Etale homotopy theory (after Artin-Mazur, Friedlander et al.)

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Lecture 3: Applications

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Our selection of applications:

- Comparisons
- Friedlander's etale K-theory
- Sullivan's Galois symmetries in topology
- Dugger-Isaken's Sums of squares formulas
- Etale realizations of motivic spaces
- Algebraic cycles and etale cobordism
- Rational points and homotopy fixed points

For more applications see Friedlander's great book.

- Comparison theorems:

Let X be a connected pointed scheme of finite type over $\mathbb{C}$.

Denote by X_{cl} the homotopy type of X in the classical topology.

There is a canonical map $\varepsilon: X_{cl} \to X_{et}$ in pro-H.

Comparison theorems:

Generalized Riemann Existence Theorem:
The map $\varepsilon: X_{cl} \to X_{et}$ becomes an isomorphism in pro-H after profinite completion.

For X geometrically unibranch:

$$\hat{X_{cl}} \approx X_{et} \text{ in pro-H.}$$

Comparison theorems:

For X connected and geometrically unibranch:

$\pi_1(X_{cl})^\wedge \approx \pi_1(X_{et})$ as profinite groups.

If X is geometrically unibranch and X_{cl} is simply connected:

$\pi_n(X_{cl})^\wedge \approx \pi_n(X_{et})$ for all n.

Serre's example revisited:

Let X be connected scheme over a field k of charactersitic zero. Let X_1 and X_2 be the schemes over C obtained via two different embeddings of k into C.

Then after profinite completion there is an isomorphism in pro-H:

$$\hat{X}_{1,cl} \approx \hat{X}_{2,cl}.$$

Thus the possible difference of the homotopy types of $X_{1,cl}$ and $X_{2,cl}$ vanishes after completion. To prove this we use etale homotopy theory.

Comparison of characteristics:

Let R be a discrete valuation ring with separably algebraically closed residue field k.

Let X be a smooth proper scheme over R with connected fibers X_0 and X_1.

There is a canonical isomorphism in pro-H

$$X\hat{}_{1,et} \approx X\hat{}_{0,et}$$

where ^ denotes completion away from char k.

- Friedlander's etale K-theory:

Let T be a CW-complex and $C(m)$ be the cofiber of the multiplication by m map on the circle

$$S^1 \xrightarrow{\cdot m} S^1 \to C(m).$$

The "complex K-theory" of T with Z/m-coefficients can be defined as

$$K^0(T;Z/m) = Hom_H(C(m) \wedge T, BU) \text{ and}$$
$$K^1(T;Z/m) = Hom_H(S^1 \wedge C(m) \wedge T, BU).$$

where BU is the infinite complex Grassmannian.

Friedlander's etale K-theory:

If Y is a pro-space, its (complex) K-theory is defined by

$$K^0(Y;Z/m) = Hom_{pro\text{-}H}(C(m) \wedge Y, \#BU) \text{ and}$$
$$K^1(Y;Z/m) = Hom_{pro\text{-}H}(S^1 \wedge C(m) \wedge Y, \#BU).$$

where $\#BU = \{cosk_n\ BU\}_n \in$ pro-H.

If X is a scheme of finite type over a complete discrete valuation ring with separably closed residue field, Friedlander defines the "etale K-theory of X" to be the K-theory of X_{et}.

Friedlander's etale K-theory:

There is an Atiyah-Hirzebruch spectral sequence

$$E_2 = H^*_{et}(X;Z/m) \Rightarrow K^*_{et}(X;Z/m).$$

If X is a complex variety, then

$$K^*_{et}(X;Z/m) \approx K^*(X_{cl};Z/m).$$

Galois action on etale K-theory:

Let X be a variety over a field k with absolute Galois group $G_k = Gal(\bar{k}/k)$.

There is a natural action by G_k on

$$K^*_{et}(X \times_k \bar{k}; Z/m).$$

Dwyer and Friedlander interpreted important arithmetic questions in terms of this Galois action on etale K-theory.

Algebraic vs etale K-theory:

After the first construction by Friedlander there were more sophisticated definitions of etale K-theory by Friedlander, Dwyer-Friedlander and Thomason.

