DUALITY VIA CYCLE COMPLEXES
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The standard n-simplex over a scheme X is given by $\Delta^n_X = X \times_{Spec\,\mathbb{Z}} \Delta^n_{\mathbb{Z}}$, with $\Delta^n_{\mathbb{Z}} = Spec(\mathbb{Z}[T_0, \ldots, T_n]/(\sum T_i - 1))$. The Bloch complex of relative zero cycles of X in (cohomological numbering) is the bounded above complex $\mathbb{Z}^c_X$ of sheaves on X_{et} defined as follows: for $W \to X$ étale and $i \in \mathbb{Z}$

$$(\mathbb{Z}^c_X)^i(W)$$

is given as the free abelian group on the set of integral subschemes of dimension $-i$ in Δ^{-i}_W which intersect all faces properly. The differential is given in the usual way as the alternating sum of face maps. Geißer [4] showed the following remarkable theorem.

Theorem 1. *Let $f : X \to S$ be a scheme, separated and of finite type over the spectrum of a Dedekind ring. Let A be the ring obtained from $\mathbb{Z}$ by inverting all prime numbers p such that there is a point $s \in S$ with imperfect residue field of characteristic p. Then, for every torsion sheaf F on X_{et}, there is a canonical isomorphism in $\mathscr{D}(\mathcal{A}b)$*

$$R\mathrm{Hom}^\bullet_X(F, \mathbb{Z}^c_X) \otimes_{\mathbb{Z}} A \simeq R\mathrm{Hom}^\bullet_S(Rf_! F, \mathbb{Z}^c_S) \otimes_{\mathbb{Z}} A.$$

This is an astonishingly general duality theorem. There is no assumption on the smoothness of f, even no regularity assumption on X. Moreover, the torsion of F need not be invertible on X, only the 'imperfectness' of S must be inverted.

If duality results for S are known (e.g. $S = Spec\,\mathcal{O}_K$, K a number field or a local field, or $S = Spec\,k$, k a finite, local or separably closed field) then Theorem 1 implies duality statements for the étale cohomology of X itself. Essentially all known duality theorems in higher dimension are special cases of Theorem 1. This follows from the following calculations of the cohomology of $\mathbb{Z}^c$:

Theorem 2.
- (i) *If $S = Spec\,k$ is the spectrum of a field, then $\mathbb{Z}^c_S \simeq \mathbb{Z}$ (constant sheaf placed in degree zero)*
- (ii) *If S is regular and one-dimensional, then $\mathbb{Z}^c_S \simeq \mathbb{G}_m[1]$.*
- (iii) *If X is regular of dimension d and $m \in \mathbb{N}$ is invertible on X, then $\mathbb{Z}^c_X/m \simeq \mu_m^{\otimes d}[2d]$.*
- (iv) *If X is a smooth, d-dimensional and separated scheme of finite type over a perfect field of characteristic $p > 0$, then $\mathbb{Z}^c_X/p^r \simeq \nu^d_r[d]$ (logarithmic deRham-Witt sheaf).*

Here $\simeq$ means canonical isomorphism in the derived category of étale sheaves on X.

The result of Geißer is the culminating point of a development which started with papers of Deninger [3], Nart [12], Milne [10], Spieß [13] and Moser [11]. It uses in an essential way results on motivic cohomology and higher Chow groups obtained by Bloch, Suslin, Voevodky, Rost, Levine and himself.

Talks

Talk 1: Introduction (Alexander Schmidt)

Talk 2: Higher Chow groups
Explain the definition of higher Chow groups and Chow sheaves for varieties and present the basic properties ([1] 1.3, 2.1, 3.1, 3.2, 3.4, 4.1, 6.1, 10.1). Where possible within reasonable time, present proofs or sketches of proofs. There are some mistakes in [1] which were corrected in [2], [7]. Present what is known for schemes of finite type over a 1-dimensional base following [9], [8].

Talk 2: Higher Chow sheaves mod n on regular schemes
First, for invertible n, construct the cycle map following [6] and [9] and show Theorem 2 (iii). Then introduce the logarithmic de Rham-Witt sheaves, e.g. following [5] and show Theorem 2 (iv). Extend the last result to singular varieties ([4], Prop. 2.3)

Talk 3: Duality in the p-p case
Give a proof of the p-p duality for smooth projective varieties over a perfect field following [10] Thm. 1.11. Deduce duality over a algebraically closed field and give a complete proof of [4], Thm. 4.6.

Talk 4: Etale descent for the Bloch complex
Explain the connection between higher Chow groups and motivic cohomology and formulate without proof the Beilinson-Lichtenbaum conjecture (now a theorem). Explain the method of Thomason [14] 2.8 (here we only need sheaves of abelian groups, not of spectra). Then give a complete proof of Thm. 3.1 and Thm 7.1 of [4].

Talk 5: The main theorem
Show the main theorem Thm. 4.1 and Cor. 4.7 of [4]. Then show the corresponding results over one-dimensional bases Thm. 7.5 and Prop. 7.10.

Talk 6: Applications
Duality over finite and local fields, Kato-homology sequences [4], §5, the generalization of Roitman's theorem [4] Thm. 6.1.

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