# The pro-étale topology for schemes

## Oberseminar of the AG Schmidt, Wintersemester 2017

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**Time and place:** Tuesday, 11:00-13:00, Room SR3, Mathematikon, Start: 24.10.2017

The étale topology is one of the tools which is used in algebraic geometry for purely geometrical as well as for arithmetic questions. However, working with it always carries several technical difficulties. Let us mention some of them concerning étale cohomology theories, the derived category of constructible sheaves and the étale fundamental group.

Fix a field $k$ and a prime $\ell \neq \mathrm{char}\, k$.

1. Let $k = \bar{k}$. What is often meant by étale cohomology or $\ell$-adic cohomology, namely $\mathrm{H}^*_{\acute{e}t}(X, \bar{\mathbb{Q}}_\ell)$, is not defined as cohomology of a certain sheaf in the étale topology, but rather as an inverse limit of cohomology of finite sheaves:

$$\mathrm{H}^i_{\acute{e}t}(X, \bar{\mathbb{Q}}_\ell) := \lim_n \mathrm{H}^i(X_{\acute{e}t}, \mathbb{Z}/\ell^n) \otimes_{\mathbb{Z}_\ell} \bar{\mathbb{Q}}_\ell. \qquad (\dagger)$$

   This has a disadvantage if one wants to consider not only the cohomology groups but the whole dg-algebra from which these supposedly come, e.g. in studying of the 'rational homotopy theory' of varieties. The main difficulty is that the dg-algebra has to be functorial in certain sense, and there seems to be no straight-forward way to obtain it, however the question was solved by Deligne [Del80].

2. For certain questions one wants to compute étale cohomology of a variety over a non-algebraically closed field $k$[1]. In this case even the equation (†) does not work well, e.g. there is no Leray spectral sequence for the 'fibration' $\bar{X} \to X$ with a fiber alike $K(\mathrm{Gal}(k), 1)$. The reason of this problem is mainly being that $\lim$ is not an exact functor. In some ways an *ad hoc* solution was proposed by Jannsen [Jan88] and bears the name of *continuous étale cohomology*, $\mathrm{H}^i_{cont}(X, \mathbb{Q}_\ell)$, which satisfies expected functorial properties and, in particular, for which there exist a spectral sequence $\mathrm{H}^q_{cont}(\mathrm{Gal}(k), \mathrm{H}^p_{\acute{e}t}(\bar{X}, \mathbb{Q}_\ell)) \Rightarrow H^{p+q}_{cont}(X, \mathbb{Q}_\ell)$.

3. The category of constructible sheaves in the étale topology with $\bar{\mathbb{Q}}_\ell$-coefficients is defined in a technically involved manner. One of the versions due to Ekedahl ([Eke90]) goes roughly as follows.

   A constructible $\mathbb{Z}_\ell$-sheaf is defined as a projective system $\{F_n\}_{n\in\mathbb{N}}$ where (1) $F_n$ is a constructible $\mathbb{Z}/\ell^n$-sheaf, and (2) the maps $F_n \otimes_{\mathbb{Z}/\ell^n} \mathbb{Z}/\ell^m \to F_m$ induced by transition morphisms are isomorphisms. Similarly, the derived category of constructible $\mathbb{Z}_\ell$-sheaves $D^b_c(X_{\acute{e}t}, \mathbb{Z}_\ell)$ is defined as the full subcategory of the derived category of pro-systems of constructible sheaves which is generated by those systems satisfying the (2) condition. Tensoring the obtained category by $\mathbb{Q}_\ell$ one obtains $D^b_c(X_{\acute{e}t}, \mathbb{Q}_\ell)$. Moreover, to pass to $\bar{\mathbb{Q}}_\ell$-sheaves one has to perform the analogous construction for finite extensions of $\mathbb{Q}_\ell$ and then take a 2-categorical colimit of corresponding triangulated categories.

[1]E.g. some of the Beilinson's conjectures predict that over a finitely generated field $k$ the Chow groups of a smooth variety $X$ with rational coefficients map injectively into $\mathrm{H}^*_{\acute{e}t}(X, \mathbb{Q}_\ell)$ properly defined.

4. The pro-étale fundamental group can be defined using the Galois category of local systems in the étale topology. However, one might show that already in a case of a non-normal projective curve of genus $\geq 1$ over an algebraically closed field the category of étale local systems is not big enough ([BhaSch, Example 7.4.9 due to Deligne]), or in other words $\pi_1^{ét}$ is too 'small'[2].

