Etale homotopy theory (after Artin-Mazur, Friedlander et al.)

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Lecture 1: Motivation

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Riemann, Poincare, Lefschetz:

Until the early 20th century, algebraic geometry and algebraic topology were part of the same discipline.

For example, the idea of a Riemann surface grew out of the attempt to understand integrals of rational functions over the complex numbers.

Lives grew apart:

Then algebraic topology took off to study (topologically) more complicated spaces.

Algebraic geometry underwent an algebraization: e.g. varieties over fields of positive characteristic and schemes over more general bases.

A divorce soon regreted...

Topologists developed many powerful techniques, e.g., singular cohomology, homotopy groups, ...

Can we apply these techniques to algebraic varieties?

For a variety $X \subset P^n$ over the complex numbers: take the complex points $X(C)$ and topologize it as a subspace in complex projective space.
Write $X_{cl} := X(C)$ for this topological space.

It works:

This gives a well-defined homotopy type X_{cl} for complex varieties and, in particular, singular cohomology groups, fundamental groups, etc.

For example, we can study subvarieties via their associated class in singular cohomology.

This "cycle map" is the subject of the famous Hodge conjecture.

So far so good:

What about varieties over other fields of characteristic zero?

Let K be a field of characteristic zero and X a variety over K.

There is an embedding $K \subset C$ and we can turn X into a variety X_C over C.

Now take the complex manifold $X_C(C)$ as before and get a homotopy type and topological invariants.

Running into trouble:

For varieties over fields of positive characteristic...?

Grothendieck's response: etale topology.

Let us remain modest and stick to a field K of characteristic zero for a moment.

We took an embedding $K \subset C$ and ...

Wait: there is not just one embedding $K \subset C$ and we have to make a choice!

It becomes worse:

Let X be a variety over a field K of characteristic zero.

A fundamental question:

> Does the homotopy type of X_{cl} depend on the choice of an embedding $K \subset C$?

Serre's answer:

Yes, it does!

Serre's answer:

Theorem (Serre):
There is a smooth projective variety V defined over a number field K and there are embeddgings φ and ψ of K into C such that

$$\pi_1(V_{cl}^{\varphi}) \not\approx \pi_1(V_{cl}^{\psi}).$$

Thus, even though the two complex varieties V^{φ} and V^{ψ} are conjugate, they have different homotopy types.

Serre's example:

Let us have a look at Serre's example.

k a quadratic imaginary field

R its ring of integers

Cl the ideal class group of k

K the absolute class field of k

h the class number $= \#Cl = [K:k]$.

Serre's example:

There is an elliptic curve E defined over K with $End(E)=R$.

Key observation:

Given an embedding $\varphi: K \subset C$, $\pi_1(E^{\varphi}_{cl})$ is a projective R-module of rank one. Thus $\pi_1(E^{\varphi}_{cl})$ corresponds to an element e_φ in Cl.

Conversely, every element of Cl is of the form e_φ for some embedding $\varphi: K \subset C$ and we have $e_\varphi = e_{\varphi'}$ if and only if φ' is either equal to φ or to its complex conjugate.

Serre's example:

Let p be a prime congruent -1 modulo 4 and let

$$k = Q(\sqrt{-p}).$$

Choose p such that $h > 1$, e.g., $p=23$, $h=3$.

Let E be an elliptic curve over K with $End(E)=R$.

There are embeddings φ and ψ of K in C such that

$\pi_1(E^{\varphi}_{cl})$ is a free R-module of rank one, and

$\pi_1(E^{\psi}_{cl})$ is not a free R-module (exists since $h>1$).

Serre's example:

Take the abelian variety $A = E^{(p-1)/2}$ over K.

Let S be the ring of integers of the field of pth roots of unity. In fact, S is a free R-module of rank $(p-1)/2$. Then

$\pi_1(A^{\varphi}_{cl})$ is a free S-module of rank one, and

$\pi_1(A^{\psi}_{cl})$ is not a free S-module of rank one.

Serre's example:

Now let Y be a hypersurface in CP^{p-1} given by the homogeneous equation

$$\sum_{i=1}^{p} X_i^p = 0$$

By the Lefschetz theorem, Y is simply-connecetd.

Let G be a cyclic group of order p.
Then G acts on Y by permuting the variables, and on A, since S is a quotient of Z[G].

Now we can define: V := (Y x A)/G.

Serre's example: $V := (Y \times A)/G$.

V is a smooth projective variety defined over K.

Etale locally V is a fiber bundle over Y/G with fiber A, and $V \to Y/G$ admits a section.

This implies $\pi_1(V^{\varphi}_{cl}) \approx \pi_1(A^{\varphi}_{cl}) \rtimes G$

and $\pi_1(V^{\psi}_{cl}) \approx \pi_1(A^{\psi}_{cl}) \rtimes G$.

