$, and commutes with the K-linear standard involution $z \mapsto \overline{z}$ of D_K. The symmetric elements $d = d^*$ define a four dimensional k subspace V of D_K. The restriction of the reduced norm $N(z) = z\overline{z}$ of D_K to the subspace V has values in k, and defines a nondegenerate four dimensional quadratic space over k with discriminant $\Delta(V) = \Delta_K \ mod \ (k^*)^2$. The action of D_K^* on D_K, defined by $d \mapsto gdg^*$, preserves V such that g acts as a proper orthogonal similitude with similitude factor $Norm_{K/k}(N(g))$. The induced homomorphism $Res_{K/k}(D_K^*) \to GSO(V)$ is surjective. Its kernel is the subgroup A of all elements g in the center $Res_{K/k}(K^*)$ of G, whose K-norm $gg* = 1$ is trivial. $GO(V)$ is generated by $GSO(V)$ and the reflection ϵ, where $\epsilon(d) = (\overline{d})^* = \sigma(d)$. Conjugation by ϵ acts on $GSO(V)$ via $g \mapsto \sigma(g)$.

5.1 Lemma: Suppose V is a nondegenerate quadratic space over k of dimension four and discriminant $\Delta(V)$. Let K be the algebra $k[T]/(T^2 - \Delta(V))$. There exists a central simple algebra $D_K = D \otimes_k K$ of rank 4 over K extended from a central simple algebra D of rank 4 over k, such that $GSO(V)$ is isomorphic to the quotient group G defined by

$$1 \to A \to Res_{K/k}(D_K^*) \to GSO(V) \to 1 \ ,$$

where A is the norm 1 subgroup of the center $Res_{K/k}(\mathbb{G}_m)$ of $Res_{K/k}(D_K^*)$. $GO(V)$ is isomorphic to the semidirect product $GSO(V).(\mathbb{Z}/2\mathbb{Z})$, with the action on the normal subgroup $GSO(V)$ defined by σ.

Proof: This is well know. For the convenience of the reader we sketch the proof. The classification of k-forms implies that the statement is true, at least for some central simple algebra D_K over K of degree 4. Not all k-forms arise as orthogonal groups of proper similitudes. Sufficiency of the condition, that D_K is of the form $D \otimes_k K$, is clear from the construction explained above. To show, that it is also necessary, can be done locally. Locally the classification of quadratic forms gives four possible isomorphism classes of quadratic spaces V (see [V], p. 484): The case where K splits with Witt index $m = 0$ (V is split), and the anisotropic case (Witt index $m = 2$). Witt index $m = 1$ occurs, when the discriminant is nontrivial. In this case there are two further isomorphism classes of quadratic forms. They differ by the Hasse invariant, but have isomorphic groups of proper similitudes $GSO(V)$. The corresponding algebra D_K splits. By the theorem of Hasse-Brauer-Noether, D_K must be of the form $D_K \cong D \otimes_k K$. $\square$

Fix a k-group $GSO(V)$. It is isomorphic to $G = Res_{K/k}(D_K^*)/A$, for some choice of K and D_K. Let F be an algebraic extension field of k. The natural map $D_K(F) \to G(F)$ need not be surjective

$$0 \to A(F) \to D_K^*(F) \to G(F) \to H^1(F, A) \to 0 \ .$$

However $K^*/A \cong \mathbb{G}_m$. As in section 1 of [HST], the group F^* has the same image $H^1(k, A)$ with respect to the Galois cohomology sequence of the center $K* \hookrightarrow D_K^*$

$$0 \to A(F) \to K^*(F) \to F^* \to H^1(F, A) \to 0 \ .$$

Therefore

5.2 Lemma: Suppose $GSO(V) \cong Res_{K/k}(D_K^*)/A$ as above. Then there exists an isomorphism of the groups of F-valued points $GSO(V)(F) \cong \big(D_K^*(F) \times F^*\big)/K^*(F)$ with respect to the natural maps $K^*(F) \to D_K^*(F)$ and the norm $Norm_{K/k} : K^*(F) \to F^*$. The similitude morphism lifted to the group $D_K^*(F) \times F^*$ maps (g, t) to $Norm_{K/k}(N(g)) \cdot t^2$. The group $SO(V)(F) \subset GSO(V)(F)$ is the subgroup $(D_K^0(F) \times F^*)/K^*(F)$, where $D_K^0(F)$ is the subgroup of $D_K^*(F)$ of all elements g such that $Norm_{K/k}(N(g)) = 1$.

Proof: The statement on $GSO(V)(F)$ is shown already. Note $(F^*)^2 \subset Norm_{K/k}(N(D_K^*))$. Hence for $t \in F^*$ there exists $g \in D_K^*(F)$, such that $Norm_{K/k}(N(g))t^2 = 1$; (g, t) represents an element of $SO(V, F)$. Therefore $SO(V)(F) = (D_K^0(F) \times F^*)/K^*$. $\square$

5.3 Corollary: Let k be a local field of characteristic zero. The irreducible admissible representations of $GSO(V)(k)$ correspond to pairs $(\pi^\vee, \omega)$, where $\pi^\vee$ is an irreducible admissible representation of D_K^* with central character $\omega_{\pi^\vee}$ and ω is a character of k^*, such that $\omega_{\pi^\vee} = \omega \circ Norm_{K/k}$ of $\pi^\vee$. There exist at most two nonisomorphic extensions to an irreducible representation $(\pi^\vee, \omega, \delta)$ of $GO(V, k)$. If $\sigma(\pi^\vee, \omega) \cong (\sigma(\pi^\vee), \omega)$ is not isomorphic to $(\pi^\vee, \omega)$, this extension $(\pi^\vee)^+$ is unique ($\delta = +$). Otherwise there are two extensions $(\pi^\vee)^+, (\pi^\vee)^-$ (hence $\delta = \pm$). Each of the extensions is obtained from the other as twist with the nontrivial character of $GO(V)(k)/GSO(V,k)$.

Let k still be local. Using the Jaquet-Langlands correspondence, the admissible representations $\pi^\vee$ of $D_K^*(k)$ can be related to an irreducible admissible representation π of $Gl(2,K)$. In this sense, the irreducible admissible representations of $GSO(V)(k)$ described in corollary 5.3 are uniquely characterized by its corresponding admissible irreducible representation (π,ω) of $Gl(2,K) \times k^*$. Note $\omega_\pi = \omega \circ Norm_{K/k}$, since $\omega_\pi = \omega_{\pi^\vee}$. Twisting the representation of $GO(V)(F)$ or $GSO(V)(F)$ by one dimensional characters χ using the similitude homomorphism, amounts to change $(\pi^\vee,\omega)$ into $(\pi^\vee,\omega) \otimes \chi = (\pi^\vee \otimes (\chi \circ Norm_{K/k}), \omega\chi^2)$ resp. $(\pi,\omega) \otimes \chi = (\pi \otimes (\chi \circ Norm_{K/k}), \omega\chi^2)$ for the Jaquet-Langlands lift.

5.4 Corollary: An irreducible component of the restriction of an irreducible admissible representation of $GSO(V)(k)$ to $SO(V)(k)$ determines the representation of $GSO(V)(k)$ up to a similitude character twist with a character of order two.

Proof: Suppose D_K is split, then $\pi^\vee = \pi$. Then by [LL], p.737, an irreducible constituent $\tilde{\pi}$ of the restriction of an admissible representation $(\pi^\vee,\omega)$ of $GSO(V)(k)$ to $SO(V)(k)$ uniquely determines $(\pi^\vee,\omega)$ up to a twist $(\pi,\omega) \otimes \chi$, where $\chi^2 = 1$. If D_K is not split, then $K = k^2$. Again an irreducible constituent $\tilde{\pi}$ of the restriction of an admissible representation $(\pi^\vee,\omega)$ of $GSO(V)(k)$ to $SO(V)(k)$ uniquely determines $(\pi^\vee,\omega)$ up to a twist $(\pi,\omega) \otimes \chi$, where $\chi^2 = 1$. Now this follows from [HPS], lemma 7.2 and page 92 in the nonarchimedian case. For the archimedian case this follows from [HPS], page 95. $\square$

<u>5.5 The theta correspondence</u>: Fix a nondegenerate four dimensional quadratic space over $\mathbb{Q}$. Let $GO(V)$ be its group of similitudes. The theta correspondence is a correspondence between certain irreducible automorphic representations $(\pi^\vee,\omega,\delta)$ of $GO(V,\mathbb{A})$ an certain irreducible automorphic representation $\Pi' = \theta(\pi^\vee,\omega,\delta)$ of $GSp(4,\mathbb{A})$, and vice versa. This theta lift can be described on the level of local irreducible admissible representations. See [V] and [HST]. We follow [V]. In particular our normalization is different from [HST], p.387. In our sense Π' and $(\pi^\vee,\omega,\delta)$ are associated, iff Π and the contragredient of $(\pi^\vee,\omega,\delta)$ are associated in the sense of [HST]. Our conventions imply, that Π' and $(\pi^\vee,\omega,\delta)$ have the same central character:

$$\omega = \omega_\Pi \ .$$

The lift described above is an extension of the theta lift used in [KRS]. The lift studied in [KRS] is the Howe theta correspondence in its proper sense - it is defined on the level of the groups $Sp(4,\mathbb{A})$ and $O(V,\mathbb{A})$. These two situations are related as follows

5.6 Lemma(see [V], p.480) Suppose k is a nonarchimedian local field as above. If irreducible admissible representations Π and $(\pi^\vee,\omega,\delta)$ of $GSp(4,k)$ and $GO(V)(k)$ are related by the theta correspondence, then also their twists by a character. The restrictions to $Sp(4,k)$ resp. $O(V,k)$ contain constituents, which are related by the $(Sp(4),O(V))$-Howe correspondence. Conversely, if an irreducible $Sp(4,k)$-constituent $\tilde{\Pi}$ of Π and an irreducible $O(V)(k)$-constituent $\tilde{\pi}$ of $(\pi^\vee,\omega,\delta)$ are related by the Howe correspondence, then there exists a quadratic character χ, such that $\Pi\otimes\chi$ and $(\pi^\vee,\omega,\delta)$ are related by the extended theta lift.

In the situation of theorem 4.1 there exists an automorphic representation $\tilde{\pi}$ of $O(V,\mathbb{A})$ related to a constituent $\tilde{\Pi}$ of the $Sp(4,\mathbb{A})$-restriction of Π. $\tilde{\pi}$ can be extended to an automorphic representation of $O(V,\mathbb{A})$. This amounts to extend the automorphic representation $\tilde{\pi}^\vee$ of $D_K^0(\mathbb{A})$ to the automorphic representation $\pi^\vee$ of $D_K(\mathbb{A})$, such that $\tilde{\pi}^\vee$ is a constituent under $D_K^0(\mathbb{A})$. We extend in such a way, that $\pi^\vee$ acts by $\omega\circ Norm_{K/k}$ on the center. This extension is uniquely determined up to character twist with a character $\tau\circ Norm_{K/k}$, where $\tau=\prod_v\tau_v$ is quadratic. By the last lemma there exists a quadratic character χ of $\mathbb{A}^*$ (a priori not necessarily automorphic), such that

$$\Pi' = \Pi\otimes\chi$$

and such that Π'_v is a local theta lift of $(\pi_v^\vee,\omega_v,\delta_v)$ for all primes v. Π' need not be automorphic, since the character χ need not be automorphic. We ignore this, since we only use, that

$$\zeta^S(\Pi,\chi',s)=\zeta^S(\Pi',\chi',s)$$

holds for any character χ' of $\mathbb{A}^*/\mathbb{Q}^*$ and sufficiently large finite sets S. For this it is enough to know, that χ_v is unramified for almost all places. This is well known and reviewed in greater detail in the next section.

6. The spherical lift $\Pi'(\pi,\omega)$

Suppose Π is an arbitrary irreducible cuspidal automorphic form Π of $GSp(4,\mathbb{A})$. Let S be a sufficiently large finite set of places, such that Π_v is unramified nonarchimedian for $v\notin S$. Let Π^S be the restricted tensor product over the unramified constituents $v\notin S$.

6.1 Restriction to Sp(4): The restriction of the irreducible automorphic representation Π to $Sp(4,\mathbb{A})$ need not be irreducible. Let $\tilde{\Pi}$ be one of its irreducible constituents. Similarly, let $\tilde{\Pi}^S$ denote one of the spherical constituents of Π^S. With these notations let $\Pi_i, i=1,2$ be a pair of irreducible cuspidal representations of $GSp(4,\mathbb{A})$ unramified outside S. Then the following statements are easily seen to be equivalent:

1) $\tilde{\Pi}_1^S\cong\tilde{\Pi}_2^S$
2) $\zeta^S(\Pi_1,\tau,s)=\zeta^S(\Pi_2,\tau,s)$ for some unramified character τ^S.

3) $\Pi_1^S \cong \Pi_2^S \otimes \chi^S$ for some unramified character χ^S of $(\mathbb{A}^S)^*$ (not necessarily automorphic).

Taking this into account, we see that the assignment $\Pi^S \mapsto (\tilde{\Pi}^S, \omega_\Pi^S)$ determines Π^S up to an unramified quadratic character χ^S as in 3) (a priori not necessarily automorphic).

<u>6.2 The theta correspondence</u>: We are now interested in the properties of this lift for unramified places. Choose K and D_K, such that $GSO(V)$ can be described as in the last section. Suppose S is a finite set of places of $\mathbb{Q}$ containing the archimedian place, such that $K/\mathbb{Q}$ and D_K are unramified outside places over S. Assume $v \notin S$. Then in fact the following holds ([V],p.482). Under the local theta lift unramified representations $(\pi_v^\vee, \omega_v, \delta_v)$ of $GO(V, \mathbb{Q}_v)$ correspond to unramified representations Π'_v of $GSp(4, \mathbb{Q}_v)$. Furthermore, if $(\pi_v^\vee, \omega_v, \delta_v) = (\pi_v^\vee, \omega_v)^+$ is unramified, then $\pi_v^\vee, \omega_v, -\delta_v) = (\pi_v^\vee, \omega_v)^-$ (if it exists) is not unramified. The restriction to $GSO(V, \mathbb{Q}_v)$ contains an unramified constituent. So, by abuse of notation, an irreducible unramified admissible representation of $GO(V)(\mathbb{Q}_v)$ is determined by an unramified irreducible representation $(\pi_v^\vee, \omega_v)$ of $D_K^*(\mathbb{A}^S) \times (\mathbb{A}^S)^*)/(\mathbb{A}_K^S)*$, which equals (π_v, ω_v) by the assumption $v \in S$.

<u>6.3 Satake parameters</u>: In other words, an irreducible unramified admissible representation of $GO(V)(\mathbb{A}^S)$ is uniquely characterized by a corresponding pair of admissible representations (π^S, ω^S), where π^S is an unramified irreducible representation of $Gl(2, \mathbb{A}_K^S)$ and where ω^S is an unramified character of $(\mathbb{A}^S)^*$, such that $\omega_{\pi,v} = \omega_v \circ Norm_K$. So the theta lift in the unramified case relates unramified representations

$$(\pi^S, \omega^S) \ \leftrightarrow \ (\pi^{\vee,S}, \omega^S)^+ \ \leftrightarrow \ (\Pi')^S \ .$$

In terms of Satake parameters, the relationship between the spherical local representation π_v, ω_v of $Gl(2, K_v) \times K_v{}^*$ and the spherical local representation Π'_v of $GSp(4, \mathbb{Q}_v)$ is the following:

6.4 Lemma(nonarchimedian unramified place): Let $\Pi'_v = \Pi'_v(\pi_v, \omega_v)$ be the local theta lift of some nonramified admissible irreducible representation (π_v, ω_v). Then Π'_v is an unramified admissible irreducible representation of $GSp(4, \mathbb{Q}_v)$. Suppose <u>v is K-split</u> and suppose $v \notin S$: If

- $\pi_v = \pi_{1,v} \times \pi_{2,v}$ has Satake parameters α_v, β_v and α'_v, β'_v with $\alpha_v\beta_v = \alpha'_v\beta'_v = \omega(p_v)$,

then the theta lift

- Π'_v has Satake parameters $\alpha_v, \alpha'_v, \beta_v, \beta'_v$ with $\alpha_v\beta_v = \alpha'_v\beta'_v = \omega(p_v)$.

Suppose <u>v is K-inert</u> and suppose $v \notin S$: If

- π_v is given by the Satake parameters α_v, β_v with $\alpha_v\beta_v = \omega(p_v^2)$,

then the theta lift

- Π'_v has Satake parameters $\alpha_v^{1/2}, \alpha_v^{1/2}\chi_K(p_v), \beta_v^{1/2}, \beta_v^{1/2}\chi_K(p_v)$ with $\alpha_v^{1/2}\beta_v^{1/2} = \omega(p_v)$.

Proof: See [HST], lemma 10 and 11 or [V]. $\square$

<u>Example</u>: Let K and $\overline{L}$ be different quadratic extension fields of $\mathbb{Q}$ with composite field $L = K \cdot \overline{L}$. Let σ be the involution of $K/\mathbb{Q}$. Let π be an irreducible cuspidal unitary automorphic representation of $Gl(2, \mathbb{A}_K)$, such that $\sigma(\pi) \cong \pi \otimes (\chi_{\overline{L}/K} \circ Norm_{K/\mathbb{Q}})$ and such that π^S satisfies the Ramanujan conjecture at almost all places. For a unitary character ω of $\mathbb{A}^*/\mathbb{Q}^*$ let $\Pi' = \Pi'(\pi, \omega)$ be the theta lift. This theta lift exists and is cuspidal, if $\sigma(\pi) \not\cong \pi$ and π is generic. See [V], p.507. If Π' is neither CAP nor a weak endoscopic lift, then Π' is D-critical of two-abelian type with $D = 4$. This follows immediately from the last lemma. For a partial converse see prop. 10.3.

Another consequence of the last lemma is the following

6.5 Corollary: If the irreducible representation π^S of $Res_{K/\mathbb{Q}}(Gl(2)(\mathbb{A}^S)$ is related to the irreducible representation $(\Pi')^S = \Pi'(\pi^S, \omega^S)$ via the theta lift as above, then for $v \notin S$

$$L_v(\Pi'_v, s) = L_v(\pi_v, s) \ .$$

<u>6.6 Character twists</u>: Suppose Π is an irreducible automorphic cuspidal representation of $GSp(4, \mathbb{A})$ with central character $\omega = \omega_\Pi$. Let S be a finite set of places chosen as in 6.2. Suppose, that Π^S is unramified outside S. If one of the irreducible automorphic constituents $\tilde{\Pi}$ of the restriction of Π to $Sp(4, \mathbb{A})$ is a theta lift in the sense of [KRS], then there exists an irreducible automorphic representation π of $Gl(2, \mathbb{A}_K)$ with central character $\omega_\pi = \omega \circ Norm_{K/\mathbb{Q}}$ and an irreducible automorphic representation Π' of $GSp(4, \mathbb{A})$ with central character ω, such that

$$(\Pi')^S \cong \Pi^S \otimes \chi^S \quad , \quad (\chi^S)^2 = 1 \ .$$

χ^S and Π' need not be automorphic a priori, but they are unramified. Let us temporarily consider unramified irreducible representations Π^S of $GSp(4, \mathbb{A}^S)$ etc., which are not necessarily automorphic! Similar for the other groups under consideration. Then we have canonical bijections between the following sets:

- Equivalence classes of unramified irreducible representations Π^S of $GSp(4, \mathbb{A}^S)$, where $\Pi_1^S \simeq \Pi_2^S$ iff $\Pi_1^S \cong \Pi_2^S \otimes \chi^S$ for some unramified character χ^S of $(\mathbb{A}^S)^*$ (not necessarily automorphic).
- Equivalence classes of unramified irreducible representations π^S, ω^S of $Gl(2, \mathbb{A}_K^S) \times (\mathbb{A}^S)^*$ with $\omega_{\pi^S} = \omega^S \circ Norm_{K/\mathbb{Q}}$, where $(\pi_1^S, \omega_1^S) \simeq (\pi_2^S, \omega_2^S)$ iff $\pi_1^S \cong \pi_2^S \otimes (\chi^S \circ Norm_{K/\mathbb{Q}})$ and $\omega_1^S = \omega_2^S (\chi^S)^2$ for some unramified character χ^S of $(\mathbb{A}^S)^*$ (not necessarily automorphic).

- Equivalence classes of unramified irreducible representations π^S of $Gl(2, \mathbb{A}_K^S)$ with trivial central character ω_{π^S}, where $\pi_1^S \simeq \pi_2^S$ iff $\pi_1^S \cong \pi_2^S \otimes \psi^S$ for some unramified quadratic character $\psi^S = \chi^S \circ Norm_{K/\mathbb{Q}}$, where χ^S is a character of $(\mathbb{A}^S)^*$ (not necessarily automorphic).
- Isomorphism classes of unramified irreducible representations π^S of $Gl(2, \mathbb{A}_K^S)^0/A(\mathbb{A}^S)$ with trivial central character. Here $Gl(2, \mathbb{A}_K^S)^0$ denotes the subgroup of elements $g \in Gl(2, \mathbb{A}_K^S)$ with $Norm_{K/\mathbb{Q}}(det(g)) = 1$.
- Isomorphism classes of unramified irreducible representations $\pi^{\vee,S}$ of $D_K^0(\mathbb{A}^S)/A(\mathbb{A}^S)$ (with trivial central character).
- Isomorphism classes of unramified irreducible representations $\tilde{\pi}^S$ of $SO(V, \mathbb{A}^S)$ (with trivial central character).

The first bijection is induced by the theta correspondence. This is an immediate consequence of the last lemma. The other bijections are elementary and their verification is straight forward. The last equality follows from lemma 12, since an unramified representation automatically has trivial central character, if the center is finite. Using these bijections, it is clear that the restriction of Π^S from $GSp(4, \mathbb{A})$ to $Sp(4\mathbb{A})$ determines the representation $\tilde{\pi}^S$ of $SO(V, \mathbb{A}^S)$, hence π^S and Π^S up to unramified character twists in the sense above. In particular, this implies $(\Pi')^S \cong \Pi^S \otimes \chi^S$.

6.7 Corollary: In the situation above, for $v \notin S$ the following statement are equivalent

1) Π_v is unitary, and the Ramanujan conjecture holds for Π_v
2) Π'_v is unitary, and the Ramanujan conjecture holds for Π'_v
3) π_v and ω_v are unitary, and the Ramanujan conjecture holds for π_v.

The replacement $\pi^S \mapsto \pi^S \otimes (\psi^S \circ Norm_K)$ does not have an effect on the L-series $L_K^S(\pi \times \pi^* \otimes (\chi \circ Norm_K), s)$ and $L_K^S(\sigma(\pi) \times \pi^* \otimes (\chi \circ Norm_K), s)$. Hence these L-series are uniquely determined by the representation Π.

6.8 Proposition: Suppose, that Π is a unitary irreducible cuspidal automorphic representation of $GSp(4, \mathbb{A})$. Suppose Ramanujan's conjecture holds outside a finite set of places S. Assume, that S was chosen large enough to contain the ramified places of K and π. Also assume, that the restriction of Π to $Sp(4, \mathbb{A})$ contains a theta lift. Then there exists an algebra $K/\mathbb{Q}$ of degree two and an irreducible automorphic representation π of $Gl(2, \mathbb{A}_K)$ as in 6.4, such that the order at $s = 1$ of the following meromorphic functions coincides

1) $\zeta^S(\Pi, \chi, s) \cdot \zeta^S(\Pi, \chi\chi_K, s)$
2) $L_K^S(\chi \circ Norm_K, s) \cdot L_K^S(\sigma(\pi) \times \pi^* \otimes (\chi \circ Norm_K), s)$.

Proof: We may replace $\zeta^S(\Pi, \chi, s)$ by $\zeta^S(\Pi', \chi, s)$. Suppose $v \notin S$. By assumption Ramanujan's conjecture holds for $v \notin S$. To compute the order at $s = 1$ we therefore may ignore all places $v \notin S$, where v is K-inert. On one hand these places of K have Dirichlet

K-density zero, therefore we may ignore their contribution to the order of the meromorphic function in 2). On the other hand, the K-inert places $v \notin S$ do not contribute to the order of the product $\zeta^S(\Pi', \chi, s)\zeta^S(\Pi', \chi\chi_K, s)$ at $s = 1$, since the character χ_K eliminates their asymptotic contribution. We can therefore restrict ourselves to the K-split places outside S.

Suppose $v \notin S$ splits in K. Suppose, that the Satake parameters Π_v are $\alpha_v, \alpha'_v, \beta_v, \beta'_v$ as in lemma 6.4 above. Then $\zeta_v(\Pi', \chi, s)$ and $\zeta_v(\Pi', \chi\chi_K, s)$ are equal, and each of them is a product of five Euler factors attached to the parameters

$$\{\chi_v, \chi_v \frac{\alpha'_v}{\alpha_v}, \chi_v \frac{\alpha'_v}{\beta_v}, \chi_v \frac{\beta'_v}{\alpha_v}, \chi_v \frac{\beta'_v}{\beta_v}\} \ .$$

The Satake parameters of $\pi_v = \pi_w \times \pi_{w'}$ are α_v, β_v and α'_v, β'_v. Since $\sigma(\pi_v)$ is obtained from π_v by replacing α_v, β_v with α'_v, β'_v and vice versa, the ten local Euler-factor of 2) are attached to ten parameters

$$\{\chi_v, \chi_v \frac{\alpha'_v}{\alpha_v}, \chi_v \frac{\alpha'_v}{\beta_v}, \chi_v \frac{\beta'_v}{\alpha_v}, \chi_v \frac{\beta'_v}{\beta_v}\} \ \cup \ \{\chi_v, \chi_v \frac{\beta_v}{\beta'_v}, \chi_v \frac{\alpha_v}{\beta'_v}, \chi_v \frac{\beta_v}{\alpha'_v}, \chi_v \frac{\alpha_v}{\alpha'_v}\} \ .$$

The two sets correspond to the two extension w and w' of v. Recall $\alpha_v\beta_v = \omega_v = \alpha'_v\beta'_v$. This identity implies, that the two sets are equal. This completes the proof. □

7. The adjoint L-series of π

Suppose, that the restriction of the irreducible cuspidal automorphic representation Π to $Sp(4, \mathbb{A})$ has a constituent, which is a theta lift. To Π, or more precisely to the twist Π', we associated an irreducible automorphic representation π of $Gl(2, \mathbb{A}_K)$. We consider its adjoint L-series

$$\zeta(Ad(\pi), \chi, s) \ .$$

The partial L-series $\zeta^S(Ad(\pi), \chi, s)$ is uniquely determined by the unramified twist equivalence class of Π^S. Here S is assumed to be sufficiently large, containing the ramified places of $K/\mathbb{Q}$. The adjoint L-series has the following property

7.1 Proposition: Assume K is a number field. For an irreducible automorphic representation π of $Gl(2, \mathbb{A}_K)$ and an idele class character χ of $\mathbb{A}_K/K^*$ and a finite set S of places, such that π and χ are unramified outside S

$$L^S(\pi^* \ \times \ \pi \otimes \chi, s) \ = \ \zeta^S(Ad(\pi), \chi, s) \cdot L_K^S(\chi, s) \ .$$

Suppose π_1 and π_2 are irreducible unitary automorphic representations of $Gl(2, \mathbb{A}_K)$. Let S be a set of places, such that π_1, π_2 are unramified outside S. Then

$$L^S(\pi_1 \times \pi_2, s)$$

does not vanish at $s = 1$. Suppose each representation π_1, π_2 is either cuspidal or induced from a pair of unitary characters (this is irreducible). Then the following holds: If one of the representations π_i is cuspidal or the central characters $\omega_{\pi_1} = \omega_{\pi_2}$ coincide, then $L^S(\pi_1^* \otimes \pi_2, s)$ has a pole at $s = 1$ if and only if $\pi_1 \cong \pi_2$. This pole is simple if and only if $\pi_1 \cong \pi_2$ is cuspidal. Otherwise it is of order 2 or 4.