The pro-étale topology discovered by Bhatt and Scholze [BhaSch] allows to define all the objects above (or their better behaved versions) in a most natural way. Natural here means that constructions are both straight-forwardly functorial and sometimes follow the intuition of the topology of manifolds. The goal of the seminar is to get acquainted with the techniques and the results of this paper.

We present here the main definition and several statements which will be discussed in the seminar.

**Definition** ([BhaSch, 4.1.1])**.**

i) A map $f\colon Y \to X$ of schemes is called *weakly étale* if $f$ is flat and $\Delta_f\colon Y \to Y \times_X Y$ is flat.

ii) Write $X_{proét}$ for the category of weakly étale $X$-schemes, which we give the structure of a site by declaring a cover to be one that is a cover in the fpqc topology, i.e. a family $\{Y_i \to Y\}$ of maps in $X_{proét}$ is a covering family if any open affine in $Y$ is mapped onto by an open affine in $\coprod_i Y_i$.

It will be shown that there exist a limit-preserving functor from the category of profinite sets to $X_{proét}$ ([BhaSch, Example 4.1.9]), and moreover a functor associating to a topological space a 'constant' sheaf in the pro-étale topology. If $E$ is a topological ring (e.g. $\mathbb{Q}_\ell$, $\bar{\mathbb{Q}}_\ell$), then denote by the same letter corresponding sheaf of rings in the pro-étale topology. In particular, $H^i(X_{proét}, \bar{\mathbb{Q}}_\ell)$ denotes the $i$-th right derived functor of the functor of global sections $\mathrm{R}^i\Gamma$ applied to the sheaf $\bar{\mathbb{Q}}_\ell$.

Below for simplicity assume that $X$ is a variety over a field $k$ (though the constraints in fact needed are much weaker).

**Theorem.**

1. *There exist canonical isomorphisms*

$$\mathrm{H}^i_{ét}(\bar{X}_{ét}, \bar{\mathbb{Q}}_\ell) = \mathrm{H}^i(\bar{X}_{proét}, \bar{\mathbb{Q}}_\ell), \text{ and } \mathrm{H}^i_{cont}(X_{ét}, \mathbb{Q}_\ell) = \mathrm{H}^i(X_{proét}, \mathbb{Q}_\ell).$$

2. *One might realise the derived category of abelian sheaves on the étale site $D(X_{ét})$ as the full subcategory in the corresponding category over the pro-étale site $D(X_{proét})$. Moreover, one can prove that the left-completion of the category $D(X_{ét})$ is a full subcategory of $D(X_{proét})$, and describe it internally.*

3. *Denote by $Loc_X$ the category of sheaves of sets on $X_{proét}$ which are locally constant. Any geometric point $x$ of $X$ yields a fibre functor $ev_x\colon Loc_X \to Sets$, and the automorphism group of $ev_x$ with a compact-open topology is called the pro-étale fundamental group $\pi_1^{proét}(X, x)$.*

   *Then (1) $(Loc_X, ev_x)$ is equivalent to the category of continuous representations of $\pi_1^{proét}(X, x)$ on discrete sets; (2) the pro-finite completion of $\pi_1^{proét}(X, x)$ is the étale fundamental group $\pi_1^{ét}(X, x)$; (3) the pro-discrete completion of $\pi_1^{proét}(X, x)$ is $\pi_1^{SGA3}(X, x)$ (see [SGA3, Exp. X §6]).*

[2]Or too big, as in fact there will be constructed the group $\pi_1^{proét}(X, x)$ whose pro-finite completion is $\pi_1^{ét}(X, x)$, but which has more representations.

## Schedule

All references refer to the paper [BhaSch] if not stated otherwise.
A star near the number of a talk means that presumably an additional session will be needed.

**1. Overview** (Pavel).

In this talk some of the drawbacks of working with the étale topology and sketches of their possible solutions will be recalled: definitions of $\ell$-adic cohomology, continuous étale cohomology, derived category of constructible sheaves, and, possibly, insufficiency of $\pi_1^{\acute{e}t}$ for studying local systems on non-normal varieties.

The definition of the pro-étale topology then will be given, and the main results of the paper [BhaSch] will be announced solving mentioned issues of étale topology. We will give a sketchy overview of the paper which will contain the following topics: weakly contractible topologies and replete topoi, weakly-contractibility of the pro-étale topology and pro-(étale) versus pro-étale, associating the sheaf in the pro-étale topology to a pro-finite space, and, lastly, the infinite Galois categories and complete (aka *Noohi*) topological groups.

