AG-Seminar on *The Fargues–Fontaine curve*

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> "Si la courbe étrange X existe le mot étrange est faible pour la décrire, il faudrait un terme encore plus fort qu'étrange, je n'ose pas imaginer une telle monstruosité..." [1]

Let E be a local non-archimedean field, i.e., either E is a finite extension of $\mathbb{Q}_p$ or $E = \mathbb{F}_q((\pi))$, where q is a power of p. Let F be a perfectoid field of char p, i.e., a perfect field of char p which is complete with respect to a non-archimedean, non-discrete valuation. Then Fargues and Fontaine have attached to such a pair (E, F) a curve, which has become a fundamental object in arithmetic geometry.
This curve has three incarnations:

(1) The curve $X_{E,F}$ can be defined as a scheme. It is a Noetherian, regular scheme of dimension one (hence a curve). Historically this was the first definition.
(2) There is an adic version $X^{ad}_{E,F}$, which should be thought of as a p-adic Riemann surface.
(3) The curve has an incarnation in the world of diamonds. In this world the curve has a particularly nice and simple description

$$X^{\diamondsuit}_{E,F} := \mathrm{Spd}(F)/\varphi^{\mathbb{Z}} \times \mathrm{Spd}(E),$$

where φ denotes the Frobenius.

In this AG-Seminar we want to study the Fargues–Fontaine curve. We will start with defining the adic curve and study its structure. Then we will define the schematic curve and study vector bundles on it. Our goal is to prove the classification theorem for vector bundles on the curve. We will study this in the world of schemes. There is a GAGA type result, meaning the same classification result is true for the adic curve as well.
There are many applications of this theory. Let us a mention a few of them here: Firstly, there are applications to p-adic Hodge theory. One can reprove the fundamental theorems "weakly admissible implies admissible" and "de Rham implies potentially semistable". One can use the curve to classify p-divisible groups over $\mathcal{O}_C$, for C an algebraically closed and complete extension of $\mathbb{Q}_p$. There are applications to Rapoport–Zink spaces: The curve is used to understand p-adic period domains. The theory of G-bundles on the curve gives a more conceptual understanding of the isomorphism of the Lubin–Tate and the Drinfeld-tower.
The curve is used to get a geometric perspective on non-geometric theories, e.g. one can reinterpret results from local class field theory geometrically. Furthermore the curve can be used to realise the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ as a geometric fundamental group. We will study these two applications in the later talks of the seminar.

[1] Fargues, [C, p.6].

More recently the study of families of Fargues–Fontaine curves has become a hot topic. This is because of Fargues's conjectures. These conjectures formulate a geometric realisation (in one direction, namely Galois to automorphic) of the local Langlands correspondence. At the center of these conjectures is Bun_G, the stack of G-bundles (G reductive over a local field) on a family of Fargues-Fontaine curves.

Literature

The main text on the foundations of the schematical curve is the book by Fargues and Fontaine [FF]. For the adic curve the main references are Fargues's paper [F], the lecture notes [F-CHI] and the book by Kedlaya and Liu [KL]. There exist several introductory articles, in particular the "Durham survey" by Fargues and Fontaine [FF-DUR], Fargues's exposition for the ICM [F-RIO], Colmez's preface to the book [C] and Morrow's Bourbaki survey [M].

Talks

1) Overview and background talk (Date: 26.4., Speaker: Judith Ludwig)
This talk gives an overview of the theory of the Fargues–Fontaine curve. Furthermore some background on perfectoid fields will be explained.

2) Holomorphic functions in the variable π (Date: 3.5.).
References: [FF, 1.3-1.6]
Define the rings $\mathbb{A}, \mathcal{E}, B^{b,+}$ and B^b and introduce the Frobenius action φ, [FF, Section 1.3]. For $\rho \in]0,1[$ introduce the Gaußnorms $|\cdot|_\rho$ on B^b and state their basic properties. Discuss the weak topology on $\mathbb{A}$, cf. [FF, Corollaire 1.4.15]. Introduce the Fréchet algebras B_I associated with an interval $I \subset]0,1[$, [FF, Section 1.6], as well as the rings B^+ and B. Explain how for compact intervals I the rings B_I become Banach-algebras. Discuss example 1.6.3 in [FF]. Define the Newton polygon $Newt(b)$, first for elements $b \in B^b$, then for elements $b \in B_I$. Characterise the invertible elements $b \in B_I^*$ in terms of their Newton polygon [FF, Proposition 1.6.25]. State and sketch the proof of [FF, Théorème 2.5.1], in particular discuss that for a compact intervall $I \subset]0,1[$, the rings B_I are principal ideal domains.

