Reciprocity laws for (φ_L, Γ_L)-modules over Lubin-Tate extensions

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Abstract

In the Lubin-Tate setting we study pairings for analytic (φ_L, Γ)-modules and prove an abstract reciprocity law which then implies a relation between the analogue of Perrin-Riou's Big Exponential map as developed by Berger and Fourquaux and a p-adic regulator map whose construction relies on the theory of Kisin-Ren modules generalising the concept of Wach modules to the Lubin-Tate situation.

Introduction

Classically explicit reciprocity laws or formulas usually mean an explicit computation of Hilbert symbols or (local) cup products using e.g. differential forms, (Coleman) power series etc. and a bunch of manifestations of this idea exists in the literature due to Artin-Hasse, Iwasawa, Wiles, Kolyvagin, Vostokiov, Brückner, Coleman, Sen, de Shalit, Fesenko, Bloch-Kato, Benois ... In the same spirit Perrin-Riou's reciprocity law gives an explicit calculation of the Iwasawa cohomology pairing in terms of big exponential and regulator maps for crystalline representations of $G_{\mathbb{Q}_p}$; more precisely, the latter maps are adjoint to each other when also involving the crystalline duality paring after base change to the distribution algebra corresponding to the cyclotomic situation.

The motivating question for this article is to investigate what happens if one replaces the cyclotomic $\mathbb{Z}_p$-extension by a Lubin-Tate extension L_∞ over some finite extension L over $\mathbb{Q}_p$ with Galois group $\Gamma_L = G(L_\infty/L)$ and Lubin-Tate character $\chi_{LT} : G_L \to o_L^\times$ which all arise from a Lubin-Tate formal group attached to a prime $\pi_L \in o_L$; by q we denote the cardinality of the residue field $o_L/o_L\pi_L$. We try to extend the above sketched cyclotomic picture to the Lubin-Tate case at least for L-*analytic* crystalline representations of the absolute Galois group G_L of L. As pointed out in [SV15] already, the character $\tau := \chi_{cyc} \cdot \chi_{LT}^{-1}$ plays a crucial role.

To this aim we study (φ_L, Γ_L)-modules over the Robba ring $\mathcal{R}$ with coefficients in an appropriate extension K of L which contains the period Ω of the dual of the fixed Lubin-Tate group. One ingredient is the theory of Schneider and Teitelbaum: Via Fourier theory and the Lubin-Tate isomorpshism the locally L-analytic distribution algebra $D(o_L, K)$ of the additive group of the ring of integers o_L of L with coefficients in K becomes isomorphic to the subring $\mathcal{R}^+ \subseteq \mathcal{R}$ consisting of those functions which converge on the full open unit disk, while the functions in $\mathcal{R}$ in general only converge on some annulus $r \leqslant |Z| < 1$ for some radius $0 < r < 1$. This isomorphism induces the Mellin-transform, i.e., a topological isomorphism between $D(o_L^\times, K)$ and the $D(o_L^\times, K)$-submodule $(\mathcal{R}^+)^{\psi_L=0}$ of $\mathcal{R}^+$ on which the ψ_L-operator - up to a scalar a left inverse of the Lubin-Tate φ_L-operator - acts as zero. After introducing the Robba group ring $\mathcal{R}(\Gamma_L)$ containing $D(\Gamma_L, K)$ we extend the Mellin transform to an isomorphism of $\mathcal{R}(\Gamma_L)$ and $(\mathcal{R})^{\psi_L=0}$. This is a special case of the following

Theorem 1 (Theorem 2.33)**.** *If M denotes a L-analytic (φ_L, Γ_L)-module over $\mathcal{R}$, then $M^{\psi_L=0}$ is a free $\mathcal{R}(\Gamma_L)$-module of rank $\mathrm{rk}_{\mathcal{R}}\, M$.*

A second ingredient is Serre duality on the open unit disk and more general on the character variety of the group Γ_L, which induces a residue pairing

$$\{\ ,\ \} : \Omega^1 \times \mathcal{R} \to K$$

for the differentials $\Omega^1 = \mathcal{R}dZ$ and also a pairing

$$<\ ,\ >: \mathcal{R}(\Gamma_L) \times \mathcal{R}(\Gamma_L) \to K,$$

which induces topological isomorphisms $\mathrm{Hom}_{K,cts}(\mathcal{R}(\Gamma_L), K) \cong \mathcal{R}(\Gamma_L)$ and $\mathrm{Hom}_{K,cts}(\mathcal{R}(\Gamma_L)/D(\Gamma_L, K), K) \cong D(\Gamma_L, K)$. For a L-analytic (φ_L, Γ_L)-module M we finally define on the one hand the two Iwasawa pairings

$$\{\ ,\ \}'_{M,Iw} : \check{M}^{\psi_L=0} \times M^{\psi_L=0} \to \mathcal{R}(\Gamma_L)$$

and

$$\{\ ,\ \}_{M,Iw} : \check{M}^{\psi_L=\frac{q}{\pi_L}} \times M^{\psi_L=1} \to D(\Gamma_L, K),$$

where $\check{M} := \mathrm{Hom}_{\mathcal{R}}(M, \Omega^1)$. They are linked by the commutative diagram

$$\begin{tikzcd}
\{\ ,\ \}_{M,Iw} : \check{M}^{\psi_L=\frac{q}{\pi_L}} \arrow[d, "\varphi_L - 1"'] & \times & M^{\psi_L=1} \arrow[r] \arrow[d, "\frac{\pi_L}{q}\varphi_L - 1"'] & D(\Gamma_L, K) \arrow[d, hook] \\
\{\ ,\ \}'_{M,Iw} : \check{M}^{\psi_L=0} & \times & M^{\psi_L=0} \arrow[r] & \mathcal{R}(\Gamma_L).
\end{tikzcd}$$

Now assume that M arises as $D^{\dagger}_{rig}(W)$ under Berger's equivalence of categories (see Theorem 2.71) from a L-analytic, crystalline representation W of G_L, whence $\check{M} \cong D^{\dagger}_{rig}(W^*(\chi_{LT}))$. Then, on the other hand we obtain the pairing

$$[\ ,\ ]_{D_{cris,L}(W)} : \mathcal{R}^{\psi_L=0} \otimes_L D_{cris,L}(W^*(\chi_{LT})) \times \mathcal{R}^{\psi_L=0} \otimes_L D_{cris,L}(W) \to \mathcal{R}(\Gamma_L)$$

by base extension of the usual crystalline duality pairing. The work of Kisin and Ren provides comparison isomorphisms

$$\mathrm{comp}_M : M[\frac{1}{t_{LT}}] \cong \mathcal{R}[\frac{1}{t_{LT}}] \otimes_L D_{cris,L}(W)$$

and

$$\mathrm{comp}_{\check{M}} : \check{M}[\frac{1}{t_{LT}}] \cong \mathcal{R}[\frac{1}{t_{LT}}] \otimes_L D_{cris,L}(W^*(\chi_{LT})).$$

Here $t_{LT} = \log_{LT}(Z) \in \mathcal{R}$ denotes the Lubin-Tate period which stems from the Lubin-Tate logarithm. The Lubin-Tate character χ_{LT} induces isomorphism $\Gamma_L \xrightarrow{\cong} o_L^\times$ as well as $Lie(\Gamma_L) \xrightarrow{\cong} L$, and we let $\nabla \in Lie(\Gamma_L)$ be the preimage of 1. Then the abstract reciprocity law we prove is the following statement.

Theorem 2 (Theorem 2.74)**.** *For all $x \in \check{M}^{\psi_L=0}$ and $y \in M^{\psi_L=0}$, for which the crystalline pairing is defined via the comparison isomorphism, it holds*

$$\{\frac{\nabla}{\Omega}x, y\}'_{M,Iw} = [x, y]_{D_{cris,L}(W)}.$$

As an application we show the adjointness of big exponential and regulator maps. Recall that Berger and Fourquaux have constructed for V an L-analytic representation of G_L and an integer $h \geqslant 1$ such that

- $\mathrm{Fil}^{-h} D_{cris,L}(V) = D_{cris,L}(V)$ and
- $D_{cris,L}(V)^{\varphi_L=\pi_L^{-h}} = 0$

a *big exponential map* à la Perrin-Riou

$$\Omega_{V,h} : (\mathcal{R}^+)^{\psi_L=0} \otimes_L D_{cris,L}(V) \to D^\dagger_{\mathrm{rig}}(V)^{\psi_L=\frac{q}{\pi_L}},$$

which up to comparison isomorphism is for $h = 1$ given by $f = (1 - \varphi_L)x \mapsto \nabla x$ and which interpolates Bloch-Kato exponential maps $exp_{L,V(\chi^r_{LT})}$.

On the other hand, based on an extension of the work of Kisin and Ren in the first section, we construct for a lattice $T \subseteq V$, such that $V(\tau^{-1})$ is L-analytic and crystalline and such that V does not have any quotient isomorphic to $L(\tau)$, a *regulator map* à la Loeffler and Zerbes

$$\mathcal{L}^0_V : H^1_{Iw}(L_\infty/L, T) \cong D_{LT}(T(\tau^{-1}))^{\psi_L=1} \to (\mathcal{R}^+)^{\psi_L=0} \otimes_L D_{cris,L}(V(\tau^{-1}))$$

as applying the operator

$$1 - \frac{\pi_L}{q}\varphi_L$$

up to comparison isomorphism. Then we derive from the abstract version above with $W = V(\tau^{-1})$ the following reciprocity formula

Theorem 3 (Theorem 3.2)**.** *Assume that $V^*(1)$ is L-analytic. If $\mathrm{Fil}^{-1} D_{cris,L}(V^*(1)) = D_{cris,L}(V^*(1))$ and $D_{cris,L}(V^*(1))^{\varphi_L=\pi_L^{-1}} = D_{cris,L}(V^*(1))^{\varphi_L=1} = 0$, then the following diagram commutes:*

$$\begin{array}{ccccc}
D^\dagger_{\mathrm{rig}}(V^*(1))^{\psi_L=\frac{q}{\pi_L}} & \times & D(V(\tau^{-1}))^{\psi_L=1} & \xrightarrow{\{,\}_{Iw}} & D(\Gamma_L, \mathbb{C}_p) \\
\uparrow{\scriptstyle \Omega_{V^*(1),1}} & & {\scriptstyle \Omega\mathcal{L}^0_V}\downarrow & & \| \\
(\mathcal{R}^+)^{\psi_L=0} \otimes_L D_{cris,L}(V^*(1)) & \times & (\mathcal{R}^+)^{\psi_L=0} \otimes_L D_{cris,L}(V(\tau^{-1})) & \xrightarrow{[,]} & D(\Gamma_L, \mathbb{C}_p).
\end{array}$$

While the crystalline pairing satisfies an interpolation property (Proposition 3.22) for trivial reasons, the statement that the second Iwasawa pairing interpolates Tate's cup product pairing is more subtle (Proposition 3.21). Eventually the interpolation property of Berger and Fourquaux for $\Omega_{V,h}$ combined with the adjointness of the latter with $\mathcal{L}^0_V$ implies an interpolation formula for the regulator map, which interpolates dual Bloch-Kato exponential maps see Theorem 3.26.

Notation

Let $\mathbb{Q}_p \subseteq L \subset \mathbb{C}_p$ be a field of finite degree d over $\mathbb{Q}_p$, o_L the ring of integers of L, $\pi_L \in o_L$ a fixed prime element, $k_L = o_L/\pi_L o_L$ the residue field, $q := |k_L|$ and e the absolute ramification index of L. We always use the absolute value $|\ |$ on $\mathbb{C}_p$ which is normalized by $|\pi_L| = q^{-1}$.

We fix a Lubin-Tate formal o_L-module $LT = LT_{\pi_L}$ over o_L corresponding to the prime element π_L. We always identify LT with the open unit disk around zero, which gives us a global coordinate Z on LT. The o_L-action then is given by formal power series $[a](Z) \in o_L[[Z]]$. For simplicity the formal group law will be denoted by $+_{LT}$.

The power series $\frac{\partial(X+_{LT}Y)}{\partial Y}_{|(X,Y)=(Z,0)}$ is a unit in $o_L[[Z]]$ and we let $g_{LT}(Z)$ denote its inverse. Then $g_{LT}(Z)dZ$ is, up to scalars, the unique invariant differential form on LT ([Haz] §5.8). We also let

$$\log_{LT}(Z) = Z + \dots \tag{1}$$

denote the unique formal power series in $L[[Z]]$ whose formal derivative is g_{LT}. This $\log_{LT}$ is the logarithm of LT ([Lan] 8.6). In particular, $g_{LT}dZ = d\log_{LT}$. The invariant derivation ∂_{inv} corresponding to the form $d\log_{LT}$ is determined by

$$f'dZ = df = \partial_{\mathrm{inv}}(f)d\log_{LT} = \partial_{\mathrm{inv}}(f)g_{LT}dZ$$

and hence is given by

$$\partial_{\mathrm{inv}}(f) = g_{LT}^{-1}f' \ . \tag{2}$$

For any $a \in o_L$ we have

$$\log_{LT}([a](Z)) = a \cdot \log_{LT} \qquad \text{and hence} \qquad a g_{LT}(Z) = g_{LT}([a](Z)) \cdot [a]'(Z) \tag{3}$$

([Lan] 8.6 Lemma 2).

Let T_π be the Tate module of LT. Then T_π is a free o_L-module of rank one, say with generator η, and the action of $G_L := \mathrm{Gal}(\overline{L}/L)$ on T_π is given by a continuous character $\chi_{LT} : G_L \longrightarrow o_L^\times$. Let T'_π denote the Tate module of the p-divisible group Cartier dual to LT with period Ω (depending on the choice of a generator of T'_π), which again is a free o_L-module of rank one. The Galois action on $T'_\pi \cong T_\pi^*(1)$ is given by the continuous character $\tau := \chi_{cyc} \cdot \chi_{LT}^{-1}$, where χ_{cyc} is the cyclotomic character.

For $n \geqslant 0$ we let L_n/L denote the extension (in $\mathbb{C}_p$) generated by the π_L^n-torsion points of LT, and we put $L_\infty := \bigcup_n L_n$. The extension L_∞/L is Galois. We let $\Gamma_L := \mathrm{Gal}(L_\infty/L)$ and $H_L := \mathrm{Gal}(\overline{L}/L_\infty)$. The Lubin-Tate character χ_{LT} induces an isomorphism $\Gamma_L \xrightarrow{\cong} o_L^\times$.

Henceforth we use the same notation as in [SV15]. In particular, the ring endomorphisms induced by sending Z to $[\pi_L](Z)$ are called φ_L where applicable; e.g. for the ring $\mathscr{A}_L$ defined to be the π_L-adic completion of $o_L[[Z]][Z^{-1}]$ or $\mathscr{B}_L := \mathscr{A}_L[\pi_L^{-1}]$ which denotes the field of fractions of $\mathscr{A}_L$. Recall that we also have introduced the unique additive endomorphism ψ_L of $\mathscr{B}_L$ (and then $\mathscr{A}_L$) which satisfies

$$\varphi_L \circ \psi_L = \pi_L^{-1} \cdot trace_{\mathscr{B}_L/\varphi_L(\mathscr{B}_L)} \ .$$

Moreover, projection formula

$$\psi_L(\varphi_L(f_1)f_2) = f_1\psi_L(f_2) \qquad \text{for any } f_i \in \mathscr{B}_L$$

as well as the formula

$$\psi_L \circ \varphi_L = \frac{q}{\pi_L} \cdot \mathrm{id}$$

hold. An étale (φ_L, Γ_L)-module M comes with a Frobenius operator φ_M and an induced operator denoted by ψ_M.

Let $\widetilde{\mathbf{E}}^+ := \varprojlim o_{\mathbb{C}_p}/p o_{\mathbb{C}_p}$ with the transition maps being given by the Frobenius $\varphi(a) = a^p$. We may also identify $\widetilde{\mathbf{E}}^+$ with $\varprojlim o_{\mathbb{C}_p}/\pi_L o_{\mathbb{C}_p}$ with the transition maps being given by the q-Frobenius $\varphi_q(a) = a^q$. Recall that $\widetilde{\mathbf{E}}^+$ is a complete valuation ring with residue field $\overline{\mathbb{F}_p}$ and its field of fractions $\widetilde{\mathbf{E}} = \varprojlim \mathbb{C}_p$ being algebraically closed of characteristic p. Let $\mathfrak{m}_{\widetilde{\mathbf{E}}}$ denote the maximal ideal in $\widetilde{\mathbf{E}}^+$.

The q-Frobenius φ_q first extends by functoriality to the rings of the Witt vectors $W(\widetilde{\mathbf{E}}^+) \subseteq W(\widetilde{\mathbf{E}})$ and then o_L-linearly to $W(\widetilde{\mathbf{E}}^+)_L := W(\widetilde{\mathbf{E}}^+) \otimes_{o_{L_0}} o_L \subseteq W(\widetilde{\mathbf{E}})_L := W(\widetilde{\mathbf{E}}) \otimes_{o_{L_0}} o_L$, where L_0 is the maximal unramified subextension of L. The Galois group G_L obviously acts on $\widetilde{\mathbf{E}}$ and $W(\widetilde{\mathbf{E}})_L$ by automorphisms commuting with φ_q. This G_L-action is continuous for the weak topology on $W(\widetilde{\mathbf{E}})_L$ (cf. [GAL] Lemma 1.5.3).

Sometimes we omit the index q, L, or M from the Frobenius operator, but we always write φ_p when dealing with the p-Frobenius.

1 Wach-modules à la Kisin-Ren

1.1 Wach-modules

In this section we recall the theory of Wach-modules à la Kisin-Ren [KR] (with the simplifying assumption that - in their notation - $K = L$, $m = 1$ etc.).

By sending Z to $\omega_{LT} \in W(\widetilde{\mathbf{E}}^+)_L$ (see directly after [SV15, Lem. 4.1]) we obtain an G_L-equivariant, Frobenius compatible embedding of rings

$$o_L[[Z]] \longrightarrow W(\widetilde{\mathbf{E}}^+)_L$$

the image of which we call $\mathbf{A}_L^+$, it is a subring of $\mathbf{A}_L$ (the image of $\mathscr{A}_L$ in $W(\widetilde{\mathbf{E}})_L$). The latter ring is a complete discrete valuation ring with prime element π_L and residue field the image $\mathbf{E}_L$ of $k_L((Z)) \hookrightarrow \widetilde{\mathbf{E}}$ sending Z to $\omega := \omega_{LT} \mod \pi_L$. We form the maximal integral unramified extension (= strict Henselization) $\mathbf{A}_L^{nr}$ of $\mathbf{A}_L$ inside $W(\widetilde{\mathbf{E}})_L$. Its p-adic completion $\mathbf{A}$ still is contained in $W(\widetilde{\mathbf{E}})_L$. Note that $\mathbf{A}$ is a complete discrete valuation ring with prime element π_L and residue field the separable algebraic closure $\mathbf{E}_L^{sep}$ of $\mathbf{E}_L$ in $\widetilde{\mathbf{E}}$. By the functoriality properties of strict Henselizations the q-Frobenius φ_q preserves $\mathbf{A}$. According to [KR] Lemma 1.4 the G_L-action on $W(\widetilde{\mathbf{E}})_L$ respects $\mathbf{A}$ and induces an isomorphism $H_L = \ker(\chi_{LT}) \xrightarrow{\cong} \mathrm{Aut}^{cont}(\mathbf{A}/\mathbf{A}_L)$. We set $\mathbf{A}^+ := \mathbf{A} \cap W(\widetilde{\mathbf{E}}^+)_L$.

Set $Q := \frac{[\pi_L](\omega_{LT})}{\omega_{LT}} \in \mathbf{A}_L^+$, which satisfies per definitionem $\varphi_L(\omega_{LT}) = Q \cdot \omega_{LT}$.

Following [KR] we write $\mathcal{O} = \mathcal{O}_L(\mathbb{B})$ for the ring of rigid analytic functions on the open unit disk $\mathbb{B}$ over L, or equivalently the ring of power series in Z over L converging in $\mathbb{B}$. Via sending ω_{LT} to Z we view $\mathbf{A}_L^+$ as a subring of $\mathcal{O}$. We denote by $\mathrm{Mod}_{\mathcal{O}}^{\varphi_L, \Gamma_L, an}$ the category consisting of finitely generated free $\mathcal{O}$-modules $\mathcal{M}$ together with the following data:

(i) an isomorphism $1 \otimes \varphi_{\mathcal{M}} : (\varphi_L^* \mathcal{M})[\frac{1}{Q}] \cong \mathcal{M}[\frac{1}{Q}]$.

(ii) a semi-linear Γ_L-action on $\mathcal{M}$, commuting with $\varphi_{\mathcal{M}}$ and such that the induced action on $D(\mathcal{M}) := \mathcal{M}/\omega_{LT}\mathcal{M}$ is trivial.

We note that, since $\mathcal{M}/\omega_{LT}\mathcal{M} = \mathcal{M}[\frac{1}{Q}]/\omega_{LT}\mathcal{M}[\frac{1}{Q}]$ the map $\varphi_{\mathcal{M}}$ induces an L-linear endomorphism of $D(\mathcal{M})$, which we denote by $\varphi_{D(\mathcal{M})}$. As a consequence of (1) it, in fact, is an automorphism.

The Γ_L-action on $\mathcal{M}$ is differentiable ([BSX] Lemma 3.4.13) , and the corresponding derived action of $\mathrm{Lie}(\Gamma_L)$ is L-bilinear ([BSX] Remark 3.4.15).

Similarly, we denote by $\mathrm{Mod}_{\mathbf{A}_L^+}^{\varphi_L,\Gamma_L,an}$ the category consisting of finitely generated free $\mathbf{A}_L^+$-modules N together with the following data:

(i) an isomorphism $1 \otimes \varphi_N : (\varphi_L^* N)[\frac{1}{Q}] \cong N[\frac{1}{Q}]$.

(ii) a semi-linear Γ_L-action on N, commuting with φ_N and such that the induced action on $N/\omega_{LT}N$ is trivial.

The map φ_N induces an L-linear automorphism of $D(N) := N[\frac{1}{p}]/\omega_{LT}N[\frac{1}{p}]$ denoted by $\varphi_{D(N)}$.

Obviously we have the base extension functor $\mathcal{O} \otimes_{\mathbf{A}_L^+} - : \mathrm{Mod}_{\mathbf{A}_L^+}^{\varphi_L,\Gamma_L,an} \longrightarrow \mathrm{Mod}_{\mathcal{O}}^{\varphi_L,\Gamma_L,an}$. It satisfies

$$D(\mathcal{O} \otimes_{\mathbf{A}_L^+} N) = D(N) \ . \tag{4}$$

We write $\mathrm{Mod}_{\mathcal{O}}^{\varphi_L,\Gamma_L,0}$ for the full subcategory of $\mathrm{Mod}_{\mathcal{O}}^{\varphi_L,\Gamma_L,an}$ consisting of all $\mathcal{M}$ such that $\mathcal{R} \otimes_{\mathcal{O}} \mathcal{M}$ is pure of slope 0. Here $\mathcal{R}$ denotes the Robba ring.

