Sign changes of Fourier coefficients of cusp forms supported on prime power indices

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Abstract

Let f be an integral weight, normalized, cuspidal Hecke eigenform over $SL_2(\mathbb{Z})$ with Fourier coefficients $a(n)$. Let j be a positive integer. We prove that for almost all primes p the sequence $(a(p^{jn})_{n\geq 0}$ has infinitely many sign changes. We also obtain a similar result for any cusp form with real Fourier coefficients provide that the characteristic polynomial of some generalized Hecke operator is irreducible over $\mathbb{Q}$.

1 Introduction

Let f be a normalized Hecke eigenform of integral weight k on $\Gamma_1 := SL_2(\mathbb{Z})$ with Fourier coefficients $a(n)$ $(n \geq 1)$. In [6] it was proved that each one of the sequences $(a(n^j))_{n\geq 1}$, where $j \in \{2,3,4\}$, have infinitely many sign changes. The proof is based on suitable bounds for the sums

$$\sum_{n\leq x} a(n^j) \quad \text{and} \quad \sum_{n\leq x} a(n^j)^2$$

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given in [3], [5] which in turn are obtained from a careful analysis of various L-functions attached to f.

In this paper we will show that if j is a positive integer then for almost all primes p the sequence $(a(p^{jn}))_{n\geq 0}$ has infinitely many sign changes (Theorem 1). The proof is elementary and relies on Landau's classical theorem on Dirichlet series with non-negative coefficients, on Deligne's bound for the coefficients $a(p)$, and on properties of certain Hecke operators $T_j(p)$ (which in the case $j=1$ are just the classical Hecke operators $T(p)$).

Under the hypothesis that the characteristic polynomial of $T_j(p)$ (which has rational coefficients) is irreducible over $\mathbb{Q}$, a corresponding similar result in fact is true for any non-zero cusp form of weight k on Γ_1 with real coefficients.

2 Statement of results

We say that a sequence $(c(n))_{n\geq 1}$ of real numbers has infinitely many sign changes if there exist infinitely many n such that $c(n) > 0$ and there exist infinitely many n such that $c(n) < 0$.

We let S_k be the space of cusp forms of weight k over Γ_1.

Theorem 1. *Let $f \in S_k$ be a normalized Hecke eigenform with Fourier coefficients $a(n)$ $(n \geq 1)$. Let j be a positive integer. Then there exists a positive integer $N = N(f, j)$ depending only on f and j such that for all primes $p > N$ the sequence $(a(p^{jn}))_{n\geq 0}$ has infinitely many sign changes.*

For each non-negative integer j and a prime number p we define an operator $T_j(p) = T_{j,k}(p)$ on formal power series in q (with zero constant term) by the rule

$$(1) \qquad \sum_{n\geq 1} c(n)q^n \mapsto \sum_{n\geq 1} \left(c(p^j n) + p^{j(k-1)} c\left(\frac{n}{p^j}\right) \right) q^n$$

(with the usual convention that $c(n/p^j) = 0$ if p^j does not divides n). Note that $T_0(p) = 2$ and $T_1(p) = T(p)$ is the classical p-th Hecke operator. Then $T_j(p)$ maps S_k to itself. In fact, as it is easy to see (cf. Lemma) $T_j(p)$ for any $j \geq 1$ is a monic polynomial of degree j in $T(p)$ with integer coefficients. Since S_k has a basis of cusp forms with rational Fourier coefficients, it follows that the characteristic polynomial $\chi(T_j(p), X)$ of $T_j(p)$ on S_k has rational coefficients.

