DAFRA ZOOM SEMINAR ON DERIVED CATEGORIES

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This seminar is an introduction to derived categories with a view towards the next DaFra seminar on condensed mathematics. Our main source is [GW].

Introduction. Derived categories were defined by Grothendieck and Verdier in the 1960s. They provide an important tool in modern algebra to systematically study mathematical objects (topological spaces, manifolds, varieties...) by "carefully linearizing" them. For example, instead of studying a given topological space X directly, it is (often) easier to study its cohomology groups $H^i(X,\mathbb{Z})$, $i \in \mathbb{Z}$. These groups are constructed from certain complexes of abelian groups

$$\cdots \longrightarrow C^{i-1}(X,\mathbb{Z}) \xrightarrow{d^{i-1}} C^i(X,\mathbb{Z}) \xrightarrow{d^i} \cdots , \tag{0.1}$$

where $d^i \circ d^{i-1} = 0$. One then takes the quotient $H^i(X,\mathbb{Z}) := \ker(d^i)/\operatorname{im}(d^{i-1})$. From a technical perspective, it seems better to work with the complex of abelian groups (0.1) itself, rather than its cohomology groups. Loosely speaking, we lost too much information by passing to the quotients. For example, the information on how the different groups in (0.1) were originally connected by the maps d^i. However, even if the groups $H^i(X,\mathbb{Z})$ are nice, say, finitely generated, the single terms $C^i(X,\mathbb{Z})$ in (0.1) are huge abelian groups. Also, there are many different sequences of abelian groups as in (0.1) which give rise to the same cohomology groups. Now we have too much information. This naturally leads to the derived category of abelian groups: objects are complexes of abelian groups, and two such complexes are identified if they induce the same cohomology groups[1].

In this seminar, we develop the rudiments of the theory of derived categories. Our focus will be at providing some background knowledge for the next semesters DaFra seminar on condensed mathematics.

Contents.

- Talks 1–2: Representable functors, (co)limits, additive/abelian categories
- Talks 3–5: Triangulated/derived categories and functors
- Talk 6: Sheaves on sites

This is a *first introduction* to derived categories. For full proofs, the participants are encouraged to consult further literature, see below.

Prerequisites and Remarks.

- Categorical language: categories, functors, adjoint functors, see for example [Wed, §1–2].
- It is certainly helpful (though not strictly necessary) to be familiar with the category of modules over a ring.
- We do not discuss spectral sequences in this seminar.
- Throughout the seminar we ignore set-theoretic problems.

Getting started and further literature. Chapters 1–4 in [Y] provide a nice overview of the construction of derived categories and derived functors. More details on derived categories are for example in the books [KS, Wei], on [Stacks], or in many other sources often depending on the intended applications. If you are willing to invest more time, we highly recommend [HA]. This is like a trampoline: the more you invest, the more you get out!

[1] More precisely, if there is a zig-zag of maps of complexes which induce isomorphisms on cohomology groups.

1st Meeting, 28.05.2020.

Talk 1: Limits and colimits [Stacks, 001L], [GW, (F.3)–(F.5)].

(1) Introduce presheaves of sets, prove the Yoneda lemma, define representable functors, universal objects [Stacks, 001L].

(2) Follow [GW, (F.3)–(F.5)]: Introduce (co)limits F.1, F.2, make the connection to projective/inductive limits F.3, define (co)complete categories F.4, explain how to construct (co)limits in the category of sets F.5. Give further examples of complete and cocomplete categories: topological spaces, groups, modules over a ring. Introduce filtered index sets F.8, F.9.

(3) Explain the special cases (1)–(5) in F.10. Show that a category is complete if and only if it has products and equalizers F.11.

(4) Explain that right/left adjoint functors commute with limits/colimits F.15, define right/left exact functors F.16.

Talk 2: Abelian categories [GW, (F.6)–(F.10)], [Stacks, 09SE, 01D8].

(1) Follow [GW, (F.6)–(F.10)]: Introduce additive categories F.17, prove F.19 (see [Stacks, 00ZZ, 0101]), define additive functors F.20, (co)kernels F.21 and split sequences F.23.