By $\mathrm{Mod}_L^{F,\varphi_q}$ we denote the category of finite dimensional L-vector spaces D equipped with an L-linear automorphism $\varphi_q : D \xrightarrow{\cong} D$ and a decreasing, separated, and exhaustive filtration, indexed by $\mathbb{Z}$, by L-subspaces. In $\mathrm{Mod}_L^{F,\varphi_q}$ we have the full subcategory $\mathrm{Mod}_L^{F,\varphi_q,wa}$ of weakly admissible objects. For D in $\mathrm{Mod}_L^{F,\varphi_q,wa}$ let $V_L(D) = \mathrm{Fil}^0(B_{cris,L} \otimes_L D)^{\varphi_q=1}$ where, as usual, $B_{cris,L} := B_{cris} \otimes_{L_0} L$. In order to formulate the crystalline comparison theorem in this context we also consider the category $\mathrm{Mod}_{L_0 \otimes_{\mathbb{Q}_p} L}^{F,\varphi_q}$ of finitely generated free $L_0 \otimes_{\mathbb{Q}_p} L$-modules $\mathfrak{D}$ equipped with a $(\varphi_p \otimes \mathrm{id})$-linear automorphism $\varphi_q : \mathfrak{D} \xrightarrow{\cong} \mathfrak{D}$ and a decreasing, separated, and exhaustive filtration on $\mathfrak{D}_L := \mathfrak{D} \otimes_{L_0} L$, indexed by $\mathbb{Z}$, by $L \otimes_{\mathbb{Q}_p} L$-submodules. For $\mathfrak{D}$ in $\mathrm{Mod}_{L_0 \otimes_{\mathbb{Q}_p} L}^{F,\varphi_q}$ we define, as usual,

$$V(\mathfrak{D}) := (B_{cris} \otimes_{L_0} \mathfrak{D})^{\varphi=1} \cap \mathrm{Fil}^0(B_{dR} \otimes_L \mathfrak{D}_L) \ .$$

Let $\mathrm{Rep}_{o_L,f}(G_L)$ denote the category of finitely generated free o_L-modules equipped with a continuous linear G_L-action and $\mathrm{Rep}_{o_L,f}^{cris,an}(G_L)$ the full subcategory of those T which are free over o_L and such that the representation $V := L \otimes_{o_L} T$ is crystalline and *analytic*, i.e., satisfying that, if $D_{dR}(T) := (T \otimes_{\mathbb{Z}_p} B_{dR})^{G_L}$, the filtration on $D_{dR}(T)_{\mathfrak{m}}$ is trivial for each maximal ideal $\mathfrak{m}$ of $L \otimes_{\mathbb{Q}_p} L$ which does not correspond to the identity $\mathrm{id} : L \to L$. Correspondingly we let $\mathrm{Rep}_L^{cris}(G_L)$, resp. $\mathrm{Rep}_L^{cris,an}(G_L)$, denote the category of continuous G_L-representations in finite dimensional L-vector spaces which are crystalline, resp. crystalline and analytic. The base extension functor $L \otimes_{o_L} -$ induces an equivalence of categories $\mathrm{Rep}_{o_L,f}^{cris,an}(G_L) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{\simeq} \mathrm{Rep}_L^{cris,an}(G_L)$. Here applying $\otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ to a $\mathbb{Z}_p$-linear category means applying this functor to the Hom-modules. For V in $\mathrm{Rep}_L^{cris,an}(G_L)$ we set $D_{cris,L}(V) := (B_{cris,L} \otimes_L V)^{G_L} = (B_{cris} \otimes_{L_0} V)^{G_L}$ and $D_{cris}(V) := (B_{cris} \otimes_{\mathbb{Q}_p} V)^{G_L}$. The usual crystalline comparison theorem says that D_{cris} and V are equivalences of categories between $\mathrm{Rep}_L^{cris}(G_L)$ and the subcategory of weakly admissible objects in $\mathrm{Mod}_{L_0 \otimes_{\mathbb{Q}_p} L}^{F,\varphi_q}$.

Lemma 1.1. *([ST4] Lemma 5.3 and subsequent discussion, or [KR] Cor. 3.3.1) There is a fully faithful $\otimes$-functor*

$$\begin{aligned} \widetilde{\ } : \mathrm{Mod}_L^{F,\varphi_q} &\longrightarrow \mathrm{Mod}_{L_0 \otimes_{\mathbb{Q}_p} L}^{F,\varphi} \\ D &\longmapsto \tilde{D} := L_0 \otimes_{\mathbb{Q}_p} D \ , \end{aligned}$$

whose essential image consists of all analytic *objects, i.e., those for which the filtration on the non-identity components is trivial. A quasi-inverse functor from the essential image is given by sending $\mathfrak{D}$ to the base extension $L \otimes_{L_0 \otimes_{\mathbb{Q}_p} L} \mathfrak{D}$ for the multiplication map $L_0 \otimes_{\mathbb{Q}_p} L \to L$.*

Lemma 1.1 implies that

$$D_{cris,L}(V)^{\sim} \cong D_{cris}(V) \qquad \text{for any } V \text{ in } \mathrm{Rep}_L^{cris,an}(G_L). \tag{5}$$

We denote by $\mathfrak{M}^{et}(\mathbf{A}_L)$ the category of étale (φ_q, Γ_L)-modules over $\mathbf{A}_L$ (cf. [SV15, Def. 3.7]) and by $\mathfrak{M}^{et}_f(\mathbf{A}_L)$ the full subcategory consisting of those objects, which are finitely generated free as $\mathbf{A}_L$-module. For M in $\mathfrak{M}^{et}_f(\mathbf{A}_L)$, resp. for T in $\mathrm{Rep}_{o_L,f}(G_L)$, we put $V(M) := (\mathbf{A} \otimes_{\mathbf{A}_L} M)^{\varphi_q \otimes \varphi_M = 1}$, resp. $D_{LT}(T) := (\mathbf{A} \otimes_{o_L} T)^{\ker(\chi_{LT})}$.

Having defined all of the relevant categories (and most of the functors) we now contemplate the following diagram of functors:

$$\begin{tikzcd}
 & & & \mathfrak{M}^{et}_f(\mathbf{A}_L) \arrow[d, shift right, "V"'] \arrow[d, phantom, "\simeq"] \\
\mathrm{Mod}^{\varphi_L,\Gamma_L,an}_{\mathbf{A}_L^+} \arrow[urrr, "\mathbf{A}_L \otimes_{\mathbf{A}_L^+} -"] \arrow[rr, "\simeq"] \arrow[d] \arrow[dd, bend right=60, "\mathcal{O} \otimes_{\mathbf{A}_L^+} -"'] & & \mathrm{Rep}^{cris,an}_{o_L,f}(G_L) \arrow[r, "\subseteq"] \arrow[dd, "L \otimes_{o_L} -"] & \mathrm{Rep}_{o_L,f}(G_L) \arrow[u, shift right, "D_{LT}"'] \\
\mathrm{Mod}^{\varphi_L,\Gamma_L,an}_{\mathbf{A}_L^+} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \arrow[d, "\simeq"'] & & & \\
\mathrm{Mod}^{\varphi_L,\Gamma_L,0}_{\mathcal{O}} \arrow[r, shift left, "D"] \arrow[r, phantom, "\simeq"] \arrow[d, "\subseteq"'] & \mathrm{Mod}_L^{F,\varphi_q,wa} \arrow[l, shift left, "\mathcal{M}"] \arrow[r, shift left, "V_L"] \arrow[r, phantom, "\simeq"] \arrow[d, "\subseteq"'] & \mathrm{Rep}_L^{cris,an}(G_L) \arrow[l, shift left, "D_{cris,L}"] & \\
\mathrm{Mod}^{\varphi_L,\Gamma_L,an}_{\mathcal{O}} \arrow[r, shift left, "D"] \arrow[r, phantom, "\simeq"] & \mathrm{Mod}_L^{F,\varphi_q} \arrow[l, shift left, "\mathcal{M}"] & &
\end{tikzcd}$$

The arrows without decoration are the obvious natural ones. The following pairs of functors are quasi-inverse $\otimes$-equivalences of $\otimes$-categories:

- (D_{LT}, V) by [KR] Thm. 1.6;
- $(D_{cris,L}, V_L)$ by the crystalline comparison theorem ([F1] Rem. 3.6.7) and Lemma 1.1;
- $(D, \mathcal{M})$ by [KR] Prop. 2.2.6 (or [BSX] Thm. 3.4.16) and [KR] Cor. 2.4.4, to which we also refer for the definition of the functor $\mathcal{M}$.

In particular, all functors in the above diagram are $\otimes$-functors. The second arrow in the left column, resp. the left arrow in the upper horizontal row, is an equivalence of categories by [KR]

Cor. 2.4.2, resp. by [KR] Cor. 3.3.8. The lower square and the upper triangle are commutative for trivial reasons.

We list a few additional properties of these functors.

Remark 1.2. *i. For any M in $\mathfrak{M}_f^{et}(\mathbf{A}_L)$ the inclusion $V(M) \subseteq \mathbf{A} \otimes_{\mathbf{A}_L} M$ extends to an isomorphism*

$$\mathbf{A} \otimes_{o_L} V(M) \xrightarrow{\cong} \mathbf{A} \otimes_{\mathbf{A}_L} M \ , \tag{6}$$

which is compatible with the φ_q- and Γ_L-actions on both sides.

ii. The functors D_{LT}, V, and $V(\mathbf{A}_L \otimes_{\mathbf{A}_L^+} -)$ respect exact sequences (of abelian groups).

iii. ([BSX] Prop. 3.4.14) For any $\mathcal{M}$ in $\mathrm{Mod}_{\mathcal{O}}^{\varphi_L,\Gamma_L,an}$ the projection map $\mathcal{M}[\frac{\omega_{LT}}{t_{LT}}] \longrightarrow D(\mathcal{M})$ restricts to an isomorphism $\mathcal{M}[\frac{\omega_{LT}}{t_{LT}}]^{\Gamma_L} \xrightarrow{\cong} D(\mathcal{M})$ such that the diagram

$$\begin{tikzcd}
\mathcal{M}[\frac{\omega_{LT}}{t_{LT}}]^{\Gamma_L} \arrow[r, "\cong"] \arrow[d, "\varphi_{\mathcal{M}}"'] & D(\mathcal{M}) \arrow[d, "\varphi_{D(\mathcal{M})}"] \\
\mathcal{M}[\frac{\omega_{LT}}{t_{LT}}]^{\Gamma_L} \arrow[r, "\cong"] & D(\mathcal{M})
\end{tikzcd}$$

is commutative; moreover, $\mathcal{M}[\frac{\omega_{LT}}{t_{LT}}] = \mathcal{O}[\frac{\omega_{LT}}{t_{LT}}] \otimes_L \mathcal{M}[\frac{\omega_{LT}}{t_{LT}}]^{\Gamma_L} \cong \mathcal{O}[\frac{\omega_{LT}}{t_{LT}}] \otimes_L D(\mathcal{M})$.

Now we recall that A_{cris} is the p-adic completion of a divided power envelope of $W(\widetilde{\mathbf{E}}^+)$ and let $A_{cris,L} := A_{cris} \otimes_{L_0} L$. The inclusion $W(\widetilde{\mathbf{E}}^+) \subseteq A_{cris}$ induces an embedding $\mathbf{A}_L^+ \subseteq W(\widetilde{\mathbf{E}}^+)_L \subseteq A_{cris,L}$.

We observe that $t_{LT} = \log_{LT}(\omega_{LT})$ belongs to $B_{cris,L}^\times$. Indeed, by [Co5, §III.2] we know that $\varphi_p(B_{max}) \subseteq B_{cris} \subseteq B_{max}$, whence we obtain

$$\varphi_q(B_{max} \otimes_{L_0} L) \subseteq B_{cris,L} \subseteq B_{max} \otimes_{L_0} L,$$

where the definition of B_{max} can be found in (loc. cit.). By [Co4, Prop. 9.10, Lem. 9.17,§9.7] t_{LT} and ω_{LT} are invertible in $B_{max,L} \subseteq B_{max} \otimes_{L_0} L$ (This reference assumes that the power series $[\pi_L](Z)$ is a polynomial. But, by some additional convergence considerations, the results can be seen to hold in general (cf. [GAL] §2.1 for more details)). Hence, by the above inclusions and using that $\varphi_q(t_{LT}) = \pi_L t_{LT}$, we see that t_{LT} is a unit $B_{cris,L}$. In particular, we have an inclusion $A_{cris,L}[\frac{1}{\pi_L}, \frac{1}{t_{LT}}] \subseteq B_{cris,L}$. Moreover, since $\varphi_q(\omega_{LT}) = Q\omega_{LT}$ is invertible in $\varphi_q(B_{max} \otimes_{L_0} L)$, the elements ω_{LT} and Q are units in $B_{cris,L}$ as well. In particular, we have an inclusion

$$\mathbf{A}^+[\tfrac{1}{\omega_{LT}}] \quad \subseteq B_{cris,L}. \tag{7}$$

Next we shall recall in Lemma 1.4 below that the above inclusion $\mathbf{A}_L^+ \subseteq A_{cris,L}$ extends to a (continuous) ring homomorphism

$$\mathcal{O} \to A_{cris,L}[\tfrac{1}{\pi_L}] \subseteq B_{cris,L}. \tag{8}$$

For $\alpha \in \widetilde{\mathbf{E}}^+ \cong \mathrm{proj\,lim}_n\, o_{\mathbb{C}_p}$ we denote by $\alpha^{(0)}$ as usual its zero-component.

Lemma 1.3. *The following diagram of o_{L_0}-modules is commutative*

(9)
$$\begin{tikzcd} 0 \arrow[r] & J \arrow[r] \arrow[d] & W(\widetilde{\mathbf{E}}^+) \arrow[r, "\Theta"] \arrow[d, "u"] & o_{\mathbb{C}_p} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \ker(\Theta_L) \arrow[r] & W(\widetilde{\mathbf{E}}^+)_L \arrow[r, "\Theta_L"] & o_{\mathbb{C}_p} \arrow[r] & 0, \end{tikzcd}$$

where $J := \ker(\Theta)$, $\Theta(\sum_{n\geqslant 0}[\alpha_n]p^n)) = \sum_{n\geqslant 0}\alpha_n^{(0)}p^n$ and similarly $\Theta_L(\sum_{n\geqslant 0}[\alpha_n]\pi_L^n)) = \sum_{n\geqslant 0}\alpha_n^{(0)}\pi_L^n$, while u denotes the canonical map as defined in [FF, Lem. 1.2.3], it sends Teichmüller lifts $[\alpha]$ with respect to $W(\widetilde{\mathbf{E}}^+)$ to the Teichmüller lift $[\alpha]$ with respect to $W(\widetilde{\mathbf{E}}^+)_L$.

Proof. First of all we recall from [GAL, Lem. 1.6.1] that Θ and Θ_L are continuous and show that also u is continuous, each time with respect to the weak topology, of which a fundamental system of open neighbourhoods consists of

$$U_{\mathfrak{a},m} := \{(b_0, b_1, \ldots) \in W(\widetilde{\mathbf{E}}^+) | b_0, \ldots, b_{m-1} \in \mathfrak{a}\} = \sum_{i=0}^{m-1} V_p^i([\mathfrak{a}]) + p^m W(\widetilde{\mathbf{E}}^+)$$

and similarly $U_{\mathfrak{a},m}^L := \{(b_0, b_1, \ldots) \in W(\widetilde{\mathbf{E}}^+)_L | b_0, \ldots, b_{m-1} \in \mathfrak{a}\}$ for open ideals $\mathfrak{a}$ of $\widetilde{\mathbf{E}}^+$ and $m \geqslant 0$; see §1.5 in (loc. cit.). By o_{L_0}-linearity, we see that $u(p^m W(\widetilde{\mathbf{E}}^+)) \subseteq p^m W(\widetilde{\mathbf{E}}^+)_L$. Using the relation

$$u(V_p x) = \frac{p}{\pi_L} V_{\pi_L}(u(F^{f-1}x))$$

from [FF, Lem. 1.2.3], where $V_?$ denotes the Verschiebung, one easily concludes that

$$u(V_p^i([b])) = (\frac{p}{\pi_L})^i V_{\pi_L}^i([b^{p^{i(f-1)}}]),$$

whence $u(U_{\mathfrak{a},m}) \subseteq U_{\mathfrak{a},m}^L$ and continuity of u follows.

Since the commutativity is clear on Teichmüller lifts and on p by o_{L_0}-linearity, which generate a dense ideal, the result follows by continuity. $\square$

The following lemma generalizes parts from [PR, Prop. 1.5.2.].

Lemma 1.4. *Sending $f = \sum_{n\geqslant 0} a_n Z^n$ to $f(\omega_{LT})$ induces a continuous map*

$$\mathcal{O} \to A_{cris,L}[\frac{1}{\pi_L}],$$

where the source carries the Fréchet-topology while the target is a topological o_{L_0}-module, of which the topology is uniquely determined by requiring that $A_{cris,L}$ is open, i.e., the system $p^m A_{cris,L}$ with $m \geqslant 0$ forms a basis of open neighbourhoods of 0.

Proof. First of all, the relation $J^p \subseteq pA_{cris}$ from [PR, §1.4.1, bottom of p. 96] (note that $J^p \subseteq W_p(R)$ regarding the notation in (loc. cit.) for the last object) implies easily by flat base change

(10)
$$J_L^p \subseteq pA_{cris,L}$$

with $J_L := J \otimes_{o_{L_0}} o_L$. By [GAL, Lem. 2.1.12] we know that ω_{LT} belongs to $\ker(\Theta_L)$. Now we claim that there exists a natural number r' such that $\omega_{LT}^{r'}$ lies in $W_1 := J_L + pW(\widetilde{\mathbf{E}}^+)_L$, whence for $r := pr'$ we have $\omega_{LT}^r \in W_p$ with $W_m := W_1^m$ for all $m \geqslant 0$. To this aim note that diagram (9) induces the following commutative diagram with exact lines

(11)

$$\begin{tikzcd} 0 \arrow[r] & W_1 \arrow[r] \arrow[d] & W(\widetilde{\mathbf{E}}^+) \otimes_{o_{L_0}} o_L \arrow[r, "\Theta"] \arrow[d, "\cong"'] & (o_{\mathbb{C}_p} \otimes_{o_{L_0}} o_L)/p(o_{\mathbb{C}_p} \otimes_{o_{L_0}} o_L) \arrow[r] \arrow[d, "\mu"'] & 0 \\ 0 \arrow[r] & \ker(\Theta_L) + pW(\widetilde{\mathbf{E}}^+)_L \arrow[r] & W(\widetilde{\mathbf{E}}^+)_L \arrow[r, "\Theta_L"] & o_{\mathbb{C}_p}/po_{\mathbb{C}_p} \arrow[r] & 0, \end{tikzcd}$$

where the map μ is induced by sending $a \otimes b$ to ab and a reference for the middle vertical isomorphism is [GAL, Prop. 1.1.26]. By the snake lemma the cokernel of the left vertical map is isomorphic to

$$\begin{aligned} \ker(\mu) &\subseteq \ker\Big((o_{\mathbb{C}_p} \otimes_{o_{L_0}} o_L)/p(o_{\mathbb{C}_p} \otimes_{o_{L_0}} o_L) \to \bar{k}\Big) \\ &= \ker\big(o_{\mathbb{C}_p}/po_{\mathbb{C}_p} \otimes_k o_L/po_L \to o_{\mathbb{C}_p}/\mathfrak{m}_{\mathbb{C}_p} \otimes_k o_L/\pi_L o_L\big) \\ &= \mathfrak{m}_{\mathbb{C}_p} \otimes_k o_L/po_L + o_{\mathbb{C}_p}/po_{\mathbb{C}_p} \otimes_k \pi_L o_L/po_L \end{aligned}$$

and thus consists of nilpotent elements whence the claim follows. Here $\mathfrak{m}_{\mathbb{C}_p}$ denotes the maximal ideal of $o_{\mathbb{C}_p}$.

Now let $f = \sum_{n \geqslant 0} a_n Z^n$ satisfy that $|a_n|\rho^n$ tends to zero for all $\rho < 1$. Writing $n = q_n r + r_n$ with $0 \leqslant r_n < r$, we have

$$a_n \omega_{LT}^n = a_n \omega_{LT}^{r_n} (\omega_{LT}^r)^{q_n} \in a_n W_{pq_n} \subseteq a_n p^{q_n} A_{cris,L},$$

where the last inclusion follows from (10). But $|a_n p^{q_n}| \leqslant |a_n|_p p^{1-\frac{n}{r}}$ tends to 0 for $n \to \infty$. Thus the series $\sum_{n \geqslant 0} a_n \omega_{LT}^n$ converges in $A_{cris,L}[\frac{1}{\pi_L}]$.

Moreover, since one has $sup|a_n p^{-1+\frac{n}{r}}| \leqslant p||f||_\rho$ for the usual norms $||\cdot||_\rho$ if $1 > \rho > p^{-\frac{1}{r}}$, we obtain for any m that

$$\{f|\ ||f||_\rho < p^{-m-1}\} \subseteq \{f \in \mathcal{O} | f(\omega_{LT}) \in p^m A_{cris,L}\},$$

whence the latter set, which is the preimage of $p^m A_{cris,L}$, is open. This implies continuity. $\square$

Lemma 1.5. *The big square in the middle is a commutative square of $\otimes$-functors (up to a natural isomorphism of $\otimes$-functors).*

Proof. We have to establish a natural isomorphism

$$L \otimes_{o_L} V(\mathbf{A}_L \otimes_{\mathbf{A}_L^+} N) \cong V_L(D(\mathcal{O} \otimes_{\mathbf{A}_L^+} N)) \qquad \text{for any } N \text{ in } \mathrm{Mod}_{\mathbf{A}_L^+}^{\varphi_L,\Gamma_L,an}. \tag{12}$$

In fact, we shall prove the dual statement, i.e., using (4), that

$$(L \otimes_{o_L} V(\mathbf{A}_L \otimes_{\mathbf{A}_L^+} N))^* \cong V_L(D(N))^*, \tag{13}$$

where * indicates the L-dual. From the canonical isomorphisms

$$\begin{aligned} \mathrm{Hom}_{\mathbf{A}_L,\varphi_q}(M, \mathbf{A}) &\cong \mathrm{Hom}_{\mathbf{A},\varphi_q}(\mathbf{A} \otimes_{\mathbf{A}_L} M, \mathbf{A}) \\ &\cong \mathrm{Hom}_{\mathbf{A},\varphi_q}(\mathbf{A} \otimes_{o_L} V(M), \mathbf{A}) \\ &\cong \mathrm{Hom}_{o_L}(V(M), \mathbf{A}^{\varphi_q=1}) \\ &\cong \mathrm{Hom}_{o_L}(V(M), o_L), \end{aligned}$$

where we used (6) for the second isomorphism and write M for $\mathbf{A}_L \otimes_{\mathbf{A}_L^+} N$, we conclude that the left hand side of (13) is canonically isomorphic to $\mathrm{Hom}_{\mathbf{A}_L,\varphi_q}(\mathbf{A}_L \otimes_{\mathbf{A}_L^+} N, \mathbf{A}) \otimes_{o_L} L$. Let $\mathbf{A}^+ := \mathbf{A} \cap W(\widetilde{\mathbf{E}}^+)_L$. On the one hand, by [KR] Lemma (3.2.1), base extension induces an isomorphism

$$\mathrm{Hom}_{\mathbf{A}_L^+,\varphi_q}(N, \mathbf{A}^+[\tfrac{1}{\omega_{LT}}]) \xrightarrow{\cong} \mathrm{Hom}_{\mathbf{A}_L,\varphi_q}(\mathbf{A}_L \otimes_{\mathbf{A}_L^+} N, \mathbf{A}) \ .$$

On the other hand, in [KR] Prop. (3.2.3) they construct a natural isomorphism

$$\mathrm{Hom}_{\mathbf{A}_L^+,\varphi_q}(N, \mathbf{A}^+[\tfrac{1}{\omega_{LT}}]) \otimes_{o_L} L \xrightarrow{\cong} \mathrm{Hom}_{L,\varphi_q,\mathrm{Fil}}((N/\omega_{LT}N)[\tfrac{1}{p}], B_{cris,L}) \ . \tag{14}$$

Therefore, the left hand side of (13) becomes naturally isomorphic to

$$\mathrm{Hom}_{L,\varphi_q,\mathrm{Fil}}(D(N), B_{cris,L}) \cong V_L(D(N)^*), \tag{15}$$

where the last isomorphism is straightforward. Thus the proof of (13) is reduced to the canonical identity

$$V_L(D(N)^*) \cong V_L(D(N))^*. \tag{16}$$

This can be proved in the same way as in [F1, Rem. 3.4.5 (iii), Rem. 3.6.7]: Since V_L is a rigid $\otimes$-functor, it preserves inner Hom-objects, in particular duals.

