Shifted products of Fourier coefficients of Siegel cusp forms of degree two

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1. Introduction

Let f be an elliptic cusp form of integral weight k on a congruence subgroup of $\Gamma_1 := SL_2(\mathbf{Z})$ with real Fourier coefficients $a(n)(n \geq 1)$. Let r be a fixed positive integer. Then it was proved in [5] that the sequence $(a(n)a(n+r))_{n\geq 1}$ has infinitely many non-negative as well as infinitely many non-positive terms. The proof essentially is based on a sign-change result for the Fourier coefficients of a non-zero elliptic cusp form proved in [6].

The purpose of this paper is to give a generalization of the above result in the case of a Siegel cusp form F of integral weight k on the Siegel modular group $\Gamma_2 := Sp_2(\mathbf{Z})$ of degree 2 which is a Hecke eigenform with real Fourier coefficients and which for k even is not in the Maass subspace. The proof is much more involved and uses identities of Andrianov [1] between the Fourier coefficients of F and the eigenvalues, coupled with the fact proved in [2] that the abscissa of absolute convergence of the zeta function of F formed with the eigenvalues is exactly $k - \frac{1}{2}$.

Our result will be precisely stated in sect. 2 and the proof will be given in sect. 3.

2. Statement of result

Let $S_k(\Gamma_2)$ be the space of Siegel cusp forms of integral weight k on Γ_2. Recall that each $F \in S_k(\Gamma_2)$ has a Fourier expansion

$$F(Z) = \sum_{T>0} a(T)e^{2\pi i tr(TZ)} \qquad (Z \in \mathcal{H}_2)$$

where $\mathcal{H}_2$ is the Siegel upper half-space of degree 2 and T runs over all positive definite half-integral matrices of of size 2.

We will assume that F is a non-zero eigenform of all the Hecke operators $T_n(n \geq 1)$ [1]. Note that the eigenvalues λ_n are real, since T_n is a hermitian operator on $S_k(\Gamma_2)$ with respect to the usual Petersson scalar product.

If k is even we will also suppose that F is not in the Maass subspace, i.e. F is not a Saito-Kurokawa lift of a cusp form f of weight $2k-2$ on Γ_1 [3].

We may and will also assume that the Fourier coefficients $a(T)$ of F are real. Indeeed, as is well-known $S_k(\Gamma_2)$ has a basis of functions whose Fourier coefficients are real (in fact rational) numbers, hence the Galois groups $Gal(\mathbf{C}/\mathbf{Q})$ acts on $S_k(\Gamma_2)$ by acting on the Fourier coefficients. In particular, since T_n preserves forms with rational coefficients, if τ

denotes complex conjugation, then with F also F^τ is a Hecke eigenform with eigenvalues $\lambda_n^\tau = \lambda_n$ for all n. Hence

$$\sum_{T>0} \Re(a(T))e^{2\pi i tr(TZ)}, \quad \sum_{T>0} \Im(a(T))e^{2\pi i tr(TZ)} \qquad (Z \in \mathcal{H}_2)$$

are Hecke eigenforms with Hecke eigenvalues λ_n for all n and real Fourier coefficients, and at least one of them is non-zero since $F \neq 0$.

If $T = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$ is a half-integral positive definite matrix of size 2, we identify T with the positive definite integral binary quadratic form $ax^2 + bxy + cy^2$ and denote by $D = b^2 - 4ac$ its discriminant.

If R is half-integral positive definite we define the shifted product

$$c_R(T) := a(T)a(T+R) \qquad (T > 0).$$

Since $a(T[U]) = (\det U)^k a(T)$ for $U \in GL_2(\mathbf{Z})$ (where as usual $T[U] = U'TU$), we see that

$$c_{R[U]}(T[U]) = c_R(T) \qquad (U \in GL_2(\mathbf{Z}).$$

Theorem. *Let $F \in S_k(\Gamma_2)$ be a non-zero Hecke eigenform with real Fourier coefficients $a(T)$ which for k even is not in the Maass subspace. Let $D < 0$ be a fundamental discriminant and let $\{R_1, \ldots, R_h\}$ be a fixed full set of representatives of positive definite integral binary quadratic forms of discriminant D, where $h = h(D)$ is the class number of D. Then the following hold:*

a) There exists $i_1 \in \{1, \ldots, h\}$ with the property that that there are infinitely many $T > 0$ such that $c_{R_{i_1}}(T) \geq 0$.

b) There exists $i_2 \in \{1, \ldots, h\}$ with the property that that there are infinitely many $T > 0$ such that $c_{R_{i_2}}(T) \leq 0$.