Theorem 2. *Suppose that $\chi(T_j(p), X)$ is irreducible over $\mathbb{Q}$. Let f be any non-zero cusp form in S_k with real Fourier coefficients $a(n)$ $(n \geq 1)$. Then there is a positive integer $N = N(f, j)$ depending only on f and j such that for all primes $p > N$ the sequence $(a(p^{jn}))_{n\geq 0}$ changes sign infinitely often.*

A well-known conjecture of Maeda states that $\chi(T(p), X)$ always is irreducible over $\mathbb{Q}$. One knows that this is indeed the true for all $k \leq 2000$ and for all primes $p < 2000$ [2]. One also knows that if the characteristic polynomial of the n-th Hecke operator $T(n)$ is irreducible over $\mathbb{Q}$ for some n, then the same is true for $\chi(T(p), X)$ for almost all primes [1].

As natural examples in the above context, one may take the cuspidal part (if not zero) of theta series attached to a positive definite unimodular even integral quadratic form, or the difference (if not zero) of two such theta series with the associated quadratic forms lying in the same genus.

Remark. It is certainly possible to prove Theorem 1 in a different way using the very explicit and elementary identity

$$a(p^n) = p^{(k-1/2)n} \frac{\sin(n+1)\theta_p}{\sin\theta_p}$$

where $\theta_p \in [0, \pi]$ is defined by

$$a(p) = 2p^{k-1/2} \cos\theta_p$$

and one supposes that θ_p is different from 0 and π, cf. [4], pp. 39-40. Indeed, using the known behavior of the sequence $(\sin 2\pi nt)_{n\geq 1}$ for a real number t, one sees that $(a(p^{jn}))_{n\geq 0}$ will in fact take signs plus resp. minus approximately equally often whenever $\frac{\theta_p}{2\pi}$ is irrational. On the other hand, if this quantity is rational, taking j to be the denominator leads to a sequence of fixed sign, so that there is essentially a criterion for sign changes to arise.

We do think that our more indirect proof has some advantages, since similar arguments may also work in other situations, e.g. for Siegel modular forms, where simple explicit formulas as above in general are not available. (For Siegel cusp forms of genus 2 cf. e.g. [7].) In addition, in order to prove Theorem 2 which relies on Theorem 1 it seems necessary to proceed as we did here. On the other hand, our argument do not apply in the context of Maass forms, due to the lack of any algebraicity result for eigenvalues.

3 Proofs

Let us start with a technical result.

Lemma 3. *Let $j \geq 1$ and define $T_j(p)$ by (1). Then the following holds:*

i) $T_j(p)$ maps S_k to itself. In fact, $T_j(p)$ is a monic polynomial in $T(p)$ of degree j with integral coefficients.

ii) If $g \in S_k$ is an eigenfunction of $T_j(p)$ with eigenvalue $\lambda_j(p)$ and $b(n)$ ($n \geq 1$) are the Fourier coefficients of g, then

$$\sum_{n\geq 0} b(p^{jn})X^n = \frac{b(1)}{1-\lambda_j(p)X + p^{j(k-1)}X^2}.$$

Proof. i) From (1) one immediately verifies that for all $j \geq 1$ one has

$$T_j(p)T(p) = T_{j+1}(p) + p^{k-1}T_{j-1}(p). \tag{2}$$

Hence our claim follows by induction on $j \geq 1$, together with $T_0(p) = 2$ and $T_1(p) = T(p)$.

ii) The proof works in the same way as for $j = 1$, where one finds the classical Hecke relations for the Fourier coefficients of g. □

Remark. i) From (2) one finds for the first few cases $j \in \{2, 3, 4\}$ that

$$\begin{aligned} T_2(p) &= T(p)^2 - 2p^{k-1}, \\ T_3(p) &= T(p)^3 - 3p^{k-1}T(p), \\ T_4(p) &= T(p)^4 - 4p^{k-1}T(p)^2 + 2p^{k-1}. \end{aligned}$$

ii) Note that the recursion (2) is similar to the classical Hecke recursion relating $T(p^j)T(p)$ and $T(p^{j+1})$, but with the different initial condition $T_0(p) = 2$.