In order to see that (12) is compatible with tensor products note that base change, taking L-duals or applying comparison isomorphisms are $\otimes$-compatible. Thus the claim is reduced to the tensor compatiblity of the isomorphism (14) the construction of which we therefore recall from [KR]. It is induced by a natural map

$$\mathrm{Hom}_{\mathbf{A}_L^+}(N, \mathbf{A}^+[\tfrac{1}{\omega_{LT}}]) \otimes_{o_L} L \longrightarrow \mathrm{Hom}_L((N/\omega_{LT}N)[\tfrac{1}{p}], B_{cris,L})$$

which comes about as follows. Let $f \in \mathrm{Hom}_{\mathbf{A}_L^+}(N, \mathbf{A}^+[\frac{1}{\omega_{LT}}])$. By composing f with the inclusion (7) we obtain $f_1 : N \to B_{cris,L}$. By base extension to $\mathcal{O}$ via (8) and then localization in Q the map f_1 gives rise to a map $f_2 : (\mathcal{O} \otimes_{\mathbf{A}_L^+} N)[\frac{1}{Q}] \to B_{cris,L}$. This one we precompose with the isomorphism $1 \otimes \varphi_N$ to obtain

$$f_3 : (\mathcal{O} \otimes_{\mathbf{A}_L^+,\varphi_L} N)[\tfrac{1}{Q}] \xrightarrow{\cong} (\mathcal{O} \otimes_{\mathbf{A}_L^+} N)[\tfrac{1}{Q}] \to B_{cris,L} \ .$$

Now we observe the inclusions

$$\begin{aligned}(\mathcal{O} \otimes_{\mathbf{A}_L^+,\varphi_L} N)[\tfrac{1}{Q}] \subseteq (\mathcal{O} \otimes_{\mathbf{A}_L^+,\varphi_L} N)[\tfrac{\omega_{LT}}{t_{LT}}] \supseteq (\mathcal{O} \otimes_{\mathbf{A}_L^+,\varphi_L} N)[\varphi_L(\tfrac{\omega_{LT}}{t_{LT}})] \\ = \mathcal{O} \otimes_{\mathcal{O},\varphi_L} \big((\mathcal{O} \otimes_{\mathbf{A}_L^+} N)[\tfrac{\omega_{LT}}{t_{LT}}]\big) \ .\end{aligned}$$

They only differ by elements which are invertible in $B_{cris,L}$. Therefore giving the map f_3 is equivalent to giving a map $f_4 : \mathcal{O} \otimes_{\mathcal{O},\varphi_L} \big((\mathcal{O} \otimes_{\mathbf{A}_L^+} N)[\frac{\omega_{LT}}{t_{LT}}]\big) \to B_{cris,L}$. Finally we use Remark 1.2.iii which gives the map

$$\xi : (N/\omega_{LT}N)[\tfrac{1}{p}] \xleftarrow[\mathrm{pr}]{\cong} \big((\mathcal{O} \otimes_{\mathbf{A}_L^+} N)[\tfrac{\omega_{LT}}{t_{LT}}]\big)^{\Gamma_L} \xrightarrow{\subseteq} (\mathcal{O} \otimes_{\mathbf{A}_L^+} N)[\tfrac{\omega_{LT}}{t_{LT}}] \ .$$

By precomposing f_4 with $1 \otimes \xi$ we at last arrive at a map $f_5 : (N/\omega_{LT}N)[\frac{1}{p}] \to B_{cris,L}$. From this description the compatibility with tensor products is easily checked. $\square$

Suppose that N is in $\mathrm{Mod}_{\mathbf{A}_L^+}^{\varphi_L,\Gamma_L,an}$ and put $T := V(\mathbf{A}_L \otimes_{\mathbf{A}_L^+} N)$ in $\mathrm{Rep}_{o_L,f}^{cris,an}(G_L)$. Then, by Remark 1.2.iii and Lemma 1.5, we have a natural isomorphism of $\otimes$-functors

$$\mathrm{comp} : \mathcal{O}[\tfrac{\omega_{LT}}{t_{LT}}] \otimes_{\mathbf{A}_L^+} N \xrightarrow{\cong} \mathcal{O}[\tfrac{\omega_{LT}}{t_{LT}}] \otimes_L D_{cris,L}(L \otimes_{o_L} T) \tag{17}$$

which is compatible with the diagonal φ's on both sides.

In the proof of [KR] Cor. 3.3.8 it is shown that, for any T in $\mathrm{Rep}_{o_L,f}^{cris,an}(G_L)$, there exists an $\mathbf{A}_L^+$-submodule $\mathfrak{M} \subseteq D_{LT}(T)$ which

(N1) lies in $\mathrm{Mod}_{\mathbf{A}_L^+}^{\varphi_L,\Gamma_L,an}$ with $\varphi_{\mathfrak{M}}$ and the Γ_L-action on $\mathfrak{M}$ being induced by the (φ_q, Γ_L)-structure of $D_{LT}(T)$, and

(N2) satisfies $\mathbf{A}_L \otimes_{\mathbf{A}_L^+} \mathfrak{M} = D_{LT}(T)$.

Note that property (N2) implies that $\mathfrak{M}$ is *p-saturated in* $D_{LT}(T)$, i.e., $\mathfrak{M}[\frac{1}{p}] \cap D_{LT}(T) = \mathfrak{M}$, since $\mathbf{A}_L^+$ is obviously p-saturated in $\mathbf{A}_L$.

We once and for all pick such an $N(T) := \mathfrak{M}$. This defines a functor

$$N : \mathrm{Rep}_{o_L,f}^{cris,an}(G_L) \longrightarrow \mathrm{Mod}_{\mathbf{A}_L^+}^{\varphi_L,\Gamma_L,an}$$

which is quasi-inverse to the upper left horizontal arrow in the above big diagram. Note that N is in a unique way a $\otimes$-functor by [Sa, I.4.4.2.1].

Remark 1.6. *For T in $\mathrm{Rep}_{o_L,f}^{cris,an}(G_L)$ and $N := N(T)$ in $\mathrm{Mod}_{\mathbf{A}_L^+}^{\varphi_L,\Gamma_L,an}$ we have:*

(i) *If $L \otimes_{o_L} T$ is a positive analytic crystalline representation, then N is stable under φ_N;*

(ii) *If the Hodge-Tate weights of $L \otimes_{o_L} T$ are all $\geqslant 0$, then we have $N \subseteq \mathbf{A}_L^+ \cdot \varphi_N(N)$, where the latter means the $\mathbf{A}_L^+$-span generated by $\varphi_N(N)$.*

Proof. The corresponding assertions for $\mathcal{M} := \mathcal{O} \otimes_{\mathbf{A}_L^+} N$ are contained in [BSX] Cor. 3.4.9. Let $n_1, \ldots, n_d$ be an $\mathbf{A}_L^+$-basis of N.

For (i) we have to show that $\varphi_N(n_j) \in N$ for any $1 \leqslant j \leqslant d$. Writing $\varphi_N(n_j) = \sum_{i=1}^d f_{ij} n_i$ we know from the definition of the category $\mathrm{Mod}_{\mathbf{A}_L^+}^{\varphi_L,\Gamma_L,an}$ that $f_{ij} \in \mathbf{A}_L^+[\frac{1}{Q}]$ and from the above observation that $f_{ij} \in \mathcal{O}$. This reduces us to showing that $o_L[[Z]][\frac{1}{Q}] \cap \mathcal{O} \subseteq o_L[[Z]]$. Suppose therefore that $Q^r h = f$ for some $r \geqslant 1$, $h \in \mathcal{O}$, and $f \in o_L[[Z]]$. The finitely many zeros of $Q \in o_L[[Z]]$, which are the nonzero π_L-torsion points of the Lubin-Tate formal group, all lie in the open unit disk. By Weierstrass preparation it follows that Q must divide f already in $o_L[[Z]]$. Hence $h \in o_L[[Z]]$.

For (ii) we have to show that $n_j = \sum_{i=1}^d f_{ij} \varphi_N(n_i)$, for any $1 \leqslant j \leqslant d$, with $f_{ij} \in \mathbf{A}_L^+$. For the same reasons as in the proof of (1) we have $n_j = \sum_{i=1}^d f'_{ij} \varphi_N(n_i) = \sum_{i=1}^d f''_{ij} \varphi_N(n_i)$ with $f'_{ij} \in \mathbf{A}_L^+[\frac{1}{Q}]$ and $f''_{ij} \in \mathcal{O}$. Then $\sum_{i=1}^d (f'_{ij} - f''_{ij}) \varphi_N(n_i) = 0$. But, again by the definition of the category $\mathrm{Mod}_{\mathbf{A}_L^+}^{\varphi_L,\Gamma_L,an}$, the $\varphi_N(n_i)$ are linearly independent over $\mathbf{A}_L^+[\frac{1}{Q}]$ and hence over $\mathcal{O}[\frac{1}{Q}]$. It follows that $f'_{ij} = f''_{ij} \in \mathbf{A}_L^+$. □

First we further investigate any T in $\mathrm{Rep}_{o_L,f}^{cris,an}(G_L)$ whose **Hodge-Tate weights are all** $\leqslant 0$, i.e., which is positive. For this purpose we need the ring $\mathbf{A}^+ = \mathbf{A} \cap W(\widetilde{\mathbf{E}}^+)_L$. One has the following general fact.

Lemma 1.7. *Let F be any nonarchimedean valued field which contains $o_L/\pi_L o_L$, and let o_F denote its ring of integers; we have:*

i. Let $\alpha \in W(F)_L$ be any element; if the $W(o_F)_L$-submodule of $W(F)_L$ generated by $\{\varphi_q^i(\alpha)\}_{i\geqslant 0}$ is finitely generated then $\alpha \in W(o_F)_L$.

ii. Let X be a finitely generated free o_L-module, and let M be a finitely generated $W(o_F)_L$-submodule of $W(F)_L \otimes_{o_L} X$; if M is $\varphi_q \otimes \mathrm{id}$-invariant then $M \subseteq W(o_F)_L \otimes_{o_L} X$.

Proof. i. This is a simple explicit calculation as given, for example, in the proof of [Co1] Lemma III.5. ii. This is a straightforward consequence of i. □

Proposition 1.8. *For positive T in $\mathrm{Rep}_{o_L,f}^{cris,an}(G_L)$ we have*

$$N(T) \subseteq D_{LT}^+(T) := (\mathbf{A}^+ \otimes_{o_L} T)^{\ker(\chi_{LT})} ,$$

and $N(T)$ is p-saturated in $D_{LT}^+(T)$.

Proof. By Remark 1.6.i the $\mathbf{A}_L^+$-submodule $N(T)$ of $W(\widetilde{\mathbf{E}})_L \otimes_{o_L} T$ is $\varphi_q \otimes \mathrm{id}$-invariant (and finitely generated). Hence we may apply Lemma 1.7.ii to $M := W(\widetilde{\mathbf{E}}^+)_L \cdot N(T)$ and obtain that $N(T) \subseteq (W(\widetilde{\mathbf{E}}^+)_L \otimes_{o_L} T) \cap (\mathbf{A} \otimes_{o_L} T)^{\ker(\chi_{LT})} = D_{LT}^+(T)$. Since $N(T)$ is even p-saturated in $D_{LT}(T)$, the same holds with respect to the smaller $D_{LT}^+(T)$. □

Corollary 1.9. *For positive T in $\mathrm{Rep}_{o_L,f}^{cris,an}(G_L)$ the $\mathbf{A}_L^+$-module $D_{LT}^+(T)$ is free of the same rank as $N(T)$.*

Proof. By the argument in the proof of [Co1] Lemma III.3 the $\mathbf{A}_L^+$-module $D_{LT}^+(T)$ always is free of a rank less or equal to the rank of $N(T)$. The equality of the ranks in the positive case then is a consequence of Prop. 1.8. □

Next we relate $N(T)$ to the construction in [Be] Prop. II.1.1.

Proposition 1.10. *Suppose that T in $\mathrm{Rep}_{o_L,f}^{cris,an}(G_L)$ is positive. For $N := N(T)$ we then have:*

i. N is the unique $\mathbf{A}_L^+$-submodule of $D_{LT}(T)$ which satisfies (N1) and (N2).

ii. N is also the unique $\mathbf{A}_L^+$-submodule of $D_{LT}^+(T)$ which satisfies:

(a) N is free of rank equal to the rank of $D_{LT}^+(T)$;

(b) N is Γ_L-invariant;

(c) the induced Γ_L-action on $N/\omega_{LT}N$ is trivial;

(d) $\omega_{LT}^r D_{LT}^+(T) \subseteq N$ for some $r \geqslant 0$.

Proof. Let $P = P(\mathbf{A}_L^+)$ denote the set of height one prime ideals of $\mathbf{A}_L^+$. It contains the prime ideal $\mathfrak{p}_0 := (\omega_{LT})$.

Step 1: We show the existence of a unique $\mathbf{A}_L^+$-submodule N' of $D_{LT}^+(T)$ which satisfies (a) – (d), and we show that this N' is φ_q-invariant.

Existence: We begin by observing that the $\mathbf{A}_L^+$-submodule $N := N(T)$ of $D_{LT}^+(T)$ has the properties (a), (b), and (c), but possibly not (d). In particular, the quotient $D_{LT}^+(T)/N$ is an $\mathbf{A}_L^+$-torsion module. Hence the localizations $N_\mathfrak{p} = D_{LT}^+(T)_\mathfrak{p}$ coincide for all but finitely many

$\mathfrak{p} \in P$. By [B-CA] VII.4.3 Thm. 3 there exists a unique intermediate $\mathbf{A}_L^+$-module $N \subseteq N' \subseteq D_{LT}^+(T)$ which is finitely generated and reflexive and such that $N'_{\mathfrak{p}_0} = N_{\mathfrak{p}_0}$ and $N'_{\mathfrak{p}} = D_{LT}^+(T)_{\mathfrak{p}}$ for any $\mathfrak{p} \in P \backslash \{\mathfrak{p}_0\}$. Since $\mathbf{A}_L^+$ is a two dimensional regular local ring the finitely generated reflexive module N' is actually free, and then, of course, must have the same rank as N and $D_{LT}^+(T)$. We also have $N' = \bigcap_{\mathfrak{p}} N'_{\mathfrak{p}} = N_{\mathfrak{p}_0} \cap \bigcap_{\mathfrak{p} \neq \mathfrak{p}_0} D_{LT}^+(T)_{\mathfrak{p}}$. Since $\mathfrak{p}_0$ is preserved by $\varphi_{D_{LT}^+(T)}$ and Γ_L it follows that N' is $\varphi_{D_{LT}^+(T)}$- and Γ_L-invariant. Next the identities

$$L \otimes_{o_L} N/\omega_{LT} N = N_{\mathfrak{p}_0}/\omega_{LT} N_{\mathfrak{p}_0} = N'_{\mathfrak{p}_0}/\omega_{LT} N'_{\mathfrak{p}_0} = L \otimes_{o_L} N'/\omega_{LT} N' \supseteq N'/\omega_{LT} N'$$

show that the induced Γ_L-action on $N'/\omega_{LT} N'$ is trivial. By using [B-CA] VII.4.4 Thm. 5 we obtain, for some $m_1, \ldots, m_d \geqslant 0$, a homomorphism of $\mathbf{A}_L^+$-modules $D_{LT}^+(T)/N' \longrightarrow \oplus_{i=1}^d \mathbf{A}_L^+/\mathfrak{p}_0^{m_i} \mathbf{A}_L^+$ whose kernel is finite. Any finite $\mathbf{A}_L^+$-module is annihilated by a power of the maximal ideal in $\mathbf{A}_L^+$. We see that $D_{LT}^+(T)/N'$ is annihilated by a power of $\mathfrak{p}_0$, which proves (d).

Uniqueness: Observing that $\gamma(\omega_{LT}) = [\chi_{LT}(\gamma)](\omega_{LT})$ for any $\gamma \in \Gamma_L$ ([GAL] Lemma 2.1.15) this is exactly the same computation as in the uniqueness part of the proof of [Be] Prop. II.1.1.

Step 2: We show that N' is p-saturated in $D_{LT}^+(T)$. By construction we have $(N')_{(\pi_L)} = D_{LT}^+(T)_{(\pi_L)}$. This implies that the p-torsion in the quotient $D_{LT}^+(T)/N'$ is finite. On the other hand, both modules, N' and $D_{LT}^+(T)$, are free of the same rank. Hence the finitely generated $\mathbf{A}_L^+$-module $D_{LT}^+(T)/N'$ has projective dimension $\leqslant 1$ and therefore has no nonzero finite submodule (cf. [NSW] Prop. 5.5.3(iv)).

Step 3: We show that $N' = N$. Since both, N and N', are p-saturated in $D_{LT}^+(T)$ it suffices to show that the free $\mathbf{B}_L^+$-modules $N(V) := N[\frac{1}{p}]$ and $N'(V) := N'[\frac{1}{p}]$ over the principal ideal domain $\mathbf{B}_L^+ := \mathbf{A}_L^+[\frac{1}{p}]$ coincide. As they are both Γ_L-invariant, so is the annihilator ideal $I := ann_{\mathbf{B}_L^+}(N'(V)/N(V))$. Hence, by a standard argument as in [Be] Lemma I.3.2, the ideal I is generated by an element f of the form $\omega_{LT}^{\alpha_0} \prod_{n \geqslant 1}^{s} \varphi_L^{n-1}(Q)^{\alpha_n}$ with certain $\alpha_n \geqslant 0$, $0 \leqslant n \leqslant s$, for some (minimal) $s \geqslant 0$. Since $N(V)_{(\omega_{LT})} = N'(V)_{(\omega_{LT})}$ by the construction of N', it follows that $\alpha_0 = 0$. Assuming that $M := N'(V)/N(V) \neq 0$ we conclude that $s \geqslant 1$ (with $\alpha_s \geqslant 1$), i.e., that, with $\mathfrak{p}_n := (\varphi_L^{n-1}(Q))$, we have $M_{\mathfrak{p}_s} \neq 0$ while $M_{\mathfrak{p}_{s+1}} = 0$. We claim that $(\varphi_L^* M)_{\mathfrak{p}_{s+1}} \neq 0$. First note that we have have an exact sequence

$$0 \to (\mathbf{B}_L^+)^d \xrightarrow{A} (\mathbf{B}_L^+)^d \to M \to 0 ,$$

with f dividing $\det(A) \in \mathbf{B}_L^+ \backslash (\mathbf{B}_L^+)_{\mathfrak{p}_s}^\times$, which induces an exact sequence

$$0 \to (\mathbf{B}_L^+)^d \xrightarrow{\varphi_L(A)} (\mathbf{B}_L^+)^d \to \varphi_L^* M \to 0 .$$

Since $\varphi_L(f) = \prod_{n \geqslant 2}^{s+1} \varphi_L^{n-1}(Q)^{\alpha_{n-1}}$ divides $\det(\varphi_L(A))$ we conclude that $\det(\varphi_L(A))$ belongs to $\mathfrak{p}_{s+1}$ which implies the claim.

Now consider the following diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & (\varphi_L^* N(V))[\frac{1}{Q}] \arrow[r, "\cong"] \arrow[d, hook] & N(V)[\frac{1}{Q}] \arrow[r] \arrow[d, hook] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (\varphi_L^* N'(V))[\frac{1}{Q}] \arrow[r] & N'(V)[\frac{1}{Q}] \arrow[r] & C \arrow[r] & 0.
\end{tikzcd}$$

The upper isomorphism comes from the definition of the category $\mathrm{Mod}^{\varphi_L,\Gamma_L,an}_{\mathbf{A}_L^+}$ in which N lies. The map $(\varphi_L^*N'(V))[\frac{1}{Q}] \to N'(V)[\frac{1}{Q}]$ is injective since $\varphi_L^*N' \to N'$ is the restriction of the isomorphism $\varphi_L^*D_{LT}(T) \xrightarrow{\cong} D_{LT}(T)$. By the snake lemma and as $Q \notin \mathfrak{p}_{s+1}$ we obtain an injection

$$0 \neq (\varphi_L^*M)_{\mathfrak{p}_{s+1}} \hookrightarrow M_{\mathfrak{p}_{s+1}} = 0 \ ,$$

which is a contradiction. Thus $M = 0$ as had to be shown. □

Remark 1.11. (i) *$N(o_L(\chi_{LT}^{-1})) = \omega_{LT}\mathbf{A}_L^+ \otimes_{o_L} o_L\eta^{\otimes -1}$ and $N(o_L) = \mathbf{A}_L^+$.*

(ii) *Let $o_L(\chi) = o_Lt_0$ with $\chi : G_L \to o_L^\times$ unramified. Then there exists an $a \in W(\bar{k}_L)_L^\times$ with $\sigma a = \chi^{-1}(\sigma)a$ for all $\sigma \in G_L$ by Remark 1.21 ; in particular,*

$$N(o_L(\chi)) = D_{LT}^+(o_L(\chi)) = \mathbf{A}_L^+n_0 \quad \text{for } n_0 = a \otimes t_0,$$

where Γ_L fixes n_0 and $\varphi_{N(o_L(\chi))}(n_0) = cn_0$ with $c := \frac{\varphi_L(a)}{a} \in o_L^\times$.

Proof. Each case belongs to a positive representation T: in all cases the right hand side of the equality satisfies the properties characterizing $N(T)$ in Prop. 1.10.ii (cf. [GAL] Lemma 2.1.15). □

Lemma 1.12. *For any $T \in \mathrm{Rep}^{cris,an}_{o_L,f}(G_L)$ we have:*

i. $N(T)$ is the unique $\mathbf{A}_L^+$-submodule of $D_{LT}(T)$ which satisfies (N1) and (N2);

ii. $N(T(\chi_{LT}^{-r})) \cong \omega_{LT}^rN(T) \otimes_{o_L} o_L\eta^{\otimes -r}$.

Proof. First we choose $r \geqslant 0$ such that $T(\chi_{LT}^{-r})$ is positive. Sending N to $\omega_{LT}^rN(T) \otimes_{o_L} o_L\eta^{\otimes -r} \subseteq D_{LT}(T) \otimes_{o_L} o_L\eta^{\otimes -r}$ viewed in $D_{LT}(T) \otimes_{o_L} o_L\eta^{\otimes -r} \cong D_{LT}(T(\chi_{LT}^{-r}))$ sets up a bijection between the $\mathbf{A}_L^+$-submodules of $D_{LT}(T)$ and $D_{LT}(T(\chi_{LT}^{-r}))$, respectively. One checks that N satisfies (N1) and (N2) if and only if its image does. Hence i. and ii. (for such r) are a consequence of Prop. 1.10.i. That ii. holds in general follows from the obvious transitivity property of the above bijections. □

Proposition 1.13. *Let T be in $\mathrm{Rep}^{cris,an}_{o_L,f}(G_L)$ of o_L-rank d and such that $V = L \otimes_{o_L} T$ is positive with Hodge-Tate weights $-r = -r_d \leqslant \cdots \leqslant -r_1 \leqslant 0$. Taking (17) as an identification we then have*

$$(\frac{t_{LT}}{\omega_{LT}})^r\mathcal{O} \otimes_{\mathbf{A}_L^+} N(T) \subseteq \mathcal{O} \otimes_L D_{cris,L}(L \otimes_{o_L} T) \subseteq \mathcal{O} \otimes_{\mathbf{A}_L^+} N(T) \tag{18}$$

with elementary divisors

$$[\mathcal{O} \otimes_{\mathbf{A}_L^+} N(T) : \mathcal{O} \otimes_L D_{cris,L}(L \otimes_{o_L} T)] = [(\frac{t_{LT}}{\omega_{LT}})^{r_1} : \cdots : (\frac{t_{LT}}{\omega_{LT}})^{r_d}].$$

Proof. We abbreviate $D := D_{cris,L}(V)$. By the definition of the functor $\mathcal{M}$ in [KR] we have

$$\mathcal{O} \otimes_L D \subseteq \mathcal{M}(D) \subseteq (\frac{t_{LT}}{\omega_{LT}})^{-r}\mathcal{O} \otimes_L D \ . \tag{19}$$

On the other hand, the commutativity of the big diagram before Remark 1.2 says that $\mathcal{M}(D) \cong \mathcal{O} \otimes_{\mathbf{A}_L^+} N(T)$. This implies the inclusions (18).

