Prof. Dr. Gebhard Böckle

Study group/Forschungsseminar SS 2017:

Derived Galois Deformation Rings

Thursday, 9:15 to 10:45 am, INF 205, SR2, Begin: April 20 2017

A. Venkatesh has in the past years developed a conceptual (and largely conjectural) framework to deal with modularity theorems beyond the methods begun by Taylor and Wiles that until recently all relied on a numerical coincidence and what is known as the Taylor-Wiles(-Kisin) patching. Calegari and Geraghty extended the patching to situations where the numerical coincidence no longer holds. The patching in itself appears to be a kind of trick based on some "compactness argument", or profinite"pigeon hole principle". The idea of Galatius and Venkatesh is to work in a derived or rather simplicial setting, and consider therein a deformation theory and corresponding suitable Hecke algebras (cf. [Ven16]).

The aim of the present seminar is to get a glimpse of the deformation theory involved. We will try to cover most of Sections 2-5 and 7 in [GV16], some background on model categories from [GS07] and, hopefully, in the final talk, the example from [Mor17] that in several aspects is simpler and better understood as corresponding situations over number fields.

The foundation of the seminar consists of some homotopy theory in the "Quillen style". I will try to give an overview of some relevant material on Model Categories following Goerss-Schemmerhorn in the first 3 talks of the seminar. There are strong links to homotopy groups of (nice) topological spaces on the one hand and derived categories of modules over a ring on the other. Both will appear as examples.

The second part of the seminar will begin with a review of the "classical theory" of deformations of Galois representations. After that we shall try to cover in much of the remaining seminar the foundations of the analog in the derived setting of [GV16]. This will lead to a proof that the simplicial deformation ring exists in the derived setting of op.cit. An important tool is Lurie's simplicial analog of Schlessinger's pro-representability criterion. Along the way we develop basic tools to understand some aspects of the derived ring of op.cit. The last talk on [Mor17] should give some insights into the computation of a not so difficult but still interesting derived deformation ring. We will not be able to say much, if anything, about the later parts (Sections 8-15) of [GV16] in the seminar.

1 Model categories I

Speaker: G. Böckle 20.04.

2 Model categories II

Speaker: G. Böckle 27.04.

3 Model categories III

Speaker: G. Böckle 04.05.

4 The classical theory of deformations of Galois representations

You can say some words about motivation, but be brief on that and specialize the group Π from [Gou01] right away to be a Galois group. Give the definition of the deformation functor **D** and explain the statement of Lemma 2.3, but without a detailled proof. Next, cover the topic of representability of functors and the Mayer-Vietoris Property. Explain why this property is a necessary condition for representability and make clear the significance of the category $\mathcal{C}^0$. Then, jump directly to Lecture 3 and explain the Schlessinger approach on representability. While you can be very brief on the proof of Theorem 3.1 (or even omit it completely), give as much details on the proof of Theorem 3.3 as time permits. Finish your talk with the definition of the tangent space, the connections with cohomology groups and Theorem 4.2 (you will not have enough time to give a full proof, but possibly a short outline). If there is some time left, add a remark on the dimension conjecture and the Leopold conjecture (as given by Gouvêa just before Problem 4.14).

Sources: [Gou01, Lectures 2-4]

Speaker: NN 11.05.

5 Simplicial Artin rings and Functors

This talk should cover the first half of Section 2 of [GV16] ending with Subsection 2.3. This includes Subsection 3.1 and parts of Appendix A.

Concretely begin with Subsection 2.1 that recalls basics on simplicial commutative rings, that should large have been mentioned in the first three talks above. Then discuss Appendix A on homotopy limits and colimits up to and including Example A.4. Next treat Subsection 2.2 on Simplicial Artin Rings. Insert right before Lemma 2.8 a discussion of Subsection 3.1 on Postnikov truncations. Then finish Subsection 2.2 and also cover Subsection 2.3 on functors $\mathbf{Art}_k \to s\mathbf{Sets}$ and natural transformations.

Sources: [GV16, Subsects. 2.1, 2.2, 2.3, 3.1, App. A]

Speaker: NN 18.05.

No talk on 25.05.

Christi Himmelfahrt (Holiday) 25.05.

6 (Pro-)representable Functors on simplicial Artin rings

This talk completes Section 2 of [GV16] and then treats the first half of Section 3.