(2) Introduce abelian categories F.24, explain F.25, give examples:
   (i) The category of left (or right) R-modules is abelian. Here R is an associative, unital (possibly non-commutative) ring. Its subcategory of finitely generated R-modules is abelian if and only if R is Noetherian. In particular, the category of (finitely generated) abelian groups is abelian.
   (ii) The category of topological abelian groups is not abelian. For any topological abelian group A, the identity map induces $(A, \text{discrete}) \to (A, \text{natural})$. This is a mono- and epimorphism, but not an isomorphism if A is not discrete.

(3) Introduce (exact) complexes F.26, prove F.27, F.28, give examples of left/right exact functors:
   (i) For an object X in an abelian category $\mathcal{C}$, the functor $\mathcal{C} \to (\text{AbGrps})$, $Y \mapsto \text{Hom}_{\mathcal{C}}(X, Y)$ is left exact, and the functor $\mathcal{C}^{\text{op}} \to (\text{AbGrps})$, $Y \mapsto \text{Hom}_{\mathcal{C}}(Y, X)$ is right exact.
   (ii) Let $\mathcal{C}$ be the category of R-modules where R is commutative. For any R-module X, the functor $\mathcal{C} \to \mathcal{C}$, $Y \mapsto Y \otimes_R X$ is right exact.

(5) Introduce injective/projective objects F.35, F.36, explain F.37, see [Stacks, 01D8] for the construction of injective R-modules. Mention that Grothendieck abelian categories always have enough injectives F.38, F.43.

(6) Give the four, five and snake lemma F.32–F.34 as homework.

2nd Meeting, 18.06.2020.

Talk 3: Complexes in abelian categories [GW, (F.11)–(F.16)].

(1) Follow [GW, (F.11)–(F.16)]: Introduce the category of complexes $C(\mathcal{A})$ F.46. If $\mathcal{A}$ has one of the following properties, so has $C(\mathcal{A})$: abelian, (co)complete, Grothendieck abelian. Assume throughout that $\mathcal{A}$ is an abelian category.

(2) Introduce bounded complexes F.48, introduce the functors π_n, Z^n, B^n, H^n, prove F.49. Introduce the (stupid) truncation functors, the mapping cone F.50 and give F.51.

(3) Introduce homotopies F.53, the homotopy category $K(\mathcal{A})$ F.54, homotopy equivalences F.55, quasi-isomorphisms F.60, prove F.62, F.63.

(4) Introduce the Hom complex and tensor products of complexes in $C(\mathcal{A})$, $K(\mathcal{A})$. State (and if time permits prove) F.71.

Talk 4: Triangulated categories [GW, (F.23)–(F.28)].

(1) Follow [GW, (F.23)–(F.28)]: Introduce triangulated categories[2] F.93, give F.95, F.96.

[2] A picture of the octahedral axiom would be nice.

(2) State that the homotopy category $K(\mathcal{A})$ is equipped with the structure of a triangulated category (F.24), explain how to associate to a termwise split sequence an exact triangle F.99 (this applies for example if all X^i are injective objects).
(3) Introduce triangulated functors F.100, give F.102, F.103, mention that the subcategory of bounded complexes in $K(\mathcal{A})$ is triangulated (F.26).
(4) Introduce cohomological functors F.111, give F.112, F.113.
(5) Give the reading of [Y, §§1–4] as homework.

3rd Meeting, 02.07.2020.

Talk 5: Derived categories [GW, (F.32)–(F.34), (F.38), (F.39), (F.48)].

(1) Define the derived category $D(\mathcal{A})$ via its universal property F.123.
(2) Describe objects, morphisms and the distinguished triangles in $D(\mathcal{A})$ F.124 and the embedding $\mathcal{A} \to D(\mathcal{A})$ F.126.
(3) Introduce right derived functors $RF\colon D(\mathcal{A}) \to D(\mathcal{B})$ of an additive functor $F\colon \mathcal{A} \to \mathcal{B}$ between abelian categories F.136, explain the construction of the long exact sequence in F.139. Explain why $R^0F = F$ if F is left exact. If time permits, sketch the idea for constructing derived functors Section (F.39).
(4) State F.159, give the example of RHom Section (F.48).

Talk 6: Sheaves on sites [FGA, §2.3], [Stacks, 00V1, 03CM].

