Etale homotopy theory (after Artin-Mazur, Friedlander et al.)

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Lecture 2: Construction

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Towards the idea:

Let X be a scheme of finite type over a field k.

An "etale open set" of X is an etale map $U \to X$ which is the algebraic version of a local diffeomorphism.

These etale open sets are great for defining sheaf cohomology.

Cech coverings:

There is also a more "topological way" to compute etale sheaf cohomology:

Let F be a locally constant etale sheaf on X.

Let $\{U_i \to X\}_i$ be an etale cover.

For each $n \geq 0$, form

$$U_{i0,\ldots,in} = U_{i0} \times_X \ldots \times_X U_{in}$$

Cech complex: $U_{i0,\dots,in} = U_{i0} \times_X \dots \times_X U_{in}$

Set $C^n(U_\bullet;F) := \prod H^0(U_{i0,\dots,in};F)$.

This defines a complex $C^*(U_\bullet;F)$ whose cohomology is denoted by $H^n(U_\bullet;F)$.

But: The cohomology $H^*(U_\bullet;F)$ of a single covering does not compute the sheaf cohomology of X. The coverings are not "fine" enough.

(Would need: all $U_{i0,\dots,in}$ are contractible.)

Cech cohomology:

Solution: Make coverings "finer and finer" and consider all at once.

For a variety X over a field there is an isomorphism

$$H^n(X;F) \approx \operatorname{colim}_U H^n(U_\bullet;F)$$

where the colimit ranges over all etale covers.

Observation: The global sections $H^0(U_{i0,\dots,in};F)$ only depend on the set of connected components $\pi_0(U_{i0,\dots,in})$.

The idea:

Forming all possible $U_{i0,\dots,in}$'s yields a simplicial set

$$\pi_0(U_\bullet).$$

For a variety X over a field, the colimit of the singular cohomologies of all the spaces $\pi_0(U_\bullet)$'s computes the etale cohomology of X.

A candidate for an etale homotopy type:

the "system of all spaces $\pi_0(U_\bullet)$'s".

In order to make this idea work in full generality we need some preparations.

Pro-objects:

A category I is "cofiltering" if it has two properties:

- for any $i,j \in I$ there is a k with $k \to i$ and $k \to j$;

- for any $f,g\colon i \to j$ there is an $h\colon k \to i$ with $fh=gh$.

Let C be a category. A "pro-object" $X=\{X_i\}_{\{i \in I\}}$ in C is a functor $I \to C$ where I is some cofiltering index category.

We get a category pro-C by defining the morphisms to be

$$\mathrm{Hom}\,(X, Y) = \lim_j \operatorname{colim}_i \mathrm{Hom}\,(X_i, Y_j).$$

Pro-homotopy:

Let H be the homotopy category of connected, pointed CW-complexes.

H is equivalent to the homotopy category of connected, pointed simplicial sets.

The objects of H will be called "spaces".

The objects of pro-H will be called "pro-spaces".

Pro-homotopy groups:

Let $X=\{X_i\}_{\{i\in I\}}$ be a pro-object in H.

The homotopy groups of X are defined as the pro-groups

$$\pi_n(X) = \{\pi_n(X_i)\}_{\{i\in I\}}.$$

For A an abelian group, the homology groups of X are

$$H_n(X;A) = \{H_n(X_i;A)\}_{\{i\in I\}}.$$

Cohomology of pro-spaces:

Let A be an abelian group. The cohomology groups of X are defined as the groups

$$H^n(X;A) = \operatorname{colim}_i H^n(X_i;A).$$

If A is has an action by $\pi_1(X)$, then there are also cohomology groups of X with local coefficients in A.

Completion of groups:

Let L be a set of primes and let LGr be the full subcategory finite L-groups in the category of groups Gr.

There is an L-completion functor

$$\hat{}\ : Gr \to \text{pro-}LGr$$

such that Hom (G,K) ≈ Hom (G^,K) for K in LGr.

Completion of spaces:

Let L be a set of primes and let LH be the full subcategory of H consisting of spaces whose homotopy groups are finite L-groups.

Artin and Mazur show that there is an L-completion functor

$$\hat{} : \text{pro-}H \to \text{pro-}LH$$

such that Hom (X,W) ≈ Hom (X^,W) for W in LH.

Completion and invariants:

The canonical map $X \to X^\wedge$ induces isomorphisms

- of pro-finite L-groups

$$(\pi_1(X))^\wedge \approx \pi_1(X^\wedge)$$

- of cohomology groups

$$H^n(X;A) \approx H^n(X^\wedge;A)$$

if A is a finite abelian L-group.

A warning: Isomorphisms in pro-H

A map $X \to Y$ in pro-H which induces isomorphisms on all homotopy groups is not necessarily an isomorphism in pro-H.

To see this, let $cosk_n$: $H \to H$ be the coskeleton functor which kills homotopy in dimension $\geq n$.

Let X be a space and let $X^{\#}$ be the inverse system

$$X^{\#} = \{cosk_n X\}.$$

A warning: Isomorphisms in pro-H

There is a canonical map $X \to X^{\#} = \{cosk_n X\}$ in pro-H, which induces an isomorphism on all (pro-) homotopy groups.

The inverse of this map would be an element in

$$colim_n \, Hom\,(cosk_n X, X).$$

Hence the inverse exists if and only if

$X = cosk_n X$ for some integer n.

Isomorphisms in pro-H:

This led Artin and Mazur to introduce the following notion:

A map $f:X \rightarrow Y$ in pro-H is a "#-isomomorphism" if the induced map $f^{\#}:X^{\#} \rightarrow Y^{\#}$ is an isomorphism in pro-H.