3) The adic space Y^{ad} (Date: 10.5.)
References: [F, Théorème 2.1] and [KL, Theorems 3.7.4 and 5.3.9]
Define $Y^{ad} := Y^{ad}_{E,F} := \mathrm{Spa}(\mathbb{A}, \mathbb{A}) \backslash V(\pi[\varpi_F])$. Explain that in equal characteristic Y^{ad} is the punctured unit disc over $\mathrm{Spa}(F, \mathcal{O}_F)$ (see [F-CHI, Lecture 2]), in particular that it is indeed an adic space. Show that when E is of char 0, Y^{ad} is also an adic space. For that write it as an increasing union of "annuli" $Y_I := \mathrm{Spa}(B_I, B_I^\circ)$ (for an interval $I \subset]0,1[$), and show that the Y_I are affinoid adic spaces, following the references above. Mention the alternative proof provided by [K]. Note that Y^{ad} is not affinoid, explain that the ring $\mathcal{O}_{Y^{ad}}(Y^{ad})$ of "global holomorphic functions" recovers the ring B from the previous talk. Introduce the radius function [CS, Section 3.2] and explain how to reinterpret the Y_I in this context. Finally describe how a point in $|Y^{ad}|$ corresponds to a continuous valuation on B, cf. [F, Prop. 2.6].

4) The adic curve and classical points (Date: 17.5.)
References: [F-CHI, Lecture 4], [F, Section 2.2]
This talk introduces the adic curve. For that first show that the natural Frobenius action φ on Y^{ad} is properly discontinuous [F, p. 10 after Definition 2.5]. Define the adic quasi-compact curve X^{ad} as the quotient of Y^{ad} by the Frobenius $\varphi^{\mathbb{Z}}$. Compare the curve for E of char 0 with the curve for E of char p using tilting, [F, Théorème 2.7] (see also [W, Section 3.2]).
Define primitive elements of $\mathbb{A}$, [FF, Definition 2.2.1] and introduce the subset $|Y^{ad}|^{cl}$ of "classical" points of $|Y^{ad}|$ as the set of so called *primitive elements of degree one* of $\mathbb{A}$. Explain the Weierstrass decomposition [FF, Section 2.4]. (Motivate this key result by comparing it to the equal characteristic case [F-CHI, Example 3.4]). Give the intrinsic definition of $|Y^{ad}|^{cl}$ [FF, Corollaire 2.5.4] (see also [M, Proposition 5.4]).

5) Classical points and untilts (Date: 24.5.)
References: [KL], [FF]
Continue with the study of the "classical" points. Show that $|Y^{ad}|^{cl}$ identifies with the set of equivalence classes of characteristic zero untilts of the perfectoid field F, [KL, Theorem 3.6.5], [FF, Section 2.2]. Describe the completed local ring $\widehat{\mathcal{O}}_{Y^{ad},y}$ at a classical point y, i.e., introduce $B^+_{\mathrm{dR},y}$.

Now we switch from non-archimedean to algebraic geometry.

6) Background on curves (Date: 31.5.)
References: [FF, Sections 5.1, 5.2 and 5.5]
Discuss background on complete curves in the generalized sense as explained in [FF, Section 5.1]. Explain how to construct them from *almost euclidean rings* or from graded algebras [FF, Section 5.2]. Discuss the Harder–Narasimhan formalism [FF, 5.5], [M, Section 3.1] and examples 5.5.2.1 (vector bundles) and 5.5.2.3 (isocrystals) in [FF].

7) The schematic curve I (Date: 7.6.)
References: [FF, Sections 6.5], [F-CHI, Lecture 5] and [M, Section 2.2], [FF-DUR, Section 5]
Define the schematic curve as the scheme $X := X_{E,F} := \mathrm{Proj}(P)$, where $P := \bigoplus_{d\geq 0} B^{\varphi=\pi^d}$. It was discovered by Fargues and Fontaine prior to the adic curve. (There is no schematic version of Y.) Explain that there is a morphism of ringed spaces $X^{ad}_{E,F} \to X_{E,F}$ [F-CHI, Theorem 5.1]. State [FF, Théorème 6.5.2.]. Compare the curve to $\mathbb{P}^1$. Discuss how the curve glues the period rings B_e and B^+_{dR} from p-adic Hodge Theory, see also [M, Section 2.2]. Section 5 of [FF-DUR] might be helpful when preparing this talk.

8) The schematic curve II (Date: 14.6.)
References: [FF, Section 6.2-6.5]
Show that P is graded factorial with irreducible elements of degree 1 [FF, Théorème

6.2.1]. Discuss the fundamental exact sequence, [FF, Théorème 6.4.1]. Sketch the proof of [FF, Théorème 6.5.2.].