Finally: $\pi_1(V^{\varphi}_{cl}) \not\approx \pi_1(V^{\psi}_{cl})$.

(Because: An isomorphism would imply that $\pi_1(A^{\varphi}_{cl})$ and $\pi_1(A^{\psi}_{cl})$ were isomorphic as S-modules. ⚡)

The conclusion:

The classical topology is not an intrinsic invariant of varieties defined over a field K.

Thus even in characteristic zero we need a "better" homotopy type:

the etale homotopy type.

The dream:

The "etale homotopy type" of a scheme should be a reasonable topological space which is

- defined over a base of any characteristic;

- an intrinsic invariant, i.e., only depend on the isomorphism type of the scheme;

- a space whose cohomology and fundamental group should be equal to Grothendieck's etale cohomology and etale fundamental group;

- functorial; in particular, for varieties over fields there should be a Galois action.

Why should we care:

- Grothendieck defined an etale fundamental group of schemes. But his method does not yield higher homotopy groups.

- Grothendieck's and Quillen's work on algebraic K-theory "asks" for an etale version of topological K-theory. A good candidate: "topological" K-theory of the etale homotopy type.

- Quillen's idea for a proof of the Adams conjecture, a purely topological statement.

Proofs of the Adams conjecture:

We will discuss two methods to prove the Adams conjecture (and there are more). Both involve etale homotopy theory in an essential way.

- Today: Quillen-Friedlander's approach.
Compare spaces over complex numbers with spaces in characteristic p and use the Frobenius map.

- In Lecture 3: Sullivan's approach.
Galois symmetries on profinite completions of spaces are induced by etale homotopy types.

Spherical fibrations:

Let X be a finite CW-complex and let E be an n-dimensional complex vector bundle over X.

By endowing E with a Hermitean metric and looking at vectors of length 1 in $E-0$ we get a fiber bundle

$$S(E) \to X$$

with fiber a $2n-1$-sphere S^{2n-1}.

Fiber homotopy equivalence:

We say that two fiber bundles F and F' over X

$$\begin{tikzcd}
F \arrow[rr, shift right, "f"'] \arrow[dr] & & F' \arrow[ll, shift right, "g"'] \arrow[dl] \\
& X &
\end{tikzcd}$$

are "fiber homotopy equivalent" if

there are maps f and g

and homotopy equivalences $gf \simeq id_F$ and $fg \simeq id_{F'}$ which at each time t are maps of fiber bundles.

The J-homomorphism:

Let $K(X)$ be the Grothendieck group of finite dimensional complex vector bundles over X.

Let $SF(X)$ be the Grothendieck group of spherical fibrations modulo fiber homotopy equivalence.

The functor $S(-)$ induces the J-homomorphism

$$J: K(X) \to SF(X).$$

The Adams conjecture:

Let ψ^k be the kth Adams operation on $K(X)$. It is a functorial ring homomorphism. For a line bundle L, it is $\psi^k(L)=L^k$ in $K(X)$.

Adams' conjecture: Let E be a complex vector bundle over a finite CW-complex X and k an integer. Then there is an integer n such that $k^n(\psi^k E-E)$ maps to zero under J.

(In fact, Adams conjectures also the case of real vector bundles.)

The Quillen-Friedlander approach:

Let us assume we already knew there is a CW-complex V_{et} which represents the etale homotopy type for every reasonable scheme V.

The idea of the proof is based on three obeservations:

Quillen's observation 1:

- Homotopy types are visible in charateristic p.

Let R be a strict henselization of Z at p, $R \subset C$ an embedding and $k = \bar{F}_p$ the closed point of R, V_R a proper smooth scheme over R.

Then there are canonical equivalences of spaces

$$V\hat{}_{C,cl} \xrightarrow{\sim} V\hat{}_{C,et} \xrightarrow{\sim} V\hat{}_{R,et} \xleftarrow{\sim} V\hat{}_{k,et}$$

where ^ denotes profinite completion away from p.

Quillen's observation 2:

- Frobenius maps give Adams operations.

Let V be a scheme of characteristic p and E an algebraic vector bundle over V.

Let $F: V \to V$ be the Frobenius map and write

$$E^{(p)} = F^*E.$$

Then we have an equality in $K(V)$

$$\psi^p(E) = E^{(p)}.$$

Quillen's observation 3:

- The Frobenius identifies sphere bundles.

Let E be an algebraic vector bundle over a scheme in characteristic p.

Frobenius $E \to E^{(p)}$ restricts to $E-0 \to E^{(p)}-0$

and induces an equivalence

$$(E-0)^{\wedge}_{et} \approx (E^{(p)}-0)^{\wedge}_{et}.$$

The Quillen-Friedlander proof:

First of all, since $\psi^{ab} = \psi^a \psi^b$, we can assume that $k=p$ is a prime number.