**2. Spectral spaces: w-local coverings.** *This is based on the material of Sections 2.1 and 2.2.*

In this talk topological spaces underlying affine schemes, *spectral spaces*, are to be recalled, and the construction to be given of a pro-open cover of any such space which has no furter open covers without a section (let us call it here *the covering problem*). The construction then to be realised at the level of affine schemes.

In order to do this one might need to recall the definition of the category of spectral spaces and its various properties (using e.g. [Hoc69]). In particular, the fact that $\pi_0$ of a spectral space is canonically a pro-finite set should be explained. Definiton 2.1.1 of *w-local* spectral space and the construction of Lemma 2.1.10 solve the covering problem for spectral spaces. Definition 2.2.1 and Lemma 2.2.4 and Corollary 2.2.14 solve the covering problem for affine schemes (with all the needed statements and constructions along the way, e.g. henselisation).

**3. Pro-(étale) versus pro-étale and weakly-contractibility.** *This is based on the material of Sections 2.3 and 2.4.*

In this talk weakly étale morphisms of algebras should be introduced, and it should be proven that for any weakly étale morphism $f : A \to B$ there exist a faithfully flat ind-étale $g : B \to C$ s.t. $g \circ f$ is ind-étale (Th. 2.3.4). The definition of a *w-contractible* ring should be given, and it should be proven that for any ring $A$ there is a map $A \to A'$ which is ind-étale faithfully flat, and $A'$ is w-contractible (Lemma 2.4.9). The first statement will allow to compare the étale topology and the pro-étale topology in future talks, and the second will be useful in proving nice properties (e.g. repleteness) of the pro-étale topos.

The henselisation constructions used in the previous talk and its properties should be enough to prove Theorem 2.3.4. Running ahead, and if time permits, one might define a locally weakly contractible topoi (Def. 3.2.1) and announce its various good properties (Prop. 3.2.3, 3.3.3). To claim weakly contractibility for the pro-étale topology one needs to prove Lemma 2.4.9.

**4*. On replete topoi.** *This is based on the material of Section 3.*

In this talk the notions of a locally weakly contractible topos, a replete topos, and the left-completion of the derived category of sheaves of abelian groups should be introduced. The complete modules w.r.to a ring object with an ideal behave much better in replete topoi, than in general, and this should be explained.

Repleteness of a topos is the property that allows to work with sequential limits without caring about non-exactness of limits, i.e. $\lim_{n\in\mathbb{N}} F_n = \mathrm{R}\lim_{n\in\mathbb{N}} F_n$ if transition maps $F_{n+1} \to F_n$ are surjective (Prop. 3.1.10). The left-completion of a derived category of a topos (i.e. of abelian sheaves in it) is more or less adding limits of systems of projective complexes which are non-uniformly bounded below (Def. 3.3.1). The replete topoi are already left-complete, i.e. its left-completion of the derived category is canonically identified with itself (Prop. 3.3.3). This allows to use Postnikov towers of unbounded complexes and prove cohomological descent in replete topoi (Lemma 3.3.2, Prop. 3.3.6). Finally, the topos $\mathcal{X}$ is locally weakly contractible if it has enough objects whose covers are always split (Def. 3.2.1), and if so, the topos is replete and $D(\mathcal{X})$ is compactly generated (Prop. 3.2.3).

In this talk also the notions of complete modules in the replete topoi (w.r.to a ring object in it with an ideal) are to be introduced. The main facts about them are Propositions 3.4.2 and 3.4.4 whose proof, perhaps, may be omitted, however one might also mention Lemmata 3.4.9 and 3.4.12 which will be used in the proof of Prop. 3.5.1. One applies then this study to the particular case when the ring object comes from a noetherian ring $R$ with an ideal $\mathfrak{m}$ in it. It turns out that $R$ is not derived $\mathfrak{m}$-complete (see Def. 3.5.2 and discussion before it), but the complete complexes are determined by their reductions modulo $\mathfrak{m}^n$, and, in particular, the derived categories of complete modules over $R$ and over the completion of $R$ are the same (Lemma 3.5.7, 3.5.6). It would be best to illustrate the statements involved here with their future use of relating pro-étale constructible sheaves to Ekedahl's construction of $\ell$-adic constructible sheaves (see Section 5.5).

**5. The pro-étale topology.** *This is based on the material of Section 4.*

In this talk the pro-étale topos is introduced, its weakly contractibility should be proved, examples of 'constant' sheaves arising from topological spaces should be presented. The pro-étale site of a profinite group might be a nice non-trivial example as well as providing evidence that the pro-étale topology generalises more classical notions (in this case, continuous cohomology).

