Relative rational K-theory and cyclic homology

Introduction

Topologicial K-theory was the first example of an extra-ordinary cohomology theory and furthermore right from the beginning it was extra-ordinary in the sense, that it could not be computed as the homology of an associated chain complex like in singular cohomology or de Rham cohomology. Instead one has to compute homotopy groups of an associated space. Same holds for the even more complicated algebraic K-theory (of a ring), which have been introduced by Quillen in the beginning of the 70ties. Apart from several examples as for finite fields, real or complex numbers or number fields, there was no general approach to the computation of algebraic K-theory. Even today one conjects but still does not know, if $K_{4n}(\mathbb{Z}) = 0$, for all $n \geq 2$.

Like in ordinary cohomology theories it is easier to compute cohomology with coefficients in a field instead of the integers. This leads to rational homotopy theory, which extensively was studied by Quillen in [Qui69]. By definition of Quillen's +-construction, the corresponding rational K-theory can be described as

$$K_*(A;\mathbb{Q}) = K_*(A) \otimes \mathbb{Q} = \operatorname{Prim} H_*(\mathrm{GL}(A), \mathbb{Q}), \qquad \text{for rings } A, \tag{1.1}$$

where $\operatorname{Prim}(C) = \{c \in C;\ \delta_C(c) = c \otimes 1 + 1 \otimes c\}$ is the set of primitive elements of a coalgebra C with comultiplication δ_C. The coalgebra structure on the homology is induced by the diagonal map. As the object on the right of 1.1 is still very hard to compute, driven by the hope that there could be a link between the group homology of $\mathrm{GL}(A)$ and the Lie algebra homology of $\mathfrak{gl}(A)$, one concentrated on computing the latter one. In 1984 Loday, Quillen ([LQ84]) and independently Tsygin ([Tsy83]) showed, that

$$H^\lambda_{*-1}(A) = \operatorname{Prim} H_*(\mathfrak{gl}(A)), \qquad \text{for algebras } A,$$

where $H^\lambda_*(A)$ is defined as the homology of the Connes complex, which is the Hochschild complex of A, given in degree n as the n-fold tensor product of A, modulo the cyclic group action permuting the tensor factors.

There is in fact a link between rational group and Lie algebra homology, provided the considerable group is nilpotent. Clearly $\mathrm{GL}(A)$ is far from being nilpotent, but there is a nice description of relative K-theory as the $+$-construction of the relative Volodin space $X(A, I)$, for some ideal $I \lhd A$. More precisely $X(A, I)$ is obtained from classifying spaces of subgroups of $\mathrm{GL}(A)$, which are nilpotent, if I is a nilpotent ideal. Using this observation, two years later in 1986, Goodwillie managed to construct an isomorphism

$$K_*(A, I) \otimes \mathbb{Q} \cong H^{\lambda}_{*-1}(A, I), \qquad \text{for nilpotent ideals } I \lhd A. \tag{1.2}$$

For rational A there are isomorphisms

$$H^{\lambda}_{*-1}(A, I) \xleftarrow{\sim} HC_{*-1}(A, I) \xrightarrow{B} HC^-_*(A, I),$$

where HC is the cyclic and HC^- is the negative cyclic homology of A, defined by a double complex again involving the Hochschild complex and its cyclic action. Moreover Goodwillie found out, that the isomorphism 1.2 is induced by the relative, negative Chern character

$$\mathrm{ch}^- : K_*(A, I) \longrightarrow HC^-_*(A, I).$$

This classical theorem of Goodwillie is a cornerstone in the computation of K-theory and furthermore inspired Madsen, Hsiang and Boekstedt for their construction of the cyclotomic trace map $K_*(A) \xrightarrow{\mathrm{trc}} TC_*(A)$ into topological cyclic homology [BkHM93]. Similarly to the rational result, McCarthy proved in [McC97], that

$$\mathrm{trc} : K_*(A, I; \mathbb{Z}_p) \xrightarrow{\sim} TC_*(A, I), \qquad \text{for nilpotent } I \lhd A.$$

Here the left object means p-completed K-theory and so this map measures the p-part of K-theory. Whereas rational homotopy groups can be computed as in the category of chain complexes, for the p-completed homotopy groups one needs spectra. Cyclic homology can be thought of taking the homotopy invariants of the Hochschild complex under the cyclic group action. Geometrically this action corresponds to an action of the circle $S^1 = \mathbb{R}/\mathbb{Z}$. Heuristically topological cyclic homology transfers this construction from the category of chain complexes into the category of spectra, but the construction of TC is much more involved.

There is another result of Beilinson, which is on the arXiv since last Decembre (cf. [Bei13]) and uses the classical construction of Goodwillie to obtain a quasi-isogeny between the p-completed relative K-theory and p-completed (usual) cyclic homology.

Aim

The aim of the Oberseminar is to understand the classical result of Goodwillie in detail and therefore also to provide an introduction for the workshop on Beilinson's result, which takes place from Sept. 29th until Octobre 2nd in Kleinwalsertal. As the result is rather old, the topic benefits from a good presentation in literature. Our main guide will be the book of Loday [Lod98].

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Talks

Our main source is [Lod98] and the numbers refer to the particular section, if no other reference is given.

1. Introduction.

2. Define the algebraic K-theory via the $+$-construction 11.2.2-2.5, verify its H-space-structure 11.2.9, shortly introduce rational homotopy theory (see also [Qui69]), state that $\pi_*(X, \mathbb{Q}) = \pi_*(X) \otimes \mathbb{Q} \cong \operatorname{Prim} H_*(X, \mathbb{Q})$ (without exact proof) and derive rational K-theory as $K_*(A; \mathbb{Q}) \cong \operatorname{Prim} H_*(\mathrm{GL}(A), \mathbb{Q})$ 11.2.12. Also define relative K-theory.