Begin with Subsection 2.4 on representable Functors $\mathrm{Art}_k \to s\mathbf{Sets}$. Subsection 2.5, on Approximation by Representable Functors, introduces some important technical result. Subsection 2.6 extends the definition from 2.4. to a pro-category built out of $\mathbf{Art}_k$. Introduce this category pro-$\mathbf{Art}_k$ (there seems no explicit definition in [GV16]) and discuss the other results of 2.6. Then present Subsection 3.2 on the tensor product of (pro-)simplicial rings, and end the talk by giving the examples of pro-representable functors from Subsection 3.3.

Sources: [GV16, Subsects. 2.4, 2.5, 2.6, 3.2, 3.3]

Speaker: NN 01.06.

7 The cohomology of simplicial (pro-)Artin rings and the Dold-Kan correspondence

First cover Subsections 4.1 and 4.2 from [GV16] on the cohomology of (pro-)Artin rings and cell structures on pro-rings. Then go back to Section 3 and introduce formally cohesive functors from Subsection 3.4. End the talk be presenting the first parts of Subsection 4.3, i.e. 4.3.1 and 4.3.2 up to and including Example 4.11.

Sources: [GV16, Subsects. 3.4, 4.1, 4.2, 4.3]

Speaker: NN 08.06.

No talk on 15.06.

Fronleichnam (Holiday) 15.06.

8 Tangent complexes of rings and functors

This talk is the main preparation for the following one on Lurie's simplicial analog of Schlessinger's pro-representability criterion. It covers the central part of Section 4. Continue in Subsection 4.3 with the remaining results on Γ-sets and Γ-spaces and the cover the remainder of Subsection 4.3. In the last third of the talk describe the content of Subsection 4.4 on the tangent complex of a formally cohesive functor as an Hk module spectrum..

Sources: [GV16, Subsects. 4.3, 4.4]

Speaker: NN 22.06.

9 (Presumably) No talk on 29.06.

Conference in Münster 29.06.

10 Lurie's derived Schlessinger criterion

The main result of this talk is Lurie's derived Schlessinger criterion in Subsection 4.6 which still needs 4.5 on the constructions on cohesive functors and their effect on tangent complexes as further preparation. The end of the talk should cover Subsection 4.7 on non-reduced functors and local systems. It covers an extension of Lurie's result and is necessary for most applications, since deformation functors tend not to be reduced.

Sources: [GV16, Subsects. 4.5, 4.6, 4.7]

Speaker: NN 06.07.

11 Representation functors

The aim of this talk is to cover all of Section 5, which defines and studies an "infinitesimal representation variety" functor, parametrizing representations into an algebraic group G defined over the ring of Witt vectors $W(\mathbb{F}_q)$. This is the simplicial generalization of the usual deformation functors.

Sources: [GV16, Sect. 5]

Speaker: NN 13.07.

12 Deformation-theory notation

Present Subsections 7.1, 7.2, 7.3 from [GV16], on representable functors, the tangent complex and Galois representations. In Subsection 7.3 it is suggested to skip the étale homotopy type, but to work with the more elementary setting suggested by [GV16]. Also for Subsection 7.3, look up Γ_S from 6.1 and 6.2. Omit Subsection 7.4. In Subsection 7.5, cover the results numbered 7.1, 7.2, 7.3 and 7.5, and give Definition 7.4, but possibly leave out Lemma 7.6. These results rely on earlier parts of Subsection 7. The results are used in [Mor17]

Sources: [GV16, Sect. 7, Subsects. 6.1, 6.2]

Speaker: NN 20.07.

13 Derived deformation functors and a conjecture of de Jong

Explain the results of the preprint [Mor17].

Speaker: NN 27.07.

Literatur

[Gou01] F. Q. Gouvêa, Deformations of Galois representations, in *Arithmetic algebraic geometry (Park City, UT, 1999)*, volume 9 of *IAS/Park City Math. Ser.*, pages 233–406, Amer. Math. Soc., Providence, RI, 2001, Appendix 1 by Mark Dickinson, Appendix 2 by Tom Weston and Appendix 3 by Matthew Emerton.

[GS07] P. Goerss and K. Schemmerhorn, Model categories and simplicial methods, in *Interactions between homotopy theory and algebra*, volume 436 of *Contemp. Math.*, pages 3–49, Amer. Math. Soc., Providence, RI, 2007.

[GV16] S. Galatius and A. Venkatesh, Derived Galois Deformation Rings, arXiv:1608.07236v2, 2016.

[Mor17] S. Morel, Note sur les foncteurs de déformations dérivés et la conjecture de de Jong, preprint, https://web.math.princeton.edu/∼smorel/, 2017.

[Ven16] A. Venkatesh, Derived Hecke algebra and cohomology of arithmetic groups, arXiv:1608.07234, 2016.