Darmstadt-Frankfurt Seminar

Vertex algebras

Winter 15/16

S. Möller and N. Scheithauer

In this seminar we give an introduction to the theory of vertex algebras. The main reference is [FBZ].

Introduction and definition

Introduction (N. Scheithauer)
Introduction to the subject and overview over the following talks.

Definition and properties I (N. Scheithauer)
Definition of a field (1.2.1), locality (the standard definition is 1.2.5), definition of a vertex algebra (1.3.1), subalgebras, ideals and homomorphisms (1.3.4).

Darmstadt, 15th October 2015, S214/024, 15:00–17:30

Definition and properties

Definition and properties II
Normally ordered product of fields (2.2.2), Dong's Lemma (2.3.4), Reconstruction Theorem (2.3.11) (Dong's Lemma and the Reconstruction Theorem are the main results here and should be proved), definition of a conformal vector, a conformal vertex algebra (2.5.8) and a vertex operator algebra ([FHL], 2.2.1)

Definition and properties III
Goddard's Uniqueness Theorem (3.1.1), associativity (3.2.1) (both should be proved), Borcherds' Identity (3.3.10) (the proof by contour integrals can be sketched if time permits), a short remark on operator product expansions (eq. 3.3.10) and conformal vertex algebras (3.4.3), Reconstruction Theorem (4.4.1)

Frankfurt, 29th October 2015

Lie algebras and examples of vertex algebras

Lie algebras (M. Schwagenscheidt)

Definition, examples, semisimple and simple Lie algebras, Killing form, Cartan subalgebra, Cartan decomposition, Weyl group, Dynkin diagram, classification, Serre's construction, modules, weights, universal enveloping algebra, PBW Theorem, Verma modules, finite-dimensional irreducible modules. The lecture of Carter in [CSM] gives a nice overview. Proofs can be found in [Hm] and [S].

Examples of vertex algebras I (Y. Li)

Free bosons (2.1, 2.2, 2.3) (free bosons denote the vertex algebra associated to a Heisenberg Lie algebra, the main result here is Theorem 2.3.7, more generally $S(\hat{h}^-)$ has a vertex algebra structure, cf. sections 3.5, 4.7 in [K], the vertex algebra structure follows easily from the Reconstruction Theorem), lattice vertex algebras (5.4) (the better reference here is [K], section 5.4), boson-fermion correspondence (5.3) (this should only be presented if time permits, cf. also section 5.1 in [K]).

Darmstadt, 12th November 2015, S214/024, 15:00–17:30

Further examples and modules

Examples of vertex algebras II

Affine Kac-Moody algebras and their vertex algebras (2.4), the Segal-Sugawara construction (2.5.10 and 3.4.8), the simple quotient $L_k(g)$ (4.4.3). [FZ] is also a good reference for this talk.

Modules over vertex algebras

Definition of a module (5.1.1, cf. also section 4.1 in [FHL]), Proposition 5.1.2, Remark 5.1.4, definition of a conformal module (5.1.9), description of the irreducible modules of the lattice vertex algebra V_L (5.5.5, the modules are parametrised by L'/L, if time permits the proof in [D] can be sketched) and of the affine vertex algebra $L_k(g)$ (5.5.5).

Frankfurt, 26th November 2015

Zhu's Theorem and the Verlinde Formula

Zhu's Theorem (S. Möller)

Statement of the theorem, sketch of proof and examples ([Z]).

The Verlinde Formula (S. Möller)

Statement of the theorem, sketch of proof and examples ([H]).

Darmstadt, 17th December 2015, S214/024, 15:00–17:30

The monster vertex algebra and the moonshine conjecture

The monster vertex algebra

The main reference here is [FLM]. Construction of the monster vertex algebra V^M (10.3.32), definition of the untwisted (8.5.5) and twisted (9.2.23), (9.2.27) vertex operators, the graded dimensions of V^M (Theorem 10.5.7 for $k = 1$), the monster as automorphism group of V^M (Theorem 12.3.4), the invariant bilinear form on V^M (Corollary 12.5.4). The monster vertex algebra V^M is an orbifold of the vertex algebra of the Leech lattice (cf. section 5.7.1 in [FBZ]).

The moonshine conjecture

A good reference for this talk is Borcherds' original paper [B1]. [B2] gives a nice overview. Statement of Conway and Norton's moonshine conjecture ([B1], section 1), the monster Lie algebra (sections 6 and 7), Borcherds' proof of the moonshine conjecture (sections 8 and 9). We remark that using results of Cummins and Gannon [CG] it is possible to simplify the last step of the proof.

Frankfurt, 21st January, 2016

Literatur

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[FHL] I. B. Frenkel, Y.-Z. Huang, J. Lepowsky, *On axiomatic approaches to vertex operator algebras and modules*, Mem. Amer. Math. Soc. **104** (1993)

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[Hm] J. E. Humphreys, *Introduction to Lie algebras and representation theory*, Graduate Texts in Mathematics **9**, Springer-Verlag, New York-Berlin (1972)

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[Z] Y. Zhu, *Modular invariance of characters of vertex operator algebras*, J. Am. Math. Soc. **9** (1996), 237–302