Transcendence of zeros of Jacobi forms

YoungJu Choie and Winfried Kohnen

Abstract: A special case of a fundamental theorem of Schneider asserts that if $j(\tau)$ is algebraic (where j is the classical modular invariant), then any zero z not in $\mathbf{Q}.L_\tau := \mathbf{Q} \oplus \mathbf{Q}\tau$ of the Weierstrass function $\wp(\tau, \cdot)$ attached to the lattice $L_\tau = \mathbf{Z} \oplus \mathbf{Z}\tau$ is transcendental. In this note we generalize this result to holomorphic Jacobi forms of weight k and index $m \in \mathbf{N}$ with algebraic Fourier coefficients.

Keywords: Jacobi forms, zeros, transcendency

1. Introduction

For $\tau \in \mathcal{H}$, the complex upper half-plane, and $z \in \mathbf{C} \setminus L_\tau$ we set

$$\wp(\tau, z) := \frac{1}{z^2} + \sum_{\omega \neq 0} \Big(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \Big)$$

where ω runs over all non-zero points in the lattice $L_\tau := \mathbf{Z} \oplus \mathbf{Z}\tau$. Thus for τ fixed, $\wp(\tau, z)$ is the Weierstrass $\wp$ -function for the lattice L_τ. It is well-known that $\wp(\tau, z)$ is a meromorphic Jacobi form of weight 2 and index zero [5].

Let

$$g_2(\tau) := 60 \sum_{(m,n) \neq 0} \frac{1}{(m\tau + n)^4}, \quad g_3(\tau) := 140 \sum_{(m,n) \neq 0} \frac{1}{(m\tau + n)^6}$$

be the Weierstrass invariants attached to L_τ and let

$$j(\tau) := \frac{12^3 g_2^3(\tau)}{g_2^3(\tau) - 27 g_3^2(\tau)}$$

be the classical j-invariant.

Assume that $j(\tau)$ is algebraic and let $\omega \in \mathbf{C}^*$ be such that $\omega^{-4} g_2(\tau), \omega^{-6} g_3(\tau) \in \overline{\mathbf{Q}}$ which clearly is possible. Indeed, if $g_2(\tau) = 0$ our assertion is trivial. In the other case we choose ω with $\omega^{-4} g_2(\tau) \in \overline{\mathbf{Q}}^*$ and use the definition of $j(\tau)$, with numerator and denominator multiplied by ω^{-12} which then implies that $\omega^{-6} g_3(\tau) \in \overline{\mathbf{Q}}$ as well.

Note that the quantities $\omega^{-4} g_2(\tau)$ and $\omega^{-6} g_3(\tau)$ are the Weierstrass invariants of the lattice $\mathcal{L} = \mathbf{Z}\omega \oplus \mathbf{Z}\omega' = \omega L_\tau$ with $\tau = \frac{\omega'}{\omega}$ and the Weierstrass $\wp$-function of $\mathcal{L}$ is given by $\wp_{\mathcal{L}}(u) = \omega^{-2} \wp(\tau, z)$ with $z = \frac{u}{\omega}$.

Then a fundamental theorem of Schneider [7] applied to $\wp_{\mathcal{L}}(u)$ asserts that if z is algebraic and not in $\mathbf{Q}.L_\tau := \mathbf{Q} \oplus \mathbf{Q}\tau$, then the value $\wp_{\mathcal{L}}(u)$ is necessarily transcendental. In particular, as a very special case if $j(\tau)$ is algebraic, then any zero z not in $\mathbf{Q}.L_\tau$ of $\wp(\tau, \cdot)$ is transcendental. (For an explicit description of the zeros of $\wp(\tau, \cdot)$ we refer to [4,5]).

The purpose of this note is to show that the latter statement generalizes to arbitrary Jacobi forms of weight k and index $m \in \mathbf{N}$ with algebraic Fourier coefficients. Although the proof is rather straightforward and follows basically the same line of arguments as in [3, sect. 3], we think that the statement so far has not been given explicitly in the literature. Regarding transcendence results for elliptic modular forms we refer to [2, Cor. 2] and [6].

