Prof. Dr. Gebhard Böckle 8. April 2013

Study group/Forschungsseminar SS 2013:

(φ, Γ)-modules over the Robba ring and trianguline representations

Wednesdays, 9:15 to 10:45 am, INF 368, room 248[1]

There has been a long history of expressing properties of algebraic varieties in terms of Galois representations. A case investigated from the beginning was that of ℓ-adic Tate-modules of abelian varieties over global fields. Then Grothendieck et al. developed ℓ-adic étale cohomology that allows one to attach Galois representations to algebraic varieties. However only the work of Deligne on purity in his proof of the Weil conjectures provided a large supply of strictly compatible systems Galois representations for general varieties – or even some motives. Given any smooth projective variety X over a number field K and some $i \in \{0, \ldots, 2\dim X\}$, for any prime ℓ one can attach a Galois representation

$$\rho_\ell \colon G_K := \mathrm{Gal}(K^{\mathrm{alg}}/K) \longrightarrow \mathrm{GL}_{d_i}(\mathbb{Q}_\ell) \cong \mathrm{Aut}_{\mathbb{Q}_\ell}(H^i_{\mathrm{et}}(X_{K^{\mathrm{alg}}}, \mathbb{Q}_\ell))$$

for $d_i = \dim H^i_{\mathrm{et}}(G_{K^{\mathrm{alg}}}, \mathbb{Q}_\ell)$ that is independent of ℓ. Moreover these representations are tied together by the following compatibility: For any primes ℓ, ℓ' and all finite places v of K that do not divide neither ℓ, nor ℓ', nor the finite set S of places of K of bad reduction of X, one has equality of characteristic polynomials

$$\mathrm{charpol}(\rho_\ell(\mathrm{Frob}_v)) = \mathrm{charpol}(\rho_{\ell'}(\mathrm{Frob}_v)) \in \mathbb{Q}[T].$$

A particular case studied by Deligne in relation to the Ramanujan-Petersson conjecture was that of Galois representation attached to modular forms. Happily, this fit together with the beginning of Langlands' conjectures and there arouse immediately the following questions:

(a) Can one formulate a compatibility at the places v dividing S but not ℓ?

(b) Can one formulate a compatibility even at the places dividing ℓ?

(c) How do these local descriptions fit together with the local classification of automorphic representations?

The first question led to the definition of Weil-Deligne representations – and while (a) is known in many examples, in general, this question is still wide open. The second question led to the development of p-adic Hodge theory (in an attempt to better understand the comparison between étale, de Rham and crystalline cohomology). Fontaine's period rings play a key role. Part (c) is also answered in many cases – for instance for Hilbert modular forms, or for some cases of automorphic forms for unitary groups. But again much is left open. We shall try to review some of the above in the initial talks of the seminar. Some references are [Ca86, Del74, Fa87, Sai09].

The main emphasis of the seminar will be on a further and rather recent development. For a single geometric object, it seems natural to study all its cohomological realizations, such as the compatible system above. However if one fixes a modular curve and considers modular forms and their Galois representations for various weights, it is also natural to study p-adic properties. One motivation for this is the study of congruences of modular forms, and for this, let us fix one prime ℓ and call it p from now on. Important work on p-adic congruences dates back to Serre and Katz in the 1970's. A crucial

[1] Begin: April 17 2013

point in the further developments was certainly the conjecture of Serre on 2-dimensional mod p Galois representations and their relation modular forms from the 1980's. Edixhoven, Mazur, Ribet et al. soon developed what is now called weight and level lowering for mod p congruences between modular forms. The conjecture of Serre was ultimately proved by Khare and Wintenberger with important contributions by Kisin around 2006. A second, independent, crucial development was the study of ordinary families of modular forms by Hida which caused Mazur to investigate universal deformation rings of mod p Galois representations. This new perspective on modular forms then led to the Gouvêa-Mazur conjecture and its partial solution by Coleman who constructed p-adic families of modular forms beyond Hida's. Having the answers to the questions of the previous paragraph for a single Galois representation attached to a modular form in mind, one has now the following natural extension and variants:

(d) Can one describe the variation of the restriction of the family of Galois representations of a Hida or Coleman family to a decomposition group (above a prime p or a prime $q \neq p$)?

(e) Classical Galois representations are, essentially by definition, p-adically dense in p-adic families. Are they (or the p-adic) families (Zariski) dense in the set of all Galois representations (of a certain kind)?

(f) Is there a p-adic theory of automorphic representations that is capable to vary in families and that one can compare with the variation of a corresponding family of p-adic Galois representations?