Concerning the second part of the assertion we first of all note that, although $\mathcal{O}$ is only a Bezout domain, it does satisfy the elementary divisor theorem ([ST1] proof of Prop. 4.4). We may equivalently determine the elementary divisors of the $\mathcal{O}$-module $\mathcal{M}(D)/(\mathcal{O} \otimes_L D)$. The countable set $\mathbb{S}$ of zeros of the function $\frac{t_{LT}}{\omega_{LT}} \in \mathcal{O}$ coincides with the set of nonzero torsion points of our Lubin-Tate formal group, each occurring with multiplicity one. The first part of the assertion implies that the module $\mathcal{O}$-module $\mathcal{M}(D)/(\mathcal{O} \otimes_L D)$ is supported on $\mathbb{S}$. Let $\mathcal{M}_z(D)$, resp. $\mathcal{O}_z$, denote the stalk in $z \in \mathbb{S}$ of the coherent sheaf on $\mathbb{B}$ defined by $\mathcal{M}(D)$, resp. $\mathcal{O}$. The argument in the proof of [BSX] Prop. 1.1.10 then shows that we have

$$\mathcal{M}(D)/(\mathcal{O} \otimes_L D) = \prod_{z \in \mathbb{S}} \mathcal{M}_z(D)/(\mathcal{O}_z \otimes_L D) \ .$$

The ring $\mathcal{O}_z$ is a discrete valuation ring with maximal ideal $\mathfrak{m}_z$ generated by $\frac{t_{LT}}{\omega_{LT}}$. We consider on its field of fractions $\mathrm{Fr}(\mathcal{O}_z)$ the $\mathfrak{m}_z$-adic filtration and then on $\mathrm{Fr}(\mathcal{O}_z) \otimes_L D$ the tensor product filtration. By [Kis] Lemma 1.2.1(2) (or [BSX] Lemma 3.4.4) we have

$$\mathcal{M}_z(D) \cong \mathrm{Fil}^0(\mathrm{Fr}(\mathcal{O}_z) \otimes_L D) \qquad \text{for any } z \in \mathbb{S},$$

and this isomorphism preserves $\mathcal{O}_z \otimes_L D$. At this point we let $0 \leqslant s_1 < \ldots < s_m < r$ denote the jumps of the filtration $\mathrm{Fil}^\bullet D$, i.e., the r_j but without repetition. We write

$$D = D_1 \oplus \ldots \oplus D_m \quad \text{such that} \quad \mathrm{Fil}^{s_i} D = D_i \oplus \ldots \oplus D_m \ .$$

For the following computation let, for notational simplicity, R denote any L-algebra which is a discrete valuation ring with maximal ideal $\mathfrak{m}$. We compute

$$\begin{aligned}
\mathrm{Fil}^0(\mathrm{Fr}(R) \otimes_L D) &= \sum_{j \in \mathbb{Z}} \mathfrak{m}^{-j} \otimes_L \mathrm{Fil}^j D = \sum_{j=0}^{r} \mathfrak{m}^{-j} \otimes_L \mathrm{Fil}^j D \\
&= \sum_{i=1}^{m} \mathfrak{m}^{-s_i} \otimes_L \mathrm{Fil}^{s_i} D = \sum_{i=1}^{m} \sum_{j=i}^{m} \mathfrak{m}^{-s_i} \otimes_L D_j \\
&= \sum_{j=1}^{m} (\sum_{i=1}^{j} \mathfrak{m}^{-s_i}) \otimes_L D_j = \sum_{j=1}^{m} \mathfrak{m}^{-s_j} \otimes_L D_j \ .
\end{aligned}$$

Hence we obtain

$$\mathrm{Fil}^0(\mathrm{Fr}(R) \otimes_L D)/(R \otimes_L D) = \oplus_{j=1}^m \mathfrak{m}^{-s_j}/R \otimes_L D_j \cong \oplus_{j=1}^m R/\mathfrak{m}^{s_j} \otimes_L D_j \ .$$

By combining all of the above we finally arrive at

$$\begin{aligned}
\mathcal{M}(D)/(\mathcal{O} \otimes_L D) &= \prod_{z \in \mathbb{S}} \mathcal{M}_z(D)/(\mathcal{O}_z \otimes_L D) \cong \prod_{z \in \mathbb{S}} \mathrm{Fil}^0(\mathrm{Fr}(\mathcal{O}_z) \otimes_L D)/(\mathcal{O}_z \otimes_L D) \\
&\cong \prod_{z \in \mathbb{S}} (\oplus_{j=1}^m \mathcal{O}_z/\mathfrak{m}_z^{s_j} \otimes_L D_j) = \oplus_{j=1}^m (\prod_{z \in \mathbb{S}} (\mathcal{O}_z/(\tfrac{t_{LT}}{\omega_{LT}})^{s_j} \mathcal{O}_z \otimes_L D_j)) \\
&= \oplus_{j=1}^m (\prod_{z \in \mathbb{S}} \mathcal{O}_z/(\tfrac{t_{LT}}{\omega_{LT}})^{s_j} \mathcal{O}_z) \otimes_L D_j = \oplus_{j=1}^m \mathcal{O}/(\tfrac{t_{LT}}{\omega_{LT}})^{s_j} \mathcal{O} \otimes_L D_j \ .
\end{aligned}$$

$\square$

For a first application of this result we recall the comparison isomorphism

$$\begin{tikzcd} N(V) \arrow[d, "\subseteq"'] & & \\ N(V)[\frac{1}{Q}] \arrow[r, "\subseteq"] & \mathcal{O}[\frac{\omega_{LT}}{t_{LT}}] \otimes_{\mathbf{A}_L^+} N(V) \arrow[r, "\mathrm{comp}", "\cong"'] & \mathcal{O}[\frac{\omega_{LT}}{t_{LT}}] \otimes_L D_{cris,L}(V) \end{tikzcd} \tag{20}$$

for any T in $\mathrm{Rep}_{o_L,f}^{cris,an}(G_L)$, $V := L \otimes_{o_L} T$, and $N(V) := N(T)[\frac{1}{\pi_L}]$. The left horizontal inclusion comes from the fact that $\frac{t_{LT}}{\omega_{LT}}$ is a multiple of Q in $\mathcal{O}$. In particular, we have the commutative diagram

$$\begin{tikzcd} & N(V) \arrow[dl, "\varphi_{N(V)}"'] \arrow[d, "\varphi_{N(V)}"] \arrow[r, "\mathrm{comp}"] & \mathcal{O}[\frac{\omega_{LT}}{t_{LT}}] \otimes_L D_{cris,L}(V) \arrow[d, "\varphi_L \otimes \varphi_{cris}"] \\ N^{(\varphi)}(V) \arrow[r, "\subseteq"] & N(V)[\frac{1}{Q}] \arrow[r, "\mathrm{comp}"] & \mathcal{O}[\frac{\omega_{LT}}{t_{LT}}] \otimes_L D_{cris,L}(V) \end{tikzcd}$$

where φ_{cris} denotes the q-Frobenius on $D_{cris,L}(V)$ and where

$$N^{(\varphi)}(V) := \text{the } \mathbf{A}_L^+\text{-submodule of } N(V)[\tfrac{1}{Q}] \text{ generated by the image of } N(V) \text{ under } \varphi_{N(V)}.$$

We note that, since Q is invertible in $\mathbf{A}_L$, $N^{(\varphi)}(V)$ can also be viewed as the $\mathbf{A}_L^+$-submodule of $D_{LT}(V) = \mathbf{A}_L \otimes_{\mathbf{A}_L^+} N(V)$ generated by the image of $N(V)$ under $\varphi_{D_{LT}(V)}$. from this one easily deduces (use the projection formula for the ψ-operator) that the map $\psi_{D_{LT}(V)}$ on $D_{LT}(V)$ restricts to an operator

$$\psi_{N(V)} : N^{(\varphi)}(V) \longrightarrow N(V) \ .$$

Corollary 1.14. *Assume that the Hodge-Tate weights of V are all in $[0, r]$. Then we have*

$$\mathrm{comp}(N(V)) \subseteq \mathcal{O} \otimes_L D_{cris,L}(V), \ \ \mathrm{comp}(N^{(\varphi)}(V)) \subseteq \mathcal{O} \otimes_L D_{cris,L}(V), \ \ \textit{and} \tag{21}$$

$$\mathrm{comp}(N^{(\varphi)}(V)^{\psi_{N(V)}=0}) \subseteq \mathcal{O}^{\psi_L=0} \otimes_L D_{cris,L}(V) \ . \tag{22}$$

Proof. Apply Proposition 1.13 to $T(\chi_{LT}^{-r})$, then divide the resulting (left) inclusion in (18) by t_{LT}^r and tensor with $o_L(\chi_{LT}^r)$. This gives the first inclusion by Lemma 1.12 upon noting that $t_{LT}^r D_{cris,L}(L \otimes_{o_L} T) \otimes_L L\eta^{\otimes -r} = D_{cris,L}(L \otimes_{o_L} T(\chi_{LT}^{-r}))$. The second inclusion easily derives from the first by using that the map comp is compatible with the φ's.

For the third inclusion we consider any element $x = \sum_i f_i \varphi_{N(V)}(x_i) \in N^{(\varphi)}(V)$, with $f_i \in \mathbf{A}_L^+$ and $x_i \in N(V)$, such that $\psi_{N(V)}(x) = \sum_i \psi_L(f_i) x_i = 0$. We choose an L-basis $e_1, \ldots, e_m$ of $D_{cris,L}(V)$ and write $\mathrm{comp}(x_i) = \sum_j f_{ij} \otimes e_j$ with $f_{ij} \in \mathcal{O}$. Then

$$0 = \mathrm{comp}(\psi_{N(V)}(x)) = \sum_i \psi_L(f_i) \mathrm{comp}(x_i) = \sum_i \sum_j \psi_L(f_i) f_{ij} \otimes e_j$$

and it follows that

$$\psi_L(\sum_i f_i \varphi_L(f_{ij})) = \sum_i \psi_L(f_i) f_{ij} = 0 \ ,$$

i.e., that $\sum_i f_i \varphi_L(f_{ij}) \in \mathcal{O}^{\psi_L=0}$. On the other hand we compute

$$\begin{aligned}
\mathrm{comp}(x) &= \sum_i f_i \varphi_{N(V)}(x_i) = \sum_i f_i (\varphi_L \otimes \varphi_{cris})(\mathrm{comp}(x_i)) \\
&= \sum_i \sum_j f_i (\varphi_L(f_{ij}) \otimes \varphi_{cris}(e_j)) \\
&= \sum_j \big(\sum_i f_i \varphi_L(f_{ij}) \big) \otimes \varphi_{cris}(e_j) \ .
\end{aligned}$$

□

Corollary 1.15. *In the situation of Proposition 1.13 we have*

$$D_{cris,L}(V) \cong \Big(\mathcal{O} \otimes_{\mathbf{A}_L^+} N(T) \Big)^{\Gamma_L} .$$

Proof. We set $\mathcal{M} := \mathcal{O} \otimes_{\mathbf{A}_L^+} N(T)$ and identify $D(\mathcal{M})$ and $D_{cris,L}(V)$ based on Lemma 1.5 and using (18). The proof of [KR, Prop. (2.2.6)] combined with Remark 1.2 iii. implies the commutativity of the following diagram

$$\begin{tikzcd}
D(\mathcal{M}) \arrow[r, hook, "incl."'] \arrow[dr, hook, "incl."'] & \mathcal{O}[\frac{\omega_{LT}}{t_{LT}}] \otimes_L D(\mathcal{M}) \arrow[r, "\xi", "\cong"'] & \mathcal{M}[\frac{\omega_{LT}}{t_{LT}}] \\
& \mathcal{M}(D(\mathcal{M})) \arrow[u, hook, "incl."'] \arrow[r, "\cong"] & \mathcal{M}, \arrow[u, hook, "incl."']
\end{tikzcd}$$

in which the right vertical map is the canonical inclusion while the left vertical map stems from the definition of the functor $\mathcal{M}$ as in (19) (which also implies the commutativity of the left triangle). Taking Γ_L-invariants and using the fact that the upper line induces the isomorphism $D(\mathcal{M}) \cong \mathcal{M}[\frac{\omega_{LT}}{t_{LT}}]^{\Gamma_L}$ in Remark 1.2 (iii) the result follows. □

Corollary 1.16. *In the situation of Proposition 1.13 we have $Q^r N(V) \subseteq N^{(\varphi)}(V)$.*

Proof. In the present situation $\varphi_{N(V)} : N(V) \to N(V)$ is an semilinear endomorphism of $N(V)$ by Remark 1.6(i). Then $\varphi_{\mathcal{O} \otimes_{\mathbf{A}_L^+} N(V)} = \varphi_L \otimes \varphi_{N(V)} : \mathcal{O} \otimes_{\mathbf{A}_L^+} N(V) \to \mathcal{O} \otimes_{\mathbf{A}_L^+} N(V)$ is an endomorphism as well. The corresponding linearized maps are

$$\begin{aligned}
\varphi_{N(V)}^{lin} : \mathbf{A}_L^+ \otimes_{\mathbf{A}_L^+, \varphi_L} N(V) &\xrightarrow{\cong} N^{(\varphi)}(V) \subseteq N(V) \\
f \otimes x &\longmapsto f \varphi_{N(V)}(x)
\end{aligned}$$

and

$$\begin{aligned}
\varphi_{\mathcal{O} \otimes_{\mathbf{A}_L^+} N(V)}^{lin} = \mathrm{id}_{\mathcal{O}} \otimes \varphi_{N(V)}^{lin} : \mathcal{O} \otimes_{\mathbf{A}_L^+, \varphi_L} N(V) = \mathcal{O} \otimes_{\mathbf{A}_L^+} \big(\mathbf{A}_L^+ \otimes_{\mathbf{A}_L^+, \varphi_L} N(V) \big) \\
\longrightarrow \mathcal{O} \otimes_{\mathbf{A}_L^+} N(V) \ .
\end{aligned}$$

Since $\mathcal{O}$ is flat over $\mathbf{A}_L^+[\frac{1}{\pi_L}]$ it follows that

$$\mathcal{O} \otimes_{\mathbf{A}_L^+} N(V) / \mathrm{im}(\varphi_{\mathcal{O} \otimes_{\mathbf{A}_L^+} N(V)}^{lin}) = \mathcal{O} \otimes_{\mathbf{A}_L^+} (N(V)/N^{(\varphi)}(V)) \ .$$

But $\mathcal{O}$ is even faithfully flat over $\mathbf{A}_L^+[\frac{1}{\pi_L}]$. Hence the natural map

$$N(V)/N^{(\varphi)}(V) \longrightarrow \mathcal{O} \otimes_{\mathbf{A}_L^+} N(V)/\operatorname{im}(\varphi^{lin}_{\mathcal{O}\otimes_{\mathbf{A}_L^+} N(V)})$$

is injective. This reduces us to proving that

$$Q^r(\mathcal{O} \otimes_{\mathbf{A}_L^+} N(V)) \subseteq \operatorname{im}(\varphi^{lin}_{\mathcal{O}\otimes_{\mathbf{A}_L^+} N(V)}) \ .$$

As for any object in the category $\mathrm{Mod}_{\mathcal{O}}^{\varphi_L,\gamma_L,an}$, we do have

$$Q^h(\mathcal{O} \otimes_{\mathbf{A}_L^+} N(V)) \subseteq \operatorname{im}(\varphi^{lin}_{\mathcal{O}\otimes_{\mathbf{A}_L^+} N(V)}) \ .$$

for some sufficiently big integer h. On the other hand, (18) says that

$$(\tfrac{t_{LT}}{\omega_{LT}})^r \operatorname{comp}\big(\mathcal{O} \otimes_{\mathbf{A}_L^+} N(V)\big) \subseteq \mathcal{O} \otimes_L D_{cris,L}(V) \subseteq \operatorname{comp}\big(\mathcal{O} \otimes_{\mathbf{A}_L^+} N(V)\big) \ .$$

Since φ_{cris} is bijective we can sharpen the right hand inclusion to

$$\mathcal{O} \otimes_L D_{cris,L}(V) \subseteq \operatorname{comp}\big(\operatorname{im}(\varphi^{lin}_{\mathcal{O}\otimes_{\mathbf{A}_L^+} N(V)})\big) \ .$$

It follows that $(\frac{t_{LT}}{\omega_{LT}})^r(\mathcal{O} \otimes_{\mathbf{A}_L^+} N(V)) \subseteq \operatorname{im}(\varphi^{lin}_{\mathcal{O}\otimes_{\mathbf{A}_L^+} N(V)})$. Since the greatest common divisor of Q^h and $(\frac{t_{LT}}{\omega_{LT}})^r$ is $Q^{\min(h,r)}$ we finally obtain that $Q^r(\mathcal{O} \otimes_{\mathbf{A}_L^+} N(V)) \subseteq \operatorname{im}(\varphi^{lin}_{\mathcal{O}\otimes_{\mathbf{A}_L^+} N(V)})$. $\square$

Corollary 1.17. *In the situation of Proposition 1.13 we have, with regard to an $\mathbf{A}_L^+$-basis of $N := N(T)$ and with $s := \sum_{i=1}^d r_i$, that*

$$\det(\varphi_N : N(T) \to N(T)) = \det(\varphi_{N(V)} : N(V) \to N(V)) = Q^s$$

up to an element in $o_L^\times \cdot (\varphi_L - 1)\big((\mathbf{A}_L^+)^\times\big)$.

Proof. Note first that N is φ_N-stable by Remark 1.6(i). Moreover, the determinant of φ_N acting on $N(V)$ equals the determinant of $\varphi_L \otimes \varphi_N$ acting on $\mathcal{O} \otimes_{\mathbf{A}_L^+} N(T)$, since we can take for both an $\mathbf{A}_L^+$-basis of $N(T)$. Since $\varphi_L(\frac{t_{LT}}{\omega_{LT}}) = \frac{\pi_L}{Q}\frac{t_{LT}}{\omega_{LT}}$, by proposition 1.13 the latter determinant equals $(\frac{\pi_L}{Q})^{-s}$ multiplied by the determinant of $\varphi_L \otimes \mathrm{Frob}$ acting on $\mathcal{O} \otimes_L D_{cris,L}(V)$. The latter is equal to the determinant of Frob on $D_{cris,L}(V)$, which is π_L^s up to a unit in o_L since the filtered Frobenius module $D_{cris,L}(V)$ is weakly admissible. This shows the claim up to an element in $o_L^\times \cdot (\varphi_L - 1)(\mathcal{O}^\times)$. But $\mathcal{O}^\times = \pi_L^{\mathbb{Z}} \times (\mathbf{A}_L^+)^\times$ by [Laz1] (4.8). Hence $(\varphi_L - 1)(\mathcal{O}^\times) = (\varphi_L - 1)\big((\mathbf{A}_L^+)^\times\big)$.

$\square$

1.2 The determinant of the crystalline comparison isomorphism

Let T be any object in $\mathrm{Rep}_{o_L,f}^{cris,an}(G_L)$ of o_L-rank d and such that $V = L \otimes_{o_L} T$ has Hodge-Tate weights $-r = -r_d \leqslant \cdots \leqslant -r_1$; we set $s := \sum_{i=1}^d r_i$, $N := N(T)$ and $\mathcal{M} = \mathcal{O} \otimes N$. Consider the integral lattice

$$\mathcal{D} := \mathcal{D}(T) \subseteq D_{cris,L}(V)$$

which is defined as the image of $N/\omega_{LT}N \subseteq D(N)$ under the natural isomorphisms $D(N) \cong D(\mathcal{M}) \cong D_{cris,L}(V)$ arising from Lemma 1.5 and (4). Then with $N(-)$ also $\mathcal{D}(-)$ is a $\otimes$-functor. The aim of this subsection is to prove the following result.

Proposition 1.18. *With regard to bases of T and $\mathcal{D}$ the determinant of the crystalline comparison isomorphism*

$$B_{cris,L} \otimes_L V \cong B_{cris,L} \otimes_L D_{cris,L}(V)$$

belongs to $t_{LT}^s W(\bar{k}_L)_L^\times$.

We write $\bigwedge V$ for the highest exterior power of V over L.

Remark 1.19. *If V is L-analyic (Hodge-Tate, crystalline), then so is $\bigwedge V$.*

Since $D_{cris,L}$ is a tensor functor, we are mainly reduced to consider characters $\rho : G_L \to L^\times$, for which we denote by V_ρ its representation space.

Remark 1.20. (i) *If V_ρ is Hodge-Tate, then ρ coincides on an open subgroup of the inertia group I_L of G_L with*

$$\prod_{\sigma \in \Sigma_L} \sigma^{-1} \circ \chi_{\sigma L, LT}^{n_\sigma},$$

for some integers n_σ, where Σ_L denotes the set of embeddings of L into $\bar{L}$ and $\chi_{\sigma L,LT}$ is the Lubin-Tate character for σL and $\sigma(\pi_L)$.

(ii) *If, in addition, V_ρ is L-analytic, then ρ coincides on an open subgroup of the inertia group I_L with χ_{LT}^n for some integer n.*

Proof. This follows from [Se0] III.A4 Prop. 4 as well as III.A5 Theorem 2 and its corollary. □

Remark 1.21. *Let ρ be a crystalline (hence Hodge-Tate) and L-analytic character. We then have:*

(i) *If ρ factorizes through $G(L'/L)$ for some discretely valued Galois extension L' of L, then the determinant of the crystalline comparison isomorphism for V_ρ belongs to $(W(\bar{k}_L)_L[\frac{1}{p}])^\times$ (with respect to arbitrary bases of V and $D_{cris,L}(V)$.)*

(ii) *If ρ has Hodge-Tate weight $-s$, then the determinant of the crystalline comparison isomorphism for V_ρ lies in $t_{LT}^s(W(\bar{k}_L)_L[\frac{1}{p}])^\times$.*

(iii) *ρ is of the form $\chi_{LT}^n \chi^{un}$ with an integer n and an unramified character χ^{un}.*

Proof. We shall write K_0 for the maximal absolutely unramified subextension of K, any algebraic extension of $\mathbb{Q}_p$. Taking $G_{L'}$-invariants of the comparison isomorphism shows that the latter is already defined over

$$B_{cris,L}^{G_{L'}} = (L \otimes_{L_0} B_{cris})^{G_{L'}} = L \otimes_{L_0} (B_{cris})^{G_{L'}} = L \otimes_{L_0} \widehat{L_0'} \subseteq W(\bar{k}_L)_L[\tfrac{1}{p}],$$

whence (i). Using Remark 1.20 (ii) and applying (i) to $\rho\chi_{LT}^{-n}$ gives (ii). By the same argument it suffices to prove (iii) in the case of Hodge-Tate weight 0. Then its period lies in the completion of the maximal unramified extension of L by (i), whence the claim that ρ is unramified follows, as the inertia subgroup of G_L must act trivially. □

By Proposition 1.8 we have

$$N(T) \subseteq D_{LT}^+(T) \subseteq \mathbf{A}^+ \otimes_{o_L} T$$

if T is positive. Using (N2) and the isomorphism

$$\mathbf{A} \otimes_{\mathbf{A}_L} D_{LT}(T) \cong \mathbf{A} \otimes_{o_L} T$$

we obtain a canonical injection

$$\mathbf{A}^+ \otimes_{\mathbf{A}_L^+} N(T) \hookrightarrow \mathbf{A}^+ \otimes_{o_L} T. \tag{23}$$

Proposition 1.22. *If T is positive, then the determinant of (23) with respect to bases of $N(T)$ and T is contained in $\omega_{LT}^s(\mathbf{A}_L^+)^\times \cdot W(\bar{k}_L)_L^\times$.*

Proof. Let $M \in M_d(\mathbf{A}^+)$ be the matrix of a basis of $N(T)$ with respect to a basis of T and $P \in M_d(\mathbf{A}_L^+)$ the matrix of φ_L with respect to the same basis of $N(T)$. Then we have $\varphi_L(M) = MP$. By Corollary 1.17 we have $\det(P) = Q^s \varphi_L(f) f^{-1} u$ for some $f \in (\mathbf{A}_L^+)^\times$ and $u \in o_L^\times$. But $Q = \varphi_L(\omega_{LT})\omega_{LT}^{-1}$. We deduce that

$$\varphi_L(\det(M)) = \varphi_L(\omega_{LT}^s f)(\omega_{LT}^s f)^{-1} u \det(M) \text{ , i.e., that } (\omega_{LT}^s f a)^{-1} \det(M) \in \mathbf{A}^{\varphi_L = 1} = o_L \ .$$

with $a \in W(\bar{k}_L)_L^\times$ such that $\varphi_L(a)/a = u$. It follows that $\det(M) \in \omega_{LT}^s o_L (\mathbf{A}_L^+)^\times \cdot W(\bar{k}_L)_L^\times$. But we also have $\det(M) \in \mathbf{A}^\times$. Hence we finally obtain $\det(M) \in \omega_{LT}^s o_L(\mathbf{A}_L^+)^\times \cdot W(\bar{k}_L)_L^\times \cap \mathbf{A}^\times = \omega_{LT}^s (\mathbf{A}_L^+)^\times \cdot W(\bar{k}_L)_L^\times$.