The next three talks are devoted to classifying vector bundles on $X = X_{E,F}$, when F is algebraically closed.

9) Classification of Vector bundles - statement (Date: 21.6.)
References: [FF, Sections 5.6.4, 8.2 and 8.6]
For a fixed closed point x of X, explain how to construct vector bundles on X from pairs (M, N), where N is a $\hat{\mathcal{O}}_{X,x}$-modules and M is a certain module of B_e, the ring underlying the affine open $X\backslash\{x\} = \mathrm{Spec}(B_e)$ [FF, Prop. 5.3.2 and Section 8.2.1]. For a rational number $\lambda \in \mathbb{Q}$, introduce the vector bundles $\mathcal{O}(\lambda)$ and their basic properties [FF, Section 5.6.4]. Give a geometric interpretation of the fundamental exact sequence [FF, Section 8.2.1.3]. Discuss the link with isocrystals [FF, Section 8.2.3]. State the classification theorem [FF, Théorème 8.2.10]. Use the classification theorem to show that the curve is geometrically simply connected [FF, Théorème 8.6.1]. Mention there is a GAGA type result, i.e., that there is a similiar classification theorem in the adic setting.

10) Classification of Vector bundles - Proof - Part 1 (Date: 28.6.)
References: [FF-DUR, Section 6.3]
This is the first talk on the proof of the classification theorem. We will restrict ourselves to the rank 2 case. Explain the dévissage that reduces the proof of the classification theorem to a statement about non-vanishing of global sections for certain extensions of vector bundles, i.e., prove [FF-DUR, Propositions 6.12 and 6.13]. Show how this statement is implied by understanding modifications of vector bundles, i.e., state Theorems 6.20 and 6.27 of [FF-DUR] and show how to deduce Proposition 6.13 (for rank 2 vector bundles) from them.

11) Classification of Vector bundles - Proof - Part 2 (Date: 5.7.)
References: [FF-DUR, Sections 6.3.4 and 6.4.5]
This talk finishes the proof of the classification theorem. The task is to prove Theorems 6.20 and 6.27 of [FF-DUR] on the modifications of vector bundles. For this talk some knowledge on p-divisible groups is desirable, as these modifications are studied in terms of the Hodge-de-Rham and the Hodge-Tate periods of p-divisible groups.

In the last few talks we study some applications of the theory.

12) p-adic Hodge Theory (Date: 12.7.)
References: [FF, Sections 9 and 10]
Let $E = \mathbb{Q}_p$ and $K/\mathbb{Q}_p$ be a finite extension and let $F = (\widehat{\overline{K}})^\flat$. Describe the action of $G_K = \mathrm{Gal}(\overline{K}/K)$ on $X_{E,F}$, in particular show that ∞, the closed point on $X_{E,F}$ corresponding to $\widehat{\overline{K}}$ is the unique point whose orbit is finite [FF, Section 10.1.1].

Show how to realize p-adic Galois representations as vector bundles on the curve. [FF, Section 10.1.4]. Discuss crystalline B_e-representations [FF, 10.2] and filtered φ-modules [FF, Section 10.5] and deduce the fundamental result "weakly admissible implies admissible" from the classification theorem of vector bundles [FF, Section 10.5.6].

13) G-bundles and local class field theory (Date: 19.7.)
References: [F2], [F-SLC]
Show that the Brauer group of the curve vanishes [F2, Théorème 2.2]. Then describe the classification of G-bundles on the curve in terms of the Kottwitz set $B(G)$ [F2, Théorème 5.1] (without proof). Show how this can be used to compute the Brauer group $Br(E)$ of the local field E, [F2, Théorème 2.6]. Sections 4.-6. of [F-SLC] might also be useful when preparing this talk.

14) $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ as a geometric fundamental group (Date: 26.7.)
References: [W]
Let $C/\mathbb{Q}_p$ be complete and algebraically closed. Let D be the open unit disk centered at 1, considered as a rigid space over C. Let $\tilde{D} := \varprojlim_{x\mapsto x^p} D$ and define $\tilde{D}^* := \tilde{D}\backslash\{1\}$. It is a perfectoid space and comes equipped with an action of $\mathbb{Q}_p^*$. Consider the quotient $Z := \tilde{D}^*/\mathbb{Q}_p^*$. This is no longer a perfectoid space, but it is a nice sheaf on the category of perfectoid spaces in char p, in fact it is a diamond. Explain the main result of [W], i.e., use the Fargues–Fontaine curve to realize $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ as the étale fundamental group of Z.

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