(1) Recall presheaves of sets on $\mathcal{C}$, see Talk 1 (1). Explain that one can also consider presheaves with values in (abelian) groups/rings etc. [Stacks, 00V1].
(2) Introduce sheaves on a topological space X [Stacks, 006S]: here $\mathcal{C}$ is the category of open subsets of X. Explain that the sheaf condition can be formulated via an equalizer diagram [FGA, 2.3.3]. Give the example of constant sheaves $\underline{A}$.
(3) Follow [FGA, §2.3]: Introduce Grothendieck topologies and sites[3] 2.3.1, give examples 2.26–2.28. Another example relevant for the next DaFra: $\mathcal{C}$ is the category of compact Hausdorff spaces, coverings are finite families of jointly surjective maps[4].
(4) Introduce sheaves on sites 2.3.3, equivalence of topologies 2.3.5, prove 2.49, give example 2.50.
(5) Introduce sheafification 2.63, state and sketch 2.64. A slick way to define the sheafification is to apply twice the functor $F \mapsto F^+$, where $F^+(U) = \operatorname{colim}_{(U_i \to U)} F((U_i \to U)_i)$ where $(U_i \to U)_{i\in I}$ runs over the category of all covering of U and where

$$F((U_i \to U)_i) := \{(s_i)_i \in \prod_{i\in I} F(U_i) \mid s_{i|U_i\times_U U_j} = s_{j|U_i\times_U U_j}\},$$

see also [Stacks, 00W1]. Note that F is a sheaf if and only if $F \to F^+$ is an isomorphism. For an object S in $\mathcal{C}$, give the examples:
   (i) The constant sheaf of sets $\underline{S}$ which is the sheafification of $\mathcal{C}^{\mathrm{op}} \to (\mathrm{Sets})$, $T \mapsto \operatorname{Hom}_{\mathcal{C}}(T, S)$.
   (ii) The constant sheaf of abelian groups $\mathbb{Z}[\underline{S}]$ which is the sheafification of $\mathcal{C}^{\mathrm{op}} \to (\mathrm{AbGrps})$, $T \mapsto \mathbb{Z}[\operatorname{Hom}_{\mathcal{C}}(T, S)]$.
(6) State that Abelian (pre-)sheaves on a site (P-)Ab($\mathcal{C}$) are a Grothendieck abelian category [Stacks, 03CM]. In particular, we have for every additive functor $F\colon \mathrm{Ab}(\mathcal{C}) \to \mathcal{B}$, with $\mathcal{B}$ any abelian category, the right derived functor $RF\colon D(\mathrm{Ab}(\mathcal{C})) \to D(\mathcal{B})$.
(7) Denote by $D(\mathbb{Z})$ the derived category of abelian groups.
   (i) For S in $\mathcal{C}$, introduce the section functor $\mathcal{A} \to (\mathrm{AbGrps})$, $\mathcal{F} \mapsto \Gamma(S, \mathcal{F}) := \mathcal{F}(S)$. This induces a functor $K(\mathcal{A}) \to D(\mathbb{Z})$, and by Talk 5 (5) we obtain the derived sections functor $D(\mathcal{A}) \to D(\mathbb{Z})$, $\mathcal{F} \mapsto R\Gamma(S, \mathcal{F})$.
   (ii) For each $\mathcal{F}'$ in $D(\mathcal{A})$, there is the derived Hom functor $D(\mathcal{A}) \to D(\mathbb{Z})$, $\mathcal{F} \mapsto \operatorname{RHom}(\mathcal{F}', \mathcal{F})$, see also Talk 5 (5).

[3] For simplicity, we assume that $\mathcal{C}$ is finitely complete so that all fibre products exist.
[4] Note that every surjection of compact Hausdorff spaces is a quotient map.

Show that $\mathrm{Hom}_{\mathcal{A}}(\mathbb{Z}[\underline{S}], \text{-}) = \Gamma(S, \text{-})$, and deduce that $R\Gamma(S, \text{-}) = \mathrm{RHom}(\mathbb{Z}[\underline{S}], \text{-})$.

References

[FGA] Fantechi et. al.: *Fundamental Algebraic Geometry – Grothendieck's FGA explained*, American Mathematical Society. 3

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