□

Remark 1.23. *For $T = o_L(\chi)$ with unramified χ as in Remark 1.11 the map (23) maps the basis n_0 to $a \otimes t_0$.*

Lemma 1.24. *If T is positive, then we have:*

i. $\mathcal{O} \otimes_{o_L} \mathcal{D}(T) = \mathcal{O} \otimes_L D_{cris,L}(V) \subseteq \mathrm{comp}(\mathcal{O} \otimes_{\mathbf{A}_L^+} N(T))$;

ii. the determinant of the inclusion in i. with respect to bases of $\mathcal{D}(T)$ and $N(T)$ belongs to $(\frac{t_{LT}}{\omega_{LT}})^s (\mathbf{A}_L^+)^\times$.

iii. for $T = o_L(\chi)$ with unramified χ as in Remark 1.11: $\mathrm{comp}(n_0) = \varphi_L(a) \otimes t_0 = ca \otimes t_0 \in D_{cris,L}(V)$ *with $c = \frac{\varphi_L(a)}{a} \in o_L^\times$; in particular, the element $a \otimes t_0$ is a basis of $\mathcal{D}(T)$.*

Proof. By construction the comparison isomorphism (17) is of the form

$$\mathrm{comp} = \mathrm{id}_{\mathcal{O}[\frac{\omega_{LT}}{t_{LT}}]} \otimes_L \mathrm{comp}_0$$

with

$$\mathrm{comp}_0 : \big(\mathcal{O} \otimes_{\mathbf{A}_L^+} N[\tfrac{\omega_{LT}}{t_{LT}}]\big)^{\Gamma_L} \xrightarrow[\mathrm{pr}]{\cong} N/\omega_{LT} N[\tfrac{1}{p}] = D(N) \xrightarrow{\cong} D_{cris,L}(V)$$

the right hand arrow being the natural isomorphism from Lemma 1.5. For positive T we know in addition from the proof of Lemma 1.15 that $\big(\mathcal{O} \otimes_{\mathbf{A}_L^+} N\big)^{\Gamma_L} = \big(\mathcal{O} \otimes_{\mathbf{A}_L^+} N[\frac{\omega_{LT}}{t_{LT}}]\big)^{\Gamma_L}$. We deduce that

$$\mathrm{comp}(\mathcal{O} \otimes_{\mathbf{A}_L^+} N) \supseteq \mathcal{O} \otimes_L \mathrm{comp}_0(\big(\mathcal{O} \otimes_{\mathbf{A}_L^+} N\big)^{\Gamma_L}) = \mathcal{O} \otimes_L D_{cris,L}(V) \ .$$

By Proposition 1.13 we know that the determinant in ii. is of the form $(\frac{t_{LT}}{\omega_{LT}})^s f(\omega_{LT})$ with $f(\omega_{LT}) \in \mathcal{O}^\times$. On the other hand, if we base change the inclusion in i. to $L = \mathcal{O}/\omega_{LT}\mathcal{O}$ then we obtain the base change from o_L to L of the isomorphism $\mathcal{D} \cong N/\omega_{LT}N$. By our choice of bases the determinant of the latter lies in $o_L^\times$. Since evaluation in zero maps $(\frac{t_{LT}}{\omega_{LT}})^s f(\omega_{LT})$ to $f(0)$ it follows that $f(0)$ belongs to $o_L^\times$ and hence ([Laz1] (4.8)) that $f(\omega_{LT})$ belongs to $(\mathbf{A}_L^+)^\times$.

Now we prove iii.: By the above description of comp_0 we have to show that the image $\bar{n}_0 \in D(N(T))$ of n_0 is mapped to $ca \otimes t_0$ under the natural isomorphism from Lemma 1.5. Since under the crystalline comparison isomorphisms these elements are sent to $a \otimes (a^{-1} \otimes \bar{n}_0) \in B_{cris,L} \otimes_L V_L(D(N))$ and $ca \otimes t_0 \in B_{cris,L} \otimes_{o_L} T$, respectively, it suffices to show that the map (12) sends $a^{-1} \otimes n_0 \in L \otimes_{o_L} V(\mathbf{A}_L \otimes_{\mathbf{A}_L^+} N)$ (which corresponds to t_0 under the canonical isomorphism $T \cong V(\mathbf{A}_L \otimes_{\mathbf{A}_L^+} N)$) to $(ca)^{-1} \otimes \bar{n}_0 \in V_L(D(N))$. Dualizing, this is equivalent to the claim that the map (13) sends the dual basis $\delta_{a^{-1}\otimes n_0} \in (L \otimes_{o_L} V(M))^*$ of $a^{-1} \otimes n_0$ to $\delta_{(ca)^{-1}\otimes \bar{n}_0} \in V_L(D(N))^*$. Note that the isomorphism

$$(L \otimes_{o_L} V(M))^* \cong L \otimes_{o_L} \mathrm{Hom}_{\mathbf{A}_L,\varphi_q}(\mathbf{A}_L \otimes_{\mathbf{A}_L^+} N, \mathbf{A}) \cong L \otimes_{o_L} \mathrm{Hom}_{\mathbf{A}_L^+,\varphi_q}(N, \mathbf{A}^+[\tfrac{1}{\omega_{LT}}])$$

sends $\delta_{a^{-1}\otimes n_0}$ to $a\delta_{n_0}$. Thus it suffices to show that the map (14) sends $a\delta_{n_0}$ to $ca\delta_{\bar{n}_0}$ in $\mathrm{Hom}_{L,\varphi_q,\mathrm{Fil}}((N/\omega_{LT}N)[\frac{1}{p}], B_{cris,L})$, since the latter corresponds under (15) to $ca \otimes \delta_{\bar{n}_0} \in V_L(D(N)^*)$ which in turn corresponds to $\delta_{(ca)^{-1}\otimes\bar{n}_0}$ under (16).

If $f = a\delta_{n_0}$, which is the map which sends n_0 to a, then - in the notation of the proof of Lemma 1.5 - f_1 and f_2 share this property, while f_3 (and hence f_4) sends $c^{-1}n_0$ to a, because $\varphi_N(c^{-1}n_0) = c^{-1}\varphi_L(a)a^{-1}n_0 = n_0$. Then f_5 sends $c^{-1}\bar{n}_0$ to a, because $\xi(c^{-1}\bar{n}_0) = c^{-1}n_0$. Altogether this means, that $a\delta_{n_0}$ is mapped to $\varphi_L(a)\delta_{n_0} = ca\delta_{\bar{n}_0}$ as claimed. □

Proof of Proposition 1.18. The functor $D_{cris,L}(-)$ on crystalline Galois representations is a $\otimes$-functor and commutes with exterior powers, and the crystalline comparison isomorphism is compatible with tensor products and exterior powers. The analogous facts hold for the functor $N(-)$ and hence for the functor $\mathcal{D}(-)$ (by base change). The case of the functor $N(-)$ reduces, by using the properties (N1) and (N2) in Lemma 1.12 (i), to the case of the functor $D_{LT}(-)$. Here the properties can easily be seen by the comparison isomorphism (6).

Upon replacing T by its highest exterior power we may and do assume that the o_L-module T has rank 1. In addition by twisting T if necessary with a power of χ_{LT} we may and do assume that T is positive with $s = 0$, i.e., unramified by 1.21. In this case it is clear that - using the notation of Lemma 1.24 iii. - the crystalline comparison isomorphism sends t_0 to $a \otimes t_0$. Since the latter is also a basis of $\mathcal{D}(T)$ by the same Lemma, the proposition follows. □

1.3 Non-negative Hodge-Tate weights

Now assume that for T in $\mathrm{Rep}_{o_L,f}^{cris,an}(G_L)$ the **Hodge-Tate weights are all** $\geqslant 0$ and set $N := N(T)$. By [SV15] Remark 3.2.i.-ii. the map ψ_L preserves $\mathbf{A}_L^+$. It follows that $\psi_{D_{LT}(T)}$ maps $\mathbf{A}_L^+ \cdot \varphi_N(N)$ - and hence N by Remark 1.6.i(2) - into N. The following lemmata generalize those of [B, Appendix A].

Lemma 1.25. *For $m \geqslant 1$, there exists $Q_m \in o_L[[Z]]$ such that*

$$\psi_L(\frac{1}{\omega_{LT}^m}) = \frac{\pi_L^{m-1} + \omega_{LT}Q_m(\omega_{LT})}{\omega_{LT}^m}.$$

Proof. According to the paragraph after Remark 2.1 in [SV15] combined with Remark 3.2 ii. in (loc. cit.) we have that

$$h(\omega_{LT}) := \omega_{LT}^m \psi_L(\frac{1}{\omega_{LT}^m}) = \psi_L(\frac{[\pi_L]^m}{\omega_{LT}^m}) \in \mathbf{A}_L^+.$$

Obviously there exists $Q_m \in o_L[[Z]]$ such that

$$h(\omega_{LT}) - h(0) = \omega_{LT} Q_m(\omega_{LT}).$$

Thus the claim follows from

$$\begin{aligned} h(0) = \varphi_L(h(\omega_{LT}))_{|\omega_{LT}=0} &= \varphi_L \circ \psi_L(\frac{[\pi_L]^m}{\omega_{LT}^m})_{|\omega_{LT}=0} = \pi_L^{-1} \sum_{a \in LT_1} \left(\frac{[\pi_L]^m (a +_{LT} \omega_{LT})}{(a +_{LT} \omega_{LT})^m} \right)_{|\omega_{LT}=0} \\ &= \pi_L^{-1} \sum_{a \in LT_1} \left(\frac{[\pi_L](\omega_{LT})}{a +_{LT} \omega_{LT}} \right)^m_{|\omega_{LT}=0} = \pi_L^{m-1}, \end{aligned}$$

because $\left(\frac{[\pi_L](\omega_{LT})}{a +_{LT} \omega_{LT}}\right)_{|\omega_{LT}=0} = \pi_L$ for $a = 0$ and $= 0$ otherwise. □

Lemma 1.26. *We have*

$$\psi_{D_{LT}(T)}(\pi_L D_{LT}(T) + \omega_{LT}^{-1} N(T)) \subseteq \pi_L D_{LT}(T) + \omega_{LT}^{-1} N(T)$$

and, for $k \geqslant 1$,

$$\psi_{D_{LT}(T)}(\pi_L D_{LT}(T) + \omega_{LT}^{-(k+1)} N(T)) \subseteq \pi_L D_{LT}(T) + \omega_{LT}^{-k} N(T).$$

Proof. By Remark 1.6 (2) we can write any $x \in N(T)$ in the form $x = \sum a_i \varphi_N(x_i)$ with $a_i \in \mathbf{A}_L^+$ and $x_i \in N(T)$. Therefore $\psi_{D_{LT}(T)}(\omega_{LT}^{-(k+1)} x) = \sum \psi_L(\omega_{LT}^{-(k+1)} a_i) x_i$ by the projection formula. Since ψ_L preserves $\mathbf{A}_L^+$ and is o_L-linear we conclude by Lemma 1.25 that $\psi_L(\omega_{LT}^{-(k+1)} a_i)$ belongs to $\pi_L \mathbf{A}_L + \omega_{LT}^{-k} \mathbf{A}_L^+$, whenever $k \geqslant 1$, from which the second claim follows as $\psi_{D_{LT}(T)}(\pi_L D_{LT}(T)) \subseteq \pi_L D_{LT}(T)$ by o_L-linearity of $\psi_{D_{LT}(T)}$. For $k = 0$ finally, $\psi_L(\omega_{LT}^{-1} a_i)$ belongs to $\omega_{LT}^{-1} \mathbf{A}_L^+$, from which the first claim follows. □

Lemma 1.27. *If* $k \geqslant 1$ *and* $x \in D_{LT}(T)$ *satisfies* $\psi_{D_{LT}(T)}(x) - x \in \pi_L D_{LT}(T) + \omega_{LT}^{-k} N(T)$, *then* x *belongs to* $\pi_L D_{LT}(T) + \omega_{LT}^{-k} N(T)$.

Proof. Since $D_{LT}(T)/\pi_L D_{LT}(T)$ is a finitely generated (free) $k_L((\omega_{LT}))$-module there exists an integer $m \geqslant 0$ such that $x \in \pi_L D_{LT}(T) + \omega_{LT}^{-m} N(T)$; let l denote the smallest among them. Assume that $l > k$. Then Lemma 1.26 shows that

$$\psi_{D_{LT}(T)}(x) \in \pi_L D_{LT}(T) + \omega_{LT}^{-(l-1)} N(T).$$

Hence $\psi_{D_{LT}(T)}(x) - x$ would belong to $\pi_L D_{LT}(T) + \omega_{LT}^{-l} N(T)$ but not to $(\pi_L D_{LT}(T) + \omega_{LT}^{-(l-1)} N(T))$, a contradiction to our assumption. It follows that $l \leqslant k$, and we are done. □

Lemma 1.28. *It holds* $D_{LT}(T)^{\psi_{D_{LT}(T)}=1} \subseteq \omega_{LT}^{-1} N(T)$, *i.e.*,

$$D_{LT}(T)^{\psi_{D_{LT}(T)}=1} = \left(\omega_{LT}^{-1} N(T)\right)^{\psi_{D_{LT}(T)}=1}.$$

Proof. By induction on $k \geqslant 1$ we will show that $D_{LT}(T)^{\psi_{D_{LT}(T)}=1} \subseteq \pi_L^k D_{LT}(T) + \omega_{LT}^{-1}N(T)$, i.e., writing $x = \pi_L^k y_k + n_k \in D_{LT}(T)^{\psi_{D_{LT}(T)}=1}$ the sequence n_k will π_L-adically converge in $\omega_{LT}^{-1}N(T)$ with limit x.

In order to show the claim assume $x \in D_{LT}(T)^{\psi_{D_{LT}(T)}=1}$. As in the previous proof there exists some minimal integer $m \geqslant 0$ such that $x \in \pi_L D_{LT}(T) + \omega_{LT}^{-m}N(T)$. Then $m = 1$ and we are done since otherwise Lemma 1.27 implies that m can be decreased by 1. This proves the claim for $k = 1$.

By our induction hypothesis we can write $x \in D_{LT}(T)^{\psi_{D_{LT}(T)}=1}$ as $x = \pi_L^k y + n$ with $y \in D_{LT}(T)$ and $n \in \omega_{LT}^{-1}N(T)$. The equation $\psi_{D_{LT}(T)}(x) = x$ implies that $\psi_{D_{LT}(T)}(n) - n = \pi_L^k(\psi_{D_{LT}(T)}(y) - y)$. In the proof of Lemma 1.26 we have seen that $\psi_{D_{LT}(T)}(n) - n \in \omega_{LT}^{-1}N(T)$. Note that $\pi_L^k D_{LT}(T) \cap \omega_{LT}^{-1}N(T) = \pi_L^k \omega_{LT}^{-1}N(T)$ because $\mathbf{A}_L/\omega_{LT}^{-1}\mathbf{A}_L^+$ has no π_L-torsion. Therefore $\psi_{D_{LT}(T)}(y) - y \in \omega_{LT}^{-1}N(T)$, whence y, by Lemma 1.27, belongs to $\pi_L D_{LT}(T) + \omega_{LT}^{-1}N(T)$ so that we can write $x = \pi_L^k(\pi_L y' + n') + n = \pi_L^{k+1}y' + (\pi_L^k n' + n)$ as desired. $\square$

Set $V := T \otimes_{o_L} L$.

Lemma 1.29. *If $D_{cris,L}(V)^{\varphi_q=1} \neq 0$, then V has the trivial representation L as quotient, i.e., the co-invariants V_{G_L} are non-trivial.*

Proof. Let $W = V^*$ be the L-dual of V. Then, by [SV15, (51)] we have

$$(V_{G_L})^* \cong H^0(L, W) \cong D_{cris,L}(W)^{\varphi_q=1} \cap (B_{dR}^+ \otimes_L W)^{G_L} = D_{cris,L}(W)^{\varphi_q=1} \neq 0,$$

because $(B_{dR}^+ \otimes_L W)^{G_L} = (B_{dR} \otimes_L W)^{G_L} \supseteqq D_{cris,L}(W)$ since the Hodge-Tate weights of W are $\leqslant 0$. $\square$

Lemma 1.30. *If V does not have any quotient isomorphic to the trivial representation L, then $D_{LT}(T)^{\psi_{D_{LT}(T)}=1} \subseteq N(T)$, i.e.,*

$$D_{LT}(T)^{\psi_{D_{LT}(T)}=1} = N(T)^{\psi_{D_{LT}(T)}=1} .$$

Proof. Because of Lemma 1.28 it suffices to show that $(\omega_{LT}^{-1}N(T))^{\psi_{D_{LT}(T)}=1} \subseteq N(T)$. Let $e_1, \ldots, e_d$ be a basis of $N := N(T)$ over $\mathbf{A}_L^+$. Then, by Remark 1.6 (ii) there exist $\beta_{ij} = \sum_{\ell \geqslant 0} \beta_{ij,\ell}\omega_{LT}^\ell \in \mathbf{A}_L^+$ such that $e_i = \sum_{j=1}^d \beta_{ij}\varphi_N(e_j)$. Now assume that $\omega_{LT}^{-1}n = \sum_{i=1}^d \alpha_i e_i = \sum_{i,j} \alpha_i\beta_{ij}\varphi_N(e_j)$ belongs to $(\omega_{LT}^{-1}N)^{\psi_{D_{LT}(T)}=1}$ with $\alpha_i = \sum_{\ell \geqslant -1} \alpha_{i,\ell}\omega_{LT}^\ell \in \omega_{LT}^{-1}\mathbf{A}_L^+$. By the projection formula this implies, for $1 \leqslant j \leqslant d$,

$$\alpha_j = \psi_L(\sum_{i=1}^d \alpha_i\beta_{ij}) \equiv \omega_{LT}^{-1}\sum_{i=1}^d \alpha_{i,-1}\beta_{ij,0} \mod \mathbf{A}_L^+$$

because $\psi_L(\omega_{LT}^{-1}) \equiv \omega_{LT}^{-1} \mod \mathbf{A}_L^+$ by Lemma 1.25, whence

$$\varphi_L(\omega_{LT})\varphi_L(\alpha_j) \equiv \sum_{i=1}^d \alpha_{i,-1}\beta_{ij,0} \mod \omega_{LT}\mathbf{A}_L^+.$$

It follows from the definition of β_{ij} that

$$\varphi_N(n) = \sum_j \varphi_L(\omega_{LT})\varphi_L(\alpha_j)\varphi_N(e_j) \equiv \sum_{j,i} \alpha_{i,-1}\beta_{ij,0}\varphi_N(e_j) \equiv \sum_i \alpha_{i,-1}e_i \equiv n \mod \omega_{LT}N,$$

i.e., that $D_{cris,L}(V) \cong N/\omega_{LT}N[\frac{1}{p}]$ (by (4) and Lemma 1.5) contains an eigenvector for φ_q with eigenvalue 1, if $\omega_{LT}^{-1}n$ does not belong to N. Now the result follows from Lemma 1.29. $\square$

2 (φ_L, Γ_L)-modules over the Robba ring

2.1 Robba rings of character varieties

2.1.1 The additive case

Let $L \subseteq K \subseteq \mathbb{C}_p$ be a complete intermediate extension and let $G := o_L$ denote the additive group o_L viewed as a locally L-analytic group. The group of K-valued locally analytic characters of G is denoted by $\widehat{G}(K)$. It is shown in [ST2] §2 that there is a one dimensional smooth connected rigid analytic group variety $\mathfrak{X}$ over L which "represents the character group $\widehat{G}$". We have the rings $\mathcal{O}_K^{\leqslant 1}(\mathfrak{X}) \subseteq \mathcal{O}_K^b(\mathfrak{X}) \subseteq \mathcal{O}_K(\mathfrak{X})$ of bounded by 1, of bounded, and of all K-valued global holomorphic functions on $\mathfrak{X}$, respectively. We note that $\mathcal{O}_K(\mathfrak{X})^\times = \mathcal{O}_K^b(\mathfrak{X})^\times$ by [BSX] Lemma 2.3.

For any $a \in o_L$ the map $g \longmapsto ag$ on G is locally L-analytic. This induces an action of the multiplicative monoid $o_L \backslash \{0\}$ on the topological vector space of locally analytic functions $C^{an}(G, K)$ given by $f \longmapsto a^*(f)(g) := f(ag)$. Obviously, with $\chi \in \widehat{G}(K)$ also $a^*(\chi)$ is a character in $\widehat{G}(K)$. In this way we obtain actions of the ring o_L on these groups. In fact, this action on character groups comes from an o_L-action on the rigid character variety $\mathfrak{X}$ (cf. [BSX]). Moreover, from the action on $\mathfrak{X}$ we derive a translation action by the multiplicative monoid $o_L \backslash \{0\}$ on the Fréchet algebra $\mathcal{O}_K(\mathfrak{X})$, which will be denoted by $(a, F) \longmapsto a_*(F)$. Note that this action respect the subrings $\mathcal{O}_K^{\leqslant 1}(\mathfrak{X}) \subseteq \mathcal{O}_K^b(\mathfrak{X})$.

The continuous dual of the locally convex K-vector space $C^{an}(G, K)$ is the Fréchet algebra $D(G, K)$ of locally analytic distributions on G. The $o_L \backslash \{0\}$-action on $C^{an}(G, K)$ dualizes into an action on $D(G, K)$ denoted by $(a, \lambda) \longmapsto a_*(\lambda) = \lambda \circ a^*$. By [ST2] Thm. 2.3 we have the Fourier isomorphism

$$
\begin{aligned}
(24) \qquad D(G, K) &\xrightarrow{\cong} \mathcal{O}_K(\mathfrak{X}) \\
\lambda &\longmapsto F_\lambda(\chi) = \lambda(\chi) \ .
\end{aligned}
$$

One easily checks that this isomorphism is $o_L \backslash \{0\}$-equivariant. In the following we will denote the endomorphism $(\pi_L)_*$ on both sides by φ_L. The Fourier isomorphism maps the Dirac distribution δ_g, for any $g \in G$, to the evaluation function $\mathrm{ev}_g(\chi) := \chi(g)$. Of course, these functions are units in $\mathcal{O}_K^{\leqslant 1}(\mathfrak{X})$.

