TEMKIN'S $char(X)$-DESINGULARIZATION - SS 2016

Introduction

In [Gro65][Sec. 7.9] a resolution of singularities of a reduced scheme X is defined as a proper birational morphism $f : X' \to X$, where X' is regular. The relation between admitting resolution of singularities and being a quasi excellent scheme are moreover inquired proving on one hand that the first property implies the latter, and conjecturing on the other that the reverse implication should hold, namely, that every quasi excellent scheme should admit resolution of singularities. In characteristic zero this follows from the celebrated work of Hironaka, while in positive characteristic the best result until now is for schemes of dimension at most 3 (see [CP09]). Nevertheless, de Jong in [dJ96] provided a weaker resolution result, based on alterations, namely one can resolve X by a proper morphism $f : X' \to X$ where some (finite) field extension $K(X')/K(X)$ is allowed. Even if this result does not provide a positive answer to the resolution of singularities, it still has been proven to be extremely powerful in many applications. Building on this result, some improvements were made, the most important for our purposes being the refining theorem of Gabber [IT14b][Thm. 2.1 and 2.4] stating that if l is a prime number different from the residue characteristic at each point of X, then the alteration can be chosen in such a way that $[K(X') : K(X)]$ is prime to l. While working out the proof of Gabber theorem, Temkin found a way to give a sharper result, namely in [Tem15][Thm. 4.3.1] he has proved that if $\mathrm{char}(X)$ denotes the set of all residual characteristics of X, then the alteration can be taken so that for every $l \in \mathrm{char}(X)$ the order $[K(X') : K(X)]$ is prime to l, every time X is of finite type over a 3-dimensional scheme (or any category of schemes, closed under alterations, admitting resolution of singularities).

There are two main new ingredients in the proof of Temkin's result. The first one is *tame distillation*, that can be thought as follows: every finite extension of fields L/K can be split in a purely inseparable extension L/L' followed by a separable one L'/K, while tame distillation's goal is to split the extension in the other way around (that is a separable extension followed by an inseparable one), up to replacing L by a finite extension. This is proven to hold also for valued fields (of height one) with the notion of separable and inseparable replaced by tame and wild, respectively, by extending a theorem of Pank to non henselian local rings. The second ingredient, in order to globalize the tame distillation construction to a whole set of valuations, is to use (relative) Riemann-Zariski spaces. Aside tame distillation and Riemann-Zariski spaces, the proof of the main result of Temkin follows quite strictly the one of [IT14b][Thm. 3.5], so let us give an idea of the proof of the latter as well: let X be of finite type over some 3-dimensional scheme S, then similarly to de Jong the proof goes by induction on the relative dimension of X over S. The focus of the proof is hence on the relative curve case: in [dJ96] the goal is to reduce to a situation where the singularities are of a specific type and then resolve them explicitly. In [IT14b], the reduction is to the case where the singularities are of the kind X'/G, where G acts tamely on X', and these can be resolved thanks to the central theorem of Gabber [IT14b][Thm. 1.1].

What is lost in Temkin's result is the control of the automorphism group of the alteration over the starting scheme X. Namely, de Jong proved in addition that one can perform the construction of the alteration $X' \to X$ in such a way that $X'/G \to X$ is radicial, where $G = \mathrm{Aut}_X(X')$. In Temkin's theorem this is not the case anymore: the use of Gabber theorem at every step of the induction on the dimension is paid by losing the grasp on $\mathrm{Aut}(K(X')/K(X))$.

Structure of the talks. Even though Gabber's theorem [IT14b][Thm. 1.1] is the main tool in order to get rid of the prime to $\mathrm{char}(X)$ part of the alteration, due the length and technicality of its proof, we will take it as a black box for the whole seminar and prove it in the last talk. The first eight talks are intended to provide all (remaining) material that is needed as prerequisite: log geometry will be needed in order to resolve the last very mild singularities that are left after the use of Gabber's theorem and three talks revolve around valuations and are devoted to the new ingredients in Temkin's proof.