All of this is a very active branch of research on Galois representations. Even presenting what is currently known seems far too much for one seminar. The present seminar's aim is to understand one particular aspect, namely the role played by (φ, Γ)-modules over the Robba ring $\mathcal{R}$ over a p-adic field K. The subcategory of étale (φ, Γ)-modules over $\mathcal{R}$ is isomorphic to the category of p-adic Galois representations of G_K. At least for $K = \mathbb{Q}_p$, the variation of the decomposition group at p in a p-adic family can be studied nicely be considering the corresponding family of such (φ, Γ)-modules. These modules also play a central role in Colmez' proof of the p-adic local Langlands conjecture. Moreover they were useful in providing local answers to question (e). Some references are [Ch10, Col10a, Col96, Em11, Hi86].

1 INTRODUCTION

The introductory talk will consist of repeating the above in greater detail.

Speaker: Gebhard Böckle 17.04.

2 COMPATIBLE SYSTEMS OF GALOIS REPRESENTATIONS, WEIL-DELIGNE REPRESENTATIONS AND THE LOCAL LANGLANDS CORRESPONDENCE

This should be a crash course on the above themes: Begin with the notion of a Weil-Deligne representation. Explain briefly how to obtain it from a p-adic Galois representation at a decomposition group not above p, ∞ and leave a black box at p. Then define the notion of strictly compatible system of Galois representation. Give one or two examples.
In the second half, pick up the theme of automorphic representations from the first talk. Describe the local representations for $\mathrm{GL}_2(\mathbb{Q}_p)$, and then present the local Langlands correspondence. This would fill some gaps from the first lecture.

Literature:
[Ge09], [Bö12], [Kh01] and references therein, [BH06]

Speaker: Konrad Fischer 24.04.

3 P-ADIC HODGE THEORY AND ITS GEOMETRIC ORIGINS

One could spend more than a semester on this topic. But in this talk – for which 90 minutes are perhaps not sufficient? – one should focus on some central aspects.
I suggest to start with the introduction of the categories of filtered vector spaces and filtered (φ, N)-modules. If the speaker is ambitious, he/she could include descent data. A good reference might be [BC09, 4.1.1, 8.2.5], [FO07, § 6.4] or [GhM09, § 2]. Perhaps one can mention some important properties of these categories – in writing or orally.
Then introduce Fontaine's formalism to pass from p-adic Galois representations to vector spaces with supplementary structure via some rings. [BC09, § 5] or [FO07, § 2.1].
Introduce Fontaine's rings $B_?$ for $? \in \{\mathrm{dR}, \mathrm{crys}, \mathrm{st}\}$ *without defining them* – but explain their basic structures and some elementary properties – for much more see [FO07, 5.2, 6.1]. The properties one should focus on are those that are relevant to the structure of the vector spaces $D_?(V)$ for a p-adic Galois representation V. [BC09, § 6, § 8] or [FO07, 5.2.5,]. In particular one should explain, and this follows rather easily from properties of the ring that crys $\Rightarrow$ st $\Rightarrow$ pst $\Rightarrow$ dR and crys $\Rightarrow$ pcrys $\Rightarrow$ pst and state the rather deep theorem dR $=$ pst.
Finally it would be great to relate the abstract properties captured by filtered vector spaces and filtered φ- or (φ, N)-modules to cohomology theories of smooth projective varieties X over a p-adic ring K. This also gives a good motivation for the various comparison theorems of which Fontaine's formalism is a very deep abstraction: The most basic comparison is the de Rham isomorphisms. The de Rham cohomology is defined for any such variety X, as is p-adic étale cohomology with $\mathbb{Q}_p$ coefficients. If X has good reduction, then its crystalline cohomology is naturally a K_0-vector space with a Frobenius φ. If X has semistable reduction, one can define log-structures on it and then crystalline cohomology with schemes with log-structures provides one with a K_0-vector space carrying a (φ, N)-structure. A good reference for much of the material and in particular on the last point is perhaps [Sav11]. Unfortunately it does not contain any references. One could also consider the deep article by Tsuji [Ts99] if one wants to understand more about this.

Literature:
[BC09], [FO07], [GhM09], but also [Ber02, I,II], [Ber12, 3.1,3.2] and [Ber12].

Speaker: Tommaso Centeleghe 08.05.

4 (φ, Γ)-MODULES OVER $\mathcal{E}$ AFTER FONTAINE.

