

# THE TATE AND ARTIN CONJECTURES FOR SUPERSINGULAR K3 SURFACES

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## 1. INTRODUCTION

Recent work of Maulik [Mau12], Charles [Cha12] and Madapusi-Pera [MP13] completed the Tate conjecture for K3 surfaces  $X/\mathbb{F}_q$  which states that the following map is an isomorphism

$$\mathrm{NS}(X) \otimes \mathbb{Q}_l \rightarrow H_{et}^2(\overline{X}/\overline{\mathbb{F}}_q, \mathbb{Q}_l(1))^{\mathrm{Gal}_{\overline{\mathbb{F}}_q}}.$$

Work of Nygaard and Nygaard-Ogus in the 80's proved the conjecture for K3 surfaces whose formal Brauer group has finite height. The remaining case of infinite height is that of supersingular K3 surfaces which is conjectured by Artin to be precisely that where the rank of the Neron-Severi group obtains the maximal possible value of 22. These are strange surfaces and such phenomena are specific to positive characteristic. Since the rank of  $\mathrm{NS}(X)$  is bounded above by 22 which is the rank of  $H_{et}^2$ , to complete the proof of Tate's Conjecture, it is sufficient to prove Artin's conjecture. Although there is a lot of it, we aim to introduce some of the necessary background theory on cycles and K3 surfaces (in particular the theory in positive characteristic) so as to cover Artin's proof in the elliptic K3 case and then discuss at least some of the geometric aspects of Maulik's proof, which is essentially a degeneration argument to the elliptic case.

## 2. SCHEDULE

**Talk 1: 16 April, Introduction. Speaker: Frank Gounelas.** Introductory talk outlining the material and setting the speakers for the talks.

### INTRODUCTION TO THE TATE CONJECTURE.

**Talk 2: 23 April, Cycles and cohomology. Speaker: Katharina Hübner.** Basic definitions on cohomology, cycles and equivalence relations, the chern class map in  $\ell$ -adic cohomology following chapter 23 of Milne's notes, VI.9 of [Mil80] and [Tat64]. [Mil12], [Mil80], [Tat64].

**Talk 3: 30 April, Tate's conjecture. Speaker: Alexander Schmidt.** Tate's original paper: the statement of the conjecture, link to poles of zeta functions and proof for Fermat hypersurfaces (with some conditions) [Tat65] (or [Tat64] which is identical), [Tat94], [Mil07], [Shi79].

### THE CASE OF ELLIPTIC K3 SURFACES.

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*Date:* April 16, 2013.

**Talk 4: 7 May, Intro to complex K3 surfaces. Speaker: Nithi Rungtanapiroom.** Over  $\mathbb{C}$ , computations of invariants, Hodge diamond, (quasi-)polarisations, bits and pieces from chapters 1,2 of Huybrechts' notes [Huy13], [Bea96], [BHPVdV04].

**Talk 5: 14 May, Artin-Mazur. Speaker: Johannes Schmidt.** This will be a general talk introducing some basics of crystalline cohomology, formal Brauer groups and a general discussion of the result  $\text{rk NS}(X) \leq b_2 - 2 \text{ht } \hat{\text{Br}}X$  following Artin-Mazur. [AM77], [III94], [BO78] or the stacks project.

**Talk 6: 21 May, Supersingular K3s and Artin's conjecture. Speaker: Malte Witte.** Positive characteristic K3s. Artin's conjecture (see Artin's original paper),  $\mathbb{F}_p, \overline{\mathbb{F}}_p, \overline{k}$ , even rank and specialisation, Deligne's lifting theorem (Hassett's notes for a good summary). [Has10], [Art74], [ASD73], [Del81].

**Talk 7: 28 May, Artin's proof for K3s with elliptic fibrations I. Speaker: Jochen Gärtner.** Sections 0, 1, 2 of Artin's paper proving that elliptic supersingular K3s have  $\rho = 22$ . More specifically: Some more general bits and pieces on formal groups, Theorem 1.1 until Lemma 1.6 inclusive. [Art74], [AM77], [ASD73].

**Talk 8: 4 June, Overflow/Break.**

**Talk 9: 11 June, Artin's proof for K3s with elliptic fibrations II. Speaker: Jakob Stix.** Sections 0, 1, 2 of Artin's paper proving that elliptic supersingular K3s have  $\rho = 22$ : More specifically: Theorem 1.7 and the calculations of section 2. [Art74], [AM77], [ASD73].

#### THE CASE OF SUPERSINGULAR K3 SURFACES.

**Talk 10: 18 June, Kuga-Satake construction. Speaker: Aprameyo Pal.** Hodge structures, Clifford algebras, the associated Kuga-Satake abelian variety, moduli of abelian varieties and their Hodge bundle, Kuga-Satake morphism [Huy13], [vG00], [Nyg83], [Del72].

**Talk 11: 25 June, Maulik's proof I: Outline. Speaker: TBC.** Moduli of K3s and the Hodge bundle, quasi-polarised locus, outline of Maulik's proof Assuming 3.1 (Borcherds' work) and 5.10 (the p-adic Hodge theory). [Huy13], [Mau12], [Riz06].

**Talk 12: 2 July, Maulik's proof II: Smooth model. Speaker: Armin Holschbach.** Section 4 of the paper: spreading out to a good model over a DVR, semistable models, mixed characteristic MMP. [Mau12] and references therein.

**Talk 13: 9 July, Overflow/Break.**

**Talk 14: 16 July, Maulik's proof III: Positivity of the Hodge bundle. Speaker: TBC.** Section 5 of the paper (assuming 5.10): relative Kuga-Satake and comparing with the Hodge bundle of  $\mathcal{A}_g$  and completion of main theorem (section 7). [Mau12].

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