DELIGNE'S "LE GROUPE FONDAMENTAL DE LA DROITE PROJECTIVE MOINS TROIS POINTS" - WS 2016/17

Introduction

This seminar is devoted to go through (part of) the Deligne's paper "Le groupe fondamental de la droite projective moins trois points", in which he starts proposing a framework in which the motivic fundamental group could be defined. The idea behind it is that exactly as all cohomology theories with their various features (included gradings, weights and so on) should be in the motivic philosophy a shadow (i.e. the image through the so called realization functors) of a more generic object (i.e. a motive), similarly all the group actions on these comologies (e.g. the Galois action) should be shadows of the action of a more generic group action (i.e. of the motivic fundamental group). More specifically, motives should form an abelian category and should satisfy all properties that are known for cohomology theories, included Künneth formula and Poincaré duality that should translate (and actually induce these properties via the realization functor) in a tensor structure and a dualizing functor on the category of motives. In particular, in characteristic zero, this category should have the structure of a $\mathbb{Q}$-linear Tannaka category and the choice of a realization functor gives a fiber functor inducing an equivalence of the category of motives with the representations of an affine $\mathbb{Q}$-group scheme, the motivic fundamental group (note that although different fiber functors may give rise to non isomorphic Tannaka dual groups, they all become isomorphic on $\overline{\mathbb{Q}}$).

This motivic fundamental group, however, can only be constructed here in its prounipotent part. Worse than this, in this paper there is no definition of what a motive should be, more precisely it should be some object in the category of realization systems (see Talk 3) having "geometric origin", but this last property is not defined. In a later paper with Goncharov (see [Del05]) this will be solved, using the category of so called unramified mixed Tate motives, at least over simple varieties like $\mathbb{G}_m$ and $\mathbb{P}^1 - \{0, \infty, \mu_N\}$, where μ_N are the N-th roots of unity.

Still, what makes [Del89] very interesting is the recollection of all other theories that can be attached to a smooth variety and the relations between them, and prepares a machinery ready to apply to any sensible definition of motive.

Structure of the talks. The article consists of more than 200 pages, thus forcing us to skip some parts of it. In particular we will not go specifically through chapters recalling basic material (for example Ch.4 and Ch.9) and those collecting examples (for example Ch.2 and Ch.14). You are, though, strongly encouraged to use anything you need from these chapter, and use any example that you may find enlightening for the understanding of your talk. Note that in the article what is here called *Chapter* is referred as *Paragraphe*, everything refers to [Del89] unless otherwise specified.

Talks

Talk 1 - Overview (Giulia). Present an overview of the paper and cover very concisely Ch. 10, to try to understand what should the motivic fundamental group try to generalize, namely recall what are the topological, the étale and the algebraic fundamental groups.

Talk 2 - Chapter 6. We cover this chapter first as it contains all the Tannaka business we'll deal with in later chapters. Before going through the material of this chapter, please recall in a concise and friendly manner (i.e. without drawing the hexagon axiom on the blackboard for example) what is a Tannakian category and the Tannaka duality (see for example [DM82, Thm. 2.11]).

Talk 3 - Chapter 1. Here is given the definition of realization system (Def. 1.9) and the main theorem that they form a Tannakian category. This chapter is mainly definitions and we need the one of realization systems that are lisse (Def. 1.17) or have integral coefficients (Def. 1.23), it would be nice if you can avoid just a list of definitions and help understand why one is interested in such properties (the examples in Ch. 2 can be of great help in this).

Talk 4 - Chapter 5. In this chapter are described the notion of ring and module in a Tannaka category and how to "do algebraic geometry" in it. There are quite some examples in the chapter that should allow this talk not to degenerate into a series of dry definitions. In any case next talk will translate all this in our setting of realization system.

Talk 5 - Chapter 7. Here all the generic nonsense of Ch. 5 is translated into the world of lisse realization systems over some open of $\operatorname{Spec}\mathbb{Z}$. One could take these translations as definitions but this way it would be hopefully more clear why things are defined the way they are.

Talk 6 - Chapter 11. In this chapter and in the following some properties of the various fundamental groups are discussed, properties that we expect the motivic fundamental group to mirror. In particular here is described the crystalline action on the group associated to nilpotent flat connections (see 10.30 (ii) for the definition of it).

Talk 7 - Chapter 12. On the same line of the previous talk, here we see how to get a weight filtration on the (Lie algebra of) the group associated to nilpotent flat connections.

Talk 8 - Chapter 13. In this chapter the previous two are used to define the motivic fundamental group. Note that one should think about it only as the unipotent quotient of the "true" motivic fundamental group and is defined only for smooth varieties admitting a proper and smooth compactification $\overline{X}$ over (some open of) the ring of integers of a number field, such that $H^1(\overline{X}, \mathscr{O}_{\overline{X}}) = 0$, for example rational varieties. In Ch. 14 you can find the example of $\mathbb{G}_m$, which may clarify some aspects of the construction.

Talk 9 - Chapter 15, part 1. In order to make sense of taking "paths centered near 0 and passing around 0 and 1" we must make precise sense of "near 0", hence this chapters reviews how one can choose as a base point for the fundamental group a tangent vector at a point at the border, rather than a point on the topological space/manifold/scheme itself. As the chapter is quite long, it is convenient that in this first part one only present the classical and profinite theories.

Talk 10 - Chapter 15, part 2. Continuing along the lines of the previous talk, present the algebraic theory of base points at infinity and then how the motivic theory builds from the previous ones.

Talk 11 - Chapter 16. Here finally the case of $\mathbb{P}^1 - \{0, 1, \infty\}$ is treated. This chapter is the more lengthy and it may be a good idea to split the talk in two.

References

[Del05] Goncharov Alexander B. Deligne Pierre. "Groupes fondamentaux motiviques de Tate mixte". fre. In: *Annales scientifiques de l'École Normale Supérieure* 38.1 (2005), pp. 1–56. URL: http://eudml.org/doc/82653.

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[DM82] P. Deligne and J. Milne. Tannakian Categories*, in Hodge cycles, motives, and Shimura varieties*. Vol. 900. Lecture Notes in Mathematics. Available at http://www.jmilne.org/math/xnotes/tc.html. Springer–Verlag, Berlin–New York, 1982, pp. ii+414.