They all come equipped with natural maps

$$K^*_{alg}(X;Z/l^n) \to K^*_{et}(X;Z/l^n)$$

where l is a prime invertible on X.

Algebraic vs etale K-theory:

There is a "Bott element" β in K_{alg} whose image in K_{et} is invertible.

Thomason: If X is a smooth quasi-projective variety over a field of characteristic $\neq l$ of finite mod-l etale cohomological dimension, then

$$K_*^{alg}(X;Z/l^n)[\beta^{-1}] \to K_*^{et}(X;Z/l^n)$$

is an isomorphism.

Algebraic vs etale K-theory:

Without inverting β in K_{alg} there is the "Quillen-Lichtenbaum conjecture":

If X is a smooth variety over a field and n is invertible in k, then the natural map

$$K_i^{alg}(X;Z/n) \to K_i^{et}(X;Z/n)$$

is an isomorphism for $i-1 \geq$ mod-n etale cohomological dimension of X.

Note: The "Quillen-Lichtenbaum conjecture" follows from the "Bloch-Kato conjecture".

- Sullivan and Galois symmetries in topology:

Let us have a second look at the (complex version of the) Adams conjecture:

Let $BU(n)$ be the Grassmannian of complex n-planes, BU be the infinite complex Grassmannian.

Let BG be the classifying space of (stable) spherical fibrations.

Sullivan and Galois symmetries in topology:

Adams: For all k, the map

$$J \circ (\psi^k - 1) : BU(n) \to BU \to BG[1/k]$$

is null-homotopic, i.e., homotopic to a constant map.

First step: As in Lecture 1, it suffices to consider the p-completed maps (for each p with $(k,p)=1$)

$$J \circ (\psi^k - 1) : BU(n)\hat{} \to BU\hat{} \to BG(S_p\hat{}).$$

Sullivan's amazing idea:

Interpret the Adams operations as "Galois symmetries" on profinitely completed homotopy types of classifying spaces.

Galois symmetries in topology:

The complex projective n-space P^n is defined over Q and we know

$$P^n(C)^\wedge \approx P^n_{et}.$$

The absolute Galois group Gal_Q of Q acts on P^n_{et} and this defines an action of Gal_Q on $P^n(C)^\wedge$.

Concretely: $\sigma \in Gal_Q$ acts on $\pi_2(P^n(C)^\wedge)=Z_p$ by multiplication with $\chi(\sigma)$ where χ denotes the cyclotomoic character.

Galois symmetries in topology:

Just seen: $\sigma \in Gal_Q$ acts on $\pi_2(P^n(C)^\wedge)$ via $\chi(\sigma)$.

This is a surprising fact, since the action of Gal_Q on $P^1(C)$ is "wildly discontinuous". Only after completion we obtain a nice action.

Key fact: The etale homotopy type tells us how to read off the action on finite covers.

Galois symmetries in topology:

In the same way: There is a nice action of Gal_Q on $P^\infty(C)$^ ($\approx K(Z_p,2)$) and on $BU(n)$^:

Concretely: $\sigma \in Gal_Q$ acts on $BU(n)$^ such that

$$\sigma(c_i) = \chi(\sigma)^{-i} \cdot c_i$$

on cohomology, where c_i is the ith Chern class.

Galois symmetries in topology:

Choose $\sigma \in Gal_Q$ such that $\chi(\sigma) = k^{-1} \in Z_p^\times$. Then

$$\sigma : BU(n)\hat{} \to BU(n)\hat{} \text{ with}$$

$$\sigma(c_i) = k^i \cdot c_i.$$

Key observation: This σ is an "unstable version" of the Adams operation ψ^k. (Use splitting principle and compute the effect on line bundles.)

This is very remarkable: Without completions, ψ^k is an endomorphism of BU and not $BU(n)$.

The conclusion of the proof:

We conclude: the diagram

$$\begin{array}{ccc} BU(n-1)\hat{} & \overset{\sigma=\psi^k}{\rightarrow} & BU(n-1)\hat{} \\ i\downarrow & & \downarrow i \\ BU(n)\hat{} & \underset{\sigma=\psi^k}{\rightarrow} & BU(n)\hat{} \end{array}$$

is homotopy commutative and cartesian.