Proof: See [AC], p.200 for the case, where either π_1 or π_2 is cuspidal. Now suppose $\pi_i \cong Ind(\chi_i, \chi_i')$ is Eisenstein for characters χ_i, χ_i' of $\mathbb{A}_K^*/K^*$. If these characters are unitary outside S or for a single $v \notin S$, they are unitary characters (The kernel of the idele norm is a compact subgroup of $\mathbb{A}_K^*/K^*$). A pole for $L^S(\pi_1^* \times \pi_2, s)$ at $s = 1$ forces $\chi_1 = \chi_2$ - up to permutations of χ_i, χ_i'. Then $\omega_{\pi_1} = \omega_{\pi_1}$ implies $\chi_1\chi_1' = \chi_1\chi_1'$, hence $\chi_1' = \chi_2'$. This implies $\pi_1 \cong \pi_2$. The converse is obvious. □

7.2 Remark: Proposition 7.1 will be applied for the irreducible automorphic representations π of $Gl(2, \mathbb{A}_K)$. This representation may not be cuspidal. If it is cuspidal, it is unitary, since its central character is obtained from the central character ω_Π or ω_Π^S. This central character is unitary by assumption. If π is Eisenstein, it is a constituent of $Ind(\chi, \chi')$. Here χ, χ' are idele class characters for K. We later apply 7.1 in the situation of corollary 6.7. Hence π^S satisfies the Ramanujan conjecture. Therefore χ^S, χ'^S are unitary. This implies that χ, χ' are unitary characters. It also implies that $\pi \cong Ind(\chi, \chi')$. Therefore π is unitary, since these induced representations are irreducible and unitary.

7.3 Corollary: In the situation of proposition 6.8 we have

1) $\zeta^S(\Pi, 1, s)$ does not vanish at $s = 1$.
2) $\zeta^S(\Pi, 1, s)$ has a pole at $s = 1$ if and only if $\sigma(\pi) \cong \pi$.

Proof: $\zeta^S(\Pi, \chi_K, s)$ and $L_K(1, s)$ have simple poles at $s = 1$ (theorem 4.2). $L^S(\sigma(\pi) \times \pi^*, s)$ does not vanish at $s = 1$ (proposition 7.1). Therefore proposition 6.8 implies 1). If $\zeta^S(\Pi, 1, s)$ has a pole at $s = 1$, then we can choose $\chi_K = 1$. Then proposition 6.8 implies that $L_K(1, s) \cdot L^S(\sigma(\pi) \times \pi^*, s)$ has a pole of order two at $s = 1$. Therefore $L^S(\sigma(\pi) \times \pi^*, s)$ has a simple pole at $s = 1$. Proposition 7.1 implies $\sigma(\pi) \cong \pi$. The converse is similar. □

8. Theta lifts in the D-critical case

Let Π denote a D-critical representation of $GSp(4,\mathbb{A})$. Let K be a rank two algebra over $\mathbb{Q}$, and let π be an irreducible automorphic representation of $Gl(2,\mathbb{A})$. Let $\Pi' = \Pi'(\pi,\omega)$ be an irreducible automorphic representation on $GSp(4,\mathbb{A})$ related to π via a theta lift as explained in the last sections. Let S be a finite set of places containing the archimedian place, the places where K, Π, Π' and π are ramified. Let T be a set of density zero containing S, for which the Satake parameters of Π_v have the form $\nu_v, \varepsilon_v\nu_v, \mu_v, \varepsilon_v\mu_v$ of Π_v for $v \notin T$. Suppose $\Pi'^S \cong \Pi^S \otimes \chi^S$, for some unramified character χ^S of $\mathbb{A}^S$ (not necessarily automorphic). Under these assumptions the relation between π and Π can be specified in terms of the Satake parameters.

<u>8.1 Assumption</u>: To simplify the formulas we may assume $\chi^S = 1$. Under this simplifying assumption $\Pi = \Pi'$ the parameters t_v in the following formulas becomes 1:

For <u>K-split</u> places $v \notin T$ the Satake parameters of $\pi_v = \pi_w \times \pi_{w'}$ are

a) $(t_v\nu_v, t_v\mu_v)$ for π_w and $(t_v\nu_v, t_v\mu_v)$ for $\pi_{w'}$, if v splits completely in $K\cdot L$.

b) $(t_v\nu_v, t_v\mu_v)$ for π_w and $(-t_v\nu_v, -t_v\mu_v)$ for $\pi_{w'}$, if v splits in K but not in L.

For <u>K-inert cases</u> places $v \notin T$ the Satake parameters of π_v are

c) $(t_v\nu_v^2, t_v\mu_v^2)$, if v does not split in K.

The adjoint L-series $\zeta^S(Ad(\pi),\chi,s)$ does not change under character twists of π, hence it is uniquely determined by Π.

8.2 Lemma: Let π be the an automorphic form on $Gl(2,\mathbb{A}_K)$ related to the D-critical automorphic representation Π as explained above. Then for $v \notin T$ the local L-factor of $\zeta^S(Ad(\pi),1,s)$ is

$$\zeta_w(Ad(\pi_w),1,s)^{-1} = (1-\frac{\nu_v}{\mu_v}p_v^{-s})(1-p_v^{-s})(1-\frac{\mu_v}{\nu_v}p_v^{-s}) \ ,$$

if v splits in K. For both extensions w of the place v we have this same local factor. If v is inert in $K/\mathbb{Q}$, the local L-factor is

$$\zeta_v(Ad(\pi_v),1,s)^{-1} = (1-\frac{\nu_v^2}{\mu_v^2}p_v^{-2s})(1-p_v^{-2s})(1-\frac{\mu_v^2}{\nu_v^2}p_v^{-2s}) \ .$$

Proof: Clear from Lemma 6.4 and the definition 3.2 of D-critical representations. $\square$

8.3 Corollary: Let K, Π and π be as in the last lemma. Let ψ be an idele class character of $\mathbb{A}_K^*/K^*$. Then asymptotically for $s \to 1^+$

$$log\ \zeta^S(Ad(\pi),\psi,s) \sim \sum_{v\ K-split} Ad_v\cdot(\psi_v+\psi_v')\cdot p_v^{-s} \ .$$

The sum is over all places $v \notin T$ of $\mathbb{Q}$, which are split in K, and ψ_v, ψ'_v denote the values of $\psi_w(\pi_w)$ for the two extensions w of v in K.

Applied for $\psi = \chi \circ Norm_K$, this gives $\psi_v + \psi'_v = 2 \cdot \chi_v(p_v)$.

9. The numbers $n_K(\Pi)$

Let Π be a D-critical, cuspidal representation of $GSp(4, \mathbb{A})$, which is unitary. Let the notations be as in the last section. In particular let K be a quadratic $\mathbb{Q}$-algebra, let π be an irreducible automorphic representation of $Gl(2, \mathbb{A}_K)$ related to Π. Let $L/\mathbb{Q}$ be the Galois extension with Galois group Δ of order D, which is attached to Π. Let KL denotes the field generated by K and L, if K is a field. Otherwise define KL to be L. Then we have

9.1 Lemma : For D-critical representations Π and characters χ of Δ, the following functions have the same asymptotic behaviour at $s \to 1^+$

1) $log\ \zeta^S(\Pi, \chi, s)\zeta^S(\Pi, \chi\chi_K, s)\zeta^S(Ad(\pi), \chi \circ Norm_K, s)$
2) $log\ L^S(\pi \times \pi^* \otimes (\chi \circ Norm_K))L^S(\sigma(\pi) \times \pi^* \otimes (\chi \circ Norm_K))$
3) $\sum_{v\ K-split,\ L-split} 4 \cdot (Ad_v + 1)p_v^{-s}$.

In particular, the orders of the meromorphic functions

$\zeta^S(\Pi, \chi, s) \cdot \zeta^S(\Pi, \chi\chi_K, s) \cdot \zeta^S(Ad(\pi), \chi \circ Norm_K, s)$

$L^S\big(\pi \times \pi^* \otimes (\chi \circ Norm_K)\big) \cdot L^S\big(\sigma(\pi) \times \pi^* \otimes (\chi \circ Norm_K)\big)$

at $s = 1$ coincide. This order is an integer $1 \leq n_K(\Pi) \leq 16/[KL : \mathbb{Q}]$, which does not depend on the choice of the character χ of Δ.

Proof: Proposition 6.8 and 7.1 imply, that both meromorphic functions have the same order $n_K(\Pi, \chi)$ at $s = 1$. To show that this number is independent of $\chi \in \hat{\Delta}$, we compare the logarithm in 1) and 2) with the sum in 3), which obviously is independent of χ. That 1), 2) and 3) have the same asymptotic behaviour at $s = 1$ is seen as follows:

For $\chi, \chi\chi_K$ with $\chi \in \hat{\Delta}$ we use the asymptotic formula $log\ \zeta^S(\Pi, \chi, s)\ \sim\ \sum_v w_v \cdot \chi(p_v)p_v^{-s}$ obtained in section 3. The weights w_v are equal to $w_v = 1 + \varepsilon_v \cdot (Ad_v + 1)$. Furthermore $\varepsilon_v = \pm 1$ and $\varepsilon = 1$, if and only if v splits in L. We get for the asymptotic behaviour of $log(\zeta^S(\Pi, \chi, s)\zeta^S(\Pi, \chi\chi_K, s))$

$$\sim \sum_{v\ K-split,\ not\ L-split} -2 \cdot Ad_v\chi_v(p_v)p_v^{-s} + \sum_{v\ K-split,\ L-split} 2 \cdot (Ad_v + 2)p_v^{-s}\ .$$

On the other hand corollary 8.3 implies

$$log\ \zeta^S(Ad(\pi), \chi \circ Norm_K, s) \sim \sum_{v\ K-split} 2 \cdot Ad_v\chi_v(p_v)p_v^{-s}$$

for $\psi = \chi \circ Norm_k, \chi \in \hat{\Delta}$. In both cases the sum is over a subset of the places of $\mathbb{Q}$ omitting a set of $\mathbb{Q}$-density zero. Both formulas combined give 3). It only remains to verify the estimate for $n_K(\Pi)$. The right side in formula 3) is a sum over set of $\mathbb{Q}$-primes of Dirichlet-density $[KL:\mathbb{Q}]^{-1}$. By remark 3.3, property iv) we get

$$n_K(\Pi) \leq [KL:\mathbb{Q}]^{-1} \cdot 4(Ad_v + 1) \leq \frac{16}{[KL:\mathbb{Q}]} \ .$$

This implies the upper estimate for $n_K(\Pi)$. On the other hand there is at least one pole for $\zeta^S(\Pi, \chi_K, s)$. $\zeta^S(Ad(\pi,1,1) \neq 0$ (Corollary 7.2). Therefore $n_K(\Pi) \geq 1$, by the definition of 9.1. □

9.2 Corollary: In the situation of 9.1 assume, that Δ is an abelian group. Then $ord_{s=1} \prod_{\chi \in \hat{\Delta}} L^S(\pi \times \pi^* \otimes \chi \circ Norm_K) \cdot L^S(\sigma(\pi) \times \pi^* \otimes \chi \circ Norm_K) = D \cdot n_K(\Pi)$.

9.3 Corollary: In the situation of 9.1 assume $KL = L$ and, that Δ is an abelian group. Then

1) $ord_{s=1} \prod_{\chi \in \hat{\Delta}} \zeta^S(Ad(\pi), \chi \circ Norm_K, s) = \frac{D}{2} \cdot n_K(\Pi) - 2$.
2) $ord_{s=1} \prod_{\chi \in \hat{\Delta}} \zeta^S(\Pi, \chi, s) \zeta^S(\Pi, \chi\chi_K, s) = \frac{D}{2} \cdot n_K(\Pi) + 2$.

The order of $ord_{s=1} \prod_{\chi \in \hat{\Delta}} \zeta^S(\Pi, \chi, s)$ at $s = 1$ is $\frac{D}{4} n_K(\Pi) + 1$. Hence there exist at least $\frac{D}{4} n_K(\Pi) + 1$ characters χ of Δ, for which $\zeta^S(\Pi, \chi, s)$ has a pole at $s = 1$.

Proof: By property 1) and 2) of lemma 9.1, the sum of both formulas in corollary 9.3 adds to the formula of corollary 9.2. So it is enough to prove the second of the two formulas. The assumption $KL = L$ implies $\chi_K \in \hat{\Delta}$, hence by lemma 4.4

$$log \prod_{\chi \in \hat{\Delta}} \zeta^S(\Pi, \chi, s) \zeta^S(\Pi, \chi\chi_K, s) \ \sim \ 2 \cdot D \cdot \sum_{v \ L-split} (Ad_v + 2) \cdot p_v^{-s} \ =: \ \kappa \cdot log\ \zeta^S(s) \ .$$

Corollary 9.2 implies $D \cdot \sum_{v \ L-split} 4 \cdot (Ad_v + 1) p_v^{-s} \ \sim \ D \cdot n_K(\Pi) \cdot log\ \zeta^S(s)$, using lemma 9.1. Eliminating the Ad_v-terms in these two equations we obtain

$$(2 \cdot \kappa - D \cdot n_K(\Pi)) \cdot log\ \zeta^S(s) \ \sim \ (8D - 4D) \cdot \sum_{v \ L-split} p_v^{-s} \ = \ 4 \cdot log\ \zeta^S(s) \ .$$

This implies the second claim $\kappa = \frac{D}{2} n_K(\Pi) + 2$. The assertion on the number of poles is an immediate consequence of formula 2), since according to theorem 4.2 the poles of $\zeta^S(\Pi, \chi, s)$ at $s = 1$ are simple poles. □

10. Nondegenerate D-critical representations of abelian type

Proposition 10.1: Suppose Π is a D-critical irreducible automorphic representation, which is nondegenerate of abelian type with the corresponding $\mathbb{Q}$-algebra K of degree two and an irreducible automorphic representation π related to Π. Let σ be the involution of the algebra $K/\mathbb{Q}$. Then π is not of CM-type, i.e.

1) $\pi \cong \pi \otimes \chi$ for a character χ of $\mathbb{A}_K^*/K^*$ implies $\chi = 1$.

Furthermore if $D \geq 4$, then

2) $\sigma(\pi)$ is not isomorphic to π.
3) If $D \geq 8$ or $D = 4$ and K is a field not contained in L, then $\sigma(\pi) \ncong \pi \otimes \chi$ for all characters χ of A_K^*/K^* of finite order.
4) If $D = 4$ and K is a field contained in L, then $\sigma(\pi) \cong \pi \otimes (\chi \circ Norm_K)$ holds for the two characters $\chi \neq 1, \chi_K$ of the abelian group $\Delta = Gal(L/\mathbb{Q})$.

Proof: Existence of π follows from proposition 4.1. Proof of 1): Suppose χ satisfies $\pi \cong \pi \otimes \chi$. Compare central characters. This implies $\chi^2 = 1$. It is enough to show $\chi_w = 1$ for all K-places w outside a set of K-density zero. Hence discard places, which are K-inert or lie above places v in the exceptional set T of definition 3.2. $\pi \cong \pi \otimes \chi$ either implies $\chi_w = 1$ or $\chi_w = \mu_v/\nu_v = \nu_v/\mu_v$, using 8.1. Hence $\chi_w = 1$, since Π is nondegenerate by assumption. Therefore $(\mu_v/\nu_v)^2 \neq 1$ for $v \notin T$. (See definition 3.2).

Proof of 2): For $D \geq 4$ the set of $\mathbb{Q}$-places v, which split in K but not in L, has positive Dirichlet density. For such a place $v \notin T$ choose a place w of K over v. Using assumption 8.1 without restriction of generality, π_w has Satake parameters ν_v, μ_v and $\sigma(\pi)_w$ has Satake parameters $-\nu_v, -\mu_v$. Since Π is nondegenerate, $\pi \cong \sigma(\pi)$ is impossible.

Proof of 3): Consider K-places w as in the proof of 2). By the assumptions this set of K-places has K-density $\geq 3/4$. Furthermore for w and $w' = \sigma(w)$ over a $\mathbb{Q}$-place v the assumption $\sigma(\pi) \cong \pi \otimes \chi$ and 8.2 either implies $(-\nu_v, -\mu_v) = (\nu_v \chi_w, \mu_v \chi_w)$ or $(-\mu_v, -\nu_v) = (\nu_v \chi_w, \mu_v \chi_w)$. Hence $\chi_w = -1$ or $\chi_w = -\mu_v/\nu_v = -\nu_v/\mu_v = \chi_w^{-1}$. Thus $\chi_w^2 = 1$. Since this holds for a set of K-density $> 1/2$, χ is quadratic $\chi^2 = 1$. Since π is nondegenerate $\chi_w = -1$ for a set of K-density $> 1/2$. This is impossible for a quadratic character χ.

Statement 4): By assumption $K \subset L$ and $[L : K] = 2$. The quadratic idele class character $\chi_{L/K}$ of $\mathbb{A}_K^*/K^*$ has the form $\chi_{FL/K} = \chi \circ Norm_{K/\mathbb{Q}}$, where χ is one of the two characters of $Gal(L/\mathbb{Q})$ different from $\chi \neq 1, \chi_K$. Without restriction of generality assume 8.1. This implies for all K-places w over places $v \notin T$, which split in K, that $\sigma(\pi)_w \cong \pi_w \otimes (\chi_v \circ Norm_K)$ by 8.1 a) and b). Since this is a set of K-places of K-density 1, and since π is unitary satisfying the Ramanujan conjecture (see definition 3.2 and corollary 6.7), this implies that $L^S(\pi^* \times \sigma(\pi) \otimes (\chi^{-1} \circ Norm_K), s)$ has the same order at $s = 1$ as

$\zeta^S(Ad(\pi), 1, s)L_K(s)$. If π is cuspidal $\sigma(\pi) \cong \pi \otimes \chi \circ Norm_K$ follows from proposition 7.1. If π is not cuspidal, it is a constituent of an induced representation $Ind(\Psi_1, \Psi_2)$, where Ψ_i are characters of $\mathbb{A}_K^*/K^*$. Since the Ramanujan conjecture holds for π, these are unitary characters. Since $\chi^2 = 1$, proposition 7.1 implies $\sigma(Ind(\Psi_1, \Psi_2)) \cong Ind(\Psi_1, \Psi_2) \otimes (\chi \circ Norm_K)$. This implies $\sigma(\pi) \cong \pi \otimes (\chi \circ Norm_K)$, since $\pi \cong Ind(\Psi_1, \Psi_2)$ is irreducible in this case. $\square$

10.2 Corollary: Suppose Π is D-critical, nondegenerate of abelian type with $D \geq 4$. Then K is always a field. $\zeta^S(\Pi, 1, s)$ does not have a pole at $s = 1$.

Proof: $\chi_K = 1$ is equivalent to $\sigma(\pi) \cong \pi$ by 7.3.2. This contradicts $D \geq 4$, because of proposition 10.1.2. $\square$

10.3 Proposition: Suppose Π is D-critical, nondegenerate of two-abelian type with $D \geq 4$. Then $D = 4$ and K is a subfield of L. Furthermore

1) $\sigma(\pi) \ncong \pi$
2) $\sigma(\pi) \cong \pi \otimes \chi_{L/K}$.

For $\chi \in \hat{\Delta}$ the functions $\zeta^S(\Pi, \chi, s)$ do not vanish at $s = 1$. There are $n_K(\Pi) + 1$ characters $\chi \in \hat{\Delta}$, for which $\zeta^S(\Pi, \chi, s)$ has a pole at $s = 1$. The number $n_K(\Pi)$ is one 1, if π is a cuspidal representation. $n_K(\Pi) = 2$ if π is of Eisenstein type. There is no pole for $\chi = 1$.

$$\begin{tikzcd}[column sep=small] & L \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\ K' \arrow[dr, dash] & K'K \arrow[d, dash] & K \arrow[dl, dash] \\ & \mathbb{Q} & \end{tikzcd}$$

Proof: We have to exclude case 3) of proposition 10.1. Since $D \geq 4$ there exists a character χ of Δ, such that $\psi = \chi \circ Norm_K$ is nontrivial. Proposition 10.1.1 and 10.1.3 imply, that the order $n_K(\Pi)$ of $L^S(\pi \times \pi^* \otimes (\chi \circ Norm_K)) \cdot L^S(\sigma(\pi) \times \pi^* \otimes (\chi \circ Norm_K))$ at $s = 1$ is zero (proposition 7.1). This contradicts $n_K(\Pi) \geq 1$ (proposition 9.1). Hence case 3) of proposition 10.1 is excluded. $D = 4$ and K is a subfield of L.

Lemma 9.3 with $D = 4$ implies, that the number of character $\chi \in \hat{\Delta}$, for which $\zeta^S(\Pi, \chi, s)$ has a pole at $s = 1$, is $n_K(\Pi) + 1$. The trivial character does not contribute (Corollary 10.2). Since $D = 4$, we get $n_K(\Pi) \leq 3$. The order of $L^S(\pi \times \pi^*) \cdot L^S(\sigma(\pi) \times \pi^*, s)$ at $s = 1$ is $n_K(\Pi)$ by definition (lemma 9.1). The order of $L^S(\sigma(\pi) \times \pi^*, s)$ at $s = 1$ is trivial (proposition 7.1 and proposition 10.1.2). Therefore $n_K(\Pi)$ is the order of $L^S(\pi \times \pi^*, s)$ at $s = 1$. This number is 1 for cuspidal π, and 2 or 4 in the case where π is an Eisenstein representation (proposition 7.1). Since $n_K(\Pi) \leq 3$, $n_K(\Pi) = 1$ or $= 2$. Since $\zeta^S(\Pi, 1, 1) \neq 0$ (corollary 7.3.1) the number of poles for $\zeta^S(\Pi, \chi, s), \chi \in \hat{\Delta}$ at $s = 1$ is $n_K(\Pi) + 1$. $\square$

11. The Proof of Theorem I

Let Π be a cuspidal irreducible unitary automorphic representation of $GSp(4,\mathbb{A})$, whose archimedean component Π_∞ belongs to the discrete series representations of weight (k_1,k_2). For the proof of theorem I we may assume, that Π is neither a CAP-representation nor a weak endoscopic lift, since these two cases were reduced to the $Gl(2)$-situation. Then Ramanujan's conjecture holds for all unramified Π_v by [W]. Consider the λ-adic Galois representation W obtained from the third etale cohomology groups of Siegel modular threefolds. By lemma 2.2 the theorem I holds except in two critical cases. These are studied in appendix B. The results of appendix B imply, that either theorem I. holds or the representation Π is D-critical. See proposition 3.1 and definition 3.2. More precisely, Π is either D-critical and nondegenerate of two-abelian type with $D\geq 4$ or D-critical of CM-type with $D\geq 8$. By proposition 4.1 this implies, that the restriction of Π to $Sp(4,\mathbb{A})$ contains an irreducible constituent $\tilde{\Pi}$, which is a theta lift attached to some irreducible representation $\tilde{\pi}^+$ of $O(V,\mathbb{A})$. Here V was some nondegenerate four dimensional quadratic space over $\mathbb{Q}$. Attached to V is a quadratic $\mathbb{Q}$-algebra K, a simple central algebra D over $\mathbb{Q}$ and the K-algebra $D_K = D\otimes_{\mathbb{Q}} K$.

As explained in section 5, we find a finite set of places S, an irreducible unramified representation Π'^S of $GSp(4,\mathbb{A}^S)$, a quadratic character χ^S of $(\mathbb{A}^S)^*$, such that $\Pi'^S\cong \Pi^S\otimes\chi^S$, and Π'^S is a theta lift of $(\pi^S,\omega^S,\delta^S)$. Although Π' need not be automorphic, it allows to compute the L-series $\zeta^S(\Pi,\tau,s)$. These L-series do not depend on character twists of Π. So it is enough to know the Satake parameters of Π'_v at the unramified places, in order to compute $\zeta^S(\Pi,\tau,s)$. In fact, $\Pi'^S=\prod_{v\notin S}\Pi'_v$ is described in lemma 6.4, again up to some twist. This determines $\zeta^S(\Pi,\tau,s)$ in terms of the automorphic representation $(\pi^S)^\vee=\pi^S$. π^S satisfies Ramanujan's conjecture, since this was true for Π^S (corollary 6.7). This allows to study the degree five L-series $\zeta^S(\Pi,\tau,s)$ attached to Π in terms of π. Its analytic behaviour at $s=1$ depends only on a set of places of density one. That Π is D-critical implies strong restrictions. See proposition 6.8 and corollary 7.3 and corollary 8.3 and the complete section 9, in particular lemma 9.1. With these results at hand, it was shown in proposition 10.3, that for nondegenerate D-critical representations Π of two-abelian type the corresponding algebra K is a subfield of L (see definition 3.2) of degree $D=[L:K]=4$, such that conjugation by the nontrivial substitution $\sigma\in Gal(K/\mathbb{Q})$ gives

$$\sigma(\pi)\ncong\pi \quad , \quad \sigma(\pi)\cong\pi\otimes\chi_{L/K} \ .$$

A similar statement is proved in the D-critical cases of CM-type in appendix C. The proof in this case is more involved (although similar). The result in the D-critical cases of CM-type is, that again K is a subfield of L. But $Gal(L/\mathbb{Q})$ is either dihedral or elementary abelian of order $D=8$. See also lemma 4.5. Again

$$\sigma(\pi)\ncong\pi \quad , \quad \sigma(\pi)\cong\pi\otimes\chi_{F/K}$$

for some character $\chi_{F/K}$ attached to a quadratic extension $K\subset F$ in L. Hence, in all the D-critical cases to consider, we get $\sigma(\pi)\cong\pi\otimes\chi$ for some character χ of $\mathbb{A}_K^*/K^*$. π_∞ is related to $\pi_\infty^\vee$, which occurs in a nontrivial theta correspondence $\Pi'_\infty\leftrightarrow(\pi_\infty^\vee,\omega_\infty,\delta_\infty)$. Hence

$$\sigma(\pi^\vee)\cong\pi^\vee\otimes\chi\ .$$

Now consider the archimedian place. Since $\Pi'_\infty\cong\Pi_\infty\otimes\chi_\infty$ belongs to the discrete series, we can ask whether the theta lift allows to match discrete series Π'_∞ on $GSp(4,\mathbb{R})$ with representations $(\pi_\infty^\vee,\omega_\infty,\delta_\infty)$, for which $\sigma_\infty(\pi_\infty^\vee)\cong\pi_\infty^\vee\otimes\chi_\infty$ holds. The answer is, that this is impossible. If $K_\infty=\mathbb{R}^2$ splits in the archimedian case the theta correspondence is completely known, at least on the level of the dual pair $Sp(4,\mathbb{R}),O(V,\mathbb{R})$. This suffices for our purposes, since discrete series on $GSp(4,\mathbb{R})$ corresponds to discrete series on $Sp(4,\mathbb{R})$, except for the usual split up into subrepresentions. So this case can be excluded in the proof of theorem I, since

11.1 Lemma: Suppose $K_\infty=\mathbb{R}^2$. Let $\tilde\Pi_\infty$ be an irreducible representation of $Sp(4,\mathbb{R})$ in the discrete series, which is the local theta lift of an irreducible representation $(\tilde\pi_\infty,\delta_\infty)$ of $O(V,\mathbb{R})$, where $\tilde\pi_\infty$ is an irreducible representation of $SO(V,\mathbb{R})$. Then $\tilde\pi_\infty$ is in the discrete series of $SO(V,\mathbb{R})$ and $\sigma_\infty(\pi_\infty)\ncong\tilde\pi_\infty\otimes\chi_\infty$ holds for all characters χ_∞.