Theorem (Artin-Mazur): A map $f:X \rightarrow Y$ in pro-H is a #-isomorphism if and only if f induces an isomorphism

$$\pi_n(f): \pi_n(X) \xrightarrow{\approx} \pi_n(Y) \text{ for all } n \geq 0.$$

#-isomorphisms and a Whitehead theorem:

Let $X \to Y$ be map in pro-H and L a set of primes. Then $f\hat{}: X\hat{} \to Y\hat{}$ is a #-isomorphism if and only if f induces isomorphisms

- $\pi_1(X)\hat{} \approx \pi_1(Y)\hat{}$ and
- $H^n(Y;A) \approx H^n(X;A)$ for every $n \geq 0$ and every $\pi_1(Y)$-twisted coefficient group A which is a finite abelian L-group such that the action of $\pi_1(Y)$ factors through $\pi_1(Y)\hat{}$.

Completion vs homotopy (continued):

The canonical map $X \to X\hat{}$ induces a group homomorphism for every n

$$(\pi_n(X))\hat{} \to \pi_n(X\hat{}).$$

For $n \geq 2$, this map is in general not an isomorphism.

But: Suppose that X is simply-connected and all $\pi_n(X)$'s are "L-good" groups. Then

$$(\pi_n(X))\hat{} \approx \pi_n(X\hat{}) \text{ for all } n.$$

(There are improvements by Sullivan.)

Towards hypercoverings:

Let X be a connected, pointed scheme.

We assume that X is locally connected for the etale topology, i.e., if $U \to X$ is etale, then U is the coproduct of its connected components.

For example: X is locally noetherian.

Cech coverings:

Let $U \to X$ be an etale covering.

We can form the Cech covering associated to $U \to X$.
This is the simplicial scheme $U_\bullet = \mathrm{cosk}_0(U)_\bullet$

$$U \;\substack{\leftarrow \\ \leftarrow \\ \rightarrow}\; U\times_X U \;\substack{\rightarrow \\ \leftarrow \\ \leftarrow \\ \leftarrow \\ \rightarrow}\; U\times_X U\times_X U \;\substack{\leftarrow \\ \leftarrow \\ \leftarrow \\ \leftarrow}\; U\times_X U\times_X U\times_X U \quad \ldots$$

i.e. U_n is the $n+1$-fold fiber product of U over X.

From Cech- to hypercoverings:

To form a Cech covering, we take an etale map $U \to X$ and then we mechanically form $U_\bullet$.

In other words, each U_n is determined by $U \to X$.

Drawbck: Cech coverings are often not fine enough to provide the correct invariants.

Problem: We have no flexibility for forming U_n.

Idea: Choose coverings in each dimensions for forming $U_\bullet$.

Hypercoverings:

- Take an etale covering $U \to X$ and set $U_0 := U$.

- Form $U_0 \times_X U_0$ and choose an etale covering $U_1 \to U_0 \times_X U_0$. $U_0 \times_X U_0$ is in fact equal $(\mathrm{cosk}_0 U_0)_1$.

- Turn U_1 into a simplicial object $(\mathrm{cosk}_1 U_1)_\bullet$ and choose an etale covering $U_2 \to (\mathrm{cosk}_1 U_1)_2$.

...

- Continuing this process leads to a hypercovering of X.

Hypercoverings:

A "hypercovering" of X is a simplicial object $U_\bullet$ in the category of schemes etale over X such that

- $U_0 \to X$ is an etale covering;
- For every $n \geq 0$, the canonicial map $U_{n+1} \to (\operatorname{cosk}_n U_\bullet)_{n+1}$ is an etale covering.

All hypercoverings of X form a category.

But: This category is not cofiltering !

Homotopy and hypercoverings:

Solution: We take homotopy classes of maps as morphisms.

The category $HR(X)$ of hypercoverings of X and simplicial homotopy classes of maps between hypercoverings as morphisms is cofiltering.

Verdier's theorem: Let F be an etale sheaf on X. Then for every $n \geq 0$ there is an isomorphism

$$H^n(X;F) \approx \operatorname{colim}_{U \in HR(X)} H^n(F(U_\bullet)).$$

The etale homotopy type:

The "etale homotopy type" X_{et} of X is the pro-space

$$HR(X) \to H$$

$$U_\bullet \mapsto \pi_0(U_\bullet).$$

The etale homotopy type is a functor from the category of locally noetherian schemes to pro-H.

Note: Since we had to take homotopy classes of maps of hypercoverings, X_{et} is only a pro-object in the homotopy category H.

Etale homology and cohomology:

Let F be a locally constant etale sheaf of abelian groups on X. Then F corresponds uniquely to a local coefficient group on X_{et}.

The cohomology of X_{et} is the etale cohomology of X:

$H^n_{et}(X;F) \approx H^n(X_{et};F)$ for all $n \geq 0$ and every locally constant etale sheaf F on X.

Etale homotopy groups:

The etale homotopy groups are defined as

$$\pi_n(X) := \pi_n(X_{et}) \text{ for all } n \geq 0.$$

In general: $\pi_1(X_{et})$ is different from the profinite etale fundamental group of Grothendieck in SGA 1 (but it is the one of SGA 3).

For: $\pi_1(X_{et})$ takes all etale covers into account, not just finite ones.

Etale homotopy groups:

But: If X is "geometrically unibranch", i.e., the integral closure of its local rings is again local, then X_{et} is a pro-object in the category H_{fin} of spaces with finite homotopy groups.

In this case: $\pi_n(X_{et})$ is profinite and $\pi_1(X_{et})$ equals Grothendieck's etale fundamental group in SGA 1.

For example: every normal scheme (local rings are integrally closed) is a geometrically unibranch.

We achieved our goal:

- The etale homotopy type is an intrinsic topological invariant of X.

- It contains the information of known etale topological invariants.