After establishing all preliminary results, the last talks are devoted to prove the Key Theorem, which in turn establish the induction step of the Main Theorem.

Bibliographical note: The references [IT14a], [Ill] and [IT14b]can be found at `http://www.cmls.polytechnique.fr/perso/orgogozo/travaux_de_Gabber/GTG/GTG.pdf`.

Talks

Talk 1 - Overview (Giulia). Describe Grothendieck conjecture on resolution of singularities and describe the state of the art previous Temkin's results. Namely, explain the result of de Jong and the general strategy of its proof, give an overview of Temkin's proof and explain the fundamental differences between the two results.

Talk 2 - Log geometry. We need to introduce the notions that will be used in next talk in order to obtain the desingularization functor. Define monoids, sharp, integral and fine monoids and the saturation of a monoid. Define what is a prelog and log structure, define the notion of log smooth, log regular and fs log schemes, define strict morphisms. Give an example using toric varieties (and quickly recall the definition of toric variety) and explain how a normal strict divisor over a regular scheme naturally induces a log structure. Define what is the spectrum of a monoid. A good reference is [Ogu06]. Explain what are log and saturated blow ups and clarify [IT14a][Lemma 3.4.6] and [IT14a][Lemma 3.1.8].

Talk 3 - The monoidal desingularization functor. The goal of this talk is to prove that any functorial desingularization in characteristic 0 (as the one provided in [IT14b][Thm. 2.3.10]) induces a desingularization functor for log regular log schemes, that is to prove the existence of the desingularization functor of [IT14b][Thm. 3.4.9].

This talk should cover as much of [IT14a][Sec. 3] regarding resolution of log regular log schemes. The philosophy of the procedure is explained in [IT14a][Ss. 3.1] (and it is hidden after [IT14a][2.3.11] but $\tilde{\mathcal{F}}$ is simply the normalized resolution functor). As the desingularization procedure is functorial, one possibility for this talk is to stick to the affine case (see the beginning of [IT14b][p. 80] and [IT14a][Rmk. 2.3.5 (ii)]): this allows to avoid the definition of monoschemes and fans. In this case one need to prove [IT14a][Thm. 3.2.20] only for spectra of fine monoids and then use the sharpening functor to obtain (locally) a monodial desingularization functor (putting together [IT14a][Lemma 3.3.4], [IT14a][Rmk. 3.3.5], [IT14a][Rmk. 3.3.17] and [IT14a][Thm. 3.3.16]). After having obtained the monoidal desingularization functor, prove [IT14a][Thm. 3.4.9] in order to get functorial resolution of log regula log schemes.

Talk 4 - General Prerequisites. State and prove [Ill][Prop. 2.1]. State Raynaud-Gruson flattening by blow-ups theorem (we only need the simpler version as in [GW10][Thm. 14.141]). Define the relative version of semistable multipuncutred curves (see [Tem10] for the definitions) and state Temkin's semi-stable modification theorem [Tem10][Thm. 1.5,1.6], unfortunately proof are too long to be given. Give all missing definitions and state Gabber's theorem ([IT14b][Thm. 1.1]), which we will take as a black box.

Talk 5 - Valuations and Pank's theorem. The goal of this talk is to prove [Tem15][Cor. 2.5.6]. Recall what a valuation of rank 1, and what tame and wild extensions are (you can refer to [Tem15][Ss. 2.1]).State Pank's theorem ([Tem15][Thm. 2.4.9]), go through [Tem15][Ss. 2.5], especially state and prove [Tem15][Thm. 2.5.5] and [Tem15][Cor. 2.5.6].

Talk 6 - The space of valuations. There are two part of this talk: the goal of the first one is to prove [Tem15][Thm. 2.6.6] (and of the material in [Tem15][Ss. 2.6]), the one of the second is to go through [Tem15][Ss. 3.1], in particular explain how to put a topology on $Val_K(X)$ ([Tem15][Lemma 3.1.3]), the functoriality of its construction, state and prove [Tem15][Lemma 3.1.6]. Define the tame and unramified locus and prove that they are open ([Tem15][Lemma 3.2.8]).