Thus, twisting by ψ^k does not change the corresponding spherical fibration. This completes the sketch of Sullivan's proof of the Adams conjecture.

- Sums of squares:

Let k be a field. A "sums-of-squares formula" of type $[r,s,n]$ is an identity of the form

$$(x_1^2 + \dots + x_r^2)\cdot(y_1^2 + \dots + y_s^2) = z_1^2 + \dots + z_n^2$$

where each z_i is a bilinear expression in the x's and y's with coefficients in k.

For $k=R$ such an identity corresponds to an "axial map"

$$RP^{r-1} \times RP^{s-1} \to RP^{n-1}.$$

Sums of squares:

This relates sums-of-squares formulas over R to embedding problems of projective space in Euclidean space.

Hopf: Z/2-cohomology yields obstructions to existence of sums-of-squares formulas over R.

Davis found improved results using BP-theory.

Sums of squares in positive characteristic:

Dugger and Isaksen: The topological methods of Davis can be transferred to positive characteristic.

Etale realizations and BP-theory for pro-spaces: The topological obstructions do not depend on the field k ($\mathrm{char}\, k \neq 2$).

- Etale realizations of motivic spaces:

Is there an etale homotopy type for Voevodsky's motivic spaces?

There are at least two constructions:

- Schmidt's extension of Artin-Mazur's etale type

- Isaksen's extension of Friedlander's etale type

Schmidt's geometric etale realization:

A "motivic space over S" is a simplicial sheaf in the Nisnevich topology over Sm_S.

An "etale hypercovering" of a motivic space M is a "local trivial fibration" $U_\bullet \to M$ in "the" etale model structure of simplicial sheaves.

The etale homotopy type of M is the pro-object

$$\pi Triv_{/M} \to H$$

$$U_\bullet \mapsto \pi_0(U_\bullet).$$

Schmidt's geometric etale realization:

This defines a functor

$$ht: H_{s,et}(Sm_S) \to \text{pro-}H.$$

But: in general, the map $A^1_S \to S$ does not induce an isomorphism of etale fundamental groups.

The functor ht only factors through A^1-localization if we complete away from the residue characteristics.

Isaksen's "rigidified" etale realization:

Isaksen extends Friedlander's etale topological type to motivic spaces.

The etale type of simplicial presheaves on Sm_S is the formal extension of a colimit preserving functor of the etale type of schemes.

Using a Z/l-model structure, the etale type becomes a left Quillen functor from motivic spaces to the pro-category of simplicial sets.

- Algebraic cycles and etale cobordism:

Let X be a smooth projective complex variety.

Our goal: Understand all closed subvarieties of X, at least up to a suitable notion of equivalence.

Let $Z^p(X)$ be the free abelian group generated by codimension p irreducible closed subsets in X.
Its elements are called "cycles".

Denote $CH^p(X):=Z^p(X)/_{\sim rat}$ for cycles modulo "rational equivalence".

The cycle map:

$$CH^p(X) \xrightarrow{cl_H} H^{2p}(X;Z)$$

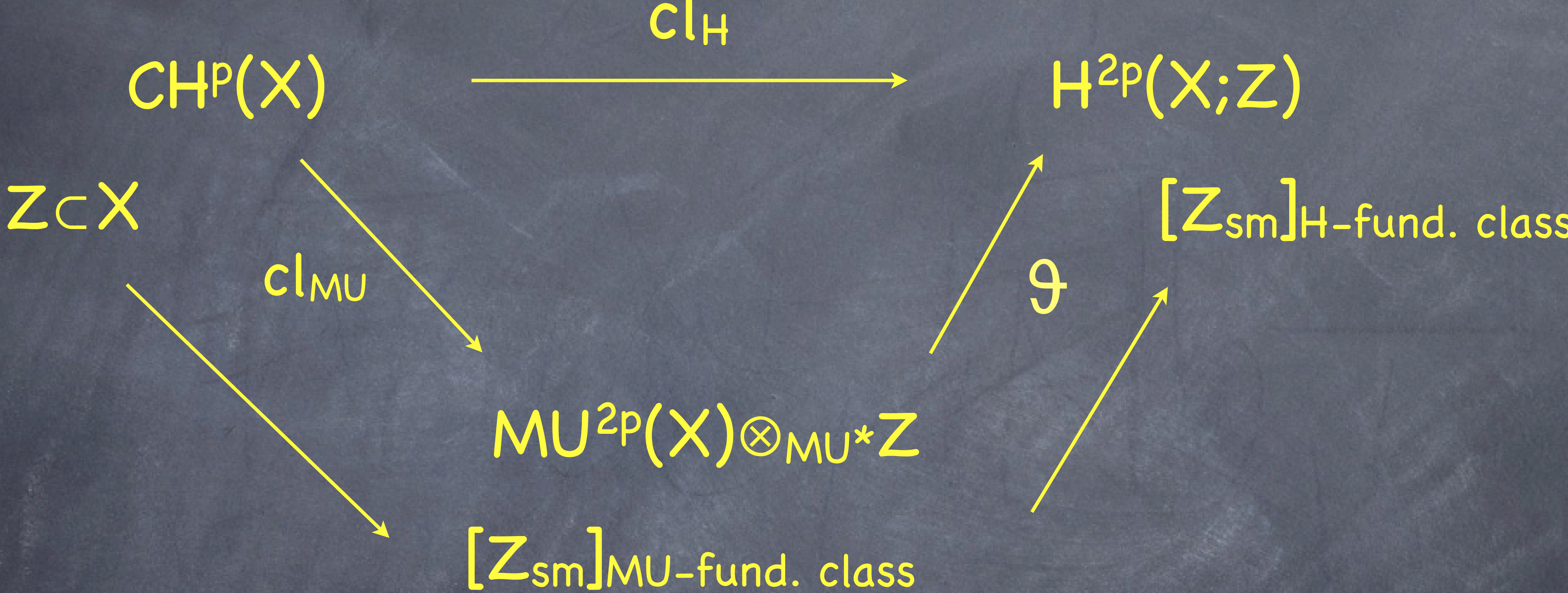
$$Z \subset X \longrightarrow [Z_{sm}]_{fund}$$

$H^{2p}(X;Z)$ denotes the singular cohomology of the complex manifold X_{cl} associated to X,

and $[Z_{sm}]_{fund}$ denotes the fundamental class of a desingularization Z_{sm} of Z.

In the 1990's Totaro showed that cl_H factors via a quotient of complex cobordism $MU^*(X_{cl})$:

Totaro's factorization:



This is diagram commutes.

Consequences:

$$\begin{array}{ccc} CH^p(X) & \xrightarrow{cl_H} & H^{2p}(X;Z) \\ & {\scriptstyle cl_{MU}} \searrow \quad \nearrow {\scriptstyle \vartheta} & \\ & MU^{2p}(X)\otimes_{MU^*}Z & \end{array}$$

- A topological obstruction on the image of cl_H: image of cl_H is contained in image of ϑ.
In particular, all odd degree cohomology operations must vanish on the image of cl_H.
- More importantly: We can study the kernel of cl_H by finding elements in the kernel of ϑ that are in the image of cl_{MU}; good candidates are polynomials in Chern classes.
Totaro used this method to find important new examples of elements in the Griffiths group.

Algebraic cycles and etale cobordism:

Now let X be a smooth projective variety over a finite field k of characterstic p and l a prime $\neq p$.

There is an etale version of the cycle map

$$CH^i(X) \xrightarrow{cl_{Het}} H^{2i}_{et}(X; Z_l(i))$$

$$Z \subset X \longrightarrow [Z] \text{“etale fund. class”}$$

Integral Tate “conjecture”: Is

$$CH^i(X) \otimes Z_l \xrightarrow{cl_{Het}} H^{2i}(X_{\bar{k}}; Z_l(i))^{G_k}$$

surjective? The answer is “no” as we will explain now.