3. Introduce the Volodin space $X(A)$ 11.2.13, sketch, that $X(A)$ is acyclic 11.2.14 (see [Sus81] for that), and sketch its connection to algebraic K-theory 11.2.15. Introduce the rel. Volodin-construction $X(A, I)$ 11.3.3 and present its connection to relative K-theory $K_*(A, I) \cong \pi_*(X(A, I)^+)$ 11.3.4-11.3.9 Derive from that $K_*(A, I; \mathbb{Q}) \cong \operatorname{Prim} H_*(X(A, I), \mathbb{Q})$.

4. Introduce the additive Volodin-construction 11.3.10. Sketch [Qui69] Appendix B.2.1 and use it together with the proof of B.4.5 loc. cit. to show that $H_*(\operatorname{Prim} H) = \operatorname{Prim} H_*(H)$, for a rational DGH H. Use the spectral sequence 11.3.10.1 (without proving its existence) to derive the long exact sequence 11.3.11.

5. Show $H_*(N, \mathbb{Q}) = H_*(\mathfrak{n}, \mathbb{Q})$, for nilpotent groups N and its associated rational Lie algebra $\mathfrak{n}$ 11.3.14, see also [Pic78]. Explain why $\operatorname{Prim} H_*(X(A, I), \mathbb{Q}) \cong \operatorname{Prim} H_*(x(A, I))$, for rational A with nilpotent $I \lhd A$ (Theorem 11.3.15).

6. Introduce Hochschild-homology $HH_*(A)$ 1.1.1, 1.1.3, sketch its Morita invariance 1.2.0-1.2.4 and define the Connes complex and $H^\lambda(A)$ 2.1.5. Introduce cyclic homology: Define the bar complex 1.1.11, the cyclic homology groups $HC_*(A)$ 2.1.0-2.1.4 and explain the isomorphism $HC_*(A) \xrightarrow{\sim} H_*^\lambda(A)$ for rational A 2.1.5. Also present the relative versions. Explain why Morita invariance also holds for cyclic homology.

7. Define negative and periodic cyclic homology $HC_*^-(A)$ 5.1.1-5.1.3 and the Connes exact sequence 5.1.5.

   $$\dots \xrightarrow{B} HC_n^-(A) \xrightarrow{I} HC_n^{per}(A) \xrightarrow{S} HC_n(A) \xrightarrow{B} HC_{n-1}^-(A) \longrightarrow \dots$$

   Explain how derivations operate on cyclic homology 4.1.4, 4.1.5, 4.1.7-4.1.10 and prove that rationally $B : HC_n(A, I) \xrightarrow{\sim} HC_{n-1}^-(A, I)$, for nilpotent $I \lhd A$ 4.1.14, 4.1.15.

8. Theorem of Loday, Quillen, Tsygan I: Recall the homology of Lie algebras via the CE-complex 10.1.3, present the adjoint action 10.1.7 and prove the proposition about coinvariants 10.1.8. Apply this for $\mathfrak{gl}_r A$: Compute the $\mathrm{GL}_r(k)$-invariants of $\mathfrak{gl}_r(A)^{\otimes n}$ 9.1.4, introduce the trace map T and its dual T^* 9.2.1, 9.2.3-9.2.5, and explain how it induces an isomorphism $(\Lambda^n \mathfrak{gl}(A))_{\mathfrak{gl}(k)} \xrightarrow{\sim} (k[S_n] \otimes A^{\otimes n})_{S_n}$ 10.2.10.2.

9. Theorem of Loday, Quillen, Tsygan II: Verify the Hopf-algebra-structure on $H_*(\mathfrak{gl}A)$ 10.2.3, compute its primitive part 10.2.15-10.2.18 and explain, why $\operatorname{Prim} H_*(\mathfrak{gl}(A)) \cong H_*^\lambda(A)$ 10.2.3 and why this isomorphism is given by θ of 10.2.19. Use the long exact sequence of relative H_*^λ and 11.3.11 of talk 4 to deduce $\operatorname{Prim} H_*(x(A,I)) \cong H_*^\lambda(A,I)$.

10. Briefly introduce cyclic homology of cyclic modules/sets (see 2.5.1, 6.1.1/6.1.2 and 7.1.1) and explain the canonical cyclic structure on the nerve BG of a group G 7.3.3. and 7.4.7. Compute its cyclic homology 7.4.8 as a direct summand of the cylic homology of a group algebra 7.4.4/7.4.5. Define the Dennis trace $H_*(\mathrm{GL}(A), \mathbb{Z}) \longrightarrow HH_*(A)$ and the Chern character $H_*(\mathrm{GL}(A), \mathbb{Z}) \longrightarrow HC_*^-(A)$ 8.4.1-8.4.5 (see also [Ros94] sect. 6.2.11).

11. Define the negative Chern character $\mathrm{ch}^- : K_*(A) \xrightarrow{h} H_*(\mathrm{GL}(A), \mathbb{Z}) \longrightarrow HC_*^-(A)$ (8.4.4, 8.4.5 and 11.4.1/11.4.2) and its relative version 11.4.5-11.4.8. Show 11.4.11 to deduce $\mathrm{ch}^- : K_*(A, I; \mathbb{Q}) \xrightarrow{\sim} HC_*^-(A, I)$, for rational A and nilpotent $I \lhd A$.

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