ON SIGN CHANGES OF EIGENVALUES OF SIEGEL CUSP FORMS OF GENUS 2 IN PRIME POWERS

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Abstract. We prove that under mild assumptions, there are infinitely many sign changes in the sequence of eigenvalues of a genus 2 Siegel cuspidal eigenform supported on p^{jn} for a fixed $j \geq 1$.

1. Introduction

Sign changes of eigenvalues of elliptic modular forms has by now been well studied. In the case of Siegel modular forms, W. Kohnen proved in [8] that the sequence $\{\lambda_F(n)\}_{n\geq 1}$ of eigenvalues of a cuspidal eigen form F of genus 2 for $\mathrm{Sp}_2(\mathbf{Z})$ (where F is not in the Maass space for the weight k even), has infinitely many sign changes, using the analytic properties of the spinor zeta function attached to F and Landau's theorem on Dirichlet series. This is in contrast to the situation when F is in the Maass space (i.e., F is a Saito-Kurokawa lifting) if and only if $_F(n) > 0$ for all n; see Breulmann's result in [4]. Recently, the work of Royer et. al. [14] proves a rather strong result that the elements of the sequence $\{\lambda_F(n)\}$ are positive or negative equally often.

Variants and refinements of the aforementioned results also exist, especially our interest is in the context of the sequence $\{\lambda_F(n)\}$ restricted to primes and prime powers; namely in [5], one of the authors proved in particular that the natural density of primes p such that $\lambda_F(p) \lesseqgtr 0$ is positive. Moreover an appendix by Kowalski and Saha to the paper of Royer et. al [14] alluded to above also says that this density is one.

Towards refinements, Pitale proved in [13] that if F is not in the Maass space, then already there are infinitely many sign changes in the sequence $\{\lambda_F(p^n)\}_{n\geq 1}$, for infinitely many primes p. Moreover the dichotomy of the eigenvalues being

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positive or changing signs was explained from the point of view of representation theory by distinguishing the form F as being a 'theta-lift' versus a non theta-lift. See [13] for more details.

We should remark here that a similar result was proved by Kohnen and Martin [9] on eigenforms for $\mathrm{SL}_2(\mathbf{Z})$: there are infinitely many sign changes in the sequence $\{\lambda_F(p^{jn})\}_n$ for a fixed prime p and an integer $j \geq 1$, where one has to add the hypothesis that 4 does not divide j.

The purpose of this paper is to prove a generalisation of the result loc. cit. for the case of Siegel cusp forms of genus 2. Even though our arguments are similar in spirit to [9], namely that we analyze the p-Euler factors of the respective zeta functions attached to F, the details are naturally more involved due to the difference in the degree of the p-Euler factors. We now proceed to state our main result precisely.

For a positive integer k we denote by $S_k(\Gamma_2)$ the space of Siegel cusp forms of weight k and genus 2 for the full Siegel modular group $\Gamma_2 := \mathrm{Sp}_2(\mathbf{Z})$. If $F \in S_k(\Gamma_2)$ is a Hecke eigenform, then for a prime p and a non-negative integer n we denote by $\lambda_F(p^n)$ the eigenvalue of F under the Hecke operator $T(p^n)$. For these notation and definitions, see the next section.

Theorem 1.1. *Let $f \in S_k(\Gamma_2)$ be a Hecke eigenform and assume that for k even f is not contained in the Maass subspace. Let j be a fixed positive integer not divisible by 4. Then the set of primes p such that the sequence $\{\lambda_F(p^{jn})\}_{n\geq 0}$ has infinitely many sign changes has natural density* 1.

2. Notations and preliminaries

For basic facts about Siegel modular forms we refer to [1, 7]. We briefly recall some facts about eigenvalues of an Hecke eigenform $F \in S_k(\Gamma_2)$. To start with, for $n \in \mathbb{N}$, one defines the Hecke operator $T(n)$ on $S_k(\Gamma_2)$ by

$$T(n)F = \sum_{\gamma\in\Gamma_2\backslash\Delta_{2,n}} F \mid_k \gamma,$$

where $\Delta_{2,n}$ is the set of integral symplectic similitudes of size 4 and scale n. Moreover

$$(F \mid_k \gamma)(Z) := (\det\gamma)^{k/2}\det(CZ+D)^{-k}F(AZ+B)(CZ+D)^{-1}),$$

for $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Delta_{2,n}$, and Z denotes an element of the Siegel upper half plane of degree 2.

Then it is well known that the operators $T(n)$ are self-adjoint with respect to the Petersson inner product on $S_k(\Gamma_2)$ and form a commutative algebra over $\mathbf{Z}$. By a theorem of Baily [3] (see also [6]), the space $S_k(\Gamma_2)$ has a basis consisting of modular forms with integral Fourier coefficients. Using this one can prove (see [10]) that if $F \in S_k(\Gamma_2)$ is an eigenfunction with $T(n)F = \lambda_F(n)F$ for all n, then $\lambda_F(n)$ are totally real algebraic integers.