Etale cobordism (Q.):

Let $\hat{MU}$ be the "pro-l-completion" of MU.

For a variety X over an alg. closed field we define the l-adic etale cobordism of X to be

$$\hat{MU}^n_{et}(X) := Hom_{\hat{SH}}(\Sigma^\infty(\hat{X}_{et}), \Sigma^n \hat{MU})$$

where $\hat{SH}$ is the stable l-adic homotopy category of profinite spectra.

An l-adic factorization (Q.): X smooth projective over $k=\bar{k}$, $l \neq \mathrm{char}\, k$.

$$CH^i(X) \xrightarrow{cl_{Het}} H^{2i}_{et}(X;\mathbb{Z}_l)$$

$Z \subset X \mapsto [Z]_{\text{etale H-fund. class}}$

$$CH^i(X) \xrightarrow{cl_{MUet}} \hat{MU}^{2i}_{et}(X) \otimes_{\hat{MU}^*} \mathbb{Z}_l \xrightarrow{\vartheta_{et}} H^{2i}_{et}(X;\mathbb{Z}_l)$$

$Z \subset X \mapsto [Z]_{\text{etale MU-fund. class}} \mapsto [Z]_{\text{etale H-fund. class}}$

Note: The construction of cl_{MUet} uses that there are "tubular neighborhoods" in etale homotopy.

Consequences:

$$CH^i(X) \xrightarrow{cl_{Het}} H^{2i}_{et}(X;Z_l)$$

$$cl_{MU}: CH^i(X) \to \hat{MU}^{2i}_{et}(X) \otimes_{\hat{MU}^*} Z_l, \qquad \vartheta_{et}: \hat{MU}^{2i}_{et}(X) \otimes_{\hat{MU}^*} Z_l \to H^{2i}_{et}(X;Z_l)$$

- A topological obstruction on the image of cl_{Het}: image of cl_{Het} is contained in image of ϑ_{et}.
In particular, all odd degree cohomology operations must vanish on the image of cl_{Het}.

- Cycles of Atiyah and Hirzebruch provide counter-examples to the integral version of the Tate conjecture for varieties over finite fields.

- Rational points and homotopy fixed points:

The ideas in this final section have been developed by Friedlander, Pal, Harpz-Schlank, Q., Wickelgren and others.

Let X be a connected smooth projective variety over a field k. Let $X(k)$ be the set of rational points.

The functoriality of the etale homotopy type gives a natural map

$$X(k) \to \mathrm{Hom}_{\hat{H}_{k_{et}}}(k_{et}, X_{et})$$
$$(k \to X) \mapsto (k_{et} \to X_{et})$$

where $\hat{H}_{k_{et}}$ is a suitable homotopy category of "profinite spaces" over k_{et}.

Rational points and homotopy fixed points:

The etale homotopy type k_{et} is equivalent to the classifying space BG_k of the absolute Galois group G_k of k.

This shows there is a natural map

$$X(k) \to \mathrm{Hom}_{\hat{H}_{BG_k}}(BG_k, X_{et}).$$

We interpret this set as the set $\pi_0((X_{et})^{hG_k})$ of connected components of the "continuous homotopy fixed points of X_{et}".

Rational points and homotopy fixed points:

Note: Different authors use different ways to get the set $\pi_0((X_{et})^{hG_k})$.

One may think of this set as a "homotopical approximation to $X(k)$".

Pal and Harpaz-Schlank use the map

$$X(k) \to \pi_0((X_{et})^{hG_k})$$

to reinterpret obstructions to the existence of rational points in terms of etale homotopy theory.

Rational points and homotopy fixed points:

Fundamental question: Is the map

$$X(k) \to \mathrm{Hom}_{\hat{H}_{BG_k}}(BG_k, X_{et}) = \pi_0((X_{et})^{hG_k}) \text{ surjective?}$$

For X a connected smooth projective curve of genus ≥ 2 over a number field, this question is equivalent to Grothendieck's "section conjecture".

The hope: Homotopy methods give us a chance to understand the set $\pi_0((X_{et})^{hG_k})$ and the above map. But so far, we don't know if this works.

Thank you!