2. Statement of result and proof

For basic facts on Jacobi forms we refer to [5]. Recall that a Jacobi form of weight $k \in \mathbf{Z}$ and index $m \in \mathbf{N}$ is a holomorphic function $\phi : \mathcal{H} \times \mathbf{C} \to \mathbf{C}$ such that

$$i) \quad \phi\Big(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\Big) = (c\tau + d)^k \, exp\Big(2\pi i m \frac{cz^2}{c\tau + d}\Big)\phi(\tau, z) \quad \Big(\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1 := SL_2(\mathbf{Z})\Big),$$

$$ii) \quad \phi(\tau, z + \lambda\tau + \mu) = exp(-2\pi i m(\lambda^2\tau + 2\lambda z))\phi(\tau, z) \quad (\forall (\lambda, \mu) \in \mathbf{Z}^2),$$

and
iii) ϕ has a Fourier expansion of the form

$$\phi(\tau, z) = \sum_{n \geq 1, r \in \mathbf{Z}, r^2 \leq 4mn} c(n, r) q^n \zeta^r \quad (q = e^{2\pi i \tau}, \zeta = e^{2\pi i z}).$$

If $c(n, r) = 0$ for $r^2 = 4mn$ then ϕ is called a Jacobi cusp form. If all the Fourier coefficients $c(n, r)$ are algebraic numbers, we say that ϕ has an algebraic Fourier expansion.

We recall that for given τ the function $\phi(\tau, \cdot)$, if not identically zero, has exactly $2m$ zeros in any fundamental domain for the action of the lattice L_τ on $\mathbf{C}$.

In a similar way one can define meromorphic Jacobi forms and generalize the notion of an algebraic Fourier expansion. For details see [1] or [3, sect. 2].

Theorem. *Let $\tau_0 \in \mathcal{H}$ be fixed and suppose that $j(\tau_0)$ is algebraic. Let $\phi(\tau, z)$ be a Jacobi form of weight k and index m with an algebraic Fourier expansion and suppose that $\phi(\tau_0, \cdot)$ is not the zero function. Then any zero z_0 of $\phi(\tau_0, \cdot)$ not contained in $\mathbf{Q}.L_{\tau_0}$ is transcendental.*

Proof. We define the "first Weber function" by

$$w(\tau, z) := \Big(-2^7 \cdot 3^5 \cdot \frac{g_2(\tau) g_3(\tau)}{g_2^3(\tau) - 27 g_3^2(\tau)}\Big) \wp(\tau, z) \qquad (\tau \in \mathcal{H}, z \in \mathbf{C}).$$

Note that $(2\pi i)^{-12}(g_2^3(\tau) - 27g_3^2(\tau))$ is the classical discriminant function $\Delta(\tau)$.

A basic result of [1] says that the field of meromorphic Jacobi forms of weight zero and index zero with an algebraic Fourier expansion is equal to $\overline{\mathbf{Q}}(j, w)$.

Let $\phi_{10,1}(\tau, z)$ be the unique Jacobi cusp form of weight 10 and index 1 whose Taylor expansion around $z = 0$ is of the form

$$(2\pi i)^2\Delta(\tau)z^2 + \mathcal{O}(z^4).$$

The function $\phi_{10,1}(\tau, \cdot)$ vanishes doubly at $z = 0$ and hence nowhere else outside of L_τ. Also $\phi_{10,1}$ has rational Fourier coefficients. (For all this see [5] or [3, sect. 2].)

Let

$$\psi := \frac{\phi^{24}}{\phi_{10,1}^{24m}\Delta^{2k-20m}}.$$

Then ψ is a meromorphic Jacobi form of weight zero and index zero with an algebraic Fourier expansion, hence can be written as a quotient of two polynomials in w with coefficients in $\overline{\mathbf{Q}}(j)$.

We observe that the pole set of $\psi(\tau, \cdot)$ is contained in L_τ, for any $\tau \in \mathcal{H}$. By assumption, $z_0 \notin L_{\tau_0}$. For z close to z_0 we then obtain an equation

$$\psi(\tau_0, z) = \frac{P(j(\tau_0), w(\tau_0, z))}{Q((j(\tau_0), w(\tau_0, z))} \tag{1}$$

where (after clearing denominators) $P(j, w)$ and $Q(j, w)$ are polynomials in w with coefficients in $\in \overline{\mathbf{Q}}[j]$, and the denominator in (1) is non-zero.