Let $\mathcal{H}_p$ denote the p-part of the Hecke algebra $\mathcal{H}$ generated by the $T(n)$. There is an isomorphism

$$\mathrm{Hom}_{\mathbf{C}}(\mathcal{H}_p, \mathbf{C}) \simeq (\mathbf{C}^\times)^3/W \tag{2.1}$$

where W is the Weyl group generated by the automorphisms of $(\mathbf{C}^\times)^3$ defined by

$$(\beta_0, \beta_1, \beta_2) \mapsto (\beta_0, \beta_2, \beta_1), \tag{2.2}$$

$$(\beta_0, \beta_1, \beta_2) \mapsto (\beta_0\beta_1, \beta_1^{-1}, \beta_2), \tag{2.3}$$

$$(\beta_0, \beta_1, \beta_2) \mapsto (\beta_0\beta_2, \beta_1, \beta_2^{-1}). \tag{2.4}$$

The Satake p-parameters $(\alpha_{0.p}, \alpha_{1,p}, \alpha_{2,p})$ of an Hecke eigenform F correspond under the isomorphism (2.1) to the homomorphism $T \mapsto \lambda_F(T)$, where $\lambda_F(T)$ is the eigenvalue of F under the Hecke operator T. To this data, one can attach the so-called spinor L-function $Z_F(s)$ defined by an Euler product

$$Z_F(s) = Z_{F,p}(p^{-s})^{-1},$$

where $Z_{F,p}(X)$ is the local spinor polynomial defined in terms of the p Satake parameters:

$$Z_{F,p}(X) = (1 - \alpha_0 X)(1 - \alpha_0\alpha_1 X)(1 - \alpha_0\alpha_2 X)(1 - \alpha_0\alpha_1\alpha_2 X). \tag{2.5}$$

Here and in the rest of the paper, we drop the subscript p in the Satake parameters, as p would be fixed but arbitrary.

In this paper we would be interested in $Z_{F,p}(p^{-s})$. In this connection let us recall for any prime p the identity as a power series in X:

$$\sum_{n\geq 0} \lambda_F(p^n)X^n = \frac{1 - p^{2k-4}X^2}{Z_{F,p}(X)}.$$

Moreover, one has the explicit identity in terms of eigenvalues:

(2.6)

$$Z_{F,p}(X)=1-\lambda_F(p)X+(\lambda_F(p)^2-\lambda_F(p^2)-p^{2k-4})X^2-\lambda_F(p)p^{2k-3}X^3+p^{4k-6}X^4.$$

3. Proof of Theorem 1.1

Let $j\in\mathbf{N}$ be fixed and let $\zeta:=e^{2\pi i/j}$ be a primitive j-th root of unity. Then the usual orthogonality relations

$$\sum_{\mu=0}^{j-1}\zeta^{\mu\nu}=\begin{cases} j & \text{if } \nu\equiv 0 \pmod{j}\\ 0 & otherwise\end{cases},$$

imply that

$$\sum_{n\geq 0}\lambda_F(p^{jn})X^{jn}=\frac{1}{j}\sum_{\mu=0}^{j-1}\frac{1-p^{2k-4}\zeta^{2\mu}X^2}{Z_{F,p}(\zeta^\mu X)}.$$

Setting $X=p^{-s}$ ($s\in\mathbf{C}$) and

$$L_{p,j}(s):=\sum_{n\geq 0}\lambda_F(p^{jn})p^{-jns}$$

we thus obtain an identity between formal Dirichlet series

$$L_{p,j}(s)=\frac{1}{j}\sum_{\mu=0}^{j-1}\frac{1-\zeta^{2\mu}p^{2k-4-2s}}{Z_{F,p}(\zeta^\mu p^{-s})}.$$

Note that the right hand side in fact gives a meromorphic continuation of $L_{p,j}(s)$ to the whole s-plane.

As mentioned in the Introduction, according to [14], the set D_F of primes p such that $\lambda_F(p)\neq 0$ has natural density 1.

Assume now that the coefficients of $L_{p,j}(s)$ do not change signs infinitely often. Then by a classical theorem of Landau [11] either $L_{p,j}(s)$ converges for all s or $L_{p,j}(s)$ has a singularity at the real point of its abscissa of convergence.

We will disprove the first assertion for all p and will disprove the second assertion for all but a finite number of p in D_F, under the hypothesis that j is not divisible by 4. The assertion of the Theorem then will follow.

Regarding the first, we note that $L_{p,j}(s)$ is a rational function in $X=p^{-s}$ where the denominator is a polynomial in X of degree $4j$ and the numerator is a polynomial in X of degree $\leq 4j-2$, with constant term 1 and so in particular is not the zero polynomial. Hence not every complex zero of the denominator polynomial can cancel against a complex zero of the numerator polynomial. We also note that $X = 0$ is not a zero of the denominator. Therefore $L_{p,j}(s)$ has poles and so cannot converge everywhere.