Proof: Since K_∞ splits, the connected component $SO(V,\mathbb{R})^0$ of $SO(V,\mathbb{R})$ in the analytic topology is either $SO(2,2)\cong Sl(2,\mathbb{R})^2/\pm$ or $SO(4)\cong(H^1)^2/\pm$, where H^1 is the group of Hamilton quaternions of norm one. In these cases, the statement of the lemma can be found in [Pr] and [KV]. In fact, checking of these cases is tedious. Therefore we at least include some detailed references:

For the first case see [Pr], theorem 3.6.1 and 3.3.1, where it is shown, that $\tilde\pi_\infty$ has to be in the discrete series, and is not $\sigma_\infty=\epsilon_\infty$ invariant. The ϵ_∞-invariant discrete series are listed in [Pr], (2.5.35-2.5.38). See also [Pr] 2.5.3. Since V is split the orthogonal group $O(V)$ is isomorphic to $O(2,2)$ in the sense of [Pr], (2.1.4). By [Pr] thm. 3.6.1 (case 3.3.1) a discrete series representation of $Sp(4,\mathbb{R})$ ([Pr] (2.4.42)) corresponds to discrete series representations of $Sl(2,\mathbb{R})^2/\pm$ (the connected component of $O(2,2)(\mathbb{R})$ of index 4) after restriction. Only those discrete series representations appear in the image, which are not conjugation invariant under the twisted action of the nontrivial element $\epsilon_\infty\in O(2,2)(\mathbb{R})/SO(2,2)(\mathbb{R})$. Note that $diag(1,1,1,-1)$ is a representative in $O(2,2)(\mathbb{R})$ for ϵ_∞, and that conjugation by this representative is easily studied on K_∞ types for $SO(2,\mathbb{R})\times SO(2,\mathbb{R})$ embedded into $O(2,2)(\mathbb{R})$ as in [Pr], (2.1.4). It replaces the type (m,n) by $(m,-n)$. This easily allows to determine the ϵ-invariant discrete series representations of $SO(2,2)$. In the notation of [Pr], (2.5.3) the ϵ_∞-conjugation invariant discrete series representations of $SO(2,2)(\mathbb{R})$ are $\pi_{m+1,0}, \pi_{0,n-1}$, which extend to the irreducible representations $\pi^p_{m+1,0}, \pi^p_{m+1,0}$ $(p = 0,1)$ of $O(2,2)(\mathbb{R})$. Exactly these representations are not in the image of a discrete series representation of $Sp(4,\mathbb{R})$, under the Howe lift. We like to understand this in terms of the representation π on $Gl(2,\mathbb{R}) \times Gl(2,\mathbb{R})$. It defines a representation of the quotient group $GSO_{2,2}(\mathbb{R})$. Its restriction to $SO_{2,2}(\mathbb{R})$ has to be described in the notations of [Pr]. We can identify $SO(2,2)$ and $SO_{2,2}$ using notation of [Pr]. Then $SO(2,2)(\mathbb{R})$ corresponds to the group $SO_{2,2}(\mathbb{R})$ defined by all pairs $(g_1, g_2) \in Gl(2,\mathbb{R})$ with $det(g_1) = det(g_2) = \pm 1$ modulo $(g_1, g_2) \sim (-g_1, -g_2)$ embedded in $Gl(4,\mathbb{R})$ by

$$(g_1, g_2) \mapsto diag(g_1, det(g_1)(g_1')^{-1}) \cdot \begin{pmatrix} a & 0 & 0 & b \\ 0 & a & -b & 0 \\ 0 & -c & d & 0 \\ c & 0 & 0 & d \end{pmatrix} \quad , \quad g_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} .$$

Using this isomorphism our representative for ϵ_∞ in $O(2,2)(\mathbb{R})$ corresponds to the matrix

$$\varepsilon = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

in $O_{2,2}(\mathbb{R})$. Obviously conjugation by ε of the image of (g_1, g_2) flips the variables $g_1, g_2 \mapsto g_2, g_1$. Thus the representation $\pi = \pi_1 \times \pi_2$ of $SO_{2,2}(\mathbb{R})$ with $\pi_1 \cong \pi_2 \otimes \chi_\infty$ is invariant under conjugation by ε up to isomorphism. This gives a contradiction to what was explained above. The case V split is therefore understood.

The definite case was studied in [KV]. For the convenience of the reader, we again sketch the situation: If $\tilde{\pi}_\infty$ comes from a pair $\pi^\vee_{1,\infty} \times \pi^\vee_{2,\infty}$ of representations $\pi^\vee_{i,\infty}$ of H^1 of dimension $dim(\pi^\vee_{i,\infty}) = d_i$, then $\pi_{1,\infty} \times \pi_{2,\infty}$ belongs to the discrete series of $Sl(2,\mathbb{R})^2$ of weight $r_i = d_i + 1$ for $\pi_{i,\infty}$ respectively (analog of Jaquet-Langlands lift). Therefore we may assume $r_1 \geq r_2 \geq 2$ and furthermore $r_1 \equiv r_2(2)$, since the central characters coincide on the diagonal embedded subgroup $\{\pm 1\} \subset Sl(2,\mathbb{R})^2$. The theta lift $\tilde{\Pi}_\infty$ of $\pi^\vee_\infty$ is a holomorphic/antiholomorphic discrete series or a limit of the holomorphic/antiholomorphic discrete series representations of $Sp(4,\mathbb{R})$ of weight $k_1 \geq k_2$, where

$$k_1 = \frac{1}{2}(r_1 + r_2) \quad , \quad k_2 = \frac{1}{2}(r_1 - r_2) + 2 \ .$$

Note $k_1 \geq k_2 \geq 2$. $\tilde{\Pi}_\infty$ belongs to the discrete series, if and only if $k_2 \geq 3$ respectively $r_1 > r_2$. In fact this amounts to or at least implies the condition $\sigma_\infty(\tilde{\pi}_\infty) \ncong \tilde{\pi}_\infty \otimes \chi_\infty$, since σ_∞ permutes the weights r_1, r_2. For these statements see [KV], p.26 and p.28(8.2). However notation in [KV] is different, since these authors use a different way to parametrize representations of $SO(4,\mathbb{R})$. $\square$

For the proof of theorem I we are left with the case $K_\infty \cong \mathbb{C}$. Although the theta correspondence has not been worked out completely for the pair $(GSp(4,\mathbb{R}), GO(3,1))$,

enough information is provided by [HST] in this situation. This allows to complete the proof of theorem I by the next lemma. Once more, $\sigma_\infty(\pi_\infty) \cong \pi_\infty \otimes \chi_\infty$ gives a contradiction, since a restriction of π_∞ lifts to a constituent $\tilde{\Pi}_\infty$ of the discrete series representation Π_∞. $\tilde{\Pi}_\infty$ can be extended to a theta lift Π'_∞. Since $\tilde{\Pi}_\infty$ belongs to the discrete series, also Π'_∞ belongs to the discrete series. This contradicts the next lemma.
□

11.2 Lemma: Suppose $K_\infty = \mathbb{C}$. Let Π'_∞ be an irreducible representation of $GSp(4,\mathbb{R})$ contained in the discrete series. Assume that Π'_∞ is a nontrivial theta lift of an irreducible representation $(\pi_\infty, \omega_\infty, \delta_\infty)$ of $GO(3,1)$ or $GO(1,3)$. Then the irreducible representation π_∞ of $Gl(2,\mathbb{C})$ can not satisfy $\sigma_\infty(\pi_\infty) \cong \pi_\infty \otimes \chi_\infty$ for a character χ_∞.

Proof: Suppose $\sigma_\infty(\pi_\infty) \cong \pi_\infty \otimes \chi_\infty$ holds for a character χ_∞. Note $\omega_{\pi_\infty} = \omega_{\sigma_\infty(\pi_\infty)}$, since π_∞ and $\sigma_\infty(\pi_\infty)$ have the same central character $\omega_\infty \circ Norm_{\mathbb{C}/\mathbb{R}}$. Therefore $\chi_\infty^2 = 1$, since $\omega_{\sigma_\infty(\pi_\infty)} = \omega_{\pi_\infty \otimes \chi_\infty} = \omega_{\pi_\infty}\chi_\infty^2$. But $\chi_\infty^2 = 1$ implies $\chi_\infty = 1$, since the character $\chi_\infty(z)$ is of the form $|z|^s(\frac{z}{|z|})^n$ for some $n \in \mathbb{Z}$. Therefore

$$\sigma_\infty(\pi_\infty) \cong \pi_\infty \ .$$

By a character twist with a character $\tau_\infty \circ Norm_{\mathbb{C}/\mathbb{R}}$ we may reduce to the case, where the central character of π_∞ is trivial. Then [JL], lemma 6.1 implies $\pi_\infty \cong Ind(\mu_\infty, \mu_\infty^{-1})$, where $\sigma_\infty(\mu_\infty) = \mu_\infty^{-1}$

$$\mu_\infty(z) = (\frac{z}{|z|})^n \quad , \quad n \in \mathbb{Z}$$

or $\sigma_\infty(\mu_\infty) = \mu_\infty$ where

$$\mu_\infty(z) = |z|^s \quad , \quad s \notin \mathbb{Z} \ .$$

The first case - with $n \neq 0$ - was considered in [HST] lemma 12. According to loc. cit. Π'_∞ must then have a $K_\infty = U(2)$-type of highest weight $(n+1,1)$ or $(n+1,0)$ or $(n+1,2)$ and infinitesimal Harish-Chandra parameter $(n,0;*)$. According to [Pr] 2.4.42 this can be no discrete series representation, since the infinitesimal Harish-Chandra parameter $(n,0)$ of the restriction of Π'_∞ to $Sp(4,\mathbb{R})$ is not regular. The infinitesimal character $\gamma'_{n,0}$ (see [Pr] page 30) is excluded in [Pr] 2.4.42 . This shows, that in the first case we do not obtain Π'_∞ in the discrete series. For $n = 0$ in the first case or also in the second case $\mu_\infty(z) = |z|^s$, the irreducible representation π_∞ is an induced representation, which contains the trivial representation of the connected maximal compact subgroup $SO(3) = U(2)/U(1)$ of $GL(2,\mathbb{C})/U(1)$. There are four possibilities to extend this to a representation of the maximal compact subgroup $O(3) \times O(1)$ of $GO(3,1)$. These are denoted $(0,+,+), (0,+,-), (0,-,+), (0,-,-)$ in [HST], page 395. Therefore the computation of [HST], page 395 implies, that Π'_∞ has a $U(2)$-type of weight $(1,1), (1,0), \emptyset, \emptyset$ (in the last two cases, the theta correspondence is trivial). These are Howe minimal $K_\infty = U(2)$-types arising from the theta correspondence for the pair $(O(V)(\mathbb{R}), Sp(4,\mathbb{R}))$. We show, that they can not occur in a discrete series representation $\tilde{\Pi}'_\infty$ of $Sp(4,\mathbb{R}) \subset GSp(4,\mathbb{R})$. A discrete series representation is tempered and has real infinitesimal character. Therefore [Pr] 2.3.20 and 2.3.23 can be applied with $\gamma=\lambda$, with the notations of loc. cit. For the two K_∞-types $(1,1),(1,0)$ considered we have $\|\pi'_{m,n}\|_{lambda}=m-1=0$ according to [Pr] (2.1.22). Therefore the infinitesimal character has norm $\|\lambda\|=\|\gamma\|=\|\pi'_{m,n}\|_{lambda}=0$ by [Pr] (2.3.21). Thus this K_∞-type must be a lowest K_∞-type of $\tilde{\Pi}'_\infty$ in the sense of Vogan. But discrete series representation $\tilde{\Pi}'_\infty$ of $Sp(4,\mathbb{R})$ have a unique lowest K_∞-type. According to [Pr] (2.4.44-47) the types $\pi'_{1,1},\pi'_{1,0}$ do not occur. $\square$

12. Proof of theorem II

Suppose Π is unitary and as in theorem I. Suppose the representation $\rho_{\Pi,\lambda}$, constructed in theorem I, were reducible of the form $\rho_{\Pi,\lambda}=2\cdot\rho_0$. Let $V=V_{\rho_0}$ be the two dimensional representation space of ρ_0. Then ρ_0 is E-rational and $G\subset Gl(V_{\rho_0})\cong Gl(2,k)$. In particular $det(\rho_0)$ is attached to an automorphic character ω_0 of $\mathbb{A}^*/\mathbb{Q}^*$.

Since $G\subset Gl(2)$ only case 1,2,3,5,8 in Taylor's list is possible. An immediate check gives, that Π belongs either to case 1 or 3 of Taylor's list, since in case 2,5,8 the representation $\rho_{\Pi,\lambda}$ can not be of the form $2\cdot\rho_0$.

Case 1: In this case $\overline{G}\subset G$ has index at most two. $\overline{G}=G^0\cdot N$, where N is the centralizer of G^0 in $\overline{G}$. We can compute N inside $Gl(V_{\rho_0})$. Since G^0 is a torus and acts on V_{ρ_0} by the characters $\chi_1\neq n\chi_1^{-1}$ in the notation of [T], p.298, the normalizer N and G^0 are contained in a common maximal torus of $Gl(V_{\rho_0})$. In particular $\overline{G}$ is abelian in the present situation.

Since the restriction to the subgroup $Gal(\overline{\mathbb{Q}}/\overline{L})$ of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ (of index at most two) is abelian, the representation $\rho_{\Pi,\lambda}$ is either abelian and a sum of two characters or induced from a character of $Gal(\overline{\mathbb{Q}}/\overline{L})$

$\rho_0\cong\chi_1\oplus\chi_2$, or
$\rho_0=Ind_{Gal(\overline{\mathbb{Q}},\overline{L})}^{Gal(\overline{\mathbb{Q}}/\mathbb{Q})}(\chi_3)$.

Similar as in lemma one shows, that χ_1,χ_2 are attached to characters on $\mathbb{A}^*/\mathbb{Q}^*$ resp χ_3 is attached to a grossencharacter of $\mathbb{A}^*_{\overline{L}}/\overline{L}^*$. This implies, that ρ_0 is attached to an irreducible automorphic form on $Gl(2,\mathbb{A})$. We normalize π by a twist with the idele norm character, such that $L_v(\pi,s-\frac{w}{2})=L_v(\rho_0,s)$ holds for unramified v.

Suppose Π satisfies Ramanujan's conjecture for all unramified places. If Π does not satisfy Ramanujan's conjecture, then Π is a CAP representation. For CAP representation $\rho_{\Pi,\lambda}\cong\rho_1\oplus\rho_2$, where ρ_1 is attached to a cuspidal automorphic representation and where ρ_2 is attached to an Eisenstein series (see the discussion below). Since there are no CAP representation for $Gl(2)$ this case is excluded. So return to the case where Π is not

CAP. Since $L^S(\Pi,s) = L^S(\pi,s)^2$ the representation π satisfies the Ramanujan conjecture for almost all places. By strong multiplicity one for $Gl(2)$ the representation π therefore is unitary, if it is cuspidal. In the Eisenstein case π must be a sum of two unitary grossencharacters. Again π is unitary.

$L^S(\Pi,s) = L^S(\pi,s)^2$ for a suitably finite set S implies $\zeta^S(\Pi,1,s) = \zeta^S(s)^2 L^S(\pi\otimes\pi^\vee,s)$. Since π is unitary, we can apply proposition 7.1. This gives a pole for $\zeta^S(\Pi,s)$ of order ≥ 2 at $s=1$. This contradicts theorem 4.2 and excludes the case 1.

<u>Case 3</u>: It remains the case, where Π belongs to case 3. In this case $G = Gl(2,k)$ and $\pi_0(G) = 1$. The assumption $\rho_{\Pi,\lambda} = 2\cdot\rho_0$ implies that in the unramified cases the Satake parameters of Π_v are highly restricted $\Pi_v \sim (\alpha_v,\alpha_v,\beta_v,\beta_v) \sim (\alpha_v,\beta_v,\beta_v,\alpha_v)$, or

$$\Pi_v \sim (\alpha_v,\beta_v,\alpha_v,\beta_v)\ ,$$

where α_v,β_v are the eigenvalues of $Frob_v$ in ρ_0, up to the twist factor $p_v^{-\frac{w}{2}}$. Since $G = Gl(2,k)$ we can assume $\alpha_v \neq \pm\beta_v$ for all v in a set of primes of Dirichlet density 1. Then $\alpha_v\beta_v = \omega_{\Pi,v}(p_v)$. This excludes the second alternative $\Pi_v \sim (\alpha_v,\beta_v,\alpha_v,\beta_v)$ on this set of density 1, since otherwise $\alpha_v^2 = \omega_{\Pi,v}(p_v) = \beta_v^2$ and therefore $\alpha_v = \pm\beta_v$. So we have a set T of places of density zero, outside of which Π_v satisfies condition i) and ii) and iii) of definition 3.2:

Put $L = \mathbb{Q}$ and $\nu_v = \alpha_v, \mu_v = \beta_v$ and $\zeta_v = 1$, then i) is just $\Pi_v \sim (\alpha_v,\alpha_v,\beta_v,\beta_v)$. Furthermore statement ii) - $\nu_v\mu_v = \omega_{\Pi_v}(p_v)$ - holds for $v \notin T$, since $\alpha_v\beta_v = \omega_{\Pi_v}(p_v)$ as shown above. An also Ramanujan's conjecture holds for almost all places, In other words, Π is D-critical with $D=1$ resp. $L=\mathbb{Q}$, provided Π is neither CAP nor a weak endoscopic lift. Let us assume this for the moment. Then lemma 4.4 for $D=1$ and $\chi=1$ implies, that $\zeta^S(\Pi,1,s)$ has a pole at $s=1$. More precisely $log\ \zeta^S(\Pi,1,s) \sim \sum_v (Ad_v+2)\cdot p_v^{-s}$ at $s\to 1^+$. So the restriction of Π to $Sp(4,\mathbb{A})$ contains a theta lift, attached to the algebra $K = \mathbb{Q}^2$. Π is accociated to a unitary representation π of $Gl(2,\mathbb{A}_K) = Gl(2,\mathbb{A})^2$. Since a pole of $\zeta^S(\Pi,1,s)$ is simple (theorem 4.2), we get $\sum_v (Ad_v+1)\cdot p_v^s \sim 0$ at $s\to 1^+$. Therefore $\zeta^S(\Pi,1,s)^2\zeta^S(Ad(\pi),1,s)$ has order 0 at $s=1$ by lemma 9.1. Since $\zeta^S(Ad(\pi),1,s)$ does not vanish and $\zeta^S(\Pi,1,s)$ has a pole at $s=1$, we get a contradiction. So it remains to discuss, whether Π can be CAP or a weak endoscopic lift. If it were, then $\rho_{\Pi,\lambda} \cong \rho_1\oplus\rho_2$, where ρ_i are attached to irreducible automorphic representations π_i of $Gl(2,\mathbb{A})$. If Π is a weak endoscopic lift, then $\pi_{\infty,i}$ belong to the discrete series of weight r_i with $r_1 > r_2$. By a theorem of Ribet ρ_1,ρ_2 are irreducible. Hence $\rho_1 \cong \rho_0 \cong \rho_2$. This contradicts $r_1 \neq r_2$. So Π must be a CAP representation.

<u>The CAP case</u>: Then $\rho = \rho_1\oplus\rho_2$ and one of the representation ρ_1,ρ_2 is reducible. Then $\rho_{\Pi,\lambda} = 2\cdot\rho_0$ implies, that ρ_0 is reducible

$\rho_0 \cong \chi_1\oplus\chi_2$ for Dirichlet characters χ_1,χ_2 of $\mathbb{A}^*/\mathbb{Q}^*$, not necessarily unitary.

This implies that Π is a CAP representation associated to the Borel subgroup B of $GSp(4)$. These were classification in [P] and [S]. The description given in [S] as binary theta lifts, implies that Π_∞ cannot belong to the discrete series for such CAP representations. $\square$

Now suppose that $\rho_{\Pi,\lambda}$ contains a one dimensional subrepresentation ρ_1. Still suppose Π is unitary and satisfies the assumptions of theorem I. To prove theorem II we show, that under these circumstances Π must be a CAP representation. It can not be a weak endoscopic lift $\rho_{\Pi,\lambda} = \rho_{\pi_1} \oplus \rho_{\pi_2}$, since the two dimensional representations ρ_{π_i} attached to the elliptic cusp forms are irreducible. So, if theorem II were false, we can find Π such that $\rho_{\Pi,\lambda}$ occurs as a subrepresentation of the Galois-representation W_{Π_λ} build from the third cohomology of the Shimura variety. So we are in the situation of section 2. In particular property c) of Π, as sormulated in section 2, will now be used.

Recall $\rho_{\Pi,\lambda}^\vee \otimes \omega \cong \rho_{\Pi,\lambda}$ for $\omega = \omega_\Pi \mu_l^{-w}$. Therefore the one dimensional character $\rho_2 = \rho_1^\vee \otimes \omega$ also occurs in $\rho_{\Pi,\lambda}$. Since $\rho_{\Pi,\lambda}$ is a subrepresentation of W_{Π_λ} we have $\rho_2 \neq \rho_1$ since otherwise $(\rho_1)^2 = \omega$. This is excluded by property c) of Π, formulated in section 2, since ω is induced by n. Therefore $\rho_{\Pi,\lambda} = \rho_1 \oplus \rho_2 \oplus \rho_3$ with $dim(\rho_1) = dim(\rho_2) = 1$.

Consider $\rho_{\Pi,\lambda}$ as a $\mathbb{Q}_l$-representation of dimension $4 \cdot [E_\lambda : \mathbb{Q}_l]$ in the sense of theorem III, then this representation $\rho_{\Pi,\lambda}$ is Hodge-Tate. Therefore the characters ρ_1, ρ_2 are locally algebraic and have the form $\rho_i = \chi_i \mu_l^{n_i}$ for characters χ_i of $\mathbb{A}^*/\mathbb{Q}^*$ of finite order $(i = 1, 2)$. If Π is not CAP, the representation $\rho_{\Pi,\lambda}$ is pure of weight w. Therefore $n_1 = n_2 = -w/2$ and w is even. Then $n = \mu_l^{-w} \omega_\Pi$ with ω_Π a character of finite order, we find an integer $\nu > 0$ such that $\rho_1^{2\nu} = \rho_2^{2\nu} = n^\nu$. This contradicts property c) of the representation Π (section 2) and completes the proof of theorem II. $\square$

Appendix A: Balanced representations

Let in this section $\tilde{N}$ denote a finite group, and let k denote an algebraically closed field of characteristic zero. A group of type $(\mathbb{Z}/2\mathbb{Z})^r$ is called elementary two-abelian.

Definition: Consider representations

$$\rho : \tilde{N} \to Gl(V)$$

on finite dimensional k vectorspaces V. We say that ρ has "only two eigenvalues" if for all $g \in \tilde{N}$ the matrix $\rho(g)$ has at most two different eigenvalues $\zeta_1(g), \zeta_2(g)$. Let their multiplicities be $a_1(g), a_2(g)$, with the convention $\zeta_1(g) = \zeta_2(g)$ if only one eigenvalue occurs. We put $\zeta(g) = \zeta_1(g)/\zeta_2(g)$, which is well defined in k^* up to inverse.

Let ρ be a representation with only two eigenvalues with associated projective representation

$$\overline{\rho} : \tilde{N} \to PGl(V) \ .$$

For the kernel $Z(\rho)$ of $\overline{\rho}$ we have $\rho|Z(\rho) = \chi \cdot id$ for some character of χ of $Z(\rho)$. Consider

$$0 \to Z(\rho) \to \tilde{N} \to Q(\rho) \to 0$$

$$D(\rho) = \#Q(\rho) \ ,$$

i.e let $D(\rho)$ denote the number of elements of $Q(\rho) = \tilde{N}/Z(\rho)$. If ρ is fixed, we write Z, Q, D instead of $Z(\rho), Q(\rho), D(\rho)$. We have $g \in Z(\rho)$ iff $\zeta(\rho, g) = 1$ and the order of root of unity $\zeta(\rho, g)$ is the order of the image of g in $Q(\rho)$.

Decompose $\rho \cong \bigoplus_j \rho_j$ into irreducible representations ρ_j. If ρ has only two eigenvalues, then also the representations ρ_j. The possible irreducible representations ρ_j with only two eigenvalues were classified in [T2], lemma 9 and corollary 1. There are four possible types of irreducible representations ρ_j

1. $dim(\rho_j) = 1$.
2. $dim(\rho_j) = 2$ and either ρ_j is dihedral (including D_4, i.e. induced from a character of a subgroup of index two) or the image $Q_j = Q(\rho_j)$ of $\tilde{N}$ in the associated two dimensional projective group $PGl(2, k)$ is A_4 or S_4 or A_5.
3. $dim(\rho_j) = d_j \geq 4$. In this case there is a subgroup N_j of $\tilde{N}$ of index ≥ 16. The quotient $Q_j = \tilde{N}/N_j$ is an elementary abelian two group, N_j acts by scalars under ρ_j, and the elements of $\tilde{N} \setminus N_j$ have trace zero and eigenvalues of the form $\zeta_1(n) = -\zeta_2(n)$. Thus $\zeta(\rho_j, n) = 1$ resp. $\zeta(\rho_j, n) = -1$ if $n \in N_j$ resp. $n \notin N_j$.

Definition: A representation ρ with only two eigenvalues is called <u>balanced</u> if $d = dim(\rho)$ is even and $a_1(g) = a_2(g) = d/2$ holds for all $g \neq 1$ in $Q(\rho)$. Then formally put $a_1(g) = a_2(g)$ also for $g = 1$ in $Q(\rho)$.

Example: Any two dimensional representation with at most two eigenvalues is balanced.

The main result of this appendix is

Lemma A: Suppose ρ is balanced. Then ρ is an isotypic multiple of a two dimensional representation, or the associated group $Q(\rho)$ is elementary two-abelian of order $D(\rho) \geq 4$. If $D = 4$, then ρ is a multiple of $\chi \otimes (1 \oplus \chi_1 \oplus \chi_2 \oplus \chi_1\chi_2)$ for characters χ, χ_1, χ_2 with $\chi_1^2 = \chi_2^2 = 1$.