It suffices to prove the conjecture for the Grassmannian $Gr=:V$ and the canonical bundle $E \to V$.

Crucial point: The Grassmannian and the canonical bundle can be defined as schemes over the integers.

Then we should be able to apply the observations in the following way:

The Quillen-Friedlander proof:

$$\begin{tikzcd}
K(V_{C,cl}) \arrow[d, "J"] & & K(V_C) \arrow[ll] \arrow[d, "J"] & K(V) \arrow[l] \arrow[r] \arrow[d, "J"] & K(V_k) \arrow[d, "J"] \\
SF(V_{C,cl}) \arrow[r, "\Theta_L"] & SF(V\hat{}_{C,cl}) & SF(V\hat{}_{C,et}) \arrow[l] & SF(V\hat{}_{et}) \arrow[l] \arrow[r] & SF(V\hat{}_{k,et})
\end{tikzcd}$$

Observe: An element in the kernel of Θ_L is of order p^n for some n.

It suffices to show $\Theta_L(J(\psi^p E_C - E_C)) = 0$ in $SF(V\hat{}_{C,cl})$.

For then we have $p^n J(\psi^p E_C - E_C) = 0$ in $SF(V_{C,cl})$.

The Quillen-Friedlander proof:

$$\begin{tikzcd}
K(V_{C,cl}) \arrow[d, "J"] & & K(V_C) \arrow[ll] \arrow[d, "J"] & K(V) \arrow[l] \arrow[r] \arrow[d, "J"] & K(V_k) \arrow[d, "J"] \\
SF(V_{C,cl}) \arrow[r, "\Theta_L"] & SF(V^{\wedge}_{C,cl}) & SF(V^{\wedge}_{C,et}) \arrow[l, "\approx"'] & SF(V^{\wedge}_{et}) \arrow[l] \arrow[r] & SF(V^{\wedge}_{k,et})
\end{tikzcd}$$

We need to show: $J(\psi^p(E_C)-E_C) = 0$ in $SF(V^{\wedge}_{C,cl})$.

By the comparison of classical and etale homotopy types, it suffices to show:

$$J(\psi^p(E_C)-E_C) = 0 \text{ in } SF(V^{\wedge}_{C,et}).$$

The Quillen-Friedlander proof:

$$
\begin{tikzcd}
K(V_{C,cl}) \arrow[d, "J"] & & K(V_C) \arrow[ll] \arrow[d, "J"] & K(V) \arrow[l] \arrow[r] \arrow[d, "J"] & K(V_k) \arrow[d, "J"] \\
SF(V_{C,cl}) \arrow[r, "\Theta_L"] & SF(V^{\wedge}_{C,cl}) & SF(V^{\wedge}_{C,et}) \arrow[l, "\approx"'] & SF(V_{\widehat{et}}) \arrow[l, "\approx"'] \arrow[r, "\approx"] & SF(V^{\wedge}_{k,et})
\end{tikzcd}
$$

We need to show: $J(\psi^p(E_C)-E_C) = 0$ in $SF(V^{\wedge}_{C,et})$.

Since characteristic p sees homotopy,
it suffices to show:

$$J(\psi^p(E_k)-E_k) \text{ in } SF(V_{k,\widehat{et}}).$$

The Quillen-Friedlander proof:

$$
\begin{tikzcd}
K(V_{C,cl}) \arrow[d, "J"] & & K(V_C) \arrow[ll] \arrow[d, "J"] & K(V) \arrow[l] \arrow[r] \arrow[d, "J"] & K(V_k) \arrow[d, "J"] \\
SF(V_{C,cl}) \arrow[r, "\Theta_L"] & SF(V^{\wedge}_{C,cl}) & SF(V^{\wedge}_{C,et}) \arrow[l, "\approx"'] & SF(V^{\wedge}_{et}) \arrow[l, "\approx"'] \arrow[r, "\approx"] & SF(V^{\wedge}_{k,et})
\end{tikzcd}
$$

We need to show: $J(\psi^p(E_k)-E_k) = 0$ in $SF(V^{\wedge}_{k,et})$.

By "Frobenius = Adams operation" it suffices to show:

$$J(E_k^{(p)}-E_k) \text{ in } SF(V^{\wedge}_{k,et}).$$

This holds by Observation 3 and we are done!

Friedlander's theorem:

There is a very difficult point we just assumed:

- If V is a scheme over R and E an algebraic vector bundle of dimension n, then

$$(E-0)^{\wedge}_{et} \to V^{\wedge}_{et}$$

is a (completed) $(2n-1)$-sphere fibration.

In his thesis, Friedlander proved that geometric and homotopy fibers behave well under etale homotopy types, thereby proved the Adams conjecture.