In order to do this one has to introduce affine pro-étale schemes and prove that they generate the topos $Shv(X_{pro\acute{e}t})$ in a precise sense. Definitions 4.1.1, 4.2.1, Examples 4.1.9, 4.1.10, 4.2.11, Lemma 4.2.12 might be of use. If time permits, one could explain how one can construct the pro-finite étale homotopy type of a scheme using the pro-étale topology (Remark 4.2.9). The section 4.3 covers the new version of continuous cohomology for profinite groups, and, perhaps, one might announce main advantages of this approach without going into details as this will not be used in subsequent talks.

**6*. Relations with the étale topology.** *This is based on the material of Section 5.*

There is a morphism of topoi $\nu : Shv(X_{pro\acute{e}t}) \to Shv(X_{\acute{e}t})$, and the goal of this talk is to explain its properties, and the properties of the corresponding functor between derived categories of abelian sheaves $D(X_\star) := D(X_\star, \mathbb{Z})$, ${}_{\star=\acute{e}t}$ or ${}_{\star=pro\acute{e}t}$.

In particular, one shows that $\nu^*$ is fully faithful, its image can be described (Lemma 5.1.2), it induces an isomoprhism between the categories of locally free modules of finite rank over a discrete ring $Loc_{X_{\acute{e}t}}(R) \cong^{\nu^*} Loc_{X_{pro\acute{e}t}}(R)$ (Cor. 5.1.5), and on the abelian level its image is a Serre subcategory (Cor. 5.1.9). On the level of derived categories $D(X_{pro\acute{e}t})$ has a semi-orthogonal decomposition with $D(X_{\acute{e}t})$ being one of the orthogonals (Prop. 5.2.6), the left completion of $D(X_{\acute{e}t})$ is a full subcategory of $D(X_{pro\acute{e}t})$ with a clear description (Prop. 5.3.2). The functorial properties of $Shv(X_{pro\acute{e}t})$ should be explained, paricularly base change (Lemma 5.4.3).

Using Section 5.5. one can compare Ekedahl's construction of the derived category of what should be $R$-modules on the étale site where $R$ is a complete topological ring in $\mathfrak{m}$-adic topology with the corresponding category (defined more naively) in the pro-étale sense. If time is left, one can recall étale continuous cohomology (Def. 5.6.1) and prove the comparison with the pro-étale cohomology (Prop. 5.6.2).

**7**. Derived categories of constructible sheaves.** *This is based on the material of Section 6.*

In this series of talks the categories of constructible sheaves in both the étale and the pro-étale topology are constructed and compared, as well as the 6-functors formalism is obtained in the pro-étale world. The section is very long, and many details have to be skipped. The freedom of their selection is laid down on the speaker. Perhaps, it would be best to point out all the places where statements differ from the étale topology (as is done in the paper). We now give a brief summary of the section.

Constructible sheaves in topology of manifolds are sheaves which are locally constant on some sufficiently good stratifications. This is the way one defines the notion of constructibility for torsion sheaves in the étale topology. One of the goals of the talk is to prove that this notion works well in the pro-étale topology of topologically noetherian schemes for $E$- and $\mathcal{O}_E$-sheaves where $E$ is an algebraic extension of $\mathbb{Q}_\ell$.

In order to study constructible sheaves one starts with pullbacks and pushforwards along closed immersions in Sections 6.1, 6.2. The material there is stated for pro-étale topology, however it is explained which results fail in the étale case (mainly, the fact that the pullback along the closed immersion preserves limits, Example 6.1.6). Section 6.3 covers the material of constructible complexes in étale topology, however, defining the notion with a novelty: locally constructible sheaves have to be constant with perfect values (as objects of the derived category of complexes of modules over some discrete ring $F$). In Section 6.4 this is used to prove that under uniform cohomological finiteness of schemes the subcategory of compact objects in $D(X_{\acute{e}t}, F)$ is precisely the derived category of constructible sheaves.

Using the notions of complete modules in replete topoi (see Talk 4), the constructible complexes in the pro-étale topology are defined (Def. 6.5.1). It is a tensor category (Lemma 6.5.5) which is functorial with respect to pullbacks and !-pushforwards along locally closed subsets (Lemma 6.5.8). For general pullbacks and pushforwards of a general morphisms see Lemmata 6.5.9, 6.5.11. In Section 6.6 the announced result on the characterisation of constructible sheaves as locally constant is proved (Proposition 6.6.11).