We now proceed to the proof of Theorem 1. Let f be a normalized Hecke eigenform with Fourier coefficients $a(n)$ ($n \geq 1$). Let p be a prime and assume that the sequence $(a(p^{jn}))_{n\geq 0}$ does not have infinitely many sign changes.

Invoking Landau's theorem on Dirichlet series with non-negative coefficients we deduce that the Dirichlet series

$$\sum_{n\geq 0} a(p^{jn})p^{-jns}, \quad (\Re(s) \gg 1) \tag{3}$$

(supported on powers of p^j) either must have a singularity at the real point of its abscissa of convergence or must converge for all $s \in \mathbb{C}$. We want to deduce a contradiction in both cases for p large.

We start by considering the first case. Let

$$P(X) = \sum_{n \geq 0} a(p^n) X^n.$$

Then

$$P(X) = \frac{1}{1 - a(p)X + p^{k-1}X^2} = \frac{1}{(1-\alpha_p X)(1-\beta_p X)} \tag{4}$$

where

$$\alpha_p, \beta_p = \frac{a(p) \pm \sqrt{a(p)^2 - 4p^{k-1}}}{2}. \tag{5}$$

Note that $a(p)^2 \leq 4p^{k-1}$ by Deligne's theorem, i.e. α_p and β_p are complex conjugates numbers.

Let $\zeta := e^{2\pi i/j}$ be a primitive j-th root of unity and let $\nu \in \mathbb{Z}$. Then the classical orthogonality relation

$$\sum_{\mu=0}^{j-1} \zeta^{\mu\nu} = \begin{cases} j & \text{if } \nu \equiv 0 \ (mod\ j) \\ 0 & \text{if } \nu \not\equiv 0 \ (mod\ j) \end{cases}$$

implies

$$\sum_{n \geq 0} a(p^{jn}) X^{jn} = \frac{1}{j} \sum_{\mu=0}^{j-1} P(\zeta^\mu X)$$

and hence replacing $X = p^{-s}$ we find from (4) that

$$\sum_{n \geq 0} a(p^{jn}) p^{-jns} = \frac{1}{j} \sum_{\mu=0}^{j-1} \frac{1}{(1-\alpha_p \zeta^\mu p^{-s})(1-\overline{\alpha_p}\zeta^\mu p^{-s})} \quad (\Re(s) \gg 1). \tag{6}$$

Note that the right-hand side of (6) gives a meromorphic continuation of the Dirichlet series (3) to $\mathbb{C}$.

We will investigate when one of the denominators on the right-hand side of (4) has a real zero. In this case necessarily at least one of the numbers $\alpha_p \zeta^\mu$, $\overline{\alpha_p}\zeta^\mu$ is real. Suppose that $\alpha_p \zeta^\mu = \nu \in \mathbb{R}$. Then also $\overline{\alpha_p}\zeta^{-\mu} = \nu$ and a short calculation using (5) shows that $\nu = \pm p^{k-1/2}$ and so

$$a(p) = \alpha_p + \overline{\alpha_p} = \pm p^{k-1/2}(\zeta^\mu + \zeta^{-\mu}).$$

We get the same result if we start with the condition that $\overline{\alpha_p}\zeta^\mu$ is real.

Now suppose there are infinitely many primes p for which there are integers μ_p (mod j) such that

$$a(p) = \pm p^{k-1/2}(\zeta^{\mu_p} + \zeta^{-\mu_p}). \tag{7}$$

Recall that the field

$$K_f := \mathbb{Q}(\{a(l)\}_l)$$

obtained from $\mathbb{Q}$ by adjoining all the numbers $a(l)$ (l prime) is a number field, i.e. a field of finite degree over $\mathbb{Q}$ and the degree only depends on the Hecke eigenform f. From (7) we deduce that

$$\sqrt{p}(\zeta^{\mu_p} + \zeta^{-\mu_p}) \in K_f$$

for all our relevant primes p.