Proof: The proof follows from claim 7 and 8 below. $\square$

Claim 1: Let $\rho : \tilde{N} \to Gl(V)$ be balanced of dimension d. Let Z be the kernel of $\overline{\rho}$. Then $\rho|Z = \chi \cdot id$ for some character of χ of Z. Suppose (Z, χ) is fixed at the moment. Let ρ' be a finite dimensional irreducible representation of $\tilde{N}$ of dimension d', such that $\rho'|Z = \chi \cdot id$. Then there exists an integer $1/c(Q)$ depending only on $Q = \tilde{N}/Z$, such that the multiplicity $m(\rho, \rho')$, with which ρ' appears in ρ is either zero or

$$m(\rho, \rho') = c(Q)dd' \ .$$

If $\rho \cong \bigoplus m(\rho, \rho_i)\rho_i$ is a decomposition into irreducible, nonisomorphic representations ρ_i of dimensions d_i, then

$$\sum_i d_i^2 = 1/c(Q) \ .$$

Proof: Put $D = D(\rho)$. The multiplicity $m(\rho, \rho') = D^{-1} \sum_{g \in Q} \chi_\rho(g) \overline{\chi}_{\rho'}(g)$ is zero unless ρ' appears in ρ. Then ρ' satisfies $\chi_{\rho'}(g) = a_1'(g)\zeta_1(g) + a_2'(g)\zeta_2(g)$, $a_1'(g) + a_2'(g) = d'$. We can therefore express the integer $m(\rho, \rho')$ in the form

$$D^{-1} \cdot \sum_{g \in Q} \big(a_1(g) + a_2(g)\big)\big(a_1(g)' + a_2'(g)\big) + a_1(g)a_2'(g)\big(\zeta(g) - 1\big) + a_2(g)a_1'(g)\big(\zeta(g)^{-1} - 1)\big) \ .$$

In this formula replace $\zeta = \zeta(g)$ or ζ^{-1} by the $\mathbb{Q}$-projection $[\mathbb{Q}(\zeta) : \mathbb{Q}]^{-1} trace_{\mathbb{Q}(\zeta)/\mathbb{Q}}(\zeta)$. This $\mathbb{Q}$-projection only depends on the order $m = ord(g)$ of g. It defines a weakly multiplicative function $s(m)$ on $\mathbb{N}$. For prime powers p^n

$$s(p^n) = \ 1, \ -1/(p-1), \ 0 \quad \text{for} \quad n = \ 0, \ 1, \ n \geq 2 \ .$$

In particular, the numbers $\alpha_i = (1 - s(i))/2$ satisfy $\alpha_1 = 0, \ \alpha_2 = 1$ and $0 \leq \alpha_i \leq 3/4 < 1$ for $i \geq 3$. With these notations $a_1(g) = a_2(g) = d/2$ and $a_1'(g) + a_2'(g) = d'$ implies

$$m(\rho, \rho') = dd' \cdot c(Q) \ ,$$

where $c(Q) := 1 - d(Q)$ and $d(Q) := \frac{1}{2D} \sum_{g \in Q} \Big(1 - s(ord(g))\Big)$. Obviously $c(Q) > 0$, since not all $m(\rho, \rho')$ are zero.

Let $n_i(Q)$ denote the numbers of elements in Q of order i. Then $D \cdot d(Q) = \sum n_i(Q)\alpha_i = n_2 + \frac{3}{4}n_3 + \frac{1}{2}n_4 + ...$

Decompose $\rho \cong \bigoplus_{t=1}^t m(\rho,\rho_i)\rho_i$ into irreducible non isomorphic constituents ρ_i. Then $m(\rho,\rho_i)c(Q)^{-1} = dd_i$ and $\sum_i m(\rho,\rho_i)d_i = d$ imply $\sum_{i=1}^t d_i^2 = 1/c(Q)$, hence $c(Q)^{-1}$ is an integer. Claim 1 is proved.

<u>Claim 2</u>: Let Q be a finite group of order D and define $c(Q)$ and $d(Q)$ as above. Let $n_i(Q)$ denote the numbers of elements in Q of order i. Then

a) $D/2 \geq n_2(Q)$ implies $c(Q) > 1/8$.
b) $3D/4 \geq n_2(Q) > D/2$ implies $c(Q) > 1/16$ and $c(Q) > 1/8$, if Q is a 2-Sylow group.
c) $n_2(Q) > 3D/4$ implies $Q \cong (\mathbb{Z}/2\mathbb{Z})^r$ and $c(Q) = \frac{1}{D}$.

<u>The proof of a)-c)</u>: The first statement of c) is $Q \cong (\mathbb{Z}/2\mathbb{Z})^r$. It follows from elementary group theory and the knowledge $n_2(Q) > 3D/2$ of the elements of order two. Fix $g \in Q, g^2 = 1$. Any $h \in Q$, for which $h^2 = (gh)^2 = 1$ holds, is in the centralizer $Z_Q(g)$ of g. By the assumption on $n_2(Q)$ the number of such elements h is $> D/2$. Therefore $Z_Q(g) = Q$. Thus $g \in Z(Q)$. The centre has more than $3D/4$ elements and $Q = Z(Q)$.

The statements a),b) and c) concerning the value $c(Q)$ use $c(Q) = 1 - d(Q)$ and $D \cdot d(Q) = \sum n_i(Q)\alpha_i$ with $\alpha_i \leq 3/4$ for $i \geq 3$. Hence

$$d(Q) \leq D^{-1}\big(n_2(Q) + \frac{3}{4}(D - 1 - n_2(Q))\big) \leq \frac{n_2(Q)}{4D} + \frac{3}{4}(1 - D^{-1}) \ .$$

If $n_2(Q) \leq D/2$, therefore $d(Q) < 7/8$ and $c(Q) > 1/8$. If $n_2(Q) \leq 3D/4$, therefore $d(Q) \leq 3/16 + 3/4 - 3/4D < 15/16$ and $c(Q) > 1/16$. If $n_2(Q) > D/2$ and Q is a 2-Sylow group, then $n_i(Q) \neq 0$ unless i is a power of 2. Hence

$$d(Q) = D^{-1}\big(n_2(Q) + \sum_{i>2}\frac{1}{2}n_i(Q)\big) = (2D)^{-1}\big(n_2(Q) + D - 1\big) < \frac{n_2(Q)}{2D} + \frac{1}{2} \ .$$

For $n_2(Q) \leq 3D/4$ this implies $d(Q) < 7/8$ and $c(Q) > 1/8$. If $n_2(Q) > 3D/4$, then Q is elementary two-abelian. Hence $n_i(Q) = 0$ for $i > 2$ and $d(Q) = (D-1)/D$ resp. $c(Q) = 1/D$. Claim 2 follows.

<u>Claim 3</u>: Suppose ρ is balanced and $\chi_1 \oplus \chi_2 \oplus \chi_3 \hookrightarrow \rho$ contains three nonisomorphic one dimensional representations $\rho_i = \chi_i, i = 1,2,3$. Then all quotients χ_i/χ_j are quadratic characters.

<u>Proof</u>: By a twist assume $\chi_1 = 1$. Suppose χ_2 were a character of order $N \geq 2$ and χ_3 of order $M \geq 2$. Since ρ has only two eigenvalues, $\chi_2(g) \neq 1$ and $\chi_3(g) \neq 1$ implies $\chi_2(g) = \chi_3(g)$. For simplicity reduce to $Q = (\mathbb{Z}/NM\mathbb{Z})^2$; then χ_3/χ_2 is trivial on at least $1 + (N-1)(M-1)$ elements, hence on $> NM/3$ elements of this group of order NM resp.

$> NM/2$ elements if $N \neq 2$. For $N \neq 2$ this forces $\chi_2(g) = \chi_3(g)$ for all g contradicting the assumption $\chi_2 \neq \chi_3$. Hence $N = 2$, and still this forces this χ_3/χ_2 to be quadratic. But then χ_2 is quadratic ($N = 2$) and also χ_3 is quadratic. This completes the proof.

Claim 4: Suppose ρ is balanced. Suppose $\rho_1 \oplus \rho_2 \hookrightarrow \rho$, where ρ_1 irreducible of dimension 2 and ρ_2 a one dimensional character. Then $Q = Q(\rho_1)$ is an elementary two-abelian group.

Proof: Assume $\rho_2 = 1$ by a twist. Then one eigenvalue is always $\zeta_1(g) = 1$. The representation ρ_1 is automatically balanced. Put $Q = Q(\rho_1)$ and $Z = Z(\rho_1) = Kern(\overline{\rho}_1)$. Suppose $\rho_1|Z = \chi$. By assumption $\sum_{g\in\tilde{N}} \overline{\chi}_{\rho_1}(g)$ vanishes. Furthermore $\chi_{\rho_1}(g) = 2\chi(g)$ for $g \in Z$ and $\chi_{\rho_1}(g) = 1 + \zeta(g)$ for $g \notin Z$, since 1 is always an eigenvalue. Hence

$$\#Z \cdot \sum_{g\neq 1 \in Q} (1 + \overline{\zeta}(g)) + 2 \cdot \sum_{g\in Z} \overline{\chi}(g) \ = \ 0 \ .$$

For $\chi = 1$ the left side would be > 0, hence $\chi \neq 1$. Therefore the sum of the $1 + \overline{\zeta}(g)$ extended over all $g \in Q$, including $g = 1$, is 2. Hence

$$\frac{1}{2D} \sum_{g\in Q} (1 - \overline{\zeta}(g)) \ = \ 1 - \frac{1}{D} \ ,$$

or in other words $c(Q) = 1/D$. The claim now follows from

Claim 5: Let Q be a finite group Q of order D. Then $c(Q) = \frac{1}{D}$ iff Q is elementary two-abelian.

We have $c(D) = 1 - (D)^{-1}(n_2 + \sum_{i\geq 3} \alpha_i n_i)$ with $0 \leq \alpha_i < 1$. Thus $1/D = c(Q)$ is equivalent to $\sum_{i\geq 3} \alpha_i n_i = \sum_{i\geq 3} n_i$ or $n_i = 0, i \geq 3$. This is equivalent to the fact, that Q is elementary two-abelian.

Claim 6: Suppose ρ is balanced and $Q = Q(\rho)$. If Q is elementary two-abelian, then $\zeta(g) = -1$ if $g \neq 1$ in $Q(\rho)$ and $\zeta(g) = 1$ if $g = 1$ in Q.

Claim 7: Suppose ρ is balanced with $Q = Q(\rho)$. Suppose $c(Q) > 1/8$. Then ρ is an isotypic multiple $\rho = a \cdot \rho_1$ of a two dimensional representation ρ_1, or Q is an elementary abelian two-group of order $D = D(\rho) = 4$. In the second case ρ is an isotypic multiple of $\chi \otimes (1 \oplus \chi_1 \oplus \chi_2 \oplus \chi_1\chi_2)$.

Proof: Decompose $\rho = \oplus m(\rho, \rho_i)\rho_i$ into irreducible representations ρ_i of dimension d_i. Then $c(Q) > 1/8$ or $c(Q)^{-1} \leq 7$ implies $\sum_i d_i^2 \leq 7$ by claim 1. Hence there is at most one i with $d_i = 2$, all others satisfy $d_j = 1, j \neq i$.

Hence either ρ is an isotypic multiple of an irreducible two dimensional representation. Otherwise there are two alternatives: Either ρ is a direct sum of characters (all $d_i = 1$

involving at least three, but at most seven different characters), each with the same multiplicity. Then all characters are quadratic and Q is elementary two-abelian (claim 3). Then $D = c(Q)^{-1} < 8$ is one of the two-powers $\sum_i d_i^2 = D = 1,\ 2,\ 4$. Hence $D = 4$ and the character ρ has to be of the form indicated.

On the other hand there is the case, where there is one irreducible two dimensional constituent ρ_1 with multiplicity $2a$, and there are (up two 3 distinct, at least one) characters χ_i each with multiplicity a (claim 1)

$$\rho = 2a \cdot \rho_1 \ \oplus \ \bigoplus_{i=1}^{t} a \cdot \chi_i$$

(with $1 \leq t \leq 3$). By a character twist we may assume $\chi_1 = 1$. Let us show that this second case cannot occur.

Consider the group $Q_1 = Q(\rho_1)$ of the balanced two dimensional representation ρ_1. By claim 4 the group Q_1 is elementary two-abelian. Let $Q(\rho) \to Q(\rho_1)$ be the natural surjection and U its kernel. We claim that U is trivial. Therefore $Q(\rho) \cong Q(\rho_1)$, which is elementary two-abelian. Hence $c(Q) = 1/D$ is a 2-power (by claim 5) and the assumption $c(Q) > 1/8$ implies $D \leq 4$. Since $D = \sum d_i^2 > 4$ this is a contradiction, which excludes this case. Triviality of U: Fix $\overline{g} \in U$ and choose a representative $g \in \tilde{N}$. It remains to show $\overline{g} = 1$. We obtain $4a$ times the eigenvalue $\chi(g)$ on $2a \cdot \rho_1$, and at most $3a$ eigenvalues on the sum of character spaces including 1. Therefore $\chi(g) = \chi_i(g) = 1$ for all $i = 1, .., t$, since ρ is balanced. But this implies $\rho(g) = 1$, hence $\overline{g} = 1$ in $Q(\rho)$. Hence U is trivial.

Claim 8: Suppose ρ is balanced with $Q = Q(\rho)$. Suppose $c(Q) \leq 1/8$. Then Q is elementary two-abelian and $D = D(\rho) \geq 8$.

Proof: $n_2(Q) > D/2$ holds by our assumption $c(Q) \leq 1/8$ (claim 2a). If $n_2(Q) > 3D/4$, then Q is elementary two-abelian (claim 2c) of order $D \geq 8$ (claim 5), and we are done. In the remaining case $c(Q) > 1/16$ (claim 2b). Furthermore Q can not be a 2-Sylow group (claim 2b). In fact, this leads to a contradiction:

On one hand $\sum_i d_i^2 < 16$ (claim 1), hence in particular $d_i < 4$. This implies $d_i = 2$ or $d_i = 1$. All irreducible constituents of $\rho = \bigoplus_i m_i \rho_i$ therefore have dimension $d_i \leq 2$. Since Q is not a 2-Sylow group, at least one representation ρ_i has dimension $d_i = 2$.

The natural map

$$Q(\rho) \to \prod_i Q(\rho_i)$$

has an abelian kernel K. K is contained in the center of $Q(\rho)$.

Suppose first, that one of the ρ_i is one dimensional. Then all $Q(\rho_i)$ are elementary two-abelian (claim 4). Thus $Q(\rho)$ is a central extension of a two-abelian group by an abelian group K. Therefore $Q \cong Q' \times K'$, where Q' is a 2-Sylow group and K' is odd abelian. Then $n_2(Q) > D/2$ implies $K' = 0$ and $Q = Q'$. A contradiction, since Q was shown not to be a 2-Sylow group. Therefore all ρ_i must be 2-dimensional. At most three of them occur, each with equal multiplicity, which we can assume to be one. Then claim 1 gives $m(\rho, \rho_i)c(Q)^{-1} = dd_i$, hence $c(Q)^{-1} = dd_i = 2d = 2 \cdot 2t$, where $t = 2, 3$. $t = 1$ is excluded by the assumption $c(Q) \leq 1/8$.

Each of the maps $Q(\rho) \to Q(\rho_i)$ is surjective. Therefore $n_2(Q(\rho)) > D(\rho)/2$ implies $n_2(Q(\rho_i)) > D(\rho_i)/2$. The list in [T2] therefore excludes the case $Q(\rho_i) = A_4, S_4, A_5$. There only remain two dimensional irreducible representations ρ_i. In particular, $\tilde{N}$ is not abelian. Therefore the $Q(\rho_i)$ are dihedral groups $D_{2n_i} = \langle S, N | S^2 = N^{n_i} = 1, SNS = N^{-1} \rangle$, since $Q(\rho_i)$ is not cyclic (otherwise $\tilde{N}$ is a central extension of an cyclic group by a cyclic group, hence abelian.) Hence ρ is the sum of two (or three) irreducible 2-dimensional representation of dihedral type.

Let $U_i \subset Q$ be the normal subgroup of index two belonging to the subgroup of index 2 in the group $Q(\rho_i)$ attached to ρ_i. Then the restriction of ρ to U is still balanced, and contains 2 different one dimensional characters. By claim 4 the groups U_i are therefore two-abelian. U_i is the extension of a two-abelian group by a central abelian group Z_i. Furthermore Q is an extension of a 2-group by the central abelian group Z_i. As above this implies, that $Q \cong Q' \times K'$. Here Q' is a 2-Sylow group of Q and K' is abelian of odd order. Then, as above, we show $K' = 0$ and $Q = Q'$ is a 2-Sylow group. This again gives a contradiction.

Appendix B: The Critical Cases

We use notations from [T] reviewed in section 2. In particular we use the definitions of the groups G, G^0, $\overline{G}$, N and the finite group $\tilde{N}$.

The third case of [T]: Recall $G = \overline{G}$ and $G^0 = Gl(2,k)$ in case 3. The representation s of G restricted to $G^0 = Gl(2,k)$ is a $2m$-fold copy of the two dimensional standard representation t ([T], p. 301 bottom). Therefore

$$s \cong \rho \otimes t \quad , \quad \rho(zg) = z^{-1}\rho(g) \quad , \quad z = \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} \in \mu_n$$

for a finite dimensional representation ρ of the finite group $\tilde{N}$. It is easy to see that Property e) of the representation Π and the Tchebotarev density theorem imply, that for every element $g \in \tilde{N}$ the matrix $\rho(g)$ has at most two different eigenvalues $\zeta_1(g), \zeta_2(g)$ with multiplicities $a_1(g), a_2(g)$, such that $a_1(g) + a_2(g) = 2m$. In case there is only one eigenvalue put $\zeta_1(g) = \zeta_2(g)$. The ratio

$$\zeta(g) = \zeta_1(g)/\zeta_2(g) \qquad \text{for } g \in \tilde{N}$$

is well defined up to inverse and depends only on the image of g in the quotient group $\pi_0(N) = N/Z(G^0) \cong \tilde{N}/\mu_n$. This ratio can be extended to G well defined up to inverse: for $g \in G$ with image $\overline{g}$ in $G/G^0 \cong N/Z(G^0)$ put $\zeta(g) = \zeta(\overline{g})$ and takes values in the finite set Ω, defined to be the of roots of unity of order dividing twice the order of the finite group $\pi_0(N)$

$$\zeta(g) \in \Omega \quad , \quad g \in G \ .$$

We identify this set with the corresponding finite set Ω of conjugacy classes of the matrices $diag(\zeta(g), 1)$ in $PGl(2,k)$. In this sense

$$\zeta : G \to G/G^0 \to \Omega \ \subset \ PGl(2,k)/\sim \ .$$

Any g in $G = (Gl(2,k) \times \tilde{N})/\mu_n$ is conjugate to some representative

$$g \sim \begin{pmatrix} u(g) & * \\ 0 & v(g) \end{pmatrix} \times \tilde{n}(g) \quad , \quad \tilde{n}(g) \in \tilde{N} \ .$$

The diagonal entries $u(g), v(g)$ are uniquely determined up to a permutation and multiplication by a common n-th root of unity. Their ratio $u(g)/v(g)$ is well defined up to inverse. This ratio only depends on the conjugacy class of the image of g in $G/N \cong PGl(2,k)$. Let $\Omega \subset PGl(2,k)$ be the finite set defined above. Let $G_\Omega \subset G$ be the subset of all elements g in G, whose image in $(G^0)_{ad} = PGl(2,k)$ is conjugate to an element in Ω. By [T], lemma 2 the set

$$T = T_\Omega$$

of primes v with the Frobenius elements $Frob_v$ in G_Ω has Dirichlet density zero. We may enlarge T such that it contains the archimedian and ramified places of Π.

Lemma B1: Suppose Π satisfies the assumptions a) - e). Suppose we are in case three of Taylor's list. Then there exists a set T of $\mathbb{Q}$-places of density 0, such that for $v \notin T$ the Satake parameters of Π_v are of the form

$$\Pi_v \sim (\nu_v, \nu_v\zeta(Frob_v)^{-1}, \mu_v, \mu_v\zeta(Frob_v)) \quad , \quad \nu_v\mu_v = n(Frob_v)$$

for certain numbers ν_v, μ_v, where $\zeta(g)$ is the root of unity defined above.

<u>Proof</u>: Choose $T = T_\Omega$ as above and $v \notin T$. Property e) of Π implies, that up to an arbitrary (!) permutation the Satake parameters

$$(\nu_v, \tilde{\nu}_v, \mu_v, \tilde{\mu}_v) \quad , \quad \nu_v\mu_v = \tilde{\nu}_v\tilde{\mu}_v = n(Frob_v)$$

of Π_v are of the form

$$(v(g)\zeta_1(g), v(g)\zeta_2(g), u(g)\zeta_1(g), u(g)\zeta_2(g)) \ ,$$

where $g = r(Frob_v)$ or $g = Frob_v$ for simplicity, by abuse of notation. The roots of unity $\zeta_i(g)$ are defined as above. Modulo the action of the Weyl group on the Satake parameters $(\nu_v, \tilde{\nu}_v, \mu_v, \tilde{\mu}_v)$ of Π_v, we may assume that the first Satake parameter is

$$\nu_v = v(g)\zeta_1(g) \quad , \quad g = Frob_v \ .$$

For $v \notin T$ the parameter $\mu_v = n(Frob_v)/\nu_v$ satisfies $\mu_v \neq v(Frob_v)\zeta_2(Frob_v)$, since otherwise $\zeta_1(Frob_v)\zeta_2(Frob_v)v(Frob_v)^2 = n(Frob_v) = \zeta_1(Frob_v)\zeta_2(Frob_v)u(Frob_v)^2$ due to the relation $\tilde{\nu}_v\tilde{\mu}_v = n(Frob_v)$. This would imply $u(Frob_v)/v(Frob_v) \in \Omega$, which is impossible for $v \notin T$. Therefore with in the pair $\tilde{\nu}_v, \tilde{\mu}_v$ we can assume $\tilde{\nu}_v = v(Frob_v)\zeta_2(Frob_v)$, since the Weyl group contains an element fixing ν_v, μ_v, that permutes $\tilde{\nu}_v$ and $\tilde{\mu}_v$. This proves the lemma. $\square$

Lemma B2: Suppose Π is critical in the sense, that it does not satisfy the statements of the main theorem. If it belongs to case three of Taylor's list, then the representation Π is a D-critical representation. It is nondegenerate of two-abelian type with $D \geq 4$ and corresponding Galois group

$$Gal(L : \mathbb{Q}) \ \cong \ \pi_0(G) \ .$$

<u>Proof</u>: For $T = T_\Omega$ chosen as above, the images of the Frobenius elements for $v \notin T$ still generate the finite group $\pi_0(N) = \pi_0(G)$. On the other hand $(\mu_v/\nu_v)^{2D} \neq 1$ and $(\tilde{\mu}_v/\tilde{\nu}_v)^{2D} \neq 1$ for $v \notin T$ by definition of T and $D = \#\pi_0(G)$. For $v \notin T$ therefore $\zeta(Frob_v)$ is (up to inverse) the unique ratio of Satake parameters of Π_v, that is contained in Ω.

Moreover all Satake parameters are different from each other, unless $\zeta(Frob_v) = 1$ or equivalently $r(Frob_v) \in G^0$.

These facts imply, that ρ is a balanced representation of the finite subgroup $\tilde{N}$ of N. The kernel $Z(\rho)$ of the projective representation $\overline{\rho}$ of $\tilde{N}$ attached to ρ - as defined in the appendix - is characterized by: $g \in Z(\rho)$ iff $\zeta(g) = 1$ iff $g \in Z(G^0) \cap \tilde{N} = \mu_n$. By the main result on balanced representations, proved in the appendix A, ρ is either a multiple of a 2-dimensional representation, or $Q(\rho) = \tilde{N}/Z(\rho)$ is a two abelian group of order $D \geq 4$. In the first case the representations s and r are a multiple of a 4-dimensional representation and the main theorem holds for Π with $\pi_0(G)$ a finite subgroup of $PGl(2,k)$. In the second case, the representation Π is D-critical, since $\tilde{N}/Z(\rho) = \tilde{N}/\mu_n = \pi_0(G)$ is two-abelian. Hence $\zeta(Frob_v) = \pm 1$ for $v \notin T$. The reason is, that the order of $\zeta(g)$ equals the order of the image of $g \in \tilde{N}$ in the quotient group $Q(\rho)$, which is two-abelian. In particular $\zeta(Frob_v) = 1$ iff $Frob_v \in G^0$. The corresponding field L is therefore the field attached to the finite Galois group defined by $\pi_0(G) = G/G^0$.

<u>The first case of [T]</u>: In the first case either $G = \overline{G}$ or $[G : \overline{G}] = 2$. To distinguish the two possibilities, we call these cases 1a and 1b. Recall

$$G^0 = \mathbb{G}_m^r \quad , \quad r = 1,2$$

is a torus of rank one or two over k contained in N. Therefore $N = \overline{G}$ and $\overline{G}$ is obtained from the finite subgroup $\tilde{N}$ via

$$\overline{G} = (\tilde{N} \times G^0)/\mu_n^r \ .$$

<u>G^0-eigenspaces</u>: The restriction $\overline{s}$ of the representation s of G to $\overline{G}$ decomposes

$$\overline{s} = \rho \oplus \rho' \ .$$

Here ρ and ρ' are representations of $\overline{G}$, whose restriction to G^0 are characters χ resp. χ' of the torus G^0, such that $\chi\chi' = n$. This is clear for the subtorus G^0. This extends to $\overline{G}$, since $\overline{G} = N$ is the centralizer of this subtorus. (See Taylor's list and also [T], p. 302). So ρ is the representation of $\overline{G}$ on the χ-eigenspace of G^0 and ρ' is the representation of $\overline{G}$ on the χ'-eigenspace of G^0. Property c) of Π implies $\chi^j \neq (\chi')^j$ for all powers $j \geq 1$, since otherwise $\chi^{2j} = n^j$ in contradiction to property c) of the automorphic representation Π. In particular, the character χ'/χ is not of finite order.

In the present case the finite subgroup $\tilde{N}$ of $N = \overline{G}$ is a normal subgroup with quotient $N/\tilde{N} \cong G^0/G^0[n]$. The characters χ^n, χ'^n of the torus G^0 define a faithful embedding of $G^0 \cong G^0/G^0[n]$ into $\mathbb{G}_m^2$. The composite map $\overline{G} = N \to N/\tilde{N} \to \mathbb{G}_m$ defines two different characters χ^n, χ'^n of $\overline{G}$. The characters χ^n, χ'^n of the torus G^0 therefore extend to characters of $N = \overline{G}$

$$\Phi, \Psi : \overline{G} \to k^* \ .$$

In particular, $\tau = (\chi'/\chi)^n$ extends to a character of N, which is not of finite order.

Next we show the

Claim: Both representations ρ and ρ' of $\tilde{N}$ are balanced representations (see appendix A) with at most two eigenvalues:

Let $\Omega \subset k^*$ be the set of roots of unity of order $2n$, where we temporarily - for the definition of Ω - choose $n = \#\pi_0(G)$. The set of places such that either $\tau(r(Frob_v))$ (or $n(r(Frob_v))$) is contained in Ω has Dirichlet density zero. Let T be a set of Dirichlet density zero containing these places.

For a given element in $\pi_0(N)$, there always exists an lift g in $\overline{G}$, such that $g = r(Frob_v)$ for $v \notin T$. For such $g = r(Frob_v)$ consider the eigenvalues ζ_i under the representation ρ and the eigenvalues ζ'_j under the representation ρ'. Since the order of $\pi_0(N)$ was n, we have $\zeta_1^{2n} = \zeta_2^{2n} = ... = \lambda$ and similarly $\zeta_1'^{2n} = \zeta_2'^{2n} = ... = \lambda'$. But $\lambda \neq \lambda'$, for $\tau(g) \notin \Omega$ implies $\tau(g^{2n}) \neq 1$, hence $\chi'(g^{2n})^n \neq \chi'(g^{2n})^n$ and therefore even $\lambda^n \neq \lambda'^n$. We used, that g^{2n} is contained in the torus G^0 and so $\chi(g^{2n})$ and $\chi'(g^{2n})$ are defined.