Lemma 2.1. *The endomorphism φ_L makes $\mathcal{O}_K(\mathfrak{X})$ into a free module over itself of rank equal to the cardinality of $o_L/\pi_L o_L$; a basis is given by the functions ev_g for g running over a fixed system of representatives for the cosets in $o_L/\pi_L o_L$.*

Proof. This is most easily seen by using the Fourier isomorphism which reduces the claim to the corresponding statement about the distribution algebra $D(o_L, K)$. But here the ring homomorphism φ_L visibly induces an isomorphism between $D(o_L, K)$ and the subalgebra $D(\pi_L o_L, K)$ of $D(o_L, K)$. Let $R \subseteq o_L$ denote a set of representatives for the cosets in $o_L/\pi_L o_L$. Then the Dirac distributions $\{\delta_g\}_{g \in R}$ form a basis of $D(o_L, K)$ as a $D(\pi_L o_L, K)$-module. $\square$

Lemma 2.2. *The $o_L^\times$-action on $D(G, K) \cong \mathcal{O}_K(\mathfrak{X})$ extends naturally to a (jointly) continuous $D(o_L^\times, K)$-module structure.*

Proof. In a first step we consider the case $K = L$, so that K is spherically complete. By [ST1] Cor. 3.4 it suffices to show that $C^{an}(G, K)$ as an $o_L^\times$-representation is locally analytic. This means we have to establish that, for any $f \in C^{an}(G, K)$, the orbit map $a \longmapsto a^*(f)$ on $o_L^\times$ is locally analytic. But this map is the image of the locally analytic function $(a, g) \longmapsto f(ag)$ under the isomorphism $C^{an}(o_L^\times \times G, K) = C^{an}(o_L^\times, C^{an}(G, K))$ in [ST3] Lemma A.1.

Now let K be general. All tensor products in the following are understood to be formed with the projective tensor product topology. By the universal property of the latter the jointly continuous bilinear map $D(o_L^\times, L) \times \mathcal{O}_L(\mathfrak{X}) \to \mathcal{O}_L(\mathfrak{X})$ extends uniquely to a continuous linear map $D(o_L^\times, L)\widehat{\otimes}_L \mathcal{O}_L(\mathfrak{X}) \to \mathcal{O}_L(\mathfrak{X})$. This further extends to the right hand map in the sequence of continuous K-linear maps

$$\big(K\widehat{\otimes}_L D(o_L^\times, L)\big)\widehat{\otimes}_K \big(K\widehat{\otimes}_L \mathcal{O}_L(\mathfrak{X})\big) \to K\widehat{\otimes}_L \big(D(o_L^\times, L)\widehat{\otimes}_L \mathcal{O}_L(\mathfrak{X})\big) \to K\widehat{\otimes}_L \mathcal{O}_L(\mathfrak{X}) \ .$$

The left hand map is the obvious canonical one. We refer to [PGS] §10.6 for the basics on scalar extensions of locally convex vector spaces. The same reasoning as in the proof of [BSX] Prop. 2.5.ii shows that $K\widehat{\otimes}_L \mathcal{O}_L(\mathfrak{X}) = \mathcal{O}_K(\mathfrak{X})$. It remains to check that $K\widehat{\otimes}_L D(o_L^\times, L) = D(o_L^\times, K)$ holds true as well. For any open subgroup $U \subseteq o_L^\times$ we have $D(o_L^\times, -) = \oplus_{a \in o_L^\times/U} \delta_a D(U, -)$. Hence it suffices to check that $K\widehat{\otimes}_L D(U, L) = D(U, K)$ for one appropriate U. But $o_L^\times$ contains such a subgroup U which is isomorphic to the additive group o_L so that $D(U, -) \cong D(o_L, -) \cong \mathcal{O}_-(\mathfrak{X})$. In this case we had established our claim already. $\square$

The operator φ_L has a distinguished K-linear continuous left inverse ψ_L^D which is defined to be the dual of the map

$$\begin{aligned} C^{an}(G, K) &\longrightarrow C^{an}(G, K) \\ f &\longmapsto (\pi_L)_!(f)(g) := \begin{cases} f(\pi_L^{-1} g) & \text{if } g \in \pi_L o_L, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

and then, via the Fourier transform, induces an operator $\psi_L^{\mathfrak{X}}$ on $\mathcal{O}_K(\mathfrak{X})$. One checks that for Dirac distributions we have

$$\psi_L^D(\delta_g) = \begin{cases} \delta_{\pi_L^{-1} g} & \text{if } g \in \pi_L o_L, \\ 0 & \text{otherwise.} \end{cases} \tag{25}$$

Together with Lemma 2.1 this implies the following.

Lemma 2.3. *If $R_0 \subseteq o_L$ is a set of representatives for the nonzero cosets in $o_L/\pi_L o_L$ then*

$$\ker(\psi_L^{\mathfrak{X}}) = \oplus_{g \in R_0} \operatorname{ev}_g \cdot \varphi_L(\mathcal{O}_K(\mathfrak{X})) \ .$$

We also recall the resulting projection formula

$$\psi_L^{\mathfrak{X}}(\varphi_L(F_1)F_2) = F_1 \psi_L^{\mathfrak{X}}(F_2) \qquad \text{for any } F_1, F_2 \in \mathcal{O}_K(\mathfrak{X}).$$

In order to establish a formula for the composition $\varphi_L \circ \psi_L^{\mathfrak{X}}$ we let $\mathfrak{X}[\pi_L] := \ker(\mathfrak{X} \xrightarrow{\pi_L^*} \mathfrak{X})$. Then $\mathfrak{X}[\pi_L](\mathbb{C}_p)$ is the character group of the finite group $o_L/\pi_L o_L$. The points in $\mathfrak{X}[\pi_L](\mathbb{C}_p)$ are defined over some finite extension K_1/K. For any $\zeta \in \mathfrak{X}[\pi_L](\mathbb{C}_p)$ we have the continuous translation operator

$$\begin{aligned} \mathcal{O}_{K_1}(\mathfrak{X}) &\longrightarrow \mathcal{O}_{K_1}(\mathfrak{X}) \\ F &\longmapsto ({}_\zeta F)(\chi) := F(\chi\zeta) \ . \end{aligned}$$

Proposition 2.4. *i. For any $F \in \mathcal{O}_{K_1}(\mathfrak{X})$ we have*

$$[o_L : \pi_L o_L] \cdot \varphi_L \circ \psi_L^{\mathfrak{X}}(F) = \sum_{\zeta \in \mathfrak{X}[\pi_L](\mathbb{C}_p)} {}_\zeta F \ .$$

ii. $\varphi_L(\mathcal{O}_K(\mathfrak{X})) = \{F \in \mathcal{O}_K(\mathfrak{X}) : {}_\zeta F = F$ for any $\zeta \in \mathfrak{X}[\pi_L](\mathbb{C}_p)\}$.

Proof. i. Since the functions ev_g generate a dense subspace in $\mathcal{O}_{K_1}(\mathfrak{X})$ ([ST1] Lemma 3.1 the proof of which remains valid for general K by [PGS] Cor. 4.2.6 and Thm. 11.3.5) it suffices, by the continuity of all operators involved, to consider any $F = \mathrm{ev}_g$. We compute

$$\begin{aligned}(\sum_\zeta {}_\zeta \mathrm{ev}_g)(\chi) &= \sum_\zeta \mathrm{ev}_g(\chi\zeta) = \chi(g) \sum_\zeta \zeta(g) \\ &= \begin{cases} [o_L : \pi_L o_L] \cdot \chi(g) & \text{if } g \in \pi_L o_L, \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} [o_L : \pi_L o_L] \cdot \mathrm{ev}_g(\chi) & \text{if } g \in \pi_L o_L, \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

On the other hand

$$\varphi_L(\psi_L^{\mathfrak{X}}(\mathrm{ev}_g)) = \varphi_L\Big(\begin{cases} \mathrm{ev}_{\pi_L^{-1} g} & \text{if } g \in \pi_L o_L, \\ 0 & \text{otherwise} \end{cases} \Big) = \begin{cases} \mathrm{ev}_g & \text{if } g \in \pi_L o_L, \\ 0 & \text{otherwise.} \end{cases}$$

ii. If ${}_\zeta F = F$ for any $\zeta \in \mathfrak{X}[\pi_L](\mathbb{C}_p)$ then $\varphi_L(\psi_L^{\mathfrak{X}}(F)) = F$ by i. On the other hand

$$({}_\zeta \varphi_L(F))(\chi) = \varphi_L(F)(\chi\zeta)) = F(\pi_L^*(\chi)\pi_L^*(\zeta)) = F(\pi_L^*(\chi)) = \varphi_L(F)(\chi) \ .$$

□

We have observed in the above proof that the functions ev_g, for $g \in G$, generate a dense subspace of $\mathcal{O}_K(\mathfrak{X})$. Considering the topological decomposition

$$\begin{aligned}\mathcal{O}_K(\mathfrak{X}) &= \varphi_L(\mathcal{O}_K(\mathfrak{X})) \oplus \mathcal{O}_K(\mathfrak{X})^{\psi_L^{\mathfrak{X}}=0} \\ F &= \varphi_L(\psi_L^{\mathfrak{X}}(F)) + (F - \varphi_L(\psi_L^{\mathfrak{X}}(F)))\end{aligned} \tag{26}$$

we see, using (25), that the ev_g for $g \in \pi_L o_L$, resp. the ev_g for $g \in o_L^\times$, generate a dense subspace of $\varphi_L(\mathcal{O}_K(\mathfrak{X}))$, resp. of $\mathcal{O}_K(\mathfrak{X})^{\psi_L^{\mathfrak{X}}=0}$. In view of Lemma 2.2 the obvious formula $a_*(\mathrm{ev}_g) = \mathrm{ev}_{ag}$ together with the fact, that the Dirac distributions δ_a, for $a \in o_L^\times$, generate a dense subspace of $D(o_L^\times, K)$, then imply that the decomposition (26) is $D(o_L^\times, K)$-invariant.

Lemma 2.5. *(Mellin transform) The action of $D(o_L^\times, K)$ upon $\delta_1 \in D(o_L, K)$ combined with the Fourier isomorphism induces the map*

$$\begin{aligned} D(o_L^\times, K) &\xrightarrow{\cong} D(o_L, K)^{\psi_L^D=0} \cong \mathcal{O}_K(\mathfrak{X})^{\psi_L^{\mathfrak{X}}=0} \\ \lambda &\longmapsto \qquad \lambda(\delta_1) \triangleq \lambda(\mathrm{ev}_1)\end{aligned}$$

which is a topological isomorphism of $D(o_L^\times, K)$-modules.

Proof. The disjoint decomposition into open sets $o_L = \pi_L o_L \cup o_L^\times$ induces the linear topological decomposition $D(G,K) = \varphi_L(D(G,K)) \oplus D(o_L^\times, K)$. The assertion follows by comparing this with the decomposition (26). $\square$

In the following we use the isomorphism $\chi_{LT} : \Gamma_L \xrightarrow{\cong} o_L^\times$ in order to identify the distribution algebra $D(\Gamma_L, K)$ with $D(o_L^\times, K)$. We then have obvious versions of Lemma 2.2 and Lemma 2.5 for $D(\Gamma_L, K)$.

Next we recall the construction of the Robba ring $\mathcal{R}_K(\mathfrak{X})$ [BSX] §2.1. As a quasi-Stein space $\mathfrak{X} = \bigcup_{n\geqslant 1} \mathfrak{X}_n$ has an admissible covering by an increasing sequence $\mathfrak{X}_1 \subseteq \ldots \subseteq \mathfrak{X}_n \subseteq \ldots$ of affinoid subdomains $\mathfrak{X}_n$. The complements $\mathfrak{X}\backslash\mathfrak{X}_n$ are admissible open, and the Robba ring is defined to be

$$\mathcal{R}_L(\mathfrak{X}) := \varinjlim_n \mathcal{O}_L(\mathfrak{X}\backslash\mathfrak{X}_n) \ .$$

Since any affinoid subdomain of $\mathfrak{X}$ must be contained in some $\mathfrak{X}_n$ the Robba ring $\mathcal{R}_L(\mathfrak{X})$ is equal to $\varinjlim_{\mathfrak{U}} \mathcal{O}_L(\mathfrak{X}\backslash\mathfrak{U})$ where $\mathfrak{U}$ runs through all affinoid subdomains of $\mathfrak{X}$. This shows that the initial definition does not depend on the choice of the $\mathfrak{X}_n$. In the following we always will take those $\mathfrak{X}_n$ which were defined in [BSX] after Remark 1.21. They satisfy:

(1) The system $(\mathfrak{X}\backslash\mathfrak{X}_n)_{/\mathbb{C}_p}$ is isomorphic to an decreasing system of one dimensional annuli ([BSX] Prop. 1.20). This implies:

- $\mathcal{R}_L(\mathfrak{X})$ is the increasing union of the rings $\mathcal{O}_L(\mathfrak{X}\backslash\mathfrak{X}_n)$ and contains $\mathcal{O}_L(\mathfrak{X})$;
- each $\mathcal{O}_L(\mathfrak{X}\backslash\mathfrak{X}_n)$ as well as $\mathcal{R}_L(\mathfrak{X})$ are integral domains.

(2) With $\mathfrak{X}$ each $\mathfrak{X}\backslash\mathfrak{X}_n$ is one dimensional and smooth.

(3) Each $\mathfrak{X}\backslash\mathfrak{X}_n$ is a quasi-Stein space ([BSX] Prop. 2.1). This implies:

- The $\mathcal{O}_L(\mathfrak{X}\backslash\mathfrak{X}_n)$ are naturally Fréchet algebras. We will therefore view $\mathcal{R}_L(\mathfrak{X})$ as the locally convex inductive limit of these Fréchet algebras.

(4) The action of the monoid $o_L\backslash\{0\}$ on $\mathcal{O}_L(\mathfrak{X})$ extends naturally to a continuous action on $\mathcal{R}_L(\mathfrak{X})$ ([BSX] Lemma 2.12). In fact, the $o_L^\times$-action preserves each subring $\mathcal{O}_L(\mathfrak{X}\backslash\mathfrak{X}_n)$ ([BSX] Lemma 2.10).

To say more about the $o_L^\times$-action we need to go a bit more into the technicalities. The reason behind the property (3) is that $\mathfrak{X}\backslash\mathfrak{X}_n = \bigcup_{s\leqslant s'} \mathfrak{X}(s,s')$ has an admissible covering by an increasing sequence of affinoid subdomains $\mathfrak{X}(s,s')$, which over $\mathbb{C}_p$ become isomorphic to closed annuli ([BSX] Prop. 1.20) and which are $o_L^\times$-invariant. In [BSX] Prop. 2.17 and paragraphs after Lemma 2.18 it is shown that the induced $o_L^\times$-action on the Banach spaces $\mathcal{O}_L(\mathfrak{X}(s,s'))$ is locally L-analytic. According to [ST1] Prop. 3.2 this $o_L^\times$-action therefore extends to a separately continuous action of the distribution algebra $D(o_L^\times, L)$ first on $\mathcal{O}_L(\mathfrak{X}(s,s'))$ and then, by passing to the projective limit, on $\mathcal{O}_L(\mathfrak{X}\backslash\mathfrak{X}_n)$. These actions, in fact, are jointly continuous since any separately continuous bilinear map between Fréchet spaces is jointly continuous. Passing to the inductive limit (w.r.t. n) we finally obtain a separately continuous $D(o_L^\times, L)$-action on $\mathcal{R}_L(\mathfrak{X})$.

2.1.2 The multiplicative case

It will be technically convenient to pick an open subgroup $\Gamma \subseteq \Gamma_L$ which is isomorphic to the additive group o_L. Then:

- Γ is a locally L-analytic subgroup of Γ_L with the same Lie algebra $\mathrm{Lie}(\Gamma) = \mathrm{Lie}(\Gamma_L)$, and the inclusion $\Gamma \subseteq \Gamma_L$ gives rise to an inclusion of distribution algebras $D(\Gamma, K) \subseteq D(\Gamma_L, K)$.

- Γ is a free $\mathbb{Z}_p$-module of rank d.

Similarly as in the previous section we let $\widehat{\Gamma}_L(K)$ and $\widehat{\Gamma}(K)$ denote the group of K-valued locally L-analytic characters of Γ_L and Γ, respectively.

Remark 2.6. *The restriction map $\widehat{\Gamma}_L(\mathbb{C}_p) \longrightarrow \widehat{\Gamma}(\mathbb{C}_p)$ is surjective.*

Proof. Since $\mathbb{C}_p^\times$ is divisible and hence injective as an abelian group, any abstract homomorphism $\Gamma \to \mathbb{C}_p^\times$ extends to a homomorphism $\Gamma_L \to \mathbb{C}_p^\times$. But the continuity as well as the local L-analyticity of a homomorphism $\chi : \Gamma_L \to \mathbb{C}_p^\times$ can be tested on its restriction $\chi|\Gamma$. Observe for this that the diagram

$$\begin{tikzcd} \Gamma \arrow[r, "\chi"] \arrow[d, "\gamma\cdot"'] & \mathbb{C}_p^\times \arrow[d, "\cdot\chi(\gamma)"] \\ \gamma\Gamma \arrow[r, "\chi"] & \mathbb{C}_p^\times \end{tikzcd}$$

is commutative for any $\gamma \in \Gamma_L$. $\square$

As for G in the previous section we have character varieties for Γ_L and Γ as well (cf. [ST2] Thm. 2.3, Lemma 2.4, Cor. 3.7 and [Eme] Propositions 6.4.5 and 6.4.6):

- $\widehat{\Gamma}_L(-)$ and $\widehat{\Gamma}(-)$ are, in a natural way, represented by a rigid analytic group variety $\mathfrak{X}^\times$ and $\mathfrak{X}_\Gamma^\times$, respectively, over L. Note that $\mathfrak{X}_\Gamma^\times$ is isomorphic to $\mathfrak{X}$.

- The restriction map $\rho : \mathfrak{X}^\times \longrightarrow \mathfrak{X}_\Gamma^\times$ is a finite faithfully flat covering ([Eme] proof of Prop. 6.4.5).

- $\mathfrak{X}^\times$ and $\mathfrak{X}_\Gamma^\times$ are one dimensional quasi-Stein spaces.

- $\mathfrak{X}_\Gamma^\times$ is smooth, and $\mathcal{O}_L(\mathfrak{X}_\Gamma^\times)$ is an integral domain.

- The Fourier transforms

$$D(\Gamma_L, L) \xrightarrow{\cong} \mathcal{O}_L(\mathfrak{X}^\times) \qquad \text{and} \qquad D(\Gamma, L) \xrightarrow{\cong} \mathcal{O}_L(\mathfrak{X}_\Gamma^\times)$$

  sending a distribution μ to the function $F_\mu(\chi) := \mu(\chi)$ are isomorphisms of Fréchet algebras.

There are two sources for explicit elements in the distribution algebras $D(\Gamma_L, L)$ and $D(\Gamma, L)$. First of all we have, for any group element $\gamma \in \Gamma_L$, resp. $\gamma \in \Gamma$, the Dirac distribution δ_γ in $D(\Gamma_L, L)$, resp. in $D(\Gamma, L)$. As in the previous section the corresponding holomorphic function $F_{\delta_\gamma} = \mathrm{ev}_\gamma$ is the function of evaluation in γ.

Lemma 2.7. *i. Let $\gamma \in \Gamma_L$ be any element not of finite order; then the zeros of the function* $\mathrm{ev}_\gamma - 1$ *on* $\mathfrak{X}^\times$ *are exactly the characters* χ *of finite order such that* $\chi(\gamma) = 1$.

ii. For any $1 \neq \gamma \in \Gamma$ *the zeros of the function* $\mathrm{ev}_\gamma - 1$ *on* $\mathfrak{X}_\Gamma^\times$ *all have multiplicity one.*

Proof. i. Obviously the zeros of $\mathrm{ev}_\gamma - 1$ are the characters χ such that $\chi(\gamma) = 1$. On the other hand consider any locally L-analytic character $\chi : \Gamma_L \to \mathbb{C}_p^\times$. Its kernel $H := \ker(\chi)$ is a closed locally L-analytic subgroup of Γ_L. Hence its Lie algebra $\mathrm{Lie}(H)$ is an L-subspace of $\mathrm{Lie}(\Gamma_L) \cong L$. We see that either $\mathrm{Lie}(H) = \mathrm{Lie}(\Gamma_L)$, in which case H is open in Γ_L and hence χ is a character of finite order, or $\mathrm{Lie}(H) = 0$, in which case H is zero dimensional and hence is a finite subgroup of Γ_L. If $\chi(\gamma) = 1$ then, by our assumption on γ, the second case cannot happen.

ii. (We will recall the concept of multiplicity further below.) Because of the isomorphism $\mathfrak{X}_\Gamma^\times \cong \mathfrak{X}$ it suffices to prove the corresponding assertion in the additive case. Let $0 \neq g \in o_L$ and let $\chi \in \mathfrak{X}(\mathbb{C}_p)$ be a character of finite order such that $\chi(g) = 1$. By [ST2] we have an isomorphism between $\mathfrak{X}_{/\mathbb{C}_p}$ and the open unit disk $\mathbf{B}_{/\mathbb{C}_p}$. Let $z \in \mathbf{B}(\mathbb{C}_p)$ denote the image of χ under this isomorphism. By [ST2] Prop. 3.1 and formula ($\diamond\diamond$) on p. 458, the function $\mathrm{ev}_g - 1$ corresponds under this isomorphism to the holomorphic function on $\mathbf{B}(\mathbb{C}_p)$ given by the formal power series

$$F_{gt_0'}(Z) = \exp(g\Omega \log_{LT}(Z)) - 1 \ ,$$

where $\Omega \neq 0$ is a certain period. By assumption we have $F_{gt_0'}(z) = 0$. On the other hand the formal derivative of this power series is

$$\frac{d}{dZ} F_{gt_0'}(Z) = g\Omega g_{LT}(Z)(F_{gt_0'}(Z) + 1) \ .$$

Since $g_{LT}(Z)$ is a unit in $o_L[[Z]]$ we see that z is not a zero of this derivative. It follows that z has multiplicity one as a zero of $F_{gt_0'}(Z)$. □

The other source comes from the Lie algebra $\mathrm{Lie}(\Gamma) \cong \mathrm{Lie}(\Gamma_L)$. Using the derivative $d\chi_{LT} : \mathrm{Lie}(\Gamma_L) \xrightarrow{\cong} L$ of the locally L-analytic isomorphism $\chi_{LT} : \Gamma_L \xrightarrow{\cong} o_L^\times$ we obtain the element

$$\nabla := d\chi_{LT}^{-1}(1) \in \mathrm{Lie}(\Gamma_L) \ .$$

On the other hand there is the L-linear embedding ([ST1] §2)

$$\begin{aligned} \mathrm{Lie}(\Gamma) &\longrightarrow D(\Gamma, L) \\ \mathfrak{x} &\longmapsto [f \mapsto \frac{d}{dt} f(\exp_\Gamma(t\mathfrak{x}))_{|t=0}] \ . \end{aligned}$$

We therefore may and will view ∇ always as a distribution on Γ or Γ_L.

Lemma 2.8. *The zeros of the function* F_∇ *on* $\mathfrak{X}_\Gamma^\times$ *are precisely the characters of finite order each with multiplicity one.*

Proof. Once again because of the isomorphism $\mathfrak{X}_\Gamma^\times \cong \mathfrak{X}$ it suffices to prove the corresponding assertion in the additive case. This is done in [BSX] Lemma 1.28. □

To recall from [BSX] §1.1 the concept of multiplicity used above and to explain a divisibility criterion in these rings of holomorphic functions we let $\mathfrak{Y}$ be any one dimensional smooth rigid analytic quasi-Stein space over L such that $\mathcal{O}_L(\mathfrak{Y})$ is an integral domain. Under these assumptions the local ring in a point y of the structure sheaf $\mathcal{O}_{\mathfrak{Y}}$ is a discrete valuation ring. Let $\mathfrak{m}_y$ denote its maximal ideal. The divisor $\operatorname{div}(f)$ of any nonzero function $f \in \mathcal{O}_L(\mathfrak{Y})$ is defined to be the function $\operatorname{div}(f) : \mathfrak{Y} \to \mathbb{Z}_{\geqslant 0}$ given by $\operatorname{div}(f)(y) = n$ if and only if the germ of f in y lies in $\mathfrak{m}_y^n \backslash \mathfrak{m}_y^{n+1}$. By Lemma 1.1 in (loc. cit.) for any affinoid subdomain $\mathfrak{Z} \subseteq \mathfrak{Y}$ the set

$$\{x \in \mathfrak{Z} \mid \operatorname{div}(f) > 0\} \text{ is finite.} \tag{27}$$

Lemma 2.9. *For any two nonzero functions $f_1, f_2 \in \mathcal{O}_L(\mathfrak{Y})$ we have $f_2 \in f_1\mathcal{O}_L(\mathfrak{Y})$ if and only if* $\operatorname{div}(f_2) \geqslant \operatorname{div}(f_1)$.

