Seminar: Perfectoid spaces

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The aim of this seminar is to understand Scholze's thesis [11] where he introduces perfectoid spaces to prove Deligne's weight-monodromy conjecture for complete intersections in toric varieties. Scholze spectacularly manages to switch from mixed characteristic to positive characteristic in order to deduce the conjecture from the known case of positive characteristic [5]. We will follow the exposition in [11] closely introducing perfectoid spaces and the important tilting equivalence allowing to pass to positive characteristic. Amusingly the tilting equivalence (in its early form for perfectoid algebras) will not only be used to deduce the weight-monodromy conjecture but also for proving the main theorems about perfectoid spaces.

Talk 1.a - Adic spaces as topological spaces (0.5 sessions)

Introduce the underlying topological space of an affinoid adic space following chapter 2 in [11], specifically definition 2.1 until proposition 2.12. Use example 2.20 in loc. cit. to illustrate these definitions. Also discuss the adic spectrum of an affinoid field ([11], 2.26 - 2.29). Proofs and further information can be found in the paper [8] (in greater generality).

Talk 1.b - Adic spaces as ringed spaces

This talk provides the adic spectrum of an affinoid algebra with a structure (pre)sheaf. Define this structure (pre)sheaf and then adic spaces as in [11], 2.13 - 2.18. Also state proposition 2.19 and sketch its proof ([9], 2.1). Discuss the universal mapping property ([2], 6.1.4) of the affinoid algebra $(k\langle T_1, \ldots, T_n\rangle, k^\circ\langle T_1, \ldots, T_n\rangle)$ and deduce that $\mathrm{Spa}(k\langle T_1, \ldots, T_n\rangle, k^\circ\langle T_1, \ldots, T_n\rangle)$ can be considered as a closed n-dimensional unit ball. Finally introduce adification, i.e. proposition 3.8 in [9] with $Y = \mathrm{Spec}(k)$ and $S = \mathrm{Spa}(k, k^\circ)$, and explain the adification of the scheme $\mathbb{A}^n_k$. If time permits, discuss theorem 2.21 in [11]. Discussing the relation to Berkovich spaces and formal schemes is not necessary.

Talk 2 - Perfectoid fields

This talk defines perfectoid fields and constructs their tilts, which are fields of positive characteristic. Present chapter 3 in [11] and give detailed proofs especially of lemma 3.4.

Talk 3 - The category of almost modules

Define the category of $K^{\circ a}$-modules ([11] 4.1 - 4.11) but provide more details, which can be found in [7], specifically: Prove 2.1.2 - 2.1.4 loc. cit. in order to get a feeling for almost modules. Note that in our situation $\tilde{\mathfrak{m}} \cong \mathfrak{m}$ is flat which simplifies the exposition. Using 2.2.2 explain, why the morphisms of almost modules take this form. Shortly introduce algebra objects and their modules in tensor categories as in 2.2.5 - 2.2.7. Discuss the adjoints of the functor $(-)^a$ ([7], 2.2.13 and 2.2.19 - 2.2.21). The construction of the functor $(-)_{!!}$ can be omitted.

Talk 4 - Almost etale algebras

Present the remaining part of chapter 4 in [11] starting from definition 4.12. Also present [7], 3.1.1, 3.1.4. The discussion in 3.4.44 compares the definition of an almost etale morphism with the usual notion. Do not prove 4.16 and 4.17, instead give an overview of the cotangent complex ([11], below 5.10, [10], chapter 2) and its almost version to prove theorem 3.2.18 in [7]. For this you will need 5.11, 5.12 in [11] (resp. 3.2.9, 3.2.16 in [7]). Explain that 3.2.18 in [7] is an important step in the proof of [11], 4.17 (see [7], 5.3.27).

Talk 5 - Perfectoid algebras and tilting

Prove the tilting equivalence for perfectoid algebras (theorem 5.2 in [11]). The proof of theorem 5.2 ends on page 26 with the proof of theorem 5.10.

Talk 6 - Perfectoid algebras and their etale extensions

This talk examines the behaviour of etale extensions of perfectoid algebras under tilting and proves Falting‘s almost purity theorem in positive characteristic. That is, it covers the rest of chapter 5, [11], starting from proposition 5.17.