Now suppose that $L_{p,j}(s)$ has a pole on the real axis. Then there must exist $\mu \in \{0, 1, \ldots, j-1\}$ and $\sigma_0 \in \mathbf{R}$ such that

$$Z_{f,p}(\zeta^{\mu} p^{-\sigma_0}) = 0.$$

Since the triple $(\alpha_0, \alpha_1, \alpha_2) \in (\mathbf{C}^{\times})^3$ is only determined up to the action of the Weyl group (see section 2), we may assume after an eventual re-arranging that

$$1 - \alpha_0 \zeta^{\mu} p^{-\sigma_0} = 0, \quad \text{i.e.,} \quad p^{\sigma_0} = \alpha_0 \zeta^{\mu}. \tag{3.1}$$

Recall that

$$\alpha_0^2 \alpha_1 \alpha_2 = p^{2k-3}, \tag{3.2}$$

see e.g., [2, p. 329]. Since by hypothesis F for k even is not contained in the Maass subspace, by Weissauer's theorem [15] F satisfies the Ramanujan-Petersson conjecture, i.e., one has

$$|\alpha_1| = |\alpha_2| = 1. \tag{3.3}$$

Hence from (3.2) and (3.3) we obtain

$$|\alpha_0| = p^{k-3/2}$$

and then (3.1) (bearing in mind that σ_0 is real) implies that

$$\sigma_0 = k - 3/2.$$

We thus find that

$$\alpha_0 = p^{k-3/2} \zeta^{-\mu} \quad \text{and} \quad \alpha_1 \alpha_2 = \zeta^{2\mu}. \tag{3.4}$$

From (2.6) we obtain by comparing coefficients that

$$\lambda_F(p) = \alpha_0 (1 + \alpha_1 + \alpha_2 + \alpha_1 \alpha_2) \tag{3.5}$$

and

$$\Lambda_F(p) := \lambda_F(p)^2 - \lambda_F(p^2) - p^{2k-4} = \alpha_0^2 (\alpha_1 + \alpha_2 + 2\alpha_1\alpha_2 + \alpha_1^2 \alpha_2 + \alpha_1 \alpha_2^2). \tag{3.6}$$

If we set

$$\beta := \alpha_1 \zeta^{-\mu},$$

then using (3.3) and (3.4) we see after a short calculation that (3.5) and (3.6) can be rewritten as

$$\lambda_F(p) = p^{k-3/2}(\beta + \overline{\beta} + \zeta^{\mu} + \zeta^{-\mu}) \tag{3.7}$$

and

$$\Lambda_F(p) = p^{2k-3}((\beta + \overline{\beta})(\zeta^{\mu} + \zeta^{-\mu}) + 2), \tag{3.8}$$

respectively.

We now use the assumption that j is not divisible by 4. Then

$$\zeta^{\mu} + \zeta^{-\mu} \neq 0 \tag{3.9}$$

for every $\mu \in \{0, 1, \ldots, j-1\}$, since otherwise $\cos \frac{2\pi\mu}{j} = 0$ and so $\frac{2\pi\mu}{j} = \frac{\pi}{2}$, i.e. $4\mu = j$, a contradiction.

Let K_F be the field obtained from $\mathbf{Q}$ by adjoining the eigenvalues $\lambda_F(\ell)$ and $\lambda_F(\ell^2)$ for all primes ℓ. Then by [10], K_F is a (totally real) algebraic extension of $\mathbf{Q}$ of finite degree. It follows that also $K_F(\zeta)$ is a finite extension of $\mathbf{Q}$.

From (3.8) and (3.9) we deduce that $\beta + \overline{\beta} \in K_F(\zeta)$. For $p \in D_F$, it then follows from (3.7) that

$$\beta + \overline{\beta} + \zeta^{\mu} + \zeta^{-\mu} \in K_f(\zeta) \setminus \{0\}$$

and hence

$$\mathbf{Q}(\sqrt{p}) \subset K_F(\zeta).$$

Now recall from Galois theory that if $p_1, \ldots, p_r$ are pairwise different primes, then

$$[\mathbf{Q}(\sqrt{p}_1, \ldots, \sqrt{p}_r) : \mathbf{Q}] = 2^r.$$

Thus if 4 does not divide j and r is chosen such that $2^r > [K_F(\zeta) : \mathbf{Q}]$, there cannot exist r different primes p in D_F with the property that $L_{p,j}(s)$ has a real pole. This proves our theorem.

Remark 3.1.

(i) We note, as mentioned in the introduction, that a similar reasoning as above was used in [9, p. 1924] to prove a corresponding result in the (much simpler) case of elliptic modular forms, where however in the statement one has to add the condition that 4 does not divide j. (We are indebted to NN for pointing out to us the latter remark.)

(ii) The assumption that 4 does not divide j is crucial in the above proof. We do not know at present how to deal with the case $4|j$.

(iii) It would be very interesting to investigate if the theorem could be generalized to genus greater than 2, under some suitable assumptions on the p-Satake parameters, for example the Ramanujan-Peterssson conjecture.

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