This talk should present two important results. The first is the correspondence between p-adic Galois representations of G_K for a p-adic field K and the category of (φ, Γ)-modules over the ring $\mathcal{E}$ due to Fontaine, see [Ber02, III.1-III.2 without III.2.4] and with more details [FO07, Ch. 4]. The second result is an extension of Fontaine's correspondence due to Kisin and Ren [KR01] that arises by considering Lubin-Tate groups instead of the $\mathbb{Z}_p^*$-cyclotomic tower over K. The bases of both correspondences is the field of norms of Fontaine-Wintenberger, cf. [Wi83]. For background on Lubin-Tate groups, one possible reference is Milne's script [Mi, Ch. 1] on class field theory. [KR01] also refers to some results of Colmez.

Literature:
[Ber02] for a quick overview, [FO07] for many details, perhaps also the seminar notes [Bö08, Ce08]. Then for the second part [KR01] and [Mi] for some background on Lubin-Tate groups.

Speaker: NN 15.05.

5 THE ROBBA RING $\mathcal{R}$ AND SOME P-ADIC FUNCTIONAL ANALYSIS

The aim of this talk is to introduce the ring $\mathcal{R}$, relate it to other rings of p-adic Hodge theory and to some notions of functional analysis. Berger's survey [Ber12] is very brief about $\mathcal{R}$. It gives its definition and some basic properties in 2.1 and 2.2. A reasonable summary of further properties is given in [Col10a, pp. 6-7] with more details in [Col10a, I.1]. I think it would be good to present all statements from [Col10a, pp. 6-7] with some indication of proof: interpretation of various rings as rings of functions, the maps φ and its *inverse* ψ, stuff about residues, the relations to spaces such as C^0, LA, $\mathcal{D}$, $\mathcal{D}_0$ LP_n via the Amice transform etc. A more extensive treatment of some parts of p-adic functional analysis can also be found in [Col04]. There is also a number of big related rings. It might be useful to introduce them in this talk already, since they will be used in the following one. They are described in [Ber10, § 1.1] and the relation to $\mathcal{R}$ is hinted at on the bottom of page 5 in [Ber12]. See also [Ber02, IV.2].

Literature:
[Ber02], [Ber12], [Col10a] and [Col04]. Perhaps also [Ber10].

Speaker: NN 22.05.

6 (φ,Γ)-MODULES: THE PASSAGE FROM $\mathcal{E}$ TO $\mathcal{R}$.

The aim of this talk is to describe the transition from (φ,Γ)-modules over $\mathcal{R}$ to (φ,Γ)-modules over $\mathcal{R}$. This is via overconvergent (φ,Γ)-modules. Colmez [Col10a, pp. 7-9] gives an overview of how this works. However the actual and rather difficult proofs are omitted. Included in this talk are also results on cohomology of (φ,Γ)-modules along that passage.

It is best to begin with the definition of a (φ,Γ)-module and an étale (φ,Γ)-module both over $\mathcal{R}$ as in [Ber12, 2.2.1]. See also [Col08, § 2.1]. A survey over the first step of the desired correspondence, namely from Fontaine's (φ,Γ)-modules to overconvergent ones is indicated in [Ber02, III.3]. A bit more is said in [Col10a, pp. 8,9]. Note that the transition obscures the relation of a p-adic object to its mod p reduction: it is rather straightforward how to compute the mod p reduction of a (φ,Γ)-module in the sense of Fontaine, while it is much less clear how to do that for such a module over $\mathcal{R}$.

The computation of the cohomology of (φ,Γ)-modules of Fontaine and its relation to Galois cohomology was the subject of Herr's thesis, see [He99]. The same method can be transferred from these to (φ,Γ)-modules over $\mathcal{R}$. We note that Herr gives two complexes to compute this cohomology and that both are important. A very brief survey is in [Ber02, 2.4], or alternatively [Ber12, §2.4]. A more general result, but well-written is in [Ch10, 2.1]. I am not sure how much one can prove in the time given.

Literature:

Speaker: NN 29.05.

7 *MAKE UP DAY*

At this point, we are perhaps one day behind in the seminar. So I suggest to use it to finish talk 6, so that we are back on schedule.

Speaker: NN 05.06.

8 RESULTS ON (φ,Γ)-MODULES OVER $\mathcal{R}$.

Following [Ber12, 2.3, 2.4, 3.4] define the notion of a trianguline representation, explain the results on the slope filtration by Kedlaya and introduce weights of trianguline representations. Another references is [Col08, 2.2]. In terms of slopes one can now give another characterization of the correspondence between p-adic Galois representations and isoclinic (φ,Γ)-modules over $\mathcal{R}$ of slope zero as in [Col08, 2.7].