3. Proof of Theorem

We shall give the proof only for a), since case b) can be treated in the same way mutatis mutandis.

Suppose that the assertion of a) is wrong. Then there exists $n_0 \in \mathbf{N}$ such that for all $i = 1, \ldots, h$ and all $T > 0$ with $tr(T) \geq n_0$ one has

$$c_{R_i}(T) = a(T)a(T+R_i) < 0. \tag{1}$$

We may suppose that n_0 is odd.

Letting $T = nR_i (n \geq n_0)$ in (1) we obtain

$$a(nR_i)a((n+1)R_i) < 0 \qquad (i = 1, \ldots, h; n \geq n_0). \tag{2}$$

From (2) we conclude successively that for each fixed i, all coefficients

$$a(n_0 R_i), a((n_0+2)R_i), a((n_0+4)R_i), \ldots$$

are non-zero and of the same sign. In other words, for each fixed $i \in \{1, \ldots, h\}$ the numbers $a(nR_i)$ for all n odd and n large enough are non-zero and of the same sign. Hence by a classical result of Landau, each of the Dirichlet series

$$\sum_{n \geq 1, n \equiv 1 \pmod 2} a(nR_i) n^{-s} \qquad (\sigma := \Re(s) \gg 1)$$

must converge up to the first singularity.

Let us denote by

$$Z_F(s) := \prod_p Z_{F,p}(p^{-s})^{-1} \qquad (\sigma \gg 1)$$

the spinor zeta function of F, where

$$Z_{F,p}(X) := (1 - \alpha_{0,p} X)(1 - \alpha_{0,p}\alpha_{1,p} X)(1 - \alpha_{0,p}\alpha_{2,p} X)(1 - \alpha_{0,p}\alpha_{1,p}\alpha_{2,p} X)$$

and $\alpha_{0,p}, \alpha_{1,p}, \alpha_{2,p}$ are the Satake p-parameters of F.

Since by hypothesis for k even F is not a Saito-Kurokawa lift, by [1,4,7] the function $Z_F(s)$ has holomorphic continuation to $\mathbf{C}$.

Recall that $Z_F(s)$ is related to the numbers λ_n through the equation

$$\Lambda_F(s) := \sum_{n \geq 1} \lambda_n n^{-s}$$

$$= \zeta(2s - 2k + 4)^{-1} Z_F(s) \qquad (\sigma \gg 1). \tag{3}$$

Now let χ be a character of the ideal class group of the quadratic field $\mathbf{Q}(\sqrt{D})$ and denote by $L(s, \chi) (\sigma > 1)$ its L-series. Then by [1] one has the fundamental identity between Dirichlet series

$$\frac{L(s-k+2, \chi)}{\zeta(2s-2k+4)} \sum_{i=1}^{h} \chi(R_i) D_{F,i}(s) = A_{F,\chi} \Lambda_F(s) \qquad (\sigma \gg 1) \tag{4}$$

where $\chi(R_i)$ has the obvious meaning,

$$D_{F,i}(s) := \sum_{n \geq 1} a(nR_i) n^{-s} \qquad (\sigma \gg 1)$$

and

$$A_{F,\chi} := \sum_{i=1}^{h} \chi(R_i) a(R_i).$$

If we invert (4) we find for each i that

(5) $$D_{F,i}(s) = \frac{1}{h}\zeta(2s-2k+4)\Big(\sum_{\chi} \frac{\overline{\chi}(R_i)A_{F,\chi}}{L(s-k+2,\chi)}\Big)\Lambda_F(s) \qquad (\sigma \gg 1).$$