Talk 7 - Tame distillation. The goal of this talk is to prove [Tem15][Thm. 3.3.6]. In order to do so, you need to state and prove [Tem15][Thm. 3.2.10] (you can insert [Tem15][Lemma 3.2.4] directly in the proof as it follows straight forward fro the results of the previous talk) and more or less all the material in [Tem15][Ss. 3.3].

Talk 8 - Key Theorem. State and prove the Key Theorem [Tem15][Thm. 4.2.1]. We will prove the theorem only in the separated case, so you can safely replace "pseudo-projective" with "quasi-projective" (and as a side remark state that the theorem holds also for non-separated morphisms if you wish).

Talk 9 - Main theorem. State and prove [IT14b][Lemma 3.5.2]. As in the previous talk, assume everything to be separated and replace "pseudo-projective" with "quasi-projective". The proof of the main theorem strictly follows the proof of [IT14b][Thm. 3.5]. Adapt the proof to our situation, going through Step 1-5.

Talk 10 - Proof, finished. Finish the proof of the main theorem following [IT14b][Thm. 3.5], using the Key Theorem [Tem15][Thm. 4.2.1]. Explain how the absolute result follows by proving [Tem15][Thm.1.2.4,1.2.6] (you can use for example [Tem15][Par. 4.3.2]).

Talk 11 - Gabber's theorem. Theorem [IT14b][Thm. 1.1] is a relative version of [IT14a][Thm. 1.1], and the latter follows from the first checking carefully at every step that the constructions preserve the properties we need. We will content to sketch a proof of the latter: give an overview of the proof of [IT14a][Thm. 1.1], you can find the strategy of the proof in [IT14a][Ss. 4.1] and [IT14a][Ss. 5.2], while the proof itself occupies §4 and §5 of [IT14a]. If time permits, stress which steps are more delicate to transpose in the relative setting and what should be checked.

References

[CP09] Vincent Cossart and Olivier Piltant. "Resolution of singularities of threefolds in positive characteristic. II". In: *J. Algebra* 321.7 (2009), pp. 1836–1976.

[dJ96] A.J. de Jong. "Smoothness, semi-stability and alterations". In: *Publ. Math. IHES* 83 (1996), pp. 51–93.

[Gro65] A. Grothendieck. "Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. II". In: *Inst. Hautes Études Sci. Publ. Math.* 24 (1965), p. 231.

[GW10] Ulrich Görtz and Torsten Wedhorn. *Algebraic geometry I*. Advanced Lectures in Mathematics. Schemes with examples and exercises. Vieweg + Teubner, Wiesbaden, 2010, pp. viii+615.

[Ill] Luc Illusie. "Exposé IX. Uniformisation locale première à ℓ". In: *Astérisque* 363-364 (). Travaux de Gabber sur l'uniformisation locale et la cohomologie étale des schémas quasi-excellents.

[IT14a] Luc Illusie and Michael Temkin. "Exposé VIII. Gabber's modification theorem (absolute case)". In: *Astérisque* 363-364 (2014). Travaux de Gabber sur l'uniformisation locale et la cohomologie étale des schémas quasi-excellents, pp. 103–160.

[IT14b] Luc Illusie and Michael Temkin. "Exposé X. Gabber's modification theorem (log smooth case)". In: *Astérisque* 363-364 (2014). Travaux de Gabber sur l'uniformisation locale et la cohomologie étale des schémas quasi-excellents, pp. 167–212.

[Ogu06] Arthur Ogus. "Lectures on logarithmic geometry." `https://math.berkeley.edu/~ogus/preprints/log_book/logbook.pdf`. 2006.

[Tem10] Michael Temkin. "Stable modification of relative curves". In: *J. Algebraic Geom.* 19.4 (2010), pp. 603–677.

[Tem15] Michael Temkin. "Tame distillation and desingularization by p-alterations." Aug. 2015.