Since there are only finitely many choices for μ_p mod j, going over to a subsequence of the sequence of primes as above, we conclude that there are infinitely many primes $p_1, p_2, p_3, \ldots$ and a suited μ mod j such that

$$\mathbb{Q}(\sqrt{p_1}(\zeta^{\mu} + \zeta^{-\mu}), \sqrt{p_2}(\zeta^{\mu} + \zeta^{-\mu}), \sqrt{p_3}(\zeta^{\mu} + \zeta^{-\mu}), \ldots) \subseteq K_f.$$

Taking quotients we finally deduce that

$$\mathbb{Q}(\sqrt{p_1 p_2}, \sqrt{p_1 p_3}, \ldots) \subseteq K_f.$$

However, the degree of $\mathbb{Q}(\sqrt{p_1 p_2}, \ldots, \sqrt{p_1 p_r})$ over $\mathbb{Q}$ is 2^{r-1} as is well known and easy to see. Thus letting $r \to \infty$ we obtain a contradiction.

We therefore find that for almost all primes p the right hand side of (6) has no real poles.

From Landau's theorem, for those primes p we deduce that the series (3) converges everywhere, in particular is an entire function in s. However, applying part ii) of the Lemma with $g = f$ we see that

$$\sum_{n\geq 0} a(p^{jn}) X^{jn} = \frac{1}{1 - \lambda_j(p) X^j + p^{j(k-1)} X^{2j}}$$

and the denominator on the right-hand side is a polynomial in X^j of degree 2, hence it is non-constant and so has zeros. Setting $X = p^{-s}$ we then obtain a contradiction.

Hence for almost all primes p the sequence $(a(p^{jn}))_{n\geq 0}$ must indeed have infinitely many sign changes. This proves the Theorem. $\square$

We will now prove Theorem 2. Let f be an arbitrary non-zero cusp form in S_k with real Fourier coefficients $a(n)$ $(n \geq 1)$. Let p be a prime and assume that $(a(p^{jn}))_{n\geq 0}$ does not have infinitely many sign changes. Then, as before, by Landau's Theorem the series

$$\sum_{n\geq 0} a(p^{jn})p^{-jns} \quad (\Re(s) \gg 1)$$

either has a singularity at the real point of its abscissa of convergence or must converge for all s.

Write f as a linear combination of the basis $f_1, f_2, \dots f_h$ of S_k consisting of normalized Hecke eigenforms. Then from the proof of Theorem 1 we deduce that for almost all primes the first possibility cannot hold.

Fix such a p and consider the subspace $V_p \subseteq S_k$ consisting of all cusp forms g whose Fourier coefficients $b(p^{jn})$ satisfy the bound

$$b(p^{jn}) \ll_{g,c} p^{jnc}$$

for all $n \geq 0$, for every $c \in \mathbb{R}$. Then $f \in V_p$.

On the other hand, V_p is stable under $T_j(p)$, as is immediate from the definition (1). Since $T(p)$ and so $T_j(p)$ is hermitian on S_k, if the space V_p is not the zero space it must contain an eigenfunction f_0 of $T_j(p)$.

By hypothesis, $\chi(T_j(p), X)$ is irreducible over $\mathbb{Q}$, hence $cf_0 \in \{f_1, f_2, \dots, f_h\}$ for some $c \in \mathbb{C}^*$. Indeed, the hypothesis implies that $\chi(T_j(p), X)$ is a separable polynomial over $\overline{\mathbb{Q}}$, i.e. has no multiple roots over $\overline{\mathbb{Q}}$, hence each eigenspace of $T_j(p)$ is of dimension 1. Since $T_j(p)$ commutes with any element in the Hecke algebra, f_0 must be an eigenfunction of all Hecke operators, and our assumption follows. Thus if $V_p \neq \{0\}$, we conclude that one of the normalized Hecke eigenforms f_ν $(\nu \in \{1, 2, \dots, h\})$ is contained in V_p which we have already ruled out by the argument in the final part of the proof of Theorem 1. We thus derive the assertion of Theorem 2. □

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