Talk 7 - Perfectoid spaces and tilting

In this talk it will be shown that the adic spectrum of a perfectoid affinoid k-algebra and its tilt are naturally homeomorphic and that their structure (pre)sheaves are tilts of each other. This is the content of [11], theorem 6.3.i),ii). Prove these two parts of the theorem, i.e. present the beginning of chapter 6 in [11] until corollary 6.8. The approximation lemma 6.5 deserves a detailed proof.

Talk 8 - Perfectoid spaces and cohomological triviality of $\mathcal{O}_X^{\circ a}$

Finish the proof of the missing parts iii), iv) of theorem 6.3 in [11], chapter 6, and definition 6.9 - proposition 6.14 loc. cit., concerning the sheaf property and (almost) vanishing of the cohomology of $\mathcal{O}_X^+$. Tate's acyclicity theorem ([2], 8.2.1) has to be used as a black box. If time permits explain how to translate this theorem from the language of rigid-analytic spaces to the language of adic spaces. In the end show that the process of tilting glues ([11], 6.15 - 6.17) and that fibre products of perfectoid spaces exist ([11], 6.18).

Talk 9 - Etale topology of perfectoid spaces

This talk introduces the etale and strongly etale site of a perfectoid space. Explain the necessity of introducing the strongly etale site. Follow [11], definition 7.1 until corollary 7.8.

Talk 10 - Falting's almost purity theorem

State and prove Falting's almost purity theorem in its strong version proved in [11], theorem 7.9 and deduce the consequences for the etale sites of perfectoid spaces ([11], 7.10 - 7.13). Finally present [11], 7.14 - 7.19 explaining all occuring notions like fibred topoi ([1], Exp. VI.7.1.1, 8.1).

Talk 11 - Toric varieties

Give an overview of classical toric varieties ([11], 8.1 - 8.4) focussing on the description of the cohomology $H^0(X_\Sigma, \mathcal{O}(D))$ of T-Weil divisors. More details can be found in [3], [6]. Next define toric adic and perfectoid spaces and relate them to the classical ones. Prove in detail [11], theorem 8.5 - 8.8 (omitting a proof of the analogue of the approximation lemma 6.5).

Talk 12 - The weight monodromy conjecture

This talk proves the final result of [11], namely the weight-monodromy conjecture for complete intersections in toric varieties in arbitrary characteristic ([11], theorem 9.6) by reducing it to the corresponding result in positive characteristic established by Deligne [5]. Explain shortly the relevance of the conjecture (see [4]) and then present Scholze's proof. It should become clear how Scholze manages to compare the etale cohomology of schemes in positive characteristic and characteristic 0 using the tilting equivalence of perfectoid spaces. If time permits, sketch Deligne's proof in positive characteristic.

References

[1] *Théorie des topos et cohomologie étale des schémas. Tome 2.* Lecture Notes in Mathematics, Vol. 270. Springer-Verlag, Berlin-New York, 1972. Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck et J. L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne et B. Saint-Donat.

[2] S. Bosch, U. Güntzer, and R. Remmert. *Non-Archimedean analysis*, volume 261 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1984. A systematic approach to rigid analytic geometry.

[3] David A. Cox, John B. Little, and Henry K. Schenck. *Toric varieties*, volume 124 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2011.

[4] Pierre Deligne. Théorie de Hodge. I. In *Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 1*, pages 425–430. Gauthier-Villars, Paris, 1971.

[5] Pierre Deligne. La conjecture de Weil. II. *Inst. Hautes Études Sci. Publ. Math.*, (52):137–252, 1980.

[6] William Fulton. *Introduction to toric varieties*, volume 131 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1993. The William H. Roever Lectures in Geometry.

[7] Ofer Gabber and Lorenzo Ramero. *Almost ring theory*, volume 1800 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2003.

[8] R. Huber. Continuous valuations. *Math. Z.*, 212(3):455–477, 1993.

[9] R. Huber. A generalization of formal schemes and rigid analytic varieties. *Math. Z.*, 217(4):513–551, 1994.

[10] Luc Illusie. *Complexe cotangent et déformations. I.* Lecture Notes in Mathematics, Vol. 239. Springer-Verlag, Berlin-New York, 1971.

[11] Peter Scholze. Perfectoid spaces. *Publ. Math. Inst. Hautes Études Sci.*, 116:245–313, 2012.