Next explain the passage from (φ,Γ)-modules over $\mathcal{R}$ to filtered (φ,N)-representations by reproducing [Ber12, 3.3]. The passage to de Rham representations is more subtle. For this, it is suggested to introduce B-pairs from [Ber10] and to state the equivalence between such and (φ,Γ)-modules over $\mathcal{R}$. This requires the introduction of the ring B_e. It might also be good at this point to say more about B_{dR}, e.g. [Ber02, II.2]. Try to say something meaningful about the correspondence between B-pairs and (φ,Γ)-modules over $\mathcal{R}$.

Literature:
[Ber12], [Ber02], [Ber10], [Col08].

Speaker: NN 12.06.

9 COLMEZ CLASSIFICATION OF TRIANGULINE REPRESENTATIONS OF $G_{\mathbb{Q}_p}$

The aim of the talk is to present the classification of Colmez of trianguline representations of $G_{\mathbb{Q}_p}$ of dimension 2 with as many proofs as possible. The talk should begin with a classification of 1-dimensional trianguline representations and (φ,Γ)-modules over $\mathcal{R}$. This can be found in [Col08, 3.2], [Ch10, p. 11] and (in terms of B-pairs) in [Na09, 1.45]. Next I suggest to state Colmez classification and to interpret it, see [Col08, p. 5, Thm. 0.8] and [Ber12, p. 13f.]. The remainder of the talk should be spent on proving parts of Colmez' classification:

For this one first needs to describe the extension classes of 1-dimensional representations as in [Ber12, Thm 3.5.2], [Ch10, Ch. 2], [Col08, Ch. 3]. None of the proofs is short, if fully given. From this one can try to give large parts of the proof of the main theorem. It might also be interesting to state further results on the p-adic representations classified by Colmez in terms of (φ,Γ)-modules over $\mathcal{R}$ such as those presented in [Col08, Ch. 5]. One needs to make a selection.

Literature:
[Ber12],[Ch10], [Col08], [Na09].

Speaker: NN 19.06.

10 DENSITY OF CRYSTALLINE POINTS IN LOCAL VERSAL DEFORMATION SPACES I

The aim of this and the following talk is to actually prove a theorem, namely [Ch10, Thm. A] – at least to a large extent. This includes the proof of the main result of [Col08, Ch. 6]. There are even more general results due to Nakamura. The talks are to follow [Ch10]. Both talks can freely use, but may need to recall in parts, [Ch10, Ch. 1, 2].

The first talk should cover at least [Ch10, Ch. 3]. In particular, define/construct T_d^{reg}, $F_d^{\square}$, $S_d^{\square}$, state the results 3.8 and 3.9, Theorem 3.14 and its proof, and Prop. 3.17 and Prop. 3.18 without proof.

Literature:
[Ch10], [Col08].

Speaker: NN 26.06.

11 DENSITY OF CRYSTALLINE POINTS IN LOCAL VERSAL DEFORMATION SPACES II

This talk continues the previous talk. It should cover [Ch10, Ch. 4] and in particular prove the density result of [Ch10, Theorem A]. The focus is in 4.1–4.6. The part 4.8 can be skipped if time runs out. The talk should however cover (parts of) the proof of the crucial Theorem 4.6 which is [Ch11, Thm. C].

Literature:
[Ch10] and [Ch11].

Speaker: NN 03.07.

12 BREAK OR SECOND MAKE UP DAY

The organizer is away. 10.07.

13 P-ADIC UNITARY PRINCIPAL SERIES FOR $\mathrm{GL}_2(\mathbb{Q}_P)$

This talk asks for a courageous speaker who is willing to read [Col10b] and gives us a survey on it. There is a tiny bit on this in [Ber12, 4.1].

Literature:
[Col10b]

Speaker: NN 17.07.

14 TOWARD (φ,Γ)-MODULES OVER MULTIVARIABLE ROBBA RINGS

This would be a great theme with future perspectives to finish the seminar: We saw earlier the correspondence built on Lubin-Tate theory by Kisin-Ren between (φ,Γ)-modules of a certain type and p-adic Galois representations. This should in some sense be a continuation. As suggested by Kisin-Ren, their paper when considering all embeddings $K \hookrightarrow \overline{\mathbb{Q}}_p$, should give a better theory and should allow one to view p-adic Galois representations over multivariable Robba rings. The latter program has just been started by [Ber12]. The aim of the talk is to present as much as possible from [KR01] and [Ber12] in relation to such a more general correspondence.

Literature:
[KR01] and [Ber12].

Speaker: NN 24.07.

LITERATUR

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[Col08] Pierre Colmez, *Représentations triangulines de dimension 2*. Astérisque **319** (2008), 213–258.

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