Let M be the degree of $P(j, w)$ and denote the coefficients of $P(j, w)$ by $a_m(j)$ $(0 \le m \le M)$. Since $\psi(\tau_0, z_0) = 0$, evaluating (1) at $z = z_0$ we obtain

$$0 = \sum_{m=0}^{M} a_m(j(\tau_0))w(\tau_0, z_0)^m$$

with $a_m(j(\tau_0)) \in \overline{\mathbf{Q}}$ for all m, since $j(\tau_0) \in \overline{\mathbf{Q}}$. Observe that there must be $m \ge 1$ such that $a_m(j(\tau_0)) \ne 0$. Indeed, otherwise $a_m(j(\tau_0)) = 0$ for all m, hence by (1) $\psi(\tau_0, \cdot)$ and so $\phi(\tau_0, \cdot)$ would be identically zero, a contradiction to our hypothesis.

We therefore concude that $w(\tau_0, z_0)$ is algebraic.

Suppose that $g_2(\tau_0) = 0$ or $g_3(\tau_0) = 0$ (which by the valence formula is equivalent to saying that τ_0 is Γ_1 -equivalent to $\rho := e^{2\pi i/3}$ resp. i). Then $w(\tau_0, \cdot)$ is zero outside of L_{τ_0}, and in the same way as above we conclude that $a_0(j(\tau_0)) = 0$ and so again $\phi(\tau_0, \cdot) = 0$, a contradiction. Thus we see that the pre-factor of $w(\tau_0, \cdot)$ is non-zero.

Note that this pre-factor can be written as

$$\Big(-2^7 \cdot 3^5 \cdot \frac{\omega^{-4}g_2(\tau_0) \cdot \omega^{-6}g_3(\tau_0)}{(\omega^{-4}g_2(\tau_0))^3 - 27(\omega^{-6}g_3(\tau_0))^2}\Big) \cdot \omega^{-2}$$

for any $\omega \in \mathbf{C}^*$.

Hence choosing ω such that $\omega^{-4} g_2(\tau_0)$ is contained in $\overline{\mathbf{Q}}^*$, one sees that the Weber function is essentially given by $\omega^{-2}\wp(\tau, z)$ and so $\omega^{-2}\wp(\tau_0, z_0)$ is algebraic. By the theorem of Schneider described in sect. 1, since z_0 is not contained in $\mathbf{Q}.L_{\tau_0}$, we therefore conclude that z_0 is transcendental.

Acknowledgements. The authors thank the referee for useful comments. The first author was partially supported by NRF-2015049582, NRF-NRF-2013R1A2A2A01068676.

References

[1] R. Berndt: Zur Arithmetik der elliptischen Funktionenkörper höherer Stufe, J. Reine Angew. Math. 326 (1981), 79-94

[2] J.H. Bruinier, W. Kohnen and K. Ono: The arithmetic of the values of modular functions and the divisors of modular forms, Compos. Math. 140 (2004), 552-566

[3] Y. Choie and W. Kohnen: Special values of elliptic functions at points of the divisors of Jacobi forms, Proc. AMS 131, no. 11 (2003), 3309-3317

[4] W. Duke and Ö. Imamoglu: The zeros of the Weierstrass $\wp$-function and hypergeometric series, Math. Ann. 340 (2008), 897-905

[5] M. Eichler and D. Zagier: The theory of Jacobi forms, Progress in Math. vol. 55, Birkhäuser: Boston 1985

[6] S. Gun, M. Ram Murty and P. Rath: Algebraic independence of values of modular forms, Int. J. Number Theory vol. 7, no. 4 (2011), 1069-1074

[7] T. Schneider: Arithmetische Untersuchungen elliptischer Integrale, Math. Ann. 113 (1937), 1-13

Department of Mathematics, Pohang Institute of Science and Technology, Pohang 790-784, Korea
E-Mail address: yjc@postech.ac.kr

Mathematisches Institut, Universität Heidelberg, INF 288, D-69120 Heidelberg, Germany
E-Mail address: winfried@mathi.uni-heidelberg.de