Section 6.7 is the study of the 6-functor formalism for constructible pro-étale sheaves.

In Section 6.8 the case of $E$- and $\mathcal{O}_E$-constructible sheaves is treated where $E$ is an algebraic extension of $\mathbb{Q}_\ell$. Lisse sheaves are compared with those in étale topology, and all the properties are pretty much standard, except that one should bear in mind that lisse $E$-sheaves are not just $\mathcal{O}_E$ sheaves up to inverting $\ell$ (see Prop. 6.8.4 (4), (6)), however, this is the case for the derived categories (Prop. 6.8.14, (2)).

**8*. The infinite Galois theory and the pro-étale fundamental group.** *This is based on the material of Section 7.*

In this talk the infinite Galois theory should be introduced and applied to the category of local systems in the pro-étale topology to yield the pro-étale fundamental group. Comparison of this group with the étale and prodiscrete fundamental groups then explained.

Let $G$ be a topological group. If $G$ is isomorphic to the automorphism group of a fibre functor $F$ from the category of continuous representations of $G$ in discrete sets topologized by the compact-open topology, it is called a *Noohi* group (Def. 7.1.1). It turns out that if $G$ has a basis of open neighbourhoods of 1 being open subgroups, then $G$ is Noohi iff it is complete[3] (Prop. 7.1.5). An infinite Galois category is the pair $(\mathcal{C}, F)$ where $F$ is a fibre functor from $\mathcal{C}$ to sets and some conditions are imposed so that there is hope for it to be equivalent to the pair $(\mathrm{Aut}(F)-Set, forget)$. Unfortunately, it is not clear what conditions are neccessary and sufficient for it to happen, however, if $\mathrm{Aut}(F)$ acts transitively on each set $F(X)$, then $\mathrm{Aut}(F)$ is a Noohi group, and the category $\mathcal{C}$ is equivalent to its representations (Th. 7.2.5). The infinite Galois category in this situation is called *tame* (Def. 7.2.4).

There are three candidates for the category of local systems of sets in the pro-étale topology: locally constant sheaves, locally weakly constant sheaves and étale schemes satisfying the valuative criterion of properness (Def. 7.3.1). For different definitions different properties are more easily proven, but it turns out that all three categories coincide (Lemma 7.3.9), denote it by $\mathrm{Loc}_X$. The geometric point of a locally topologically noetherian scheme $X$ defines a fiber functor $\mathrm{ev}_x$ on the category $\mathrm{Loc}_X$ which makes it an infinite Galois category which is tame (Lemma 7.4.1), and thus allows to apply the machinery above. The pro-étale fundamental group $\pi_1^{proét}(X, x) = \mathrm{Aut}(\mathrm{ev}_x)$ has $\pi_1^{ét}(X, x)$ as its profinite completion (Lemma 7.4.3), and its prodiscrete completion is $\pi_1^{SGA3}(X, x)$ (Lemma 7.4.7). Example 7.4.9 shows that $\pi_1^{proét}(X, x)$ has more representations than $\pi_1^{ét}(X, x)$ for $X$ a non-normal projective curve of genus $\geq 1$, and Lemma 7.4.10 shows that for unibranch varieties this does not happen, i.e. $\pi_1^{proét}(X, x) = \pi_1^{ét}(X, x)$.

# References


[BhaSch] Bhatt, B., Scholze, P., *The pro-étale topology for schemes*, Astérisque No. 369 (2015), 99–201.

[Eke90] Ekedahl T., *On the adic formalism*, In The Grothendieck Festschrift, Vol. II, volume 87 of Progr. Math., Birkhäuser Boston, Boston, MA, (1990): 197-218.

[Del80] Deligne P., *La conjecture de Weil. II*, Inst. Hautes Études Sci. Publ. Math., (52):137–252, 1980.

[Hoc69] Hochster, M., *Prime ideal structure in commutative rings*, Transactions of the AMS 142 (1969): 43-60.

[Jan88] Jannsen U., *Continuous étale cohomology*, Math. Ann., 280(2):207–245, 1988.

[SGA3] *Schémas en groupes. Tome III, Structure des schémas en groupes réductifs: Séminaire du Bois Marie, 1962/64, SGA 3*, (1970).


[3]There exist different notions of group completion, this one is according to two-sided uniformity. Perhaps, this should be explained in more detail: there is something about it in [Bourbaki N., *Topologie generale, §3*], but probably this is not the best reference.