On the other hand the set of eigenvalues $\{\zeta_1, .., \zeta'_1, ..\}$ has cardinality at most 4 by property e) of the underlying automorphic representation Π. In fact

$$\{\zeta_1, .., \zeta'_1, ..\} = \{\nu_v, \tilde{\nu}_v, \mu_v, \tilde{\mu}_v\} \ ,$$

where the Satake parameters of Π_v on the right furthermore satisfy the relation $\nu_v\mu_v = \tilde{\nu}_v\tilde{\mu}_v = n(Frob_v)$. Without restriction of generality we can assume $\nu_v = \zeta_1$. We claim, that there exists at most two different eigenvalues ζ'_i. Otherwise the Satake parameters would be say $\zeta_1, \zeta'_1, \zeta'_2, \zeta'_3$, and the Satake relation $\zeta_1\zeta'_2 = \zeta'_1\zeta'_3$ raised to the $2n$-th power contradicts $\lambda \neq \lambda'$. Therefore ρ' has at most two eigenvalues. The same holds for ρ, reversing the argument.

The Satake parameters must be of the form $(\zeta_i, \zeta_j, \zeta'_a, \zeta'_b)$ with $\zeta_i\zeta'_a = \zeta_j\zeta'_b$. Otherwise there is a Satake relation of the form $\zeta_i\zeta_j = \zeta'_k\zeta'_l$ or $\zeta_i\zeta_j = \zeta_k\zeta'_l$ or $\zeta_i\zeta'_j = \zeta'_k\zeta'_l$. Raising it to the n-th resp. $2n$-th power gives a contradiction to $\lambda \neq \lambda'$; note $(\zeta_i\zeta_j)^n = \lambda$. Put $\zeta(g) = \zeta_1(g)/\zeta_2(g)$ or $\zeta(g) = 1$ if there is only one eigenvalue; this number is an n-th root of unity uniquely defined up to inverse. Then the Satake parameters of Π_v are

$$(\nu_v, \tilde{\nu}_v, \mu_v, \tilde{\mu}_v) = (\nu_v, \nu_v\zeta(Frob_v)^{-1}, \mu_v, \mu_v\zeta(Frob_v)) \quad , \quad \nu_v\mu_v = \omega_\Pi(p_v) \ .$$

Furthermore

$$\zeta(Frob_v) = \zeta'(Frob_v) \ ,$$

where $\zeta'(g)$ is the correspondingly defined quotient of eigenvalues for the representation ρ' (for suitable choice of numbering). Of course, we used the freedom to normalize the Satake parameter subject to a reparametrization with respect to the action of the Weyl group. Property e) implies that the representations ρ and ρ' are balanced representations: Either $\zeta(g) = \zeta'(g) = 1$ or all 4 Satake parameters are different and occur with multiplicity m. This proves the claim. $\square$

For a given element $g \in \overline{G}$ the values $\zeta(g), \zeta'(g)$ depends up to inverse only on the image of g in the finite quotient group $\overline{G}/G^0 = \pi_0(N)$. By a density argument the above information for Frobenius elements therefore implies

$$\zeta(g) = \zeta'(g) \quad , \quad g \in \tilde{N}$$

for all g in the finite group $\tilde{N}$.

Let $\overline{\rho}, \overline{\rho}'$ be the projective representations of $\tilde{N}$ associated to ρ, ρ' with corresponding kernels $Z(\rho)$ and $Z(\rho')$. Put $Q(\rho) = \tilde{N}/Z(\rho)$ and similar for ρ'. Then $\zeta(g) = 1$ is equivalent with $g \in Z(\rho)$ or $g \in Z(\rho')$. Therefore we have an injective homomorphism $\tilde{N}/Z(\rho) \to Q(\rho) \times Q(\rho')$ whose composition with each projection is injective and surjective. Therefore $Q(\rho) \cong Q(\rho')$. Let $\pi_L(G)$ be the image of $Z(\rho)$ in $\pi_0(N) = \pi_0(\overline{G})$. In fact this group is a normal subgroup of $\pi_0(G)$. If the torus G^0 is two-dimensional, then $\pi_L(G)$ is trivial. If G^0 is the one dimensional torus, then $\pi_L(G)$ is a finite subgroup of a k-torus, hence cyclic. We get an exact sequence

$$0 \to \pi_L(G) \to \pi_0(G) \to \Delta \to 0 \ .$$

Let L be the field attached to the kernel of the surjective group homomorphism

$$Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \to \Delta \ .$$

If ρ and ρ' are m-multiples of two dimensional representations, then s and r are a m-multiple of a four dimensional induced representation. This follows from $[G : \overline{G}] \leq 2$. Therefore the main theorem holds for Π, such that Δ is a finite subgroup of $PGl(2,k)$. If ρ and ρ' are not m-multiples of two dimensional representations, the main result on balanced representations implies, that the isomorphic groups $Q(\rho) \cong Q(\rho')$ are elementary two-abelian of order $D(\rho) \geq 4$. In particular $\zeta(g) \in \{\pm 1\}$ and the representation Π is a D-critical automorphic representation with $D = [L : \mathbb{Q}]$. Obviously $D \geq D(\rho) \geq 4$. From was said above, this statement is already clear in the case 1a, when $G = \overline{G}$. Hence it remains to prove the next lemma in the case, where $\overline{G} \neq G$.

Lemma B3: Suppose Π does not satisfy the statement of the main theorem. If it belongs to case one of Taylor's list, then Π is a D-critical automorphic representation. The underlying Galois group $Gal(L/\mathbb{Q})$

$$\pi_0(G) \twoheadrightarrow \Delta = Gal(L/\mathbb{Q})$$

is a 2-group of order $D \geq 4$ and isomorphic to $\pi_0(G)$ divided by a normal cyclic subgroup. In case 1a where $G = \overline{G}$, the group $Gal(L/\mathbb{Q})$ is a elementary two-abelian group and Π is nondegenerate. In case 1b it is a metabelian group and isomorphic to $\pi_0(G)$ with the elementary two-abelian normal subgroup $\pi_0(\overline{G})$ of order $\overline{D} \geq 4$ and quotient $\pi_0(G)/\pi_0(\overline{G}) = \mathbb{Z}/2\mathbb{Z}$.

Example: The case, where $\pi_0(G)$ has 8 elements, hence $\pi_0(\overline{G}) \cong (\mathbb{Z}/2\mathbb{Z})^2$ turns out to be the most important. See lemma 4.5. Therefore we further specify the structure of $\pi_o(G)$ in this case: Since the group is of order 8, it is either abelian of type $(\mathbb{Z}/2\mathbb{Z})^3$ or $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, or isomorphic to the dihedral group of order eight generated by N, S with $S^2 = 1, N^4 = 1, N^S = N^{-1}$. (This follows from Huppert thm. 14.10, since the quaternion group can not contain a normal subgroup isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$). The case $(\mathbb{Z}/2\mathbb{Z})^3$ arises iff $g^2 = 1$ for all $g \in \pi_0(G)$. Furthermore: The dihedral group D_8 of order eight has the following normal subgroups: The center $\langle N^2 \rangle$ of order two, the subgroup $R = \langle N \rangle \cong \mathbb{Z}/4\mathbb{Z}$, and the two subgroups $P = \{1, N^2, SN, SN^3\}$ and $Q = \{1, N^2, S, SN^2\}$, which are both isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$. Using the isomorphism induced by $S \mapsto SN$, which changes the presentation, we can assume without restriction of generality, that the subgroup $Q \subset D_8$ defined above can be identified with the given group $Q = \pi_0(\overline{G}) \subset \pi_0(G)$. There are four additional subgroups of order two generated by S, SN^2, SN and SN^3, which are not normal in Δ. The conjugacy classes are $\{1\}, \{N^2\}, \{N, N^3\}, \{S, SN^2\}$ and $\{SN, SN^3\}$. The commutator group $[D_8, D_8]$ is cyclic of order two and generated by N^2. Hence D_8 has four different abelian characters $1, \chi_P, \chi_Q$ and $\chi_R = \chi_P \chi_Q$, where χ_P and χ_Q have kernels P and Q.

Proof of lemma B3: Since the lemma has already been verified in case 1a, assume now $G \neq \overline{G}$. Then $G = \langle \overline{G}, \sigma \rangle$ with $\sigma^2 \in \overline{G}$ and by definition σ acts nontrivially on G^0, which in the first case is a torus of dimension one or two over k. In particular the dimension of $T = G^0$ has to be two, for otherwise σ acts by the inversion on T. Since σ interchanges the two characters χ and χ' this would imply $n = \chi\chi' = 1$ contrary to the assumption a): $n|G^0 \neq 1$.

Hence the dimension of the torus G^0 in case 1b is two. Since $\chi \oplus \chi'$ is faithful on G^0, the morphism $(\chi, \chi') : \mathbb{G}_m^2 \cong G^0 \to \mathbb{G}_m^2$ is an isomorphism. Thus we can identify $G^0 = \mathbb{G}_m^2$ in such a way, that $\chi(t_1, t_2) = t_1, \chi'(t_1, t_2) = t_2$ holds for all $(t_1, t_2) \in (k^*)^2$.

Since $\chi' \neq \chi$ $\big(dim(G^0) = 2\big)$ and $\chi \neq \chi^{-1}$ (assumption $n \neq 1$), an argument similar as above implies, that σ interchanges the two characters and therefore $\sigma(t_1, t_2)\sigma^{-1} = (t_2, t_1)$.

Recall, that the restriction $\overline{s} \cong \rho \oplus \rho'$ of s to $\overline{G}$ decomposes. The restrictions of ρ resp. ρ' to G^0 are the characters χ resp. χ' of G^0. Put $\overline{s}^\sigma(g) = \overline{s}(\sigma g \sigma^{-1})$, then $\overline{s}^\sigma \cong \overline{s}$ by assumption. This implies $\rho^\sigma \cong \rho', \rho'^\sigma \cong \rho$, since $\chi^\sigma = \chi', \chi'^\sigma = \chi$. But σ acts nontrivially on G^0. Therefore $\overline{s} \cong \rho \oplus \rho^\sigma$.

Recall that the representation ρ of $\overline{G}$ on V_ρ was shown to be balanced and without restriction of generality we may therefore assume from now on

Assumption: ρ not to be an isotypic multiple of a 2-dimensional representation. The group $Q(\rho)$ attached to the balanced representation ρ of the group $\tilde{N}_\rho$ is elementary two-abelian of order $D = D(\rho) \geq 4$.

Consider the Zariski closure $\overline{G}_\rho$ of $\overline{G}$ in $Gl(V_\rho)$. Then $(\overline{G}_\rho)^0 \cong \mathbb{G}_m$ and $\overline{G}_\rho \cong (\mathbb{G}_m \times \tilde{N}_\rho)/\mu_n$ and the group $Q(\rho)$ is isomorphic to $\pi_0(\overline{G}_\rho)$. Namely $\overline{\rho}(g) = 1$ implies $\rho(g) \in k^* \cdot id_{V_\rho}$, hence $g \in (\mathbb{G}_m \cap \tilde{N}_\rho)(k) = \mu_n$. Now n can be chosen as $n = 2$, since $Q(\rho)$ is two-abelian. Thus

$$\overline{G}_\rho = (\mathbb{G}_m \times \tilde{N}_\rho)/\mu_2$$

$$0 \to \mu_2 \to \tilde{N}_\rho \to Q(\rho) \to 0 \ .$$

The central extension $\tilde{N}_\rho$ defines a cohomology class in $H^2(Q, \mu_2)$ for the group $Q = Q(\rho)$. Let $a(g_1, g_2)$ be a representing 2-cocycle (for the inhomogenous bar resolution), normalized such that $a(0,0) = 1$. Then $q(g) = a(g,g) \in \mu_2$ is a function $q : Q \to \mu_2$ independent of the choice of cocycle. $q(g) = 1$ holds iff $\tilde{g}^2 = 1$ for a (any) lift $\tilde{g} \in \tilde{N}_\rho$ of $g \in Q$. Note $\tilde{g}_1\tilde{g}_2 = a(g_1, g_2)(\widetilde{g_1 g_2})$. Furthermore $q(g_1g_2) = q(g_1)q(g_2)[g_1, g_2]$ is a quadratic form on Q with associated form $[g_1, g_2] = \tilde{g}_1^{-1}\tilde{g}_2\tilde{g}_1\tilde{g}_2^{-1}$, which is bilinear since it has values in the central subgroup μ_n of $\tilde{N}$.

We have a similar situation for $\rho' = \rho^\sigma$. We identify $Q(\rho)$ and $Q' = Q(\rho')$ via σ. The corresponding function q' on Q attached to the extension class of $\tilde{N}_{\rho'}$ in $H^2(Q, \mu_2)$ is $q'(g) = q(\sigma g \sigma^{-1})$.

Recall that $Q(\rho) \cong Q(\rho') \cong \pi_0(\overline{G})$. Using the cocycles defining $\tilde{N}_\rho$ and $\tilde{N}_{\rho'}$ we find a finite σ-stable subgroup $\tilde{N}$, such that

$$\overline{G} \cong (\mathbb{G}_m^2 \times \tilde{N})/\mu_2^2$$

$$0 \to \mu_2 \times \mu_2 \to \tilde{N} \to Q \to 0 \ .$$

Recall that the group $Q = Q(\rho)$ is elementary two-abelian and that $Q(\rho')$ was identified with $Q(\rho)$ by the action of σ. We also view q, q' as functions on $\overline{G}$, which factorize over the quotient $\pi_0(\overline{G})$.

The previous decomposition of $\overline{G}$ can be extended to G. We find a subgroup H, such that

$$G \cong (\mathbb{G}_m^2 \times H)/\mu_2^2$$

$$0 \to \tilde{N} \to H \to \mathbb{Z}/2\mathbb{Z} \to 0 \ .$$

For this find a representative $\sigma \in G \setminus \overline{G}$, such that $\sigma^2 \in \tilde{N}$ and put $H = \langle \tilde{N}, \sigma \rangle$. For any choice of σ we have $\sigma^2 = t \cdot g$ with $t \in G^0, g \in \tilde{N}$. By definition $\sigma t g \sigma^{-1} = tg$, hence

$t^\sigma/t = g(g^\sigma)^{-1} \in G^0 \cap \tilde{N} = \mu_2^2$. If $t = (a,b) \in (k^*)^2$, then $t^\sigma/t = (b/a, a/b)$, hence $b = \pm a$. So we can replace σ by $\tilde{\sigma} = \sigma(a^{-1},1) \in \sigma \cdot G^0$ in such a way, that $\tilde{\sigma}^2 = (1,1)$ or $(1,-1)$ is contained in $\tilde{N}$. μ_2^2 is a normal subgroup of H and $H/\mu_2^2 \cong \pi_0(G)$. The proof of lemma B3 can now completed by the next two lemmas

Lemma B4: Assume $\sigma \notin \overline{G}$. Then $s(\sigma)$ has either two different eigenvalues $t, -t$ or four different eigenvalues $t, -t, it, -it$, depending on whether $\sigma^2 \in G^0$ or not. Each eigenvalue occurs with equal multiplicity.

Proof: For any choice of $\sigma \in G$, $\sigma \notin \overline{G}$ we have

1) $(\sigma \cdot (t_1, t_2))^2 = \sigma^2 \cdot (t,t)$ for $t = t_1 t_2 \in k^*$ and the element $(t,t) \in G^0$ acts on W by the scalar $t \cdot id_W$.

2) σ^4 acts on W by a scalar. Namely $\sigma^4 \in (G^0)^\sigma = \{(t,t) \mid t \in k^*\}$. Hence the eigenvalues of $s(\sigma)$ are of the form $\pm t, \pm it$ for some $t \in k^*$.

On the other hand note, that $\sigma^2 \in G^0$ implies $\sigma \in (G^0)^\sigma$. Therefore $s(\sigma^2)$ acts by a scalar on W iff $\sigma^2 \in G^0$. Let $a,b,c,d \in \mathbb{N}$ denote the multiplicities of the four possible eigenvalues of $s(\sigma)$. $s(\sigma)$ permutes V_ρ and $V_{\rho'}$. Therefore $Tr_W(s(\sigma)) = a - b + i \cdot c - i \cdot d = 0$. This implies $a = b$ and $c = d$. Hence $s(\sigma^2)$ acts by a scalar on W, either if $a = b = 0$ or $c = d = 0$. This is the case $\sigma^2 \in G^0$. Hence $s(\sigma)$ has two different eigenvalues $t, -t$ with equal multiplicity.

In the remaining case $\sigma^2 \notin G^0$ both $a = b, c = d$ are non negative. So there are four differnt eigenvalues of type $\pm t, \pm it$. In this case $s(\sigma^2)$ has the eigenvalues t^2 and $-t^2$ with the multiplicities $2a$ and $2c$. It only remains to show $a = c$. The image of $\sigma^2 \in \overline{G}$ in $\pi_0(\overline{G}) = \overline{G}/G^0$ is non zero. In terms of the function $\zeta(g) = \zeta'(g), g \in \overline{G}$ introduced earlier, this means $\zeta(\sigma^2) \neq 1$. Therefore σ^2 has two different eigenvalues on both subrepresentations V_ρ and $V_{\rho'}$. The representations ρ, ρ' are balanced representations of $\overline{G}$. Hence each of the two different eigenvalues of σ^2 on V_ρ has equal multiplicities $dim(V_\rho)/2$ and similar for ρ'. We already know, that there are only two eigenvalues $t^2, -t^2$ of σ^2. Hence we get $a = c$, since $dim(V_\rho) = dim(V_{\rho'})$. Lemma B4 is proved. $\square$

The Satake parameters for $Frob_v \notin \overline{G}$: It remains to determine the Satake parameters of Π_v in case 1b in order to complete the proof lemma B3.

Lemma B5: Suppose the assumptions are as in lemma B2 and suppose we are in case 1b of Taylor's list. Then Π is D-critical, where $Gal(L/\mathbb{Q}) = \pi_0(G)$ is a metabelian extension of a $\mathbb{Z}/2\mathbb{Z}$ by an elementary two-abelian group $Q(\rho) = \pi_0(\overline{G})$ of order ≥ 4. There exists a set T of Dirichlet density 0, such that for $v \notin T$ the local representation Π_v has Satake parameters

$$(\nu_v, \zeta_v^{-1}\nu_v, \mu_v, \zeta_v\mu_v) \quad , \quad \nu_v\mu_v = n(Frob_v) \ ,$$

where $\zeta_v = \zeta(Frob_v) = \pm 1$ and $\zeta_v = 1$ iff $Frob_v \in G^0$.

Proof: <u>First the cases C,D and E</u>, where $Frob_v \notin \overline{G}$.

There is the <u>case C</u>, where $Frob_v^2 \notin G^0$. By lemma B4 and property e) of Π

$$\Pi_v \sim (t_v, -t_v, it_v, -it_v) \sim (t_v, \pm it_v, \mp it_v, -t_v)$$

up to replacement of i by $-i$. Thus $\nu_v/\mu_v = \pm i$. The case $(t_v, \pm it_v, -t_v, \mp it_v)$ is impossible, since it contradicts $\nu_v\mu_v = \tilde{\nu}_v\tilde{\mu}_v$. Furthermore $\zeta_v(\Pi_v, 1, s)$ is attached to the L-parameters $(1, -1, -1/i, -i, -1)$ and therefore $log\ \zeta_v(\Pi_v, 1, s) = -p_v^{-s} + O(p_v^{-2s})$.

In the next <u>case D</u> $Frob_v^2 \in G^0$ and by lemma B4 and property e) of Π

$$\Pi_v \sim (t_v, -t_v, -t_v, t_v) \sim (t_v, t_v, -t_v, -t_v) \sim (-t_v, -t_v, t_v, t_v)\ .$$

Hence $\nu_v/\mu_v = -1$ and $\zeta_v(\Pi_v, 1, s)$ is attached to the L-parameters $(1, 1, -1, -1, 1)$. Thus $log\ \zeta_v(\Pi_v, 1, s) = p_v^{-s} + O(p_v^{-2s})$.

Now the <u>case E</u>, where $Frob_v^2 \in G^0$ and

$$\Pi_v \sim (t_v, -t_v, t_v, -t_v)\ .$$

Then $\nu_v/\mu_v = 1$ and $\zeta_v(\Pi_v, 1, s)$ is attached to the L-parameters $(1, -1, -1, -1, -1)$, hence $log\ \zeta_v(\Pi_v, 1, s) = -3p_v^{-s} + O(p_v^{-2s})$.

Finally the cases <u>case A and B</u>, where $Frob_v \in \overline{G}$ and either $Frob_v \in G^0$ or $Frob_v \notin G^0$. Then according to the discussion of case 1a, the representation Π_v has Satake parameters

$$(\nu_v, \nu_v\zeta_v^{-1}, \mu_v, \mu_v\zeta_v)\ ,$$

where $\zeta_v = \zeta(Frob_v) \in \{\pm 1\}$ and $= 1$ iff $Frob_v \in G^0$. This proves lemma B5. $\square$

The argument shows, that Π is degenerate in case 1b. A substitute will be provided by the following further analysis of the cases A and B, where $Frob_v \in \overline{G}$: we use $\overline{G} = (\mathbb{G}_m^2 \times \tilde{N})/\mu_2^2$ to write $r(Frob_v)$ in the form $(t_v, t_v') \times g_v$ mod μ_2^2 with $(t_v, t_v') \in (k^*)^2$ and $g_v \in \tilde{N}$. Let $\alpha_v, \zeta_v\alpha_v$ resp. $\alpha_v', \zeta_v\alpha_v'$ be the eigenvalues of $\rho(g_v)$ resp. $\rho'(g_v)$ under the balanced representations ρ and ρ' of $\overline{G}$. Since $\tilde{N}$ is a finite group, and every element has order dividing 4, these eigenvalues are contained in $\pm 1, \pm i$. Obviously $\alpha_v^2 = q(Frob_v)$ and ${\alpha_v'}^2 = q'(Frob_v)$, where

$$q, q' : \pi_0(\overline{G}) \to \mu_2$$

are the quadratic forms on $\pi_0(\overline{G})$ extended to $\overline{G}$, that were defined earlier related to the extension $0 \to \mu_2^2 \to \tilde{N} \to \pi_0(\overline{G}) \to 0$. In particular

$$\rho(Frob_v) \sim t_v \begin{pmatrix} \alpha_v E & * \\ 0 & \alpha_v\zeta_v E \end{pmatrix}$$

$$\rho'(Frob_v) \sim t'_v \begin{pmatrix} \alpha'_v E & * \\ 0 & \alpha'_v \zeta_v E \end{pmatrix} .$$

Without restriction of generality, by changing α'_v to $\alpha'_v\zeta_v$, the Satake parameters of Π_v therefore are

$$(t_v\alpha_v, t_v\alpha_v\zeta_v, t'_v\alpha'_v, t'_v\alpha'_v\zeta_v)$$

with $n(Frob_v) = \nu_v\mu_v = (t_v\alpha_v)(t'_v\alpha'_v)$. Hence

$$t_v\alpha_v/t'_v\alpha'_v = (t_v\alpha_v)^2/n(Frob_v) = t_v^2 q(Frob_v)/n(Frob_v) = q\frac{\Psi}{n}(Frob_v)$$

$$t_v\alpha_v/t'_v\alpha'_v = n(Frob_v)(t'_v\alpha'_v)^{-2} = {t'_v}^{-2}q'(Frob_v)n(Frob_v) = q'\frac{n}{\Phi}(Frob_v) .$$

Therefore $q'q = q'/q$ is the quadratic character $(q'/q)(Frob_v) = (t_v)^2(t'_v)^2/n^2(Frob_v)$

$$qq' = \Psi\Phi/n^2$$

of $\overline{G}$, where

$$\Psi, \Phi : \overline{G} \to k^*$$

are the characters $(t,t') \times g$ mod $\mu_2^2 \mapsto t^2$ resp. $(t')^2$ of $\overline{G}$ defined earlier. Recall that $\tilde{N}$ is a central extension of μ_2 in our present situation. Also define $\Psi' : \overline{G} \to k^*$ by $\Psi' = n^2/\Psi$. Then

$$\Psi' = q'q\Phi .$$

The quadratic character $q'q$ is trivial on G^0 and may be viewed as a character of $\pi_0(\overline{G})$. Let L be the fixed field of the inverse image of G^0 in $Gal(\overline{\mathbb{Q}}:\mathbb{Q})$ and let $\overline{L}$ be the fixed field of the inverse image of $\overline{G}$ in $Gal(\overline{\mathbb{Q}}:\mathbb{Q})$.

<u>Asymptotics</u>: We now determine the <u>weights</u> w_v in the asymptotic formula

$$log\ \zeta_v(\Pi_v,\chi_v,s) = w_v \cdot \chi_v(p_v)p_v^{-s} + O(p_v^{-2s}) ,$$

where $\zeta_v(\Pi_v,\chi_v,s)$ denotes the local factor of the degree 5 standard L-series of the automorphic representation Π at a nonarchimedian unramified place $v \notin T_\Omega$ and $w_v = 1+\varepsilon_v(Ad_v+1)$, $Ad_v = \frac{\mu_v}{\nu_v} + \frac{\nu_v}{\mu_v} + 1$, where $(\nu_v, \varepsilon_v\nu_v, \mu_v, \varepsilon_v\mu_v)$ with $\varepsilon \in \{\pm 1\}$ are the Satake parameters of Π_v.

<u>The weights w_v</u>: According to the 5 possibilities A,B,C,D,E we get

A,B: In these cases $Frob_v \in \overline{G}$ and $w_v = (1+2\varepsilon_v) + \varepsilon_v q(Frob_v)(\frac{\Psi}{n}(Frob_v) + \frac{\Psi'}{n}(Frob_v))$.

C,D,E : In these cases $Frob_v \notin \overline{G}$ and $\nu_v/\mu_v = \pm i$ resp. -1 resp. 1. The weights are $-1, 1, -3$ respectively.

The weights are $w_v = -1,\ 1,\ -3$ in the cases C,D,E. See the proof of lemma 7. In case A resp. case B (v splits in L or not), we found

$$Ad_v = 1 + \frac{t_v\alpha_v}{t'_v\alpha'_v} + \frac{t'_v\alpha'_v}{t_v\alpha_v}\ ,$$

or with the notations from above

$$Ad_v = 1 + q(Frob_v)\Big(\frac{\Psi}{n}(Frob_v) + \frac{\Psi'}{n}(Frob_v)\Big)\ .$$

In the next lemma Ψ and Ψ' are shown to be nontrivial Grossencharacters of the quadratic extension field $\overline{L}$ of $\mathbb{Q}$. Therefore in a statistical sense, the weight factors behave like $w_v = 3, -1$ in the cases A and B for any character χ of finite order, since Ψ/n and Ψ'/n themselves are not of finite order and since q end ε - viewed as depending on $Frob_v$ - is constant on the cosets $Gal(\overline{\mathbb{Q}} : \overline{L})/Gal(\overline{\mathbb{Q}} : L) \cong \pi_0(\overline{G})$.

Lemma B6: Ψ/n and Ψ'/n are inverse algebraic Grossencharacters of the quadratic extension field $\overline{L}$ of $\mathbb{Q}$. These characters are not of finite order.

Proof: A λ-adic representation ρ is called E-rational, if it is unramified at almost all places, such that $trace\ \rho(Frob_v)^i \in E$ holds for all $i \in \mathbb{N}$. The following properties are obvious: If ρ is E-rational, then its restriction to a subgroup of finite index is again E-rational. If ρ and ρ' are E-rational, then also $\rho \otimes \rho'$. If $\rho = \chi \otimes \rho_0$ is E-rational and if χ is a character and ρ_0 a representation with finite image of order N, then χ is $E(\zeta_N)$-rational if ρ is E-rational and vice versa.

The representation r of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ on W (see section 2) arises from a semisimple λ-adic representation, which is E-rational. For a number field $E = E_\Pi$ as in [T], p.297 the characteristic polynomial $P_v(X)^m$ of $r(Frob_v)$ is a polynomial in $E[X]$. This follows from $L_v(\Pi_v, s) = P_v(p^{-s})^{-1}$ (property e).