If

$$D(s) = \sum_{n\geq 1} c(n) n^{-s}$$

is a formal Dirichlet series, we denote by

$$D^{\flat}(s) := \sum_{n\geq 1, n\equiv 1 \pmod 2} c(n) n^{-s}$$

its "odd" part. Using that the general coefficient of the product $D(s) = D_1(s)D_2(s)$ of two Dirichlet series $D_1(s)$ and $D_2(s)$ is the multiplicative convolution of the coefficients of $D_1(s)$ and $D_2(s)$, it follows that

(6) $$D^{\flat}(s) = D_1^{\flat}(s) D_2^{\flat}(s).$$

If we apply (6) to (5) we find that

$$D_{F,i}^{\flat}(s) = \frac{1}{h}\zeta^{\flat}(2s-2k+4)\Big(\sum_{\chi} \frac{\overline{\chi}(R_i)A_{F,\chi}}{L^{\flat}(s-k+2,\chi)}\Big)\Lambda_F^{\flat}(s) \qquad (\sigma \gg 1).$$

The first two factors on the right are holomorphic for $\sigma > k-1$. Since $Z_F(s)$ is holomorphic on $\mathbf{C}$ the same is true for $Z_F^{\flat}(s)$ which is obtained from $Z_F(s)$ by removing the Euler factor at 2. Hence by (3), $\Lambda_F^{\flat}(s)$ is holomorphic for $\sigma > k-1$.

From what we proved above we now see that for each i, the series $D_{F,i}^{\flat}(s)$ converges for $\sigma > k-1$. Note that the convergence in fact is absolute because of our hypothesis on the signs.

We now apply (6) to (4) to get

(7) $$\frac{L^{\flat}(s-k+2,\chi)}{\zeta^{\flat}(2s-2k+4)}\Big(\sum_{i=1}^{h} \chi(R_i) D_{F,i}^{\flat}(s)\Big) = A_{F,\chi}\Lambda_F^{\flat}(s) \qquad (\sigma \gg 1).$$

We claim that there exists χ such that $A_{F,\chi} \neq 0$. Indeed, if not $D_{F,i}(s)$ would be zero (in fact for each i) which contradicts (2).

We now fix χ with $A_{F,\chi} \neq 0$. From (7) we will then derive a contradiction as follows.

It was proved in [2] that the abscissa of absolute convergence of $\Lambda_F^\flat(s)$ is exactly $\sigma_a = k - \frac{1}{2} > k-1$. In fact, it was proved in [2], pp. 373-374 that for each prime q the series

$$\sum_{n\geq 1,(n,q)=1} \lambda_n n^{-s} \qquad (\sigma \gg 1)$$

has abscissa of absolute convergence equal to $k - \frac{1}{2}$. (Note that the normalizations of the L-series in [2] are different from here.)

Now choose $s_0 \in \mathbf{R}$ with $k-1 < s_0 < k - \frac{1}{2}$. Denote by $\gamma(n)$ (n odd) the general coefficient of $\Lambda_F^\flat(s)$. Then

$$\sum_{n\geq 1, n\equiv 1 \pmod 2} |\gamma(n)| n^{-s_0}$$

must diverge.

On the other hand, if $b(n)$ respectively $c(n)$ (n odd) denote the general coefficients of the Dirichlet series of the product on the right of (7), we find from (7) that

$$\begin{aligned} |A_{F,\chi}| \sum_{n\geq 1, n\equiv 1 \pmod 2} \frac{|\gamma(n)|}{n^{s_0}} &= \sum_{n\geq 1, n\equiv 1 \pmod 2} |\Big(\sum_{d|n} b(d) c(\frac{n}{d})\Big)| \frac{1}{n^{s_0}} \\ &\leq \sum_{n\geq 1, n\equiv 1 \pmod 2} \Big(\sum_{d|n} |b(d)||c(\frac{n}{d})|\Big) \frac{1}{n^{s_0}} \\ &= \Big(\sum_{n\geq 1, n\equiv 1 \pmod 2} \frac{|b(n)|}{n^{s_0}}\Big)\Big(\sum_{n\geq 1, n\equiv 1 \pmod 2} \frac{|c(n)|}{n^{s_0}}\Big) \end{aligned}$$

which is finite since $s_0 > k-1$. This gives the desired contradiction.

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