Proof. We consider the principal ideal $f_1\mathcal{O}_L(\mathfrak{Y})$. As a consequence of [BSX] Prop. 1.6 and Prop. 1.4 we have

$$f_1\mathcal{O}_L(\mathfrak{Y}) = \{f \in \mathcal{O}_L(\mathfrak{Y}) \backslash \{0\} : \operatorname{div}(f) \geqslant \operatorname{div}(f_1)\} \cup \{0\}.$$

$\square$

We now apply these results to exhibit a few more explicit elements in the distribution algebra $D(\Gamma, L)$, which will be used later on.

Lemma 2.10. *For any $1 \neq \gamma \in \Gamma$ the fraction $\frac{\nabla}{\delta_\gamma - 1}$ is a well defined element in the integral domain $D(\Gamma, L)$.*

Proof. By the Fourier isomorphism we may equivalently establish that the fraction $\frac{F_\nabla}{\mathrm{ev}_\gamma - 1}$ exists in $\mathcal{O}_L(\mathfrak{X}_\Gamma^\times)$. But for this we only need to combine the Lemmas 2.7, 2.8, and 2.9. $\square$

The next elements will only lie in the Robba ring of Γ. First of all we observe that the definition of the Robba ring $\mathcal{R}_L(\mathfrak{X})$ in the previous section was completely formal and works precisely the same way for any quasi-Stein space. Hence we have available the Robba rings $\mathcal{R}_L(\mathfrak{X}_\Gamma^\times)$ and $\mathcal{R}_L(\mathfrak{X}^\times)$. Since the morphism $\rho : \mathfrak{X}^\times \longrightarrow \mathfrak{X}_\Gamma^\times$ is finite the preimage under ρ of any affinoid subdomain in $\mathfrak{X}_\Gamma^\times$ is an affinoid subdomain in $\mathfrak{X}^\times$. The inclusion $\mathcal{O}_L(\mathfrak{X}_\Gamma^\times) \subseteq \mathcal{O}_L(\mathfrak{X}^\times)$ therefore extends to a natural homomorphism of rings

$$\rho^* : \mathcal{R}_L(\mathfrak{X}_\Gamma^\times) \longrightarrow \mathcal{R}_L(\mathfrak{X}^\times) \ .$$

Remark 2.11. *The homomorphism $\rho^* : \mathcal{R}_L(\mathfrak{X}_\Gamma^\times) \to \mathcal{R}_L(\mathfrak{X}^\times)$ is injective.*

Proof. As a finite map ρ has the property that the preimage $\rho^{-1}(\mathfrak{U})$ of an affinoid subdomain $\mathfrak{U} \subseteq \mathfrak{X}_\Gamma^\times$ is an affinoid subdomain in $\mathfrak{X}^\times$ ([BGR] Prop. 9.4.4.1). Hence, if we fix an admissible covering $\mathfrak{X}_\Gamma^\times = \bigcup_{n \geqslant 1} \mathfrak{X}_n^\times$ by an increasing sequence of affinoid subdomains $\mathfrak{X}_n^\times \subseteq \mathfrak{X}_\Gamma^\times$, then $\mathfrak{X}^\times = \bigcup_{n \geqslant 1} \rho^{-1}(\mathfrak{X}_n^\times)$ again is an admissible covering by affinoid subdomains. It follows that $\mathcal{R}_L(\mathfrak{X}^\times) = \varinjlim_n \mathcal{O}_L(\mathfrak{X}^\times \backslash \rho^{-1}(\mathfrak{X}_n^\times))$, and therefore it suffices to show the injectivity of the maps $\rho^* : \mathcal{O}_L(\mathfrak{X}_\Gamma^\times \backslash \mathfrak{X}_n^\times) \to \mathcal{O}_L(\mathfrak{X}^\times \backslash \rho^{-1}(\mathfrak{X}_n^\times))$. But this is clear since the map $\rho : \mathfrak{X}^\times \backslash \rho^{-1}(\mathfrak{X}_n^\times) \to \mathfrak{X}_\Gamma^\times \backslash \mathfrak{X}_n^\times$ is surjective by Remark 2.6. $\square$

Since $\mathfrak{X}_\Gamma^\times \cong \mathfrak{X}$ we have a list of properties for $\mathfrak{X}_\Gamma^\times$ which corresponds to the list (1) – (3) in section 2.1.1. There is an admissible covering $\mathfrak{X}_\Gamma^\times = \bigcup_{n \geqslant 1} \mathfrak{X}_n^\times$ by an increasing sequence $\mathfrak{X}_1^\times \subseteq \ldots \subseteq \mathfrak{X}_n^\times \subseteq \ldots$ of affinoid subdomains $\mathfrak{X}_n^\times$ with the following properties:

(1) The system $(\mathfrak{X}_\Gamma^\times \backslash \mathfrak{X}_n^\times)_{/\mathbb{C}_p}$ is isomorphic to an increasing system of one dimensional annuli. This implies:

   – $\mathcal{R}_L(\mathfrak{X}_\Gamma^\times)$ is the increasing union of the rings $\mathcal{O}_L(\mathfrak{X}_\Gamma^\times \backslash \mathfrak{X}_n^\times)$ and contains $\mathcal{O}_L(\mathfrak{X}_\Gamma^\times)$;
   – Each $\mathcal{O}_L(\mathfrak{X}_\Gamma^\times \backslash \mathfrak{X}_n^\times)$ as well as $\mathcal{R}_L(\mathfrak{X}_\Gamma^\times)$ are integral domains.

(2) Each $\mathfrak{X}_\Gamma^\times \backslash \mathfrak{X}_n^\times$ is a one dimensional smooth quasi-Stein space.

In particular, the $\mathcal{O}_L(\mathfrak{X}_\Gamma^\times \backslash \mathfrak{X}_n^\times)$ are naturally Fréchet algebras, and we may view $\mathcal{R}_L(\mathfrak{X}_\Gamma^\times)$ as their locally convex inductive limit. We also conclude that Lemma 2.9 applies to each $\mathfrak{X}_\Gamma^\times \backslash \mathfrak{X}_n^\times$.

We now fix a basis $b = (b_1, \ldots, b_d)$ of Γ as a $\mathbb{Z}_p$-module.

Proposition 2.12. *For any $1 \leqslant j \leqslant d$ the fraction*

$$\Xi_{b,j} := \frac{1}{\mathrm{ev}_{b_j} - 1} \cdot \prod_{i \neq j} \frac{F_\nabla}{\mathrm{ev}_{b_i} - 1}$$

is well defined in the Robba ring $\mathcal{R}_L(\mathfrak{X}_\Gamma^\times)$ and independent of j and henceforth just called Ξ_b.

Proof. The zeros of the fraction $\frac{F_\nabla}{\mathrm{ev}_{b_i} - 1}$ are precisely those finite order characters which are nontrivial on b_i. Hence the product $\prod_{i \neq j} \frac{F_\nabla}{\mathrm{ev}_{b_i} - 1}$ still has a zero in any finite order character which is nontrivial on b_i for at least one $i \neq j$. On the other hand the zeros of $\mathrm{ev}_{b_j} - 1$ are those finite order characters which are trivial on b_j (and they have multiplicity one). Since only the trivial character is trivial on all $b_1, \ldots, b_d$ we see that all zeros of $\mathrm{ev}_{b_j} - 1$ with the exception of the trivial character occur also as zeros of the product in the assertion. It follows that the asserted fraction $\Xi_{b,j}$ exists in $\mathcal{O}_L(\mathfrak{X}_\Gamma^\times \backslash \mathfrak{X}_n^\times)$ provided n is large enough so that the trivial character is a point in $\mathfrak{X}_n^\times$. Since $(\prod_{i=1}^d (\mathrm{ev}_{b_i} - 1))\Xi_{b,j} = (F_\nabla)^{d-1}$ and $\mathcal{R}_L(\mathfrak{X}_\Gamma^\times)$ is integral, we see the independence of j. $\square$

2.1.3 Twisting

Consider any locally L-analytic group G and fix a locally L-analytic character $\chi : G \to L^\times$. Then multiplication by χ is a K-linear topological isomorphism $C^{an}(G,K) \xrightarrow[\cong]{\chi\cdot} C^{an}(G,K)$. We denote the dual isomorphism by

$$Tw_\chi^D : D(G,K) \xrightarrow{\cong} D(G,K) \ ,$$

i.e., $Tw_\chi^D(\mu) = \mu(\chi -)$, and call it the twist by χ. For Dirac distributions we obtain $Tw_\chi^D(\delta_g) = \chi(g)\delta_g$.

Suppose now that G is one of the groups o_L or $\Gamma \subseteq \Gamma_L \cong o_L^\times$ of the previous subsections, and let $\mathfrak{X}_G$ denote its character variety. Then χ is an L-valued point $z_\chi \in \mathfrak{X}_G(L)$. Using the product structure of the variety $\mathfrak{X}_G$ we similarly have the twist operator

$$Tw_z^{\mathfrak{X}_G} : \mathcal{O}_K(\mathfrak{X}_G) \xrightarrow{\cong} \mathcal{O}_K(\mathfrak{X}_G) \ , \ f \longmapsto f(z-) \ .$$

As any rigid automorphism multiplication by a rational point respects the system of affinoid subdomains and hence the system of their complements. Hence $Tw_z^{\mathfrak{X}_G}$ extends straightforwardly to an automorphism $Tw_z^{\mathfrak{X}_G} : \mathcal{R}_K(\mathfrak{X}_G) \xrightarrow{\cong} \mathcal{R}_K(\mathfrak{X}_G)$. The following properties are straightforward to check:

1. Under the Fourier isomorphism Tw^D_χ and $Tw^{\mathfrak{X}_G}_{z_\chi}$ correspond to each other.

2. $Tw^{\mathfrak{X}_G}_{z_1} \circ Tw^{\mathfrak{X}_G}_{z_2} = Tw^{\mathfrak{X}_G}_{z_1 \cdot z_2}$.

3. If $\alpha : G_1 \xrightarrow{\cong} G_2$ is an isomorphism between two of our groups then, for any $z \in \mathfrak{X}_{G_2}(L)$, the twist operators $Tw^{\mathfrak{X}_{G_1}}_{\alpha^*(z)}$ and $Tw^{\mathfrak{X}_{G_2}}_z$ correspond to each other under the isomorphism $\alpha_* : \mathcal{R}_K(\mathfrak{X}_{G_1}) \xrightarrow{\cong} \mathcal{R}_K(\mathfrak{X}_{G_2})$.

2.1.4 The LT-isomorphism

We write $\mathbb{B}$ for the open unit ball over L. The Lubin-Tate formal o_L-module gives $\mathbb{B}$ an o_L-action via $(a, z) \mapsto [a](z)$. If $\mathcal{O}_K(\mathbb{B})$ is the ring of power series in Z with coefficients in K which converge on $\mathbb{B}(\mathbb{C}_p)$, then the above o_L-action on $\mathbb{B}$ induces an action of the monoid $o_L \backslash \{0\}$ on $\mathcal{O}_K(\mathbb{B})$ by $(a, F) \mapsto F \circ [a]$. Similarly as before we let φ_L denote the action of π_L. Next we consider the continuous operator

$$\begin{aligned} tr : \mathcal{O}_K(\mathbb{B}) &\longrightarrow \mathcal{O}_K(\mathbb{B}) \\ f(z) &\longmapsto \sum_{y \in \ker([\pi_L])} f(y +_{LT} z) \ . \end{aligned}$$

Coleman has shown (cf. [SV15, §2]) that $tr(Z^i) \in \operatorname{im}(\varphi_L)$ for any $i \geqslant 0$. Hence, since φ_L is a homeomorphism onto its image, we have $\operatorname{im}(tr) \subseteq \operatorname{im}(\varphi_L)$ and hence, since φ_L is injective, we may introduce the K-linear operator

$$\psi_L : \mathcal{O}_K(\mathbb{B}) \longrightarrow \mathcal{O}_K(\mathbb{B}) \qquad \text{such that } \pi_L^{-1} tr = \varphi_L \circ \psi_L.$$

One easily checks that ψ_L is equivariant for the $o_L^\times$-action and satisfies the projection formula $\psi_L(f_1 \varphi_L(f_2)) = \psi_L(f_1) f_2$ as well as $\psi_L \circ \varphi_L = \frac{q}{\pi_L}$.

Furthermore, we fix a generator η' of T'_π as o_L-module and denote by $\Omega = \Omega_{\eta'}$ the corresponding period. In the following we assume that Ω belongs to K. From [ST2, theorem 3.6] we recall the LT-isomorphism

$$\begin{aligned} \kappa^* : \mathcal{O}_K(\mathfrak{X}) &\xrightarrow{\cong} \mathcal{O}_K(\mathbb{B}) \\ F &\mapsto [z \mapsto F(\kappa_z)], \end{aligned} \tag{28}$$

where $\kappa_z(a) = 1 + F_{\eta'}([a](z))$ with $1 + F_{\eta'}(Z) := \exp\left(\Omega \log_{LT}(Z)\right)$. It is an isomorphism of topological rings which is equivariant with respect to the action by the monoid $o_L \backslash \{0\}$ (as a consequence of [ST2, Prop. 3.1]). Moreover, Lemma 2.2 implies that the $o_L^\times$-action on $\mathcal{O}_K(\mathbb{B})$ extends to a jointly continuous $D(o_L^\times, K)$-module structure (by descent even for general K) and that the LT-isomorphism is an isomorphism of $D(o_L^\times, K)$-modules.

By the construction of the LT-isomorphism we have

$$\kappa^*(\mathrm{ev}_a) = \exp(a \Omega \log_{LT}(Z)) \in o_{\mathbb{C}_p}[[Z]] \qquad \text{for any } a \in o_L.$$

Hence Lemma 2.3 implies that

$$\kappa^*(\ker(\psi_L^{\mathfrak{X}})) = \sum_{a \in R_0} \exp(a \Omega \log_{LT}(Z)) \varphi_L(\mathcal{O}_K(\mathbb{B}))$$

where $R_0 \subseteq o_L$ denotes a set of representatives for the nonzero cosets in $o_L/\pi_L o_L$. Using that $\log_{LT}(Z_1 +_{LT} Z_2) = \log_{Lt}(Z_1) + \log_{LT}(Z_2)$ we compute

$$\begin{aligned} tr(\exp(a\Omega \log_{LT}(Z)) &= \sum_{y \in \ker([\pi_L])} \exp(a\Omega \log_{LT}(y +_{LT} Z)) \\ &= \Big(\sum_{y \in \ker([\pi_L])} \exp(a\Omega \log_{LT}(y))\Big) \exp(a\Omega \log_{LT}(Z)) \\ &= \Big(\sum_{y \in \ker([\pi_L])} \kappa_y(a)\Big) \exp(a\Omega \log_{LT}(Z)) \ . \end{aligned}$$

But the κ_y for $y \in \ker([\pi_L])$ are precisely the characters of the finite abelian group $o_L/\pi_L oL$. Hence $\sum_{y \in \ker([\pi_L])} \kappa_y(a) = 0$ for $a \in R_0$. It follows that $\kappa^*(\ker(\psi_L^{\mathfrak{X}})) = \ker(\psi_L)$. We conclude that under the LT-isomorphism ψ_L corresponds to $\frac{q}{\pi_L}\psi_L^{\mathfrak{X}}$ using the fact that we also have a decomposition

$$\mathcal{O}_K(\mathbb{B}) = \sum_{a \in o_L/\pi_L} \exp(a\Omega \log_{LT}(Z))\varphi_L(\mathcal{O}_K(\mathbb{B})).$$

In the following we denote by

$$\mathfrak{M} : D(\Gamma_L, K) \xrightarrow{\cong} \mathcal{O}_K(\mathbb{B})^{\psi_L = 0}$$

the composite

$$D(\Gamma_L, K) \cong D(o_L^\times, K) \cong \mathcal{O}_K(\mathfrak{X})^{\psi_L^{\mathfrak{X}} = 0} \cong \mathcal{O}_K(\mathbb{B})^{\psi_L = 0}$$

where the first map is induced by the character $\chi_{LT} : \Gamma_L \xrightarrow{\cong} o_L^\times$, the second one from 2.5 is induced by the Fourier isomorphism while the third one is the LT-isomorphism. By inserting the definitions we obtain the explicit formula

$$\mathfrak{M}(\lambda)(z) = \lambda(\kappa_z \circ \chi_{LT}) \ .$$

The construction of the above map $\mathfrak{M}$ is related to the pairing

$$\begin{aligned} \{\ ,\ \} : \mathcal{O}_K(\mathbb{B}) \times C^{an}(o_L, K) &\to K \\ (F, f) &\mapsto \mu(f), \quad \text{ where } \mu \in D(o_L, K) \text{ is such that } \mu(\kappa_z) = F(z), \end{aligned}$$

in [ST2, lem. 4.6] by the following commutative diagram:

$$\begin{array}{ccccc} D(\Gamma_L, K) & \times & C^{an}(\Gamma_L, K) & \xrightarrow{(\lambda, f) \mapsto \lambda(f)} & K \\ \downarrow{\scriptstyle \mathfrak{M}} & & \downarrow{\scriptstyle (\chi_{LT})_*} & & \| \\ \mathcal{O}_K(\mathbb{B})^{\psi_L = 0} & \times & C^{an}(o_L^\times, K) & & \| \\ \downarrow{\scriptstyle \subseteq} & & \downarrow{\scriptstyle extension\ by\ 0} & & \| \\ \mathcal{O}_K(\mathbb{B}) & \times & C^{an}(o_L, K) & \xrightarrow{\{\ ,\ \}} & K \end{array}$$

Remark 2.13. *For any $F \in \mathcal{O}_K(\mathbb{B})^{\psi_L = 0}$ and any $f \in C^{an}(o_L, K)$ such that $f | o_L^\times = 0$ we have $\{F, f\} = 0$.*

Proof. We have seen above that under the LT-isomorphism ψ_L corresponds, up to a nonzero constant, to $\psi_L^{\mathfrak{X}}$ and hence further under the Fourier isomorphism to ψ_L^D. It therefore suffices to show that for any $\mu \in D(o_L, K)^{\psi_L^D=0}$ we have $\mu(f) = 0$. For this we define $\tilde{f} := f(\pi_L -) \in C^{an}(o_L, K)$ and note that $(\pi_L)_!(\tilde{f}) = f$. By the definition of ψ_L^D we therefore obtain, under our assumption on μ, that $\mu(f) = \mu(f) - \psi_L^D(\mu)(\tilde{f}) = \mu(f - (\pi_L)_!(\tilde{f})) = \mu(0) = 0$. $\square$

Lemma 2.14. *For any $F \in \mathcal{O}_K(\mathbb{B})^{\psi_L=1}$ and $n \geqslant 0$ we have*

$$\mathfrak{M}^{-1}((1 - \frac{\pi_L}{q}\varphi_L)F)(\chi_{LT}^n) = \Omega^{-n}(1 - \frac{\pi_L^{n+1}}{q})(\partial_{\mathrm{inv}}^n F)_{|Z=0}.$$

Proof. Note that $(1 - \frac{\pi_L}{q}\varphi_L)F$ belongs to $\mathcal{O}_K(\mathbb{B})^{\psi_L=0}$. Let $\mathrm{inc}_! \in C^{an}(o_L, K)$ denote the extension by zero of the inclusion $o_L^\times \subseteq o_L$, and let $\mathrm{id} : o_L \to K$ be the identity function. Using the above commutative diagram the assertion reduces to the equality

$$\{(1 - \frac{\pi_L}{q}\varphi_L)F, \mathrm{inc}_!^n\} = \Omega^{-n}(1 - \frac{\pi_L^{n+1}}{q})(\partial_{\mathrm{inv}}^n F)_{|Z=0}.$$

By Remark 2.13 we may replace on the left hand side the function $\mathrm{inc}_!^n$ by the function id^n. Next we observe that $x\,\mathrm{id}^n(x) = \mathrm{id}^{n+1}(x)$. Hence, by [ST2, Lem. 4.6(8)], i.e., $\{F, xf(x)\} = \{\Omega^{-1}\partial_{\mathrm{inv}}F, f\}$ and induction, the left hand side is equal to

$$\begin{aligned}
\{(1 - \frac{\pi_L}{q}\varphi_L)F, \mathrm{id}^n\} &= \{\Omega^{-n}\partial_{\mathrm{inv}}^n((1 - \frac{\pi_L}{q}\varphi_L)F), \mathrm{id}^0\} \\
&= \{\Omega^{-n}(1 - \frac{\pi_L^{n+1}}{q}\varphi_L)(\partial_{\mathrm{inv}}^n F), \mathrm{id}^0\} \quad \text{since } \partial_{\mathrm{inv}}\varphi_L = \pi_L\varphi_L\partial_{\mathrm{inv}} \text{ by (3)} \\
&= \Omega^{-n}(1 - \frac{\pi_L^{n+1}}{q}\varphi_L)(\partial_{\mathrm{inv}}^n F)_{|Z=0} \quad \text{since } \mathrm{id}^0 \text{ is the trivial character of } o_L \\
&= \Omega^{-n}(1 - \frac{\pi_L^{n+1}}{q})(\partial_{\mathrm{inv}}^n F)_{|Z=0} \quad \text{since } [\pi_L](0) = 0.
\end{aligned}$$

$\square$

In the course of the previous proof we have seen that, for F in $\mathcal{O}_K(\mathbb{B})^{\psi_L=0}$ and $n \geqslant 0$,

$$\mathfrak{M}^{-1}(F)(\chi_{LT}^n) = \Omega^{-n}(\partial_{\mathrm{inv}}^n F)_{|Z=0}\ . \tag{29}$$

Lemma 2.15. *For any $F \in \mathcal{O}_K(\mathbb{B})^{\psi_L=0}$ and $n \geqslant 1$ we have*

$$\mathfrak{M}^{-1}(\log_{LT} \cdot F)(\chi_{LT}^n) = n\Omega^{-1}\mathfrak{M}^{-1}(F)(\chi_{LT}^{n-1})\ .$$

Proof. First, using (3), observe that

$$\psi_L(\log_{LT} \cdot F) = \psi_L(\pi_L^{-1}\varphi_L(\log_{LT}) \cdot F) = \pi_L^{-1}\varphi_L(\log_{LT})\psi_L(F) = 0\ .$$

Secondly note that $\partial_{\mathrm{inv}} \log_{LT} = 1$, i.e., $\partial_{\mathrm{inv}}^i \log_{LT} = 0$ for $i \geqslant 2$; also $\log_{LT}(0) = 0$. Using (29) twice we have

$$\begin{aligned}
\mathfrak{M}^{-1}(\log_{LT} F)(\chi_{LT}^n) &= \Omega^{-n}(\partial_{\mathrm{inv}}^n(\log_{LT} F))_{|Z=0} \\
&= \Omega^{-n}\left(\sum_{i+j=n}\binom{n}{i}(\partial_{\mathrm{inv}}^i \log_{LT})(\partial_{\mathrm{inv}}^j F)\right)_{|Z=0} \\
&= \Omega^{-n}n(\partial_{\mathrm{inv}}^{n-1}F)_{|Z=0} \\
&= n\Omega^{-1}\mathfrak{M}^{-1}(F)(\chi_{LT}^{n-1})\ .
\end{aligned}$$

$\square$

For the rest of this section we assume not only that K contains Ω but also that the action of G_L on $\mathbb{C}_p$ leaves K invariant.