The restriction $\rho \oplus \rho'$ of s to the subgroup $Gal(\overline{\mathbb{Q}}/\overline{L})$ decomposes and is E-rational. $\rho \otimes \rho'$ is of the form $n \otimes \rho_0$. Since $\overline{G} \cong (\mathbb{G}_m \times \tilde{N})/\mu_2^2$, the representation ρ_0 has finite image and n is the character defined by Π (see section 2, property a) or e)). By our assumptions on Π the character n is E-rational. Hence, if E is suitably enlarged by a root of unity, $\rho \otimes \rho'$ is E-rational.

The restriction $\rho \oplus \rho'$ of s to the subgroup $Gal(\overline{\mathbb{Q}}/\overline{L})$ decomposes and is E-rational, as well as the tensor square of this restriction. It contains two copies of the representation $\rho \otimes \rho'$ as direct summands. Hence $(\rho \otimes \rho)\ \oplus\ (\rho' \otimes \rho')$ defines a λ-adic semisimple abelian E-rational representation of the absolute Galois group of $\overline{L}$. By [He], p.113 this representation must be locally algebraic. Since $\rho \otimes \rho = \Psi \otimes (\oplus_i \chi_i)$ for finitely many quadratic characters of the two-abelian finite group $\tilde{N}/\mu_2$, Ψ is locally algebraic and then also Ψ/n. This proves the first part of the claim.

Assumption a) together with assumption c) for Π, applied for the characters χ and χ' of the torus $G^0 \subset G$, force the Grossencharacters Ψ/n and Ψ'/n to be of infinite order. This proves lemma B6. $\square$

Appendix C: Poles at $s = 1$ in the CM case

Suppose Π is D-critical of CM type. We already know, that the corresponding field extension $L/\mathbb{Q}$ is a Galois extension of degree $D = 8$ with Galois group either D_8 or $(\mathbb{Z}/2\mathbb{Z})^3$ (see lemma 4.5 and 4.6). Furthermore, Π is a theta lift of some $\pi = \pi_K$ on $Gl(2, \mathbb{A}_K)$, where $K/\mathbb{Q}$ is an algebra of degree 2 over $\mathbb{Q}$.

Proposition: Suppose Π is D-critical of CM type. Any choice of K, for which Π is a theta lift from $Gl(2\mathbb{A}_K)$, is a field and is contained in L and is different from $\overline{L}$. Fields K_i obtained in this way correspond to the characters $\chi = \chi_{K_i}$ of finite order, for which $\zeta^S(\Pi, \chi, s)$ has a pole at $s = 1$. For such K, there are three quadratic extension fields $\overline{L}K, F, F'$ in L/K. The automorphic representation $\pi = \pi_K$ on $Gl(2, \mathbb{A}_K)$ corresponding to Π satisfies

1) $\pi \cong \pi \otimes \chi_{K\overline{L}/K} \not\cong \pi$
2) $\sigma(\pi) \cong \pi \otimes \chi$ for $\chi = \chi_{F/K}$ and $\chi = \chi_{F'/K}$.

Proof: The statement is a partial summary of the lemmas C3-C5 and C3'-C5', proved in this appendix. $\square$.

In the following we first discuss the behaviour of $\zeta(\Pi, \chi, s)$ at $s = 1$ for abelian characters χ of the Galois group $Gal(L/\mathbb{Q})$. For this we extend the method used in the proof of lemmas 4.5 and 4.6. Let Δ be the Galois group of $L/\mathbb{Q}$, a group of order D. Let $X_1, ..$ be the conjugacy classes of this group. Fix an abelian character χ of Δ. Then $\chi_i = \chi(g)$ for $g \in X_i$ is well defined. We claim

Lemma C1: There are real numbers $w(X_i)$, such that for $s \to 1^+$

$$log\ \zeta^S(\Pi, \chi, s)\ \sim\ \left(\frac{1}{D}\sum_i \#(X_i)\cdot w(X_i)\cdot \chi_i\right)\cdot log\ \zeta(s)\ .$$

Proof: Simply put $w(X_i) = lim_{s\to 1^+} D\#(X_i)^{-1}\sum_{v\in T, Frob_v\in X_i}(1 + \varepsilon_v(Ad_v + 1))p_v^{-s}/log\ \zeta(s)$. Let us show, that these limits exist.

An analog was already obtained in appendix B, namely the existence of "statistical" weights for the images of Frobenius elements $g = Frob_v \in \Delta$ (they depended on whether

$Frob_v$ belongs to the cases A,B,C,D,E and they were $3, -1, -1, 1, -3$ respectively). If all $Frob_v, v \in T$ with $g = Frob_v \in X_i$ belong to the same case A-E, this implies existence of the weight $w(X_i)$ for the conjugacy class X_i. Whether $Frob_v$ belongs to case A-C, depends only on group theoretical properties of the image of $Frob_v$ in Δ. Hence it is the case D and the case E, that might cause trouble.

If Δ is abelian there are several classes, which might be a mixture of the cases D and E. On the other hand the existence of $w(X_i)$ in this case is an immediate consequence of the Tchebotarev density theorem. Simply vary $\chi \in \hat{\Delta}$. Then existence follows, and $w(X_i) = \frac{1}{D}\sum_\chi \overline{\chi}(X_i)\cdot n(\chi)$, where

$$log\ \zeta(\Pi,\chi,s) \ \sim\ n(\chi)\cdot log\ \zeta(s) \quad , \quad n(\chi) \in \mathbb{Z}\ ,$$

and $n(\chi)$ is the order of the meromorphic functions $\zeta(\Pi,\chi,s)$ at $s=1$. In the dihedral case $\Delta = D_8$ fortunately there is only one conjugacy class, which contains Frobenius elements with mixed cases D or E. Using the notations of the remark after lemma B3 in appendix B, this conjugacy class contains the two elements SN and SN^2. All other classes have well defined weights. Therefore existence of the remaining weight $w = w(\{SN, SN^3\})$ follows as in the abelian case. $\square$

For conjugacy classes of type A,B,C the weights are $w(X_i) = 3, -1, -1$ respectively. For a class with a mixture of case E and D, the weight $w = w(X_i)$ obviously satisfies $-3 \le w \le 1$.

The CM case $\Delta = D_8$: For notations see the example in appendix B3.

Classes X_i	1	$\{N^2\}$	$\{S, SN^2\}$	$\{N, N^3\}$	$\{SN, SN^3\}$
Cases	A	B	B	C	D+E
Weights $w(X_i)$	3	-1	-1	-1	w
Cardinality $\#(X_i)$	1	1	2	2	2
$\chi = 1$	1	1	1	1	1
$\chi = \chi_Q$	1	1	1	-1	-1
$\chi = \chi_P$	1	1	-1	-1	1
$\chi = \chi_R$	1	1	-1	1	-1

Lemma C2: The order $n(\chi)$ of $\zeta^S(\Pi,\chi,s)$ at $s=1$ for the characters $\chi = 1, \chi_Q, \chi_P, \chi_R$ is $0,0,1,0$ respectively. Up to a set of density zero the class $\{SN, SN^3\}$ has type D and $w=1$.

Proof: For characters χ of Δ we have $8\cdot n(\chi) = 3 - \chi(N^2) - 2\cdot\chi(S) - 2\cdot\chi(N) + 2\cdot w\cdot\chi(SN)$, by definition. This implies $-n(1) = n(\chi_Q) = n(\chi_R) = \frac{1}{4}(1-w)$ and $n(\chi_P) = \frac{1}{4}(w+3)$. Since we know $-3 \le w \le 1$ and the $n(\chi)$ are integers, either $w=1$ or $w=-3$.

Suppose $w = -3$. Then $\zeta^S(\Pi, 1, s)$ would have a zero at $s = 1$. This is impossible. Since Π is a theta lift, proposition 6.8 implies

$$ord_{s=1}(\zeta^S(\Pi, \chi, s)\zeta^S(\Pi, \chi\chi_K, s)) = ord_{s=1} L_K^S(\chi \circ Norm_K) L_K^S(\sigma(\pi) \times \pi^* \otimes (\chi \circ Norm_K), s) \ .$$

Apply this for $\chi = 1$. $\zeta^S(\Pi, \chi_K, s)$ has at most a simple pole at $s = 1$ by theorem 4.2. $\zeta^S(\Pi, 1, s)$ would have a simple zero at $s = 1$. Therefore the left hand side would be regular at $s = 1$. The right hand side has a pole of order at least one, due to the first factor $L_K(1, s)$. This is a contradiction. Hence $w = -3$ is impossible. Therefore $w = 1$. The claim is proved. $\square$

<u>Remark</u>: $\zeta^S(\Pi, \chi_P, s)$ has a pole at $s = 1$. Thus Π is related to an automorphic representation $\pi = \pi_K$ of $Gl(2, \mathbb{A}_K)$, where K is the quadratic extension field of $\mathbb{Q}$ defined by the character $\chi_K = \chi_P$ (theorem 4.2). It is a subfield of L.

Lemma C3: Suppose χ is an arbitrary character of finite order. Suppose $\zeta^S(\Pi, \chi, s)$ has a pole at $s = 1$. Then $\chi = \chi_P$.

Proof: By the remark 4.3 we know $\chi^2 = 1$. Let L_χ be the corresponding quadratic extension field of $\mathbb{Q}$. We may suppose $\chi \neq 1, \chi_P, \chi_Q, \chi_P\chi_Q$, by lemma C2. Then L and L_χ are linear disjoint fields and $Gal(LL_\chi/\mathbb{Q}) = Gal(L/\mathbb{Q}) \times Gal(L_\chi/\mathbb{Q})$. Since Π^S satisfies the Ramanujan conjecture, the lemma is an immediate consequence of the Tchebotarev density theorem and the fact, that up to a set of primes of density zero the Frobenius elements mapping to the conjugacy classes of $Gal(L/\mathbb{Q}) = D_8$ in the table above belong to the cases A, B, B, C, D respectively (lemma C2). $\square$

Corollary C4: The only possible K is given by $\chi_K = \chi_P$. $\square$

Let σ be the nontrivial automorphism of $K/\mathbb{Q}$. Consider the field extensions

$$\begin{tikzcd}[arrows=dash]
 & L \arrow[d] \arrow[dr] & \\
 & K\overline{L} \arrow[dl] \arrow[d] \arrow[dr] & F \arrow[d] \\
\overline{L} = L_Q \arrow[dr] & L_R \arrow[d] & K = L_P \arrow[dl] \\
 & \mathbb{Q} &
\end{tikzcd}$$

where $K = L_P$ is the fixed field of the subgroup $P = \{1, N^2, SN, SN^3\}$, and where L_Q is the field $\overline{L}$ attached to the subgroup Q. The composite $K\overline{L}$ is the fixed field of N^2. Let

F be the fixed field of the group element $SN \in D_8$, which is a quadratic extension of K. Conjugation by σ fixes K and permutes the quadratic extension fields $K\overline{L}, F, F'$ of K in L. Obviously it fixes $K\overline{L}$ and permutes F and $\sigma(F) = F'$. Since SN and SN^3 are conjugate, F' is the fixed field of the element $SN^3 \in D_8$. We obtain the quadratic characters $\chi_{F/K}, \chi_{F'/K}, \chi_{K\overline{L}/K}$. These are characters of $\mathbb{A}_K^*/K^*$. Note that $\chi_{K\overline{L}/K} = \chi_Q \circ Norm_{K/\mathbb{Q}}$.

Lemma C5: The representation π on $Gl(2, \mathbb{A}_K)$, related to Π, has the following CM-properties:

1) $\pi \cong \pi \otimes \chi_{K\overline{L}/K}$.
2) $\sigma(\pi) \not\cong \pi \otimes \chi_{K\overline{L}/K}$.
3) $\sigma(\pi) \cong \pi \otimes \chi$ for $\chi = \chi_{F/K}$ and $\chi = \chi_{F'/K}$.

Proof: 1) is a consequence of 3). Proposition 6.8 applied to $\chi = \chi_Q$ with $\chi\chi_K = \chi_Q\chi_P = \chi_R$ gives

$$ord_{s=1}\zeta^S(\Pi, \chi_Q, s)\zeta^S(\Pi, \chi_R, s) = ord_{s=1}L_K^S(\chi_Q \circ Norm_K)L_K^S(\sigma(\pi) \times \pi^* \otimes (\chi_Q \circ Norm_K), s) \ .$$

The left side is zero. Hence the function $L_K^S(\sigma(\pi) \times (\pi^* \otimes (\chi_Q \circ Norm_K), s)$ has a pole at $s = 1$. This implies $\sigma(\pi) \not\cong \pi \otimes (\chi_Q \circ Norm_K)$ by prop 7.1. This proves 2).

For the proof of 3) it is enough to show, that for a set of spherical places w of K of K-density 1, the representations π_w and $\pi_w \otimes \chi_{F/K,w}$ are locally isomorphic. This is enough, since the Ramanujan conjecture holds for π. Therefore the local isomorphisms imply $L^S(\pi^* \times (\pi \otimes \chi_{F/K}), s) \sim \zeta_K^S(s)\zeta^S(Ad(\pi), 1, s)$ at $s = 1$. $\zeta_K^S(s)$ has a pole at $s = 1$ and $\zeta^S(Ad(\pi), 1, 1) \neq 0$. Hence the left side has a pole at $s = 1$, which implies 3) by prop 7.1.

The set of places w of K, which lie over $K = L_P$-split primes p_v of $\mathbb{Q}$ with $v \notin T$ has K-density one. For such v there are two places w, w' of K in this set. The remarks after assumption 8.1 imply $\pi_w \times \pi_{w'} \sim (t_v\nu_v, t_v\mu_v) \times (t_v\tilde{\nu}_v, t_v\tilde{\mu}_v)$ for some t_v, if $\Pi_v \sim (\nu_v, \tilde{\nu}_v, \mu_v, \tilde{\mu}_v)$. The additional factors t_v may arise through the passage from Π to the twist Π'. We get the following list of cases

$Frob_v$	1	N^2	SN	SN^3
Cases	A	B	D	D
Π_v	$(\nu_v, \nu_v, \mu_v, \mu_v)$	$(\nu_v, -\nu_v, \mu_v, -\mu_v)$	$(\nu_v, -\nu_v, -\nu_v, \nu_v)$	$(\nu_v, -\nu_v, -\nu_v, \nu_v)$
$\chi_{F/K}$	1	-1	1	-1
$\chi_{F'/K}$	1	-1	-1	1

whereas the corresponding representation $\sigma(\pi_w \times \pi_{w'}) = \pi_{w'} \times \pi_w$ of $Gl(2, K_v) = Gl(2, \mathbb{Q}_v) \times Gl(2, \mathbb{Q}_v)$ satisfies

$\pi_w, \pi_{w'}$	$t_v(\nu_v, \mu_v), t_v(\nu_v, \mu_v)$;	$t_v(\nu_v, \mu_v), -t_v(\nu_v, \mu_v)$;	$t_v(\nu_v, -\nu_v), t_v(-\nu_v, \nu_v)$
$\pi_{w'}\pi_w$	$t_v(\nu_v, \mu_v), t_v(\nu_v, \mu_v)$;	$-t_v(\nu_v, \mu_v), t_v(\nu_v, \mu_v)$;	$t_v(-\nu_v, \nu_v), t_v(\nu_v, -\nu_v)$

In the list above the entries for Π_v and $\pi_v = \pi_w \times \pi_{w'}$ for the two conjugate elements SN and SN^3 are the same. Hence only one entry is in the last table. By inspection in the four possible cases we see $\sigma(\pi_v) \cong \pi_v \otimes \chi_{F/K,v}$. $\square$

The CM case $\Delta = (\mathbb{Z}/2\mathbb{Z})^3$:

Let Q be the subgroup of $\Delta = Gal(L/\mathbb{Q})$ with fixed field $\overline{L}$. Suppose Q is generated by N, S with $N^2 = S^2 = 1$ and suppose $\Delta = \langle Q, M\rangle$ with $M^2 = 1$. Let χ_Q be the nontrivial quadratic character of Δ, which is trivial on the subgroup Q. Let χ_N, χ_S be the characters defined by $\chi_N(N) = -1, \chi_N(M) = \chi_N(S) = 1$ and $\chi_S(S) = -1, \chi_S(N) = \chi_S(M) = 1$. The weights for $Frob_v = 1, N, S, NS, M, MN, MS, MNS$ are $3, -1, -1, -1, w_1, w_2, w_3, w_4$ with the corresponding cases $A, B, B, B, D+E, D+E, D+E, D+E$. It follows, similar to lemma C5, that

$$ord_{s=1}\zeta^S(\Pi,\chi,s)\zeta^S(\Pi,\chi\chi_Q,s) = m(\chi)$$

with $m(1) = 0, m(\chi_N) = 1, m(\chi_S) = 1, m(\chi_N\chi_S) = 1$.

Let denote $m = ord_{s=1}\zeta^S(\Pi,1,s)$ and let x, y, z denote the orders of $ord_{s=1}\zeta^S(\Pi,\chi,s)$ for $\chi = \chi_N, \chi_S, \chi_N\chi_S$. Then $(w_1+w_2+w_3+w_4)/8 = m$ and $(w_1-w_2+w_3-w_4)/8+1/2 = x$ and $(w_1+w_2-w_3-w_4)/8+1/2 = y$ and $(w_1-w_2-w_3+w_4)/8+1/2 = z$. The fact that $-3 \leq w_i \leq 1$ implies $m \in \{-1, 0\}$ and $x, y, z \in \{0, 1\}$. Suppose $m = -1$. Proposition 6.8 gives $ord_{s=1}\zeta^S(\Pi,1,s)\zeta^S(\Pi,\chi_K,s) \geq 1$. The order of $\zeta^S(\Pi,\chi_K,s)$ is at most 1. This is a contradiction, hence $m = 0$. We can then solve the inequalities for w_i above and get $0 \leq x+y+z \leq 2$ and $-2 \leq -x+y-z \leq 0$ and $-2 \leq x-y+z \leq 0$ and $-2 \leq -x-y+z \leq 0$. Therefore either $x = 1, y = 1, z = 0$ (without restriction of generality) or $x = y = z = 0$. In the first case we get $w_1 = 1, w_2 = 1, w_3 = 1, w_4 = -3$ and in the second we get $w_1 = -3, w_2 = 1, w_3 = 1, w_4 = 1$. Then for all $\mathbb{Q}$-places except from a set of density zero, the possible types are A, B, B, B, D, D, D, E resp. A, B, B, B, E, D, D, D. This proves

Lemma C3': There are exactly three characters $\chi = \chi_1, \chi_2, \chi_2$ of Δ, for which $\zeta^S(\Pi,\chi,s)$ has a pole at $s = 1$. For all other characters χ (of Δ) the order of $\zeta^S(\Pi,\chi,s)$ at $s = 1$ is zero. Furthermore

$$\hat{\Delta} = \{1, \chi_Q, \chi_1, \chi_1\chi_Q, \chi_2, \chi_2\chi_Q, \chi_3, \chi_3\chi_Q\} \ .$$

Since K is a subfield of L and the group Δ is abelian, corollary 9.3 together with lemma C3' implies

$$6 = 2 \cdot ord_{s=1} \prod_{\chi \in \hat{\Delta}} \zeta^S(\Pi,\chi,s) = n_K(\Pi) \cdot D/2 + 2 = 4 \cdot n_K(\Pi) + 2 \ ,$$

with the number $n_K(\Pi)$ defined in lemma 9.1. We get $n_K(\Pi) = 1$. Therefore corollary 9.3 implies

$$ord_{s=1} \prod_{\chi \in \hat{\Delta}} \zeta^S(Ad(\pi), \chi \circ Norm_K, s) = n_K(\Pi) \cdot D/2 - 2 = 2 \ .$$

The two cases in this product, where there are poles of $\zeta^S(Ad(\pi), \chi \circ Norm_K, s)$, must therefore be of the form $\chi \in \{\chi_4, \chi_4\chi_K\}$ for some $\chi_4 \in \hat{\Delta}$ (proposition 7.1).

Lemma 9.1 and $n_K(\Pi) = 1$ and the nonvanishing result of lemma C3' implies for all $\chi \in \hat{\Delta}$, that precisely one of the three functions $\zeta^S(\Pi, \chi, s)$ or $\zeta^s(\pi, \chi\chi_K, s)$ or $\zeta^S(Ad(\pi_K), \chi \circ Norm_K, s)$ has a pole at $s = 1$. This implies $\{\chi_i, \chi_i\chi_K\} \cap \{\chi_4, \chi_4\chi_K\} = \emptyset$ for $i = 1, 2, 3$. Hence χ_i for $i = 1, ..4$ are coset representatives with respect to $\{1, \chi_K\}$. Furthermore it implies $\chi_i\chi_K \notin \{\chi_1, \chi_2, \chi_3\}$. Since χ_K can be any of the characters χ_1, χ_2, χ_3, these properties imply

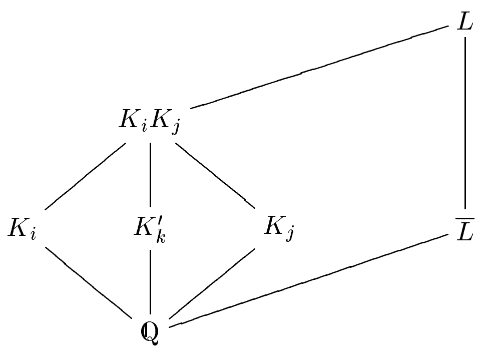
Lemma C4': $\chi_4 = \chi_1\chi_2\chi_3 = \chi_Q$.

Proof: For $1 \leq i, j \leq 3$ we can not have $\chi_i\chi_j = \chi_Q$, since otherwise $\chi_i = \chi_j\chi_Q$ contradicting lemma C3'. The product is not in $\{\chi_1, \chi_2, \chi_3\}$ either, as stated above. Therefore $\chi_1\chi_2 = \chi_3\chi_Q$ by lemma C3'. More symmetrically $\chi_1\chi_2\chi_3 = \chi_Q$. Now fix any $i \in \{1, 2, 3\}$. Then $\hat{\Delta} = \{\chi_1, \chi_2, \chi_3, \chi_4, \chi_1\chi_i, \chi_2\chi_i, \chi_3\chi_i, \chi_4\chi_i\}$. Since $\chi_k\chi_i = \chi_j\chi_Q$ or 1, lemma C3' implies $\chi_Q \in \{\chi_4, \chi_4\chi_i\}$. Since this holds for every choice of $i = 1, 2, 3$, we get $\chi_4 = \chi_Q$. $\square$.

Proposition 7.1 and the lemma C4' imply, that π has complex multiplication $\pi \cong \pi \otimes (\chi \circ Norm_K)$ for $\chi \in \hat{\Delta}$ iff χ is in the subgroup $\{1, \chi_K, \chi_Q, \chi_Q\chi_K\}$ of $\hat{\Delta}$. Since $n_K(\Pi) = 1$, lemma 9.1 implies, that we have $\sigma(\pi_K) \cong \pi_K \otimes (\chi \circ Norm_K)$ precisely for χ in the complementary coset in $\hat{\Delta}$. Therefore, since $\chi_Q = \chi_{\overline{L}}$

Lemma C5': Let σ denote the nontrivial involution of $K/\mathbb{Q}$. The automorphic representation $\pi = \pi_K$ of $Gl(2, \mathbb{A}_K)$, attached to Π and $K = K_1, K_2, K_3$ by the theta lift, has the following CM-properties

1) $\pi \cong \pi \otimes (\chi \circ Norm_K)$ for the characters χ in $\langle \chi_{\overline{L}}, \chi_K \rangle = \{1, \chi_{\overline{L}}, \chi_K, \chi_{\overline{L}}\chi_K\}$.
2) $\sigma(\pi) \cong \pi \otimes (\chi \circ Norm_K) \not\cong \pi$ for $\chi \in \hat{\Delta} - \langle \chi_{\overline{L}}, \chi_K \rangle$.



Six subfields $K_1, K_2, K_3, K'_1, K'_2, K'_3$ with $\{ijk\} = \{1, 2, 3\}$.

Appendix D: Pairings

Let G be a group, let π be an irreducible representation of G on the vectorspace V_π over a field k of characteristic zero. Let $\omega : G \to k^*$ be a one dimensional character of G. Let (V, ρ) be an isotypic multiple of V_π. In fact, we want that

1) Schur's lemma holds, and
2) that there is a natural notion of dual representation $(V, \rho)^\vee$, such that there exists a canonical isomorphism $(V, \rho) \to ((V, \rho)^\vee)^\vee$ of representations.

For instance: G is a compact group, and the representations are continuous and finite dimensional. Or G is the group of F-valued points of a reductive group over F, where F is a local nonarchimedian field, and the representations are finitely generated and admissible. Also consider $(\mathcal{G}, K)$-modules for real Lie groups, or finite dimensional algebraic representations of reductive groups over an algebraically closed field of characteristic zero. So assume that we are in one of these situations.

The Parity: Suppose, there exists an isomorphism $\psi : \rho \cong \rho^\vee \otimes \omega$ with underlying map $\psi : V \ \to \ V^\vee$ such that $\psi(\rho(g)v) = \omega(g)\rho^\vee(g)(\psi(v))$. Then we get another isomorphism by dualizing $\psi^\vee : (V^\vee)^\vee \to V^\vee$. By our assumption we can identify $(V^\vee)^\vee$ and V, so we view $\psi^\vee$ as a map from V to $V^\vee$ by abuse of notation. Obviously, $\psi^\vee$ again satisfies $\psi^\vee(\rho(g)v) = \omega(g)\rho^\vee(g)(\psi^\vee(v))$. ψ is called ϵ-symmetric, if

$$\psi^\vee = \epsilon \cdot \psi$$

holds for some constant $\epsilon = \epsilon(G, \rho, \omega, \psi)$ in k. Since $(\psi^\vee)^\vee = \psi$ the number $\epsilon(G, \rho, \omega)$ is either 1 or -1. It is called the parity of (G, ρ, ω).

In the special case, where (V, ρ) is an irreducible representation with $\rho \cong \rho^\vee \otimes \omega$, the isomorphism ψ is unique up to a constant (Schur's lemma). In particular, ψ is ϵ-symmetric for a unique parity ϵ. This parity does not depend on the choice of ψ, but only depends on G, ρ, ω. This defines $\epsilon(G, \rho, \omega)) = \epsilon(G, V, \omega)$. To determine $\epsilon(G, \rho, \omega)$ amounts to decide, whether there exists a k-bilinear nontrivial ϵ-symmetric pairing

$$\langle .,. \rangle : \ V \times V \ \to \ k \quad , \quad \langle v, v' \rangle \ = \ \epsilon \cdot \langle v', v \rangle$$

such that $\langle gv, gv' \rangle \ = \ \omega(g) \cdot \langle v, v' \rangle$ holds for all $g \in G$, and all v, v' in V. Of course, $\langle v, v' \rangle = \psi(v)(v')$.

Remark 1: $\epsilon(G, \rho, \omega)$ does not change under twisting by a one dimensional character χ of G, $\epsilon(G, \rho, \omega) = \epsilon(G, \rho \otimes \chi, \omega\chi^2)$. If G is compact and $k = \mathbb{C}$, (V, ρ) is an irreducible continuous finite dimensional representation of G with character χ_ρ. Let dg be a Haar measure of volume one, then $\int_G \omega^{-1}(g)\chi_\rho(g^2)dg$ is a real number. It is zero, unless $\rho \cong \rho^\vee \otimes \omega$. If $\rho \cong \rho^\vee \otimes \omega$, this number is $\epsilon(G,\rho,\omega)$.

<u>Remark 2</u>: Suppose $G = G_1 \times G_2$ and $(V,\rho) = (V_1,\rho_1) \otimes_k (V_2,\rho_2)$, with a given ϵ-symmetric isomorphism $\psi : \rho \cong \rho^\vee \otimes \omega$. A character ω of G can be viewed as a character of G_1, G_2 in the obvious way. We assume that ρ_2 is irreducible and that ρ_1 is a finite isotypic multiple of an irreducible representation. Then the ϵ_1-symmetric G_1-equivariant homomorphisms ψ_1 in $Hom_{G_1,\epsilon_1}(\rho_1, \rho_1^\vee \otimes \omega)$ can be identified with the $\epsilon = \epsilon_1\epsilon_2$-symmetric G-equivariant homomorphisms ψ in $Hom_{G,\epsilon}(\rho, \rho^\vee \otimes \omega)$ by the isomorphism $\psi_1 \mapsto \psi = \psi_1 \otimes \psi_2$, where $\psi_2 : \rho_2 \cong \rho_2^\vee \otimes \omega$ is the (up to a scalar) unique isomorphism for G_2. This follows from Schur's lemma by a dimension count. Hence

$$\epsilon(G,\rho,\omega,\psi) \;=\; \epsilon(G_1,\rho_1,\omega,\psi) \cdot \epsilon(G_2,\rho_2,\omega) \ .$$

Suppose (V_π, π) is an irreducible representation of G and suppose ψ is an ϵ-symmetric isomorphism as above. Suppose K is a subgroup of G, such that $\pi = \oplus_{\pi'} m(\pi')\pi'$ as a representation of K with finite multiplicities $m(\pi') > 0$ and irreducible representations π' of K. Suppose π' is such that $(\pi')^\vee \otimes \omega \cong \pi'$. Then $\epsilon(G,\pi,\omega,\psi) = \epsilon(K,\rho,\omega,\psi)$ for $\rho = m(\pi')\pi'$, where ψ, ω are obtained by restriction from G to K. In particular for $m(\pi') = 1$

$$\epsilon(G,\pi,\omega) = \epsilon(K,\pi',\omega) \ .$$

<u>The reducible case</u>: If on the other hand $(\pi')^\vee \otimes \omega \cong \pi''$ such that $\pi' \not\cong \pi''$, then $V_{\pi'} \oplus V_{\pi''}$ can be endowed both with a nondegenerate symplectic (or alternatively symmetric) nondegenerate pairing, which is K-equivariant with multiplier ω. Hence in this case no information on $\epsilon(G,\pi,\omega)$ is obtained. This also made us restrict to the case of isotypic representations (V,ρ) at the beginning.

<u>Remark 3</u>: Suppose k/k_0 is a Galois extension with Galois group Γ. Suppose G, V, π, ω, k are given as above such that (V,π) is irreducible over k. We say π has no CM, if $\pi \otimes \eta \cong \pi$ for a character η implies $\eta = 1$. Choose a basis of V. This defines a k_0-vector space structure and defines $\pi^\gamma, \gamma \in \Gamma$ such that $\gamma(\pi(g)v) = \pi^\gamma(g)(\gamma(v))$. Suppose $A_\gamma : \pi \cong \pi^\gamma \otimes \omega_\gamma$ for one dimensional characters $\omega_\gamma, \gamma \in \Gamma$, i.e $A_\gamma \pi(g) A_\gamma^{-1} = \pi^\gamma(g)\omega_\gamma(g)$ for endomorphisms $A_\gamma : V \to V$. For simplicity assume $\omega_{\tau\gamma} = (\omega_\gamma)^\tau \omega_\tau$. This is automatic, if π has no CM. Then $\tau(A_\gamma)A_\tau = A_{\tau\gamma} \cdot \lambda_{\tau,\gamma}$ for some $\lambda_{\tau,\gamma} \in k^*$. Changing the basis amounts to $A_\tau \mapsto \tau(B)A_\tau B^{-1}$. The A_τ are uniquely defined up to constants. Therefore the coboundary $\lambda_{\tau,\gamma}$ defines a unique cohomology class in the Brauer group $H^2(\Gamma, k^*)$. This class is trivial iff the A_τ can be chosen such, that they define a 1-cocycle with values in $Gl(V)$. Then by Hilbert 90 we get $A_\tau = \tau(B)B^{-1}$ for some $B \in Gl(V)$ (this is true if V is finite dimensional, but the argument carries over to the case of admissible representations). This means

$\omega_\gamma(g)\pi^\gamma(g) = \pi(g)$ for all $\gamma \in \Gamma$ for the choice of basis determined by B. In other words π is definable over the fixed field k_0 in a twisted sense. The converse is also true.

Now suppose, that in addition an isomorphism $\psi : V^\vee \otimes \omega \cong V$ of the underlying representations is given. Then together with A_γ one defines $\psi_\gamma = \psi \circ A_\gamma^{-1} : (V, \pi^\gamma) \to (V^\vee, \pi^\vee) \otimes (\omega\omega_\gamma^{-1})$. Then $\psi_\gamma \pi^\gamma(g) = \omega\omega_\gamma(g)\pi^\vee(g)\psi_\gamma$. We now obtain nondegenerate γ-sesquilinear forms $(v_1, v_2) = (\psi_\gamma(\gamma(v_1)), v_2)_{can}$ on V for fixed $\gamma \in \Gamma$, such that $(\pi(g)v_1, \pi(g)v_2) = \omega\omega_\gamma(g) \cdot (v_1, v_2)$ and $(\alpha \cdot v_1, \beta \cdot v_2) = \gamma(\alpha) \cdot \beta \cdot (v_1, v_2)$ holds for $v_1, v_2 \in V, g \in G$.

Assume now $\omega_\gamma^2 \omega^\gamma / \omega = 1$ for all $\gamma \in \Gamma$. If π has no CM, this is a consequence of $(\pi^\vee)^\gamma = (\pi^\gamma)^\vee$ and $\pi^\vee \otimes \omega \cong \pi$ and $\pi^\gamma \otimes \omega_\gamma \cong \pi$. If furthermore ψ is defined over k_0, then $\psi^\gamma = \psi$ implies that $\omega(g)\pi(g)^\bullet = \pi(g)$, where $X^\bullet = \psi^{-1}X^\vee\psi$. Note $(XY)^\bullet = X^\bullet Y^\bullet$ and $\gamma\bullet = \bullet\gamma$ and $c^\bullet = c^{-1}$ for constants $c \in k^*$. We get $A_\gamma^\bullet \pi(g) = (\omega/\omega_\gamma\omega^\gamma)(g)\pi^\gamma A_\gamma^\bullet$ from $A_\gamma\pi(g) = \omega_\gamma(g)\pi^\gamma(g)A_\gamma$. Hence by our assumptions $A_\gamma^\bullet = c_\gamma \cdot A_\gamma$ for some constants $c_\gamma \in k^*$. Applied to the definition of the cocycle $\lambda_{\tau,\gamma}$ we get $c_\gamma^\tau c_\tau \tau(A_\gamma)A_\tau = A_{\tau\gamma} \cdot \lambda_{\tau,\gamma}^{-1} c_{\tau\gamma}$. Therefore the class of $\lambda_{\tau,\gamma}$ lies in the two-torsion subgroup of $Br(k/k_0)$.

A special case: $k = \mathbb{C}$, $\Gamma = Gal(\mathbb{C}/\mathbb{R})$. Suppose $\overline{\pi} \cong \pi^\vee \cong \pi \otimes \omega^{-1}$ irreducible and $\overline{\omega} = \omega^{-1}$. Note that $\overline{\pi} \cong \pi^\vee$ means, that there exists a nondegenerate invariant hermitian symmetric form. The case we have in mind is this: $(V, \pi)^\vee \cong (V, \pi) \otimes \omega^{-1}$ and the representation π is unitary, i.e there exists an invariant positive definite invariant hermitian form on V. In this case choose the basis to be an orthonormal basis of V with respect to the hermitian form. For this choice of the basis we have $\overline{\pi}(g) = \pi^\vee(g) = \pi(g^{-1})'$ (This is true in the finite dimensional case and carries over in our situations). Hence the maps A_γ (γ complex conjugation) and ψ, as defined in this appendix, coincide up to a constant. The cocycle condition gives $\overline{A}_\gamma A_\gamma = A_\gamma \overline{A}_\gamma = \lambda \cdot E$, where λ necessarily is in $\mathbb{R}$. For a suitable choice of A_γ therefore $\lambda_\pi := \lambda \in \{\pm 1\}$. This "is" the cohomology class in $H^2(\Gamma, \mathbb{C}^*) \cong \mathbb{Z}/2\mathbb{Z}$. On the other hand $\psi = const. \cdot A_\gamma$ and $\psi^\vee = \epsilon_\pi \cdot \psi$ implies $\lambda_\pi \cdot id = \overline{A}_\gamma A_\gamma = \overline{A}_\gamma A_\gamma^\vee \cdot \epsilon$. Since $\overline{A}_\gamma A_\gamma^\vee$ is a positive definite matrix, we get $\lambda_\pi = \epsilon_\pi$. Furthermore $A' = \overline{A} = \epsilon_\pi A^{-1}$.

It is useful to give a more abstract approach: Suppose V, W are complex vectorspaces with nondegenerate hermitian-symmetric forms. Let $\phi : V \to W$ be a γ-linear map, where $\gamma \in Gal(\mathbb{C}/\mathbb{R})$. Then define the adjoint $\phi^* : W \to V$ by $\langle \phi(v), w \rangle_W = \overline{\langle v, \phi^*(w) \rangle_V}$. Then ϕ^* is γ-linear and $(\phi_1 \circ \phi_2)^* = \phi_2^* \circ \phi_1^*$, $(\phi^*)^* = \phi$. If ϕ is a γ-linear isomorphism, then also ϕ^{-1}. Hence $\phi \mapsto (\phi^*)^{-1}$ defines an involution of the group of all γ-linear automorphisms of V.

Suppose V, W is a complex vectorspace with a representation π_V, π_W of G, suppose $\langle ., . \rangle_W$ a nondegenerate hermitian-symmetric bilinear form on V, which is G invariant: $\langle \pi(g)v, \pi(g)w \rangle = \langle v, w \rangle$. Suppose the same for W. Suppose ω is a unitary character of G. Consider the χ-equivariant γ-linear homomorphism $\theta : V \to W$ such that

$$\theta \circ \pi_V(g) = \gamma(\chi(g)) \cdot \pi_W(g) \circ \theta \ .$$

We say θ is a (χ,γ)-homomorphism. The composition $\theta'\circ\theta$ of a (χ,γ)-homomorphism and a (χ',γ')-homomorphism is a $(\gamma(\chi')\chi,\gamma'\gamma)$-homomorphism. The transpose of a (χ,γ)-homomorphism is a $(\gamma(\overline{\chi}),\gamma)$-homomorphism. Furthermore, if θ is invertible, then $(\theta^*)^{-1}$ is again a (χ,γ)-homomorphisms and θ^{-1} is a $(\gamma(\overline{\chi}),\gamma)$-homomorphism. Now suppose $V=W$ and suppose γ is the complex conjugation, hence θ is an antilinear endomorphism $\theta:V\to V$, such that

$$\pi(g)\circ\theta=\overline{\chi}(g)\cdot\theta\circ\pi(g)\ .$$

Furthermore suppose $\theta^*=\epsilon\cdot\theta$ for $\epsilon\in\{\pm1\}$. Then $[v,w]=\langle\theta(v),w\rangle$ defines a $\mathbb{C}$-bilinear form such that $[\pi(g)v,\pi(g)w]=\chi(g)[v,w]$ and such that $[w,v]=\epsilon\cdot[v,w]$. In particular, if the representation π of G on V is irreducible, then Schur's lemma implies that the C-vectorspace of (χ,γ)-endomorphisms of V is one dimensional, since any two θ_1,θ_2 have product a $\mathbb{C}$-linear product $\theta_1\circ\theta_2$ commuting with $\pi(g),g\in G$. Thus $\theta^{-1}=\lambda^{-1}\cdot\theta$ and $\lambda\in\mathbb{R}^*$ and $(\theta^*)^{-1}=\rho\cdot\theta$ and $\rho\in\mathbb{R}^*$. Finally $\theta^*=\epsilon\cdot\theta$ and $\epsilon^2=1$. Then $\theta^2=\lambda\cdot id_V$ and $1=\rho\lambda\epsilon$. Replacing θ by a suitably multiple, we get a well defined $\lambda=\lambda_\pi\in\{\pm1\}$. Therefore also $\rho\in\{\pm1\}$. If $\langle .,.\rangle$ is definite, then $\rho\cdot id_V=\theta^*\theta$ is positive definite, hence $\rho=1$. Thus $\epsilon=\lambda_\pi$ is the parity $\epsilon(G,\pi,\chi)$. Of course $\theta(v)=A_\gamma^{-1}(\overline{v})$ holds in the sense above.

Now we give several examples.

<u>1.Example</u>: Let G be the group of F-valued points of a reductive group over a nonarchimedian local field, and let K be a suitable compact open subgroup. Assume that $\pi\cong\pi^\vee\otimes\omega$ holds, and that π contains the trivial representation of K with multiplicity one. If ω is trivial on K, then $\epsilon(G,\pi,\omega)=1$. Hence for unramified representations ω, π we get $\epsilon(G,\pi,\omega)=1$.

<u>2. Example</u>: Consider representations induced from a parabolic subgroup of the group G of F-valued points of a connected reductive group over a local nonarchimedian field F of characteristic zero. Let (V_π,π) be a the Langlands quotient of such an induced representation I (with respect to a tempered representation σ of the Levi subgroup). Suppose $\pi^\vee\otimes\omega\cong\pi$. Suppose $\psi_I:I\to I^\vee\otimes\omega$ is a nontrivial G-equivariant intertwining operator. Then, by [BW] XI, 2.13, its image must be the unique irreducible subrepresentation $\pi^\vee\otimes\omega$ of $I^\vee\otimes\omega$. Thus ψ_I induces an isomorphism $\psi_I:\pi\cong\pi^\vee\otimes\omega$. If $\psi_I^\vee=\epsilon\cdot\psi_I$, then $\epsilon(G,\pi,\omega)=\epsilon$. By twisting σ with unramified characters one obtains a family of induced representations I_s. Suppose Ψ_I is obtained by the restriction of a $\pm$-selfdual intertwining operator $\psi_{I_s}:I_s\to I_s^\vee\otimes\omega$ of the family: $\psi_{I_s}^\vee=\epsilon(s)\cdot\psi_{I_s}$. Then by analytic continuation the parity $\epsilon(s)$ is independent from s, such that $\epsilon=\epsilon(s)$. In fact generically, when the representation I_s is irreducible, an isomorphism $I_s^\vee\otimes\omega\cong I_s$ implies the existence of an $\pm$-selfdual intertwining operator on a dense open subset of the parameter space of the variable s.

<u>3. Example</u>: Suppose $k=\overline{\mathbb{Q}}_l$. Consider the four dimensional semisimple representation $\rho_{\Pi_0,\lambda}$ of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$, attached to a unitary irreducible automorphic representa-

tion Π_0 (theorem I). The identity $\Pi_0 \cong \Pi_0^\vee \otimes \omega_{\Pi_0}$ and the identity $L_p(\Pi_{0,p}, s - \frac{w}{2}) = det(1 - \rho_{\Pi,\lambda}(Frob_p)p^{-s})^{-1}$ implies

$$\rho_{\Pi,\lambda}^\vee \otimes (\omega_{\Pi_0} \cdot \mu_l^{-w}) \cong \rho_{\Pi,\lambda} \ .$$

Let us discuss, whether $\rho_{\Pi,\lambda}$ admits a nondegenerate equivariant symplectic pairing with multiplier $\omega = \omega_{\Pi_0}\mu_l^{-w}$.

Decompose $\rho_{\Pi,\lambda} = \oplus m_i \rho_i$ into irreducible subrepresentations. Then either $\rho_i^\vee \otimes \omega \cong \rho_j$ with $\rho_j \not\cong \rho_i$. (This is for instance the case, if ρ_i is one dimensional. This follows from property c) of Π formulated in section 2). The subrepresentation spanned by these subrepresentations ρ_i admits a symplectic pairing - for trivial reasons - as in remark 2 above. Then there is the case of two dimensional subrepresentations ρ_i, such that $\rho_i^\vee \otimes \omega \cong \rho_i$. These subrepresentations obviously admit a nondegenerate symplectic form with the multiplier ω. This immediately leaves us with the remaining nontrivial case, where $\rho_{\Pi,\lambda}$ itself is irreducible.

<u>4. Example</u>: Consider $J = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}$. For matrices $g \in Gl(N,F)$ define $g^* = (g')^{-1}$. Then $g \in GSp(2n,F)$ iff $Jg^*J^{-1} = \lambda(g)^{-1} \cdot g$ for some multiplier $\lambda(g) \in F^*$. Let κ be the matrix in $Gl(N,F)$ with entry 1 in the antidiagonal and zero else. Then $g^x = \kappa g^* \kappa$ is an involutive automorphism of $Gl(N,F)$. There exists a Borel group $B \subset Gl(N,F)$ preserved by this involution and a generic character ψ of its unipotent radical such that $\psi(n^x) = \overline{\psi}(n)$. Let $\iota = diag(1,-1,1,-1,.. \pm 1,-1,1,-1,.. \mp 1)$. Then $\iota(g) = \lambda(g)^{-1}\iota g \iota$ is an involutive automorphism of $GSp(2n,F)$. There exists a Borel group $B \subset GSp(2n,F)$ preserved by this involution and a generic character ψ of its unipotent radical such that $\psi(n^x) = \overline{\psi}(n)$.

Suppose G is split with Borel group B over the local field F. Suppose given a generic character ψ of its unipotent radical N. Suppose $g \mapsto g^x$ is an involutive automorphism of (G,B) which maps ψ to $\overline{\psi}$. Let W_ψ be the induced space $Ind_{N(F)}^{G(F)}(\psi)$ with the right action $R_\psi(g)$ of $g \in G(F)$. Let π be an admissible irreducible representation of $G(F)$. Suppose finally $\pi^x \cong \overline{\pi}$. Here $\pi^x(g) = \pi(g^x)$ is another representation of $G(F)$ on V. Similar $\overline{\pi}$ is another representation of $G(F)$ on V. This in particular applies for the cases $Gl(n)$,θ_x and unitary discrete series representations π (see [Sha]) or for all representations π of $GSp(4,F)$, were the involutive automorphism is chosen to be ι. Finally suppose π has a unique Whittaker model. So we can identify with the π isotypic component in W_ψ.

<u>Claim</u>: In this situation there exists a nontrivial antilinear endomorphism $\theta_x : V \to V$ of the representation space V of π, such that $\theta_x^2 = id_V$ and such that $\theta_x \circ \pi = \pi^x \circ \theta_x$. It is unique up to a constant.

Proof: Let $A : W_\psi \to W_{\overline{\psi}}$ be defined by $(Af)(g) = f(g^x)$ and $B : W_{\overline{\psi}} \to W_\psi$ by $(Bf)(g) = \overline{f}(g)$. Then $\theta_x = B \circ A : W_\psi \to W_\psi$ is antilinear, satisfies $\theta_x^2 = id$, maps the irreducible

π-isotypic component into itself, such that $\theta_x \circ R(g) = R^x(g) \circ \theta_x$. This is clear, since $A \circ R_\psi(g^x) = R_{\overline{\psi}}(g) \circ A$ and since $B \circ R_{\overline{\psi}}(g) = R_\psi(g) \circ B$. $\square$

<u>The case $Gl(2,F)$</u>: Suppose π is unitary. There is the antilinear nontrivial endomorphism θ_ϵ such that $\theta_\epsilon \pi(g) = \overline{\omega}_\pi(g)\pi(g)\theta_\epsilon$ with $\theta_\epsilon^2 = \epsilon(Gl(2,F),\pi,\omega_\pi)$. We define modifications: They are the antilinear nontrivial endomorphisms θ_*, θ_ι of V, such that $\theta_* = \pi(J)\theta_\epsilon$ and $\theta_\iota = \pi(\iota)\theta_*$ (where $\iota = diag(1,-1)$) and $\theta_*\pi = \pi^*\theta_*$ and $\theta_\iota\pi = \pi^\iota\theta_\iota$. Here $\pi^*(g) = \pi(g^*)$ and $\pi^\iota(g) = \pi(\iota(g))$. Obviously $\theta_\iota^2 = \theta_*^2 = \omega_\pi(-1)\theta_\epsilon^2$. Since π is generic we therefore get from the proof of the claim above $\epsilon(Gl(2,F),\pi,\omega_\pi) = \omega_\pi(-1)$. It is not difficult to see that this is also true, if π is not unitary.

<u>The case $GSp(4,F)$</u>: A variant of the Whittaker models are the generalized Whittaker models. Let $P = MN$ be the Siegel parabolic as in [PS], p.507. Suppose Π_v is a unitary irreducible admissible representation of $GSp(4,\mathbb{F})$, which has a nontrivial generalized Whittaker functional $\nu \otimes \psi_T$ attached to a nondegenerate symmetric matrix $T = T' \in M_{2,2}(F)$. According to [PS] theorem 1.1 this functional is unique, if it exists. Recall that the stabilizer of the character $\psi_T : N(F) \to \mathbb{C}$ in $M(F)$ is a semidirect product $K^* \cdot (\mathbb{Z}/2\mathbb{Z})$, where $K/\mathbb{F}$ is a quadratic algebra determined by the discriminant of T. K^* is embedded in $M(F) \cong Gl(2,F)$ by the regular representation. Furthermore the similitude factor $\lambda(y)$ of $y \in K^*$ with respect to this embedding is $Norm_{K/F}(y)$. The group $\mathbb{Z}/2\mathbb{Z}$ induces the involution σ of $K/\mathbb{Q}$, so we view the generator σ of this group $\mathbb{Z}/2\mathbb{Z}$ as an element $\sigma \in M(F) \subset GSp(4,F)$ in this way. Finally ν is a character of K^*. In such a situation we can copy the proof of the claim above.

Consider the automorphism $g^\sigma = \sigma(g\lambda(g)^{-1})\sigma$ on $GSp(4,F)$. Note that $\sigma \in M(F)$ considered above has multiplier $\lambda(\sigma) = 1$. In fact $\sigma = diag(1,-1,1,-1)$ in the first case of [PS] and $\sigma = diag(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix})$ in the second case. Therefore $g \mapsto g^\sigma$ is an involutive automorphism of $GSp(4,\mathbb{F})$. If ν is a unitary character, then one easily finds $(\nu \otimes \psi_T)(p^\sigma) = (\overline{\nu} \otimes \psi_T)(p)$ for all p in the semidirect product $N(F).K^*$. In fact, for $y \in K^* \subset M(F)$ we find $y^\sigma = \sigma(y)Norm_{K/F}(y)^{-1}$. Since $\sigma(y)y$ is in the center of $GSp(4,F)$ we necessarily have $\nu(\sigma(y)y) = \omega_{\Pi_v}(\sigma(y)y)$ and $\nu(y^\sigma) = \nu(\sigma(y)Norm_{K/F}^{-1}(y)) = \nu(y)^{-1}$. If we in addition conjugate by $diag(E,-E)$ we get the new involution $g \mapsto \tilde{\iota}(g)$ similar to the involution ι considered above. $\tilde{\iota}$ has the property $(\nu \otimes \psi_T)(\tilde{\iota}(p)) = \overline{\nu \otimes \psi_T}(p)$ for all p in the semidirect product $N(F).K^*$, still ν assumed to be unitary. This implies existence of an antilinear isomorphism $\theta_{\tilde{\iota}}$ of the representation space of Π_v, such that $\theta_{\tilde{\iota}} \circ \Pi_v = \Pi_v^{\tilde{\iota}} \circ \theta_{\tilde{\iota}}$ and $\theta_{\tilde{\iota}}^2 = id$. As in the case $Gl(2,F)$ this implies $\theta_\epsilon^2 = \omega_{\Pi_v}(-1)$. Note $\theta_\epsilon\pi(g) = \pi(g\lambda(g)^{-1})\theta_\epsilon$ and $\theta_{\tilde{\iota}} = const \cdot \pi(diag(E,-E)\sigma)\theta_\epsilon$. Hence $\theta_{\tilde{\iota}}^2 = |const|^2 \cdot \pi(diag(E,-E)\sigma)\pi(diag(E,-E)\sigma\lambda(diag(E,-E)\sigma)^{-1})\theta_\epsilon^2 = |const| \cdot \omega_\Pi(-1)\theta_\epsilon^2$, since $diag(E,-E)$ commutes with $\sigma \in M(F)$. Therefore $\epsilon(GSp(4,F),\Pi_v,\omega_{\Pi_v}) = \omega_{\Pi_v}(-1)$.

<u>5. Example</u>: Consider the special case of the group $GSp(4)$ over $\mathbb{Q}$ or $\mathbb{Q}_v$. Let $\mathcal{V}_\mu$ be a coefficient system on a Siegel modular threefold attached to a discrete series of weight

(k_1,k_2) of $GSp(4,\mathbb{R})$. Let $p: A\to M$ be the universal polarized abelian variety of genus two. Consider the cuspidal third cohomology group $H^3_P(M,\mathcal{V}_\mu)$ with $k=\overline{\mathbb{Q}}_l$. Consider the cupproduct pairing

$$H^3_P(M,\mathcal{V}_\mu)\times H^3_P(M,\mathcal{V}_\mu)\ \to\ H^6_c(M,\overline{\mathbb{Q}}(-c))\xrightarrow{tr}\overline{\mathbb{Q}}(-3-c)=\overline{\mathbb{Q}}_l(-w)\ .$$

It has parity $-(-1)^{k_1+k_2}$ and satisfies

$$tr((\sigma\times g)\cdot\eta\ \cup\ (\sigma\times g)\cdot\eta')\ =\ \mu_l(\sigma)^{-c-3}||g||^{-c}\cdot tr(\eta\cup\eta')$$

for all $\sigma\in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ and all $g\in GSp(4,\mathbb{A}_f)$.

In other words, the modified cupproduct $\eta,\eta'\mapsto tr(\eta\cup\eta'\cup\omega_0^{-1})$ defines a nondegenerate pairing of parity $-(-1)^{k_1+k_2}$

$$(W_{\Pi_f}\otimes\Pi_f)\ \otimes\ (W_{\Pi_f}\otimes\Pi_f)\ \to\ \omega_0\mu_l^{-3-c}\otimes\omega_0||.||^{-c}\ =\ \omega_\Pi\mu_l^{-3}\otimes\omega_\Pi\ .$$

Also consider the pairing $\langle.,.\rangle$, which maps $\eta,\eta'\in W_{\Pi_f}\otimes\Pi_f$ to $tr(\eta\cup\eta'\cup\omega_\Pi^{-1})$

$$(W_{\Pi_f}\otimes\Pi_f)\ \otimes\ (W_{\Pi_f}\otimes\Pi_f)\ \to\ \omega_0\mu_l^{-3}\otimes\omega_0\ .$$

It satisfies $\langle F_\infty\cdot\eta,F_\infty\cdot\eta'\rangle=-\langle\eta,\eta'\rangle$, since $F_\infty\cdot\omega_0=\omega_0(-1)\cdot\omega_0$ and $tr(F_\infty\cdot\eta,F_\infty\cdot(\eta'\cup\omega_0^{-1}))=(-1)^{c+3}$.

The cupproduct defines a ϵ-symmetric isomorphisms $\psi:(W_{\Pi_f}\otimes\Pi_f)\to(W_{\Pi_f}\otimes\Pi_f)^\vee\otimes(\omega_\Pi\mu_l^{-3}\otimes\omega_{\Pi_f})$ with $\epsilon=-(-1)^{k_1+k_2}$. In view of the discussion preceding of lemma D1, we can make - without restriction of generality - the

<u>Assumption</u>: Suppose that the representation W_{Π_f} is an isotypic multiple of an irreducible representation.

Thus the parity of the isomorphism ψ, defined by the cupproduct, is

$$\epsilon(Gal(\overline{\mathbb{Q}}/\mathbb{Q}),W_{\Pi_f},\omega_\Pi\mu_l^{-3},\psi)\cdot\epsilon(G(\mathbb{A}_f),\Pi_f,\omega_{\Pi_f})\ =\ -(-1)^{k_1+k_2}\ .$$

Furthermore $\epsilon(G(\mathbb{R}),\Pi_\infty,\omega_{\Pi_\infty})=\omega_{\Pi_\infty}(-1)=(-1)^{k_1+k_2}$. To see this, restrict to the minimal K_∞-type. K_∞ is the semidirect product of $U(2)$ and ι_∞, where conjugation by ι_∞ induces complex conjugation on the unitary group $U(2)$. The minimal K_∞-type τ occurs with multiplicity one and is induced from the irreducible representation τ_{k_1,k_2} of $U(2)$ of highest weight (k_1,k_2), where $k_1\geq k_2\geq 3$ in the holomorphic case. Similar for $\tau_{k_1,2-k_2}$ in the Whittaker case. As a representation of $U(2)$ it decomposes into the two nonisomorphic representations $\tau_{k_1,k_2}\oplus\tau^\vee_{k_1,k_2}$ resp. $\tau_{k_1,2-k_2}\oplus\tau^\vee_{k_1,2-k_2}$. Since $k_1\neq -k_2$ resp. $k_1\neq k_2-2$, the two constituents $\sigma,\sigma^{\iota_\infty}$ of τ restricted to $U(2)$ are not isomorphic. Therefore the space of $U(2)$-homomorphisms $\tau \to \tau^\vee \otimes \omega_{\Pi_\infty}$ is two dimensional: Let $\langle .,. \rangle$ be a form define by such a homomorphism. Write $V_\tau = V_\sigma \oplus V_\sigma$, such that $\iota_\infty \in K_\infty$ acts by $\iota_\infty(v_1,v_2) = (v_2,v_1)$ and $U(2)$ acts by $\sigma \oplus \sigma^{\iota_\infty}$. Choose a $\mathbb{C}$-bilinear form (v_1,v_2) on V_σ, such that $(v_1,v_2) = (v_2,v_1) = (\sigma(k)v_1, \sigma(k^{\iota_\infty})v_2)$. Then $\langle (v_1,v_2),(w_1,w_2)\rangle = \alpha\cdot(v_1,w_2) + \beta\cdot(w_1,v_2)$. Since $\iota_\infty \in K_\infty$ acts by $\iota_\infty(v_1,v_2) = (v_2,v_1)$, $\omega_{\Pi_\infty}(\iota_\infty) = \omega_{\Pi_\infty}(-1)$ implies $\alpha = \beta \cdot \omega_{\Pi_\infty}$. Therefore the parity of $\langle .,. \rangle$ is $\alpha/\beta = \omega_{\Pi_\infty}(-1)$. Finally $\omega_\infty(-1) = \omega_{\tau_{k_1,k_2}}(-1) = (-1)^{k_1+k_2}$. This implies

Lemma D1: The representation of the Galois group $Gal(\mathbb{Q}/\mathbb{Q})$ on W_{Π_f} preserves a nondegenerate bilinear form with multiplier $\omega_\Pi \mu_l^{-3}$ and parity $-\epsilon(GSp(4,\mathbb{A}),\Pi,\omega_\Pi)$. The following statements are equivalent:

1) $\epsilon(GSp(4,\mathbb{A}),\Pi,\omega_\Pi) = 1$
2) $\epsilon(Gal(\overline{\mathbb{Q}}/\mathbb{Q}), W_{\Pi_f}, \omega_{\Pi_f}\mu_l^{-3}, \psi) = -1$.

Now consider the de Rham cohomology of M. Let M be the Shimura variety of principally polarized abelian varieties of genus $g=2$ defined over the reflex field $\mathbb{Q}$. Its complex analytic points are described by the cosets $M = G(\mathbb{Q}) \setminus (X \times G(\mathbb{A}_f))$, where $G(R) = GSp(4,R)$ and where $X = H \cup -H$ is the union of the upper and lower Siegel halfspace of genus two. In fact $X = G(\mathbb{R})/Stab(iE)$ for $iE \in H$. Consider the cohomology $H^3(M,\mathcal{V}_\mu(\mathbb{C})) = H^3_B(M,\mathcal{V}_\mu)\otimes_{\mathbb{Q}}\mathbb{C}$. Complex conjugation acts on this vectorspace and sends $\eta = \eta_0 \otimes z$ to $\overline{\eta} = \eta_0 \otimes \overline{z}$ for $\eta_0 \in H^3_B(M,\mathcal{V}_\mu)$ and $z \in \mathbb{C}$. Frobenius $F_\infty \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ at infinity induces an antiholomorphic map from M to M. On the complex valued points it is descibed by $F_\infty(G(\mathbb{Q})\cdot[Z,g_f]) = G(\mathbb{Q})\cdot[\overline{Z},g_f]$, for representatives $Z \in X$ and $g_f \in G(\mathbb{A}_f)$. Here $\overline{Z}$ is the complex conjugate of the matrix $Z \in H \cup -H$. See e.g. [MS], p.309. The cohomology group $H^3(M,\mathcal{V}_\mu(\mathbb{C}))$ contains the cuspidal cohomology $H^3_P(M,\mathcal{V}_\mu(\mathbb{C}))$. It has a pure Hodge structure and decomposes into Hodge types $H^{p,q}_P(M,\mathcal{V}_\mu(\mathbb{C}))$. These are permuted by the $\mathbb{C}$-antilinear map $\eta \mapsto \overline{\eta}$ and the $\mathbb{C}$-linear map F^*_∞. They are preserved by the $\mathbb{C}$-antilinear map $\eta \mapsto F^*_\infty(\overline{\eta}) = \overline{F^*_\infty(\eta)}$, whose square is the identity. There is a decomposition into $G(\mathbb{A}_f)$-isotypic components

$$H^3_P(M,\mathcal{V}_\mu(\mathbb{C})) = \bigoplus_{\Pi_f} V_{\Pi_f} \quad , \quad V_{\Pi_f} = W^{\mathbb{C}}_{\Pi_f} \otimes_{\mathbb{C}} \Pi_f \ .$$

The $G(\mathbb{A}_f)$-action is completely decomposable, since we consider only the cuspidal part of the cohomology. By the comparison isomorphism with the etale cohomology we get the corresponding spaces W_{Π_f} over $\overline{\mathbb{Q}}_l$ as $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$-modules. The spaces $W^{\mathbb{C}}_{\Pi_f} = \bigoplus W^{p,q}_{\Pi_f}$ decompose into Hodge types, such that $F^*_\infty : W^{p,q}_{\Pi_f} \to W^{q,p}_{\Pi_f}$. Similar for complex conjugation, which sends $W^{p,q}_{\Pi_f}$ to $W^{q,p}_{\overline{\Pi}_f}$. Put $V^{p,q}_{\Pi_f} = W^{p,q}_{\Pi_f} \otimes \Pi_f$.

Recall $\Pi = \Pi_0 \otimes \|.\|^{-c/2}$ with Π_0 unitary! Hence $\overline{\Pi}_f \cong \Pi_f \otimes \omega_0^{-1}$. Consider the 0-form ω_0 in $H^0(M,\mathbb{C}) = \pi_0(M)\otimes\mathbb{C} \cong Hom_{cont}(\mathbb{Q}^* \setminus ((\mathbb{R}^*/\mathbb{R}^*_+) \times \mathbb{A}^*_f),\mathbb{C})$. We identify such continuous homomorphism with characters of $\mathbb{Q}^*\mathbb{R}^*_+ \setminus \mathbb{A}^*$, i.e. characters on $\mathbb{Q}^* \setminus \mathbb{A}^*$ of finite order.

In other words $\omega_\Pi([Z,g_f)]) = sign(Z)^c \|\nu(g_f)\|_f^{-c}\omega_{0,\infty}(sign(Z))\omega_{0,f}(\nu(g_f)) = \omega_{\Pi_f}(\nu(g_f)))$, since $\omega_{0,\infty}(-1) = \omega_{\Pi_\infty}(-1) = (-1)^c$.

Define $\theta_\infty(\eta) = \overline{\eta}\cup\omega_0$. Then $\sigma_\infty(V^{p,q}_{\Pi_f}) = V^{q,p}_{\Pi_f}$. Obviously θ_∞ is $\mathbb{C}$-antilinear (ω_0,γ)-homomorphism $\theta_\infty g^* = \overline{\omega}_0(g)g^*\theta_\infty$, such that $\theta_\infty^2 = id$. Now $V_{\Pi_f} \cong W_{\Pi_f}\otimes_C \Pi_f$. Fix a nontrivial antilinear (ω_0,γ)-automorphism θ_f of π_f, $\theta_f(v) = A_\gamma(\overline{v})$. Suitably normalized it satisfies $\theta_f^2 = \lambda_{\Pi_f} = \epsilon_{\Pi_f} \in \{\pm 1\}$. There is a notion of tensor product for γ-linear homomorphisms $\phi: V\to W$, $\phi': V'\to W'$ such that $(\phi\otimes_{\mathbb{C}}\phi')(v\otimes_{\mathbb{C}} v') = \phi(v)\otimes_{\mathbb{C}}\phi'(v')$. This is a welldefined γ-linear homomorphism from $V\otimes_{\mathbb{C}} V'$ to $W\otimes_{\mathbb{C}} W'$. By Schur's lemma we obtain an isomorphism between the $\mathbb{C}$-vectorspace of (ω_0,γ)-endomorphisms θ_∞ of V_{Π_f} and the γ-linear endomorphism θ_Π of W_{Π_f}, such that $\theta_\infty = \theta_\Pi\otimes\theta_f$. This is not a ring homomorphism unless $\lambda_{\Pi_f} = 1$, since $\theta_\infty^2 = \theta_\Pi^2\cdot\lambda_{\Pi_f}$. The morphism $\theta_\Pi: W_{\Pi_f}\to W_{\Pi_f}$ is $\mathbb{C}$-antilinear and satisfies $\Theta_\Pi^2 = \lambda_{\Pi_f}\cdot id$.

F_∞ acts on the differential form ω_0 by $F_\infty^*(\omega_0) = \omega_{\Pi_\infty}(-1)\cdot\omega_0$. Thus $F_\infty^*\circ\theta_\infty = \omega_{\Pi_\infty}(-1)\cdot\theta_\infty\circ F_\infty^*$. Furthermore $F_\infty^* = F_\infty\otimes_{\mathbb{C}} id$ (Galois action). Put $\Theta_\infty = \theta_\infty\circ F_\infty^*$. Then $\Theta_\infty = \Theta_\Pi\otimes_{\mathbb{C}}\theta_f$ with the antilinear endomorphism $\Theta_\Pi = \theta_\Pi\circ F_\infty^*$ of W_{Π_f}.

Obviously Θ_∞ is a $\mathbb{C}$-antilinear homomorphisms, which maps $V^{p,q}_{\Pi_f}$ into itself. Θ_Π is a (ω_0,γ) isomorphism with respect to the action of $G(\mathbb{A}_f)$. Finally

$$\Theta_\infty^2 = \omega_{\Pi_\infty}(-1)\cdot id\ ,$$
$$\Theta_\Pi^2 = \theta_\Pi^2\cdot(F_\infty^*)^2\cdot\omega_{\Pi_\infty}(-1) = \lambda_{\Pi_f}\omega_{\Pi_\infty}(-1)\cdot id\ .$$

Lemma D2: The following statements are equivalent:

1) $\epsilon(GSp(4,\mathbb{A}),\Pi,\omega_\Pi) = 1$
2) $\epsilon(Gal(\overline{\mathbb{Q}}/\mathbb{Q}), W_{\Pi_f}, \omega_{\Pi_f}\mu_l^{-3},\psi) = -1$.
3) $\epsilon(GSp(4,\mathbb{A}_f),\Pi_f,\omega_{\Pi_f}) = \omega_{\Pi_\infty}(-1)$.
4) $\lambda_{\Pi_f}\omega_{\Pi_\infty}(-1) = 1$
5) $\Theta_\Pi^2 = 1$ for the $G(\mathbb{A}_f)$-equivariant antilinear operator $\Theta_\Pi: W^{pq}_{\Pi_f}\to W^{pq}_{\Pi_f}$.
6) $\Theta_\Pi^2 \neq -1$.

Proof: The equivalence of 1) and 2) and 3) has been shown earlier in lemma D2. The equivalence of 4), 5) and 6) was shown above. The equivalence of 3) and 4) follows by reduction to Π_0 (twist by $\|.\|^{c/2}$), since Π_0 admits a definite hermitian invariant metric. Namely remark 3 of this section implies $\lambda_{\Pi_f} = \epsilon(GSp(4,\mathbb{A}_f),\Pi_f,\omega_{\Pi_f})$. $\square$

Since any antilinear nontrivial endomorphism of $\mathbb{C}$ has positive square, we get $\Theta_\Pi^2 = 1$ if $dim(W^{pq}_{\Pi_f}) = 1$ for at least one choice of q,p. So if $m(\Pi_\infty\Pi_f) = 1$ has multiplicity 1 for one choice of Π_∞ in the L-packet, then the properties 1)-6) are satisfied. This implies theorem IV in view of proposition 1.5.

Lemma D3: The following assertions are equivalent:

i) For all irreducible cuspidal automorphic representations Π with Π_∞ in the discrete series and $\omega_{\Pi_\infty}(-1) = 1$ we have $\epsilon(G(\mathbb{A}),\Pi,\omega_\Pi) = 1$.

ii) For all irreducible cuspidal automorphic representations Π we have $\epsilon(G(\mathbb{A}),\Pi,\omega_\Pi)=1$.
iii) For all places v and irreducible irreducible admissible representations Π_v of $GSp(4,\mathbb{Q}_v)$ we have $\epsilon(G(\mathbb{Q}_v),\Pi_v,\omega_{\Pi_v})=\omega_{\Pi_v}(-1)$.

Proof: It is enough to show i) implies ii) and ii) implies iii). We use, in the proof that the assertion iii) holds for all representations, which are not in the discrete series. This is shown below. So it remains to consider the case of discrete series representations. One can find locally generic discrete series representations for which iii) holds in the theta lift from $Gl(2,\mathbb{Q}_{v'})^2/\mathbb{Q}_{v'}^*$ (split orthogonal group case). They exist both with even and odd central characters $\omega_{\Pi_{v'}}(-1)$.

Fix some v. For the proof of iii) we can assume $v\neq\infty$, since this case is easy and was discussed already. Suppose Π_v is in the discrete series. If $\omega_{\Pi_v}(-1)=1$, then there exists a cuspidal irreducible $\Pi=\Pi_\infty\Pi_v\prod_{v'\neq v,\infty}$ with $\Pi_{v'}$ spherical and ω_{Π_∞} even. If $\omega_{\Pi_v}(-1)=-1$, choose some place $v'=\neq\infty,v$ such that $\Pi_{v'}$ is in the discrete series and has odd central character. Then one can find an irreducible cuspidal $\Pi=\Pi_\infty\Pi_v\Pi_{\tilde v}\otimes_{v'\neq v,\tilde v,\infty}\Pi_{v'}$, so that all $\Pi_{v'}$ are unramified and ω_{Π_∞} is even. This follows from global embedding techniques, used via the existence of pseudo coefficients after choosing a global central character ω, such that $\omega_{v'}=\omega_{\Pi_{v'}}$ for $v'=\infty,v,\tilde v$.

If the global result i) holds, and the local results hold outside v, then the local result $\epsilon(GSp(4,\mathbb{Q}_v,\Pi_v,\omega_{\Pi_v})=\omega_{\Pi_v}(-1)$ follows for the remaining discrete series representations Π_v at the nonarchimedian places v by the product formula $\prod_{v'}\omega_{\Pi'_v}(-1)=1$. $\square$

This part of proof is also valid for $GSp(2g)$. It should imply shows that the local statement would follow from the conjectural multiplicity one theorem for $GSp(2g)$. However, in the proof we assumed, that the local statement iii) is true for representations which are not in the discrete series. We wwill show this at least in the case $GSp(4)$.

Let Π_v be a representation of $GSp(4,\mathbb{Q}_v)$, which is not in the discrete series. Then it is either tempered or obtained as a Langlands quotient of an induced representation of a tempered representation σ of a Levi component as in example 3. A tempered representation is a constituent of a unitary induced representation. So, in the case of Langlands quotients using example 2, it remains only to consider the parity of local intertwining operators on the imaginary line by analytic continuation. Therefore consider $\Pi(\sigma)$ to be induced from the Levi component M of a proper parabolic subgroup of $GSp(4,\mathbb{Q}_v)$, where σ is a <u>unitary</u> representation of $M(\mathbb{Q}_v)$.

Let σ be an irreducible unitary discrete series representation of $M(\mathbb{Q}_v)$ and let $\Pi_v=\Pi_v(\sigma)=Ind_{P(\mathbb{Q}_v)}^{GSp(4,\mathbb{Q}_v}(\sigma\otimes 1)$ be the unitary induced representation of $GSp(4,\mathbb{Q}_v)$ on the representation space $I(\sigma)$. For nonarchimedian local fields of characteristic zero it is known, that such induced representations are multiplicity free (for an overview see e.g. [W]). So to determine the parity of the constituents, we are reduced to determine the parity of the total unitary induced representation, induced from the unitary discrete series representation σ of a Levi component of a proper parabolic subgroup P. Let $P = MN$ be its Levi decomposition. Suppose P is chosen to contain the diagonal torus, and such that $w(P) = Mw(N) = MN^-$ holds for the longest element w of the Weyl group of $GSp(4, \mathbb{Q}_v)$ represented by the matrix $J = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix} \in Gl(4, \mathbb{Q}_v)$. Then $w(M) = M$ and $M^* = M$ since $wgw^{-1} = g^*\lambda(g)$ for $g^* = (g')^{-1}$. Furthermore $w(\sigma) \cong \omega_\Pi \otimes \sigma^*$ holds, and the central character ω_σ is the restriction of the central character $\omega_{\Pi(\sigma)}$ to $M(\mathbb{Q}_v)$. There exists an isomorphism $\phi : M(\mathbb{Q}_v) \cong \prod_i Gl(n_i, \mathbb{Q}_v)$ such that $\phi(m^*) = \prod_i g_i^*$, if $\phi(m^*) = \prod_i g_i$, where again $g_i^* = (g_i')^{-1}$ in $Gl(n_i, \mathbb{Q}_v)$. Hence $\sigma^*(m) = \sigma(m^*)$ satisfies $\sigma^* \cong \sigma^\vee \cong \overline{\sigma}$ (σ is unitary). Using example 4 there exists an antilinear isomorphism θ of the representation space of σ such that $\sigma \circ \theta = \sigma^* \circ \theta$ and $\theta^2 = id$.

The intertwining operators $A(\sigma, w) : I(\sigma) \to I(\tilde{w}(\sigma))$ for $\tilde{w}$ in the Weyl group with representatives $w \in GSp(4, \mathbb{Q}_v)$ are defined, by analytic continuation, from the integrals

$$(A(\sigma, w)f)(g) = \int_{N_w} f(w^{-1}ng)dg$$

as in [Sh]. If w is in the center with $\tilde{w} = 1$ then $A(\sigma, w) = \omega_{\Pi_v}(w) \cdot id$. In particular $A(\sigma, -E) = \omega_{\Pi_v}(-1) \cdot id$. Furthermore $\Pi(\sigma \otimes \chi) = \Pi(\sigma) \otimes \chi$ for unitary characters χ of $GSp(4, \mathbb{Q}_v)$. So $\Pi(\sigma)$ and $\Pi(\sigma \otimes \chi)$ act on the the same vector space. With this identification $A(\sigma \otimes \chi, w) = A(\sigma, w)$, since there exists $n = n^- mk$ with $\lambda(m) = \lambda(k) = 1$ for $n \in N_w$. λ denotes the similitude character of $GSp(4)$.

Put $\Theta = A(\sigma^*, J) \circ I(\theta)$ with $I(\theta) : I(\sigma) \to I(\sigma^*)$ antilinear and induced by $\theta : \sigma \to \sigma^*$

$$\Theta : I(\sigma) \xrightarrow{I(\theta)} I(\sigma^*) \xrightarrow{A(\sigma^*, J)} I(w(\sigma^*)) \ .$$

Note that $w(\sigma^*) = \sigma \otimes \omega_\sigma^{-1} = \sigma \otimes \omega_\Pi^{-1}$. This is an identity, not an isomorphism! therefore $I(w(\sigma^*)) = I(\sigma \otimes \omega_\Pi^{-1}) = I(\sigma) \otimes \omega_\Pi^{-1}$. Thus Θ can be viewed as a (ω_Π, γ)-linear endomorphism

$$\Theta : I(\sigma) \to I(\sigma) \ .$$

We have $\Theta^2 = A(\sigma^*, J)I(\theta)A(\sigma^*, J)I(\theta) = A(\sigma^*, J)I(\theta^2)A(\sigma, J) = A(\sigma^*, J)A(\sigma, J) = \lambda_{\Pi(\sigma)} \cdot id$, since $I(\theta^2) = I(id) = id$. Now $A(\sigma, J) = A(\sigma \otimes \omega_\Pi^{-1}, J) = A(w(\sigma^*), J)$. Hence $\Theta^2 = A(\sigma^*, J)A(w(\sigma^*), J) = \omega_\Pi(-1)A(\sigma^*, J)A(w(\sigma^*), J^{-1})$. But

$$A(\sigma^*, J)A(w(\sigma^*), J^{-1}) = |\gamma(0, \sigma^*, r_w, \overline{\psi})|^2 \cdot id$$

is a positive multiple of the identity. This positive real number is the Plancherel measure in the sense of Shahidi [Sh], p.274 and [Sh], p.312 formula (7.8.1) for $s = 0$. Therefore $\Theta^2 = \omega_\Pi(-1) \cdot id$, hence $\epsilon(GSp(4, \mathbb{Q}_v), \Pi_v, \omega_{\Pi_v}) = \omega_{\Pi_v}(-1)$, since $\Pi(\sigma)$ is unitary. We have shown, that the next lemma holds for irreducible admissible representations, which are not in the discrete series.

Lemma D4: Let Π_v be an irreducible admissible representations $GSp(4,\mathbb{Q}_v)$. Then the parity $\epsilon(GSp(4,\mathbb{Q}_v),\Pi_v,\omega_{\Pi_v}) = \omega_{\Pi_v}(-1)$ is determined by the central character ω_{Π_v}.

Proof: It remains to consider representations Π_v in the discrete series. In fact, such representations can be globally embedded into a cuspidal irreducible automorphic representation Π of $GSp(4,\mathbb{A})$. For such Π there exists $T = T' \in M_{2,2}(\mathbb{Q})$, such that $det(T) \neq 0$ and a $\phi \in \Pi$, such that $\int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \phi(ng)\overline{\psi}_T(n)dn \neq 0$. See [S]. Here $P = MN$ is the Siegel parabolic of upper triangular block matrices. Attached to T is the quadratic algebra $K/\mathbb{Q}$ defined by the discriminant of T. $K^*(\mathbb{A}) = \mathbb{A}_K^*$ stabilizes the character ψ_T. Π is cuspidal. Therefore, by well known Sobolev techniques, the integrals $\int_{K^*\backslash\mathbb{A}_K^*}\int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \phi(yng)(\nu\otimes\psi_T)(yn)dy^*dn < \infty$ are well defined. Furthermore there exists a ν, such that one of these integrals is not zero. By Fourier's theorem ν can be chosen to be unitary. This defines a nontrivial global generalized Whittaker model of Π in the sense of [PS]. It induces a local nontrivial generalized Whittaker model of Π_v. By example 4 we get $\epsilon(\Pi_v,\omega_{\Pi_v}) = \omega_{\Pi_v}(-1)$. $\square$.

Corollary D5: The equivalent statements of D2 are true.

Furthermore $W_{\Pi_f} = Hom_{G(\mathbb{A}_f)}(\Pi_f, V_{\Pi_f})$ carries the antilinear endomorphism Θ_Π with $\Theta_\Pi^2 = id$. For $\varphi \in W_{\Pi_f}$ we have $\Theta_\Pi(\varphi) \in W_{\Pi_f}$ and the commutative diagram

$$\begin{array}{ccccc} \Pi_f & \xrightarrow{\varphi} & V_{\Pi_f} & \hookleftarrow & V \\ \downarrow\ \theta_f & & \downarrow \Theta_\infty & & \\ \Pi_f & \xrightarrow{\Theta_\Pi(\varphi)} & V_{\Pi_f} & \hookleftarrow & \Theta_\infty(V) \end{array} .$$

Consider the irreducible $G(\mathbb{A}_f)$-module $V = \varphi(\Pi_f)$. If

$$\Theta_\Pi(\varphi) = const\cdot\varphi\ ,$$

then $V = \Theta_\infty(V)$ and conversely. Furthermore $V = \Theta_\infty(V)$ is equivalent to $V\cap\Theta_\infty(V) \neq 0$. Suppose

$$V\cap\Theta_\infty(V) \neq 0 \quad \text{and} \quad V = V\cap V_{\Pi_f}^{pq}\ .$$

Under this assumption we have $\langle\eta,\overline{\eta}\cup\omega_0\rangle = tr(\eta\cup\overline{\eta}) \neq 0$ for $\eta\in V$. Hence $\langle\eta,\theta_\infty\eta\rangle \neq 0$. Since $\Theta_\infty = \theta_\infty F_\infty^*$ therefore $\langle\eta, F_\infty^*\Theta_\infty\eta\rangle = \omega_{\Pi_\infty}(-1)\cdot\langle\eta,\Theta_\infty F_\infty^*(\eta)\rangle \neq 0$. In other words the pairing $\langle.,.\rangle$ is nontrivial on $F_\infty^*(V)+V$, or rather

$$\langle V, F_\infty^*(V)\rangle \neq 0\ .$$

This means, that the pairing induced from $\langle.,.\rangle$ on W_{Π_f} with parity -1 defines a nontrivial pairing on $\mathbb{C}\cdot\varphi + \mathbb{C}\cdot F_\infty^*(\varphi) \subset W_{\Pi_f}$. This pairing is necessarily nondegenerate and odd.

The special case $\omega_0 = 1$ and $p = 0, q = 3$ is instructive, since then $H^0(M/\mathbb{C}, \Omega^3(\mathcal{V}_\mu))^{\Theta_\infty} = \Omega^3(M/\mathbb{R}, \mathcal{V}_\mu) = F^{k_1+k_2-3}(H^3(M/\mathbb{R}, \mathcal{V}_\mu(\mathbb{R}))$ (algebraic 3-form defined over $\mathbb{R}$ with values in $\mathcal{V}_\mu$). The assumptions above are satisfied, if $V \cap \Omega^3(M/\mathbb{R}, \mathcal{V}_\mu) \neq 0$ or even $V \cap \Omega^3(M/\mathbb{Q}, \mathcal{V}_\mu) \neq 0$. If this is zero, then consider $V \cap F^{k_1-1}$ for the similar conclusion.

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