The LT-isomorphism is a topological ring isomorphism

$$K\hat{\otimes}_L\mathcal{O}_L(\mathfrak{X}) = \mathcal{O}_K(\mathfrak{X}) \cong \mathcal{O}_L(\mathbb{B}) = K\hat{\otimes}_L\mathcal{O}_L(\mathbb{B})$$

(see [BSX, Prop. 2.1.5 ii.] for the outer identities).

On both sides we have the obvious coefficientwise G_L-action induced by the Galois-action on the tensor-factor K. We use the following notation:

- $\sigma \in G_L$ acting *coefficientwise* on $\mathcal{O}_K(\mathbb{B})$ is denoted by: $F \mapsto {}^{\sigma}F$; the corresponding fixed ring is $\mathcal{O}_K(\mathbb{B})^{G_L} = \mathcal{O}_L(\mathbb{B})$.
- The coefficientwise action on $\mathcal{O}_K(\mathfrak{X})$ transfers to the *twisted action* on $\mathcal{O}_K(\mathbb{B})$ by [ST2, before Cor. 3.8] given as $F \mapsto {}^{\sigma *}F := {}^{\sigma}F \circ [\tau(\sigma^{-1})]$; the corresponding fixed ring is $\mathcal{O}_K(\mathbb{B})^{G_L,*} = \mathcal{O}_L(\mathfrak{X}) = D(o_L, L)$.

Remark 2.16. *Note that the $o_L\backslash\{0\}$-action and hence the $D(o_L^\times, L)$-module structure commute with both G_L-actions. Moreover, ψ_L commutes with the G_L-actions as well.*

Recall that using the notation from [ST2, Lem. 4.6, 1./2.] the function $1 + F_{a\eta'}(Z) = \exp\big(a\Omega_{\eta'}\log_{LT}(Z)\big)$ corresponds to the Dirac distribution δ_a of $a \in o_L$ under the Fourier isomorphism.

Lemma 2.17. *Let σ be in G_L, $t' \in T'_\pi$ and $a \in o_L$. Then*

(i) $\sigma(\Omega_{t'}) = \Omega_{\tau(\sigma)t'} = \Omega_{t'}\tau(\sigma)$ *and*

(ii) ${}^{\sigma}F_{a\eta'} = F_{a\eta'} \circ [\tau(\sigma)] = F_{a\tau(\sigma)\eta'}$.

Proof. (i) The Galois equivariance of the pairing $(\ ,\ ) : T'_\pi \otimes_{o_L} \mathbb{C}_p \to \mathbb{C}_p$, from (loc. cit. before Prop. 3.1) with $(t', x) = \Omega_{t'}x$ implies that

$$\sigma(\Omega_{t'}) = \Omega_{\sigma(t')} = \Omega_{\tau(\sigma)t'}$$

while the o_L-invariance of that pairing implies that the latter expression equals $\Omega_{t'}\tau(\sigma)$.

(ii) This is immediate from (i) and the definition of F_a taking equation (3) into account. $\square$

Proposition 2.18. (i) *The LT-isomorphism restricts to an isomorphism*

$$D(o_L, K)^{G_L} = \mathcal{O}_K(\mathfrak{X})^{G_L} \cong \mathcal{O}_K(\mathbb{B})^{G_L} = \mathcal{O}_L(\mathbb{B})$$

of $D(o_L^\times, L)$-modules.

(ii) *The Mellin transform restricts to an isomorphism of $D(o_L^\times, L)$-modules*

$$D(o_L^\times, K)^{G_L} = \mathcal{O}_K(\mathfrak{X})^{G_L,\psi_L=0} \cong \mathcal{O}_L(\mathbb{B})^{\psi_L=0}.$$

Here the G_L-action on the distribution rings on the left hand sides is induced from the coefficientwise action on $\mathcal{O}_K(\mathbb{B})$ and $\mathcal{O}_K(\mathbb{B})^{\psi_L=0}$ via the LT-isomorphism and Mellin transform, respectively.

Proof. (i) and (ii) follow from passing to the fixed vectors with respect to the coefficientwise G_L-action and Remark 2.16. □

In order to express the $D(o_L^\times, L)$-module $D(o_L^\times, K)^{G_L}$ in the above proposition more explicitly we describe the previous two actions on $\mathcal{O}_K(\mathbb{B})$ now on $D(o_L, K)$:

- The coefficientwise G_L-action on $D(o_L, K) = K\hat{\otimes}_L D(o_L, L)$, which corresponds to the twisted action on $\mathcal{O}_K(\mathbb{B})$, will be written as $\lambda \mapsto {}^\sigma\lambda$.
- The G_L-action given by $\lambda \mapsto \tau(\sigma)_*({}^\sigma\lambda)$ corresponds to the coefficientwise action on $\mathcal{O}_K(\mathbb{B})$.

Note that for $\lambda \in D(o_L^\times, K)$ we have $\tau(\sigma)_*(\lambda) = \delta_{\tau(\sigma)}\lambda$, where the right hand side refers to the product of λ and the Dirac distribution $\delta_{\tau(\sigma)}$ in the ring $D(o_L^\times, K)$. Then we conclude that

$$D(o_L^\times, K)^{G_L} = \{\lambda \in D(o_L^\times, K) |\ {}^\sigma\lambda = \delta_{\tau(\sigma)^{-1}}\lambda \text{ for all } \sigma \in G_L\}.$$

2.2 (φ_L, Γ_L)-modules

2.2.1 The usual Robba ring

Let K be a complete field which contains L. We recall the definition of the Robba ring $\mathcal{R} = \mathcal{R}_K$ and construct various related rings. We let $\mathcal{R}^+ := \mathcal{R}_K^+ = \mathcal{O}_K(\mathbb{B})$ denote the Fréchet algebra of all power series in the variable Z with coefficients in K which converge on the open unit disk $\mathbb{B}$ over K. The Fréchet topology on $\mathcal{R}^+$ is given by the family of norms

$$|\sum_{i \geqslant 0} c_i Z^i|_r := \max_i |c_i| r^i \qquad \text{for } 0 < r < 1 .$$

In the commutative integral domain $\mathcal{R}^+$ we have the multiplicative subset $Z^{\mathbb{N}} = \{Z^j : j \in \mathbb{N}\}$, so that we may form the corresponding localization $\mathcal{R}^+_{Z^{\mathbb{N}}}$. Each norm $|\ |_r$ extends to this localization $\mathcal{R}^+_{Z^{\mathbb{N}}}$ by setting $|\sum_{i \gg -\infty} c_i Z^i|_r := \max_i |c_i| r^i$.

The usual Robba ring $\mathcal{R} \supseteq \mathcal{R}^+$ is constructed as follows. For any $s > 0$, resp. any $0 < r \leqslant s$, in $p^{\mathbb{Q}}$ let $\mathbb{B}_{[0,s]}$, resp. $\mathbb{B}_{[r,s]}$, denote the affinoid disk of radius s, resp. the affinoid annulus of inner radius r and outer radius s, over K. For $I = [0, s]$ or $[r, s]$ we denote by

$$\mathcal{R}^I := \mathcal{O}_K(\mathbb{B}_I)$$

the affinoid algebra of $\mathbb{B}_I$. The Fréchet algebra $\mathcal{R}^{[r,1)} := \varprojlim_{r<s<1} \mathcal{R}^{[r,s]}$ is the algebra of (infinite) Laurent series in the variable Z with coefficients in K which converge on the half-open annulus $\mathbb{B}_{[r,1)} := \bigcup_{r<s<1} \mathbb{B}_{[r,s]}$. The Banach algebra $\mathcal{R}^{[0,s]}$ is the completion of $\mathcal{R}^+$ with respect to the norm $|\ |_s$. The Banach algebra $\mathcal{R}^{[r,s]}$ is the completion of $\mathcal{R}^+_{Z^{\mathbb{N}}}$ with respect to the norm $|\ |_{r,s} := \max(|\ |_r, |\ |_s)$. It follows that the Fréchet algebra $\mathcal{R}^{[r,1)}$ is the completion of $\mathcal{R}^+_{Z^{\mathbb{N}}}$ in the locally convex topology defined by the family of norms $(|\ |_{r,s})_{r<s<1}$. Finally, the Robba ring is $\mathcal{R} = \bigcup_{0<r<1} \mathcal{R}^{[r,1)}$.

Let $1 > s \geqslant r > p^{-\frac{1}{(q-1)e}}$. Then we have a surjective map

(30)
$$\begin{aligned} \mathbb{B}_{[r,s]} &\to \mathbb{B}_{[r^q,s^q]} \\ z &\mapsto [\pi_L](z) \end{aligned}$$

according to [FX, proof of Lem. 2.6] Hence we obtain a map

(31)
$$\varphi_L^{[r^q,s^q]} : \mathcal{R}^{[r^q,s^q]} \to \mathcal{R}^{[r,s]}$$

which is isometric with respect to the supremum norm, i.e., $|\varphi_L^{[r^q,s^q]}(f)|_{[r,s]} = |f|_{[r^q,s^q]}$ for $f \in \mathcal{R}^{[r^q,s^q]}$. In particular, by taking first inverse and then direct limits we obtain a map $\varphi_L : \mathcal{R} \to \mathcal{R}$. We shall often omit the interval in $\varphi_L^{[r,s]}$ and just write φ_L.

Similarly, we obtain a continuous Γ_L-action on $\mathcal{R}$: According to (loc. cit.) we have a bijective map

(32)
$$\begin{aligned} \mathbb{B}_{[r,s]} &\to \mathbb{B}_{[r,s]} \\ z &\mapsto [\chi_{LT}(\gamma)](z) \end{aligned}$$

for any $\gamma \in \Gamma_L$, whence we obtain an isometry

(33)
$$\gamma : \mathcal{R}^{[r,s]} \to \mathcal{R}^{[r,s]}$$

with respect to the supremum norm, i.e., $|\gamma(f)|_{[r,s]} = |f|_{[r,s]}$ for $f \in \mathcal{R}^{[r,s]}$.

Finally, we extend the operator ψ_L to $\mathcal{R}$: For $y \in \ker([\pi_L])$ we have an isomorphism

$$\begin{aligned} \mathbb{B}_{[r,s]} &\to \mathbb{B}_{[r,s]} \\ z &\mapsto z +_{LT} y \end{aligned} \tag{34}$$

of affinoid varieties, because $|z +_{LT} y| = |z + y| = |z|$. Setting $tr(f) := \sum_{y \in \ker([\pi_L])} f(z +_{LT} y)$ we become a norm decreasing linear map $tr : \mathcal{R}^{[r,s]} \to \mathcal{R}^{[r,s]}$. We claim that the image of tr is contained in the (closed) image of the isometry $\varphi_L^{[r^q,s^q]}$, whence there is a norm decreasing map

$$\psi_{Col} : \mathcal{R}^{[r,s]} \to \mathcal{R}^{[r^q,s^q]},$$

such that $\varphi_L \circ \psi_{Col} = tr$. Indeed, by continuity it suffices to show that $tr(Z^i)$ belongs to the image for any $i \in \mathbb{Z}$. For $i \geqslant 0$, Coleman has shown that $tr(Z^i) = \varphi_L(\psi_{Col}(Z^i))$ with $\psi_{Col}(Z^i) \in o_L[[Z]] \subseteq \mathcal{R}^{[r^q,s^q]}$, see [SV15, §2]. For $i < 0$, we calculate

$$\begin{aligned} \varphi_L(Z^i \psi_{Col}([\pi_L](Z)^{-i} Z^i)) &= \varphi_L(Z^i) \left(\sum_{y \in \ker([\pi_L])} [\pi_L](Z)^{-i} Z^i \right) (Z +_{LT} y) \\ &= \varphi_L(Z^i) \left(\sum_{y \in \ker([\pi_L])} [\pi_L]((Z +_{LT} y))^{-i} (Z +_{LT} y)^i \right) \\ &= \varphi_L(Z^i) \left(\sum_{y \in \ker([\pi_L])} \varphi_L(Z)^{-i} (Z +_{LT} y)^i \right) \\ &= \sum_{y \in \ker([\pi_L])} (Z +_{LT} y)^i = tr(Z^i), \end{aligned}$$

whence the claim follows. We put $\psi_L^{[r,s]} := \frac{1}{\pi_L} \psi_{Col} : \mathcal{R}^{[r,s]} \to \mathcal{R}^{[r^q,s^q]}$ which induces the continuous operator $\psi_L : \mathcal{R} \to \mathcal{R}$ by taking first inverse limits and then direct limits. By definition of tr the operators $\psi_L^{[r,s]}$ and hence ψ_L satisfy the projection formula. We shall often omit the interval in $\psi_L^{[r,s]}$ and just write ψ_L.

For the rest of this subsection we assume in addition that K contains Ω. Following Colmez we set $\eta(i, Z) := F_{it_0'}(Z) + 1 = \exp(i\Omega \log_{LT}(Z))$ for $i \in o_L$ and for a fixed generator $t_0 = (t_{0,n})$ of the Tate module of LT. Recall that we have the following decompositions of Banach spaces

$$\mathcal{R}^{[r,s]} = \bigoplus_{a \in o_L \mod \pi_L^n} \varphi_L^n(\mathcal{R}^{[r^q,s^q]}) \eta(a, Z) \tag{35}$$

and hence

$$\mathcal{R} = \bigoplus_{a \in o_L \mod \pi_L^n} \varphi_L^n(\mathcal{R}) \eta(a, Z) \tag{36}$$

of LF-spaces using the formula

$$r = (\frac{\pi_L}{q})^n \sum_a \varphi_L^n \psi_L^n \left(\eta(-a, Z) r \right) \eta(a, Z). \tag{37}$$

This can easily be reduced by induction on n to the case $n = 1$. Using the definition of tr and the orthogonality relations for the character of $o_L/\pi_L o_L$, $a \mapsto \eta(a, y)$, for fixed $y \in \ker([\pi_L])$, the formula follows and, moreover, defines a continuous inverse to the continuous map

$$\begin{aligned} \mathbb{Z}[o_L] \otimes_{\pi_L o_L} \mathcal{R}^{[r^q, s^q]} &\to \mathcal{R}^{[r,s]} \\ a \otimes f &\mapsto a\varphi_L(f). \end{aligned} \tag{38}$$

Inductively, we obtain canonical isomorphisms

$$\begin{aligned} \mathbb{Z}[o_L] \otimes_{\pi_L^n o_L} \mathcal{R}^{[r^{q^n}, s^{q^n}]} &\to \mathcal{R}^{[r,s]} \\ a \otimes f &\mapsto a\varphi_L^n(f). \end{aligned} \tag{39}$$

Moreover, immediately from the definitions we have

$$\varphi_L(\eta(i,Z)) = \eta(\pi_L i, Z) \tag{40}$$

$$\sigma(\eta(i,Z)) = \eta(\chi_{LT}(\sigma)i, Z) \tag{41}$$

$$\psi_L(\eta(i,Z)) = \frac{q}{\pi_L}\eta(\frac{i}{\pi_L}, Z), \text{ if } i \in \pi_L o_L, \text{ and } 0 \text{ otherwise.} \tag{42}$$

We now introduce φ_L-modules over $\mathcal{R}$ and extend the above maps and decompositions to such modules.

Definition 2.19. *A φ_L-module M over $\mathcal{R}$ is a finitely generated free $\mathcal{R}$-module M equipped with a semilinear endomorphism φ_M such that the $\mathcal{R}$-linear map*

$$\begin{aligned} \varphi_M^{lin} : \mathcal{R} \otimes_{\mathcal{R},\varphi_L} M &\xrightarrow{\cong} M \\ f \otimes m &\longmapsto f\varphi_M(m) \end{aligned}$$

is bijective.

Technically important is the following fact (cf. [BSX] Prop. 2.4 for a more general case).

Proposition 2.20. *Let M be a φ_L-module M over $\mathcal{R}$. There exists a radius $r_0 > p^{-\frac{1}{(q-1)e}}$ and a finitely generated free $\mathcal{R}^{[r_0,1)}$-module M_0 equipped with a semilinear continuous homomorphism*

$$\varphi_{M_0} : M_0 \longrightarrow \mathcal{R}^{[r_0^{1/q},1)} \otimes_{\mathcal{R}^{[r_0,1)}} M_0$$

such that the induced $\mathcal{R}^{[r_0^{1/q},1)}$-linear map

$$\varphi_{M_0}^{lin} : \mathcal{R}^{[r_0^{1/q},1)} \otimes_{\mathcal{R}^{[r_0,1)},\varphi_L} M_0 \xrightarrow{\cong} \mathcal{R}^{[r_0^{1/q},1)} \otimes_{\mathcal{R}^{[r_0,1)}} M_0$$

is an isomorphism and such that

$$\mathcal{R} \otimes_{\mathcal{R}^{[r_0,1)}} M_0 = M$$

with $\varphi_L \otimes \varphi_{M_0}$ and φ_M corresponding to each other.

The continuity condition for the φ_{M_0}, of course, refers to the product topology on $M_0 \cong (\mathcal{R}^{[r_0,1)})^d$.

In the following we fix a φ_L-module M over $\mathcal{R}$ and a pair (r_0, M_0) as in Prop. 2.20. For any $r_0 \leqslant r \leqslant s < 1$ we then have the finitely generated free modules

$$M^{[r,1)} := \mathcal{R}^{[r,1)} \otimes_{\mathcal{R}^{[r_0,1)}} M_0 \quad \text{over } \mathcal{R}^{[r,1)}$$

and

$$M^{[r,s]} := \mathcal{R}^{[r,s]} \otimes_{\mathcal{R}^{[r,1)}} M^{[r,1)} \quad \text{over } \mathcal{R}^{[r,s]} \ .$$

They satisfy

$$M^{[r,1)} = \varprojlim_{s>r} M^{[r,s]} \qquad \text{and} \qquad M = \varinjlim_{r} M^{[r,1)} \ .$$

For $I = [r,s]$, we equip M^I with the Banach norm $|-|_{M^I}$ given by the maximum norm with respect to any fixed basis (the induced topology does not depend on the choice of basis) which is submultiplicative with respect to scalar multiplication and the norm $|-|_I$ on $\mathcal{R}^I$.

Furthermore, base change with $\mathcal{R}^{[r^{1/q},s^{1/q}]}$ over $\mathcal{R}^{[r^{1/q},1)}$ induces isomorphisms of Banach spaces

$$\varphi_{lin}^{[r,s]} = \mathcal{R}^{[r^{1/q},s^{1/q}]} \otimes_{\mathcal{R}^{[r^{1/q},1)}} \varphi_{M_0}^{lin} : \mathcal{R}^{[r^{1/q},s^{1/q}]} \otimes_{\mathcal{R}^{[r,s]},\varphi_L} M^{[r,s]} \xrightarrow{\cong} M^{[r^{1/q},s^{1/q}]}$$

and hence injective, continuous maps

$$\varphi^{[r,s]} : M^{[r,s]} \to M^{[r^{1/q},s^{1/q}]}$$

by restriction.

We define addtive, K-linear, continuous maps $\psi^{[r,s]} : M^{[r,s]} \to M^{[r^q,s^q]}$ as the composite

$$\psi^{[r,s]} : M^{[r,s]} \xrightarrow{(\varphi_{lin}^{[r^q,s^q]})^{-1}} \mathcal{R}^{[r,s]} \otimes_{\mathcal{R}^{[r^q,s^q]},\varphi_L} M^{[r^q,s^q]} \to M^{[r^q,s^q]},$$

where the last map sends $f \otimes m$ to $\psi^{[r,s]}(f)m$. By construction, it satisfies the projection formulas

$$\psi^{[r,s]}(\varphi^{[r^q,s^q]}(f)m) = f\psi^{[r,s]}(m) \qquad \text{and} \qquad \psi^{[r,s]}(g\varphi^{[r^q,s^q]}(m')) = \psi^{[r,s]}(g)m' \ , \tag{43}$$

for any $f \in \mathcal{R}^{[r^q,s^q]}$, $g \in \mathcal{R}^{[r,s]}$ and $m \in M^{[r,s]}$, $m' \in M^{[r^q,s^q]}$ as well as the formula

$$\psi^{[r,s]} \circ \varphi^{[r^q,s^q]} = \frac{q}{\pi_L} \cdot \mathrm{id}_{M^{[r^q,s^q]}} \ .$$

The decomposition (35) combined with (iterates of) $\varphi_{lin}^{[r,s]}$ gives rise to decompositions

$$M^{[r^{\frac{1}{q^n}},s^{\frac{1}{q^n}}]} = \bigoplus_{b \in (o_L/\pi_L^n)} \eta(b,Z)\varphi_L^n(M^{[r,s]}) \tag{44}$$

of Banach spaces and

$$M = \bigoplus_{b \in (o_L/\pi_L^n)} \eta(b,Z)\varphi_L^n(M) \tag{45}$$

of LF-spaces, again given by the formula

$$m = (\frac{\pi_L}{q})^n \sum_a \varphi_M\psi_M\left(\eta(-a,Z)m\right)\eta(a,Z). \tag{46}$$

2.2.2 The Robba ring of a group

In this subsection we assume that Ω is contained in K. Then we have the LT and Fourier isomorphism of Fréchet algebras

$$\mathcal{R}^+ = \mathcal{O}_K(\mathbb{B}) \xrightarrow{\cong} \mathcal{O}_K(\mathfrak{X}) \xrightarrow{\cong} D(o_L, K)$$

sending the variable Z to some element, say X, in $D(o_L, K)$. By formally substituting Z by X in $\mathcal{R}$ we define the ring extensions $\mathcal{R}^?(o_L)$ of $D(o_L, K)$, for ? an interval as before or nothing, obtaining the following commutative diagram of Fréchet algebras

$$\begin{tikzcd}
\mathcal{R}^? \arrow[r, "\cong"] & \mathcal{R}^?(o_L) \\
\mathcal{R}^+ \arrow[u, hook] \arrow[r, "\cong"] & D(o_L, K). \arrow[u, hook]
\end{tikzcd}$$

We shall often take these isomorphisms as identifications.

Recall that $L_n = L(LT[\pi_L^n])$. We set

$$\Gamma_n := G(L_\infty/L_n) = \ker\left(\Gamma_L \xrightarrow{\chi_{LT}} o_L^\times \to (o_L/\pi_L^n)^\times\right).$$

Let $n_0 \geqslant 1$ be minimal among n such that $\log_p : 1+\pi_L^n o_L \to \pi_L^n o_L$ and $\exp : \pi_L^n o_L \to 1+\pi_L^n o_L$ are mutually inverse isomorphisms and consider the map

$$\ell : \Gamma_L \to L, \quad \gamma \mapsto \log_p(\chi_{LT}(\gamma))$$

which induces an isomorphism $\ell : \Gamma_n \to \pi_L^n o_L$ for all $n \geqslant n_0$. We put $\Gamma := \Gamma_{n_0}$. The group isomorphism $\ell_n := \pi_L^{-n}\ell : \Gamma_n \cong o_L$ induces an isomorphism of Fréchet algebras

$$D(o_L, K) \xrightarrow{\cong} D(\Gamma_n, K)$$

sending the variable X to some element, say Y_n, in $D(\Gamma_n, K)$. By formally substituting Z by Y_n in $\mathcal{R}$ we define the ring extensions $\mathcal{R}^?(\Gamma_n)$ of $D(\Gamma_n, K)$, for ? an interval as before or nothing, obtaining the following commutative diagram of Fréchet algebras

$$\begin{tikzcd}
\mathcal{R}^? \arrow[r, "\cong"'] & \mathcal{R}^?(o_L) \arrow[r, "(\ell_{n,*})^{-1}", "\cong"'] & \mathcal{R}^?(\Gamma_n) \\
\mathcal{R}^+ \arrow[u, hook] \arrow[r, "\cong"'] & D(o_L, K) \arrow[u, hook] \arrow[r, "(\ell_{n,*})^{-1}", "\cong"'] & D(\Gamma_n, K). \arrow[u, hook]
\end{tikzcd}$$

for $n \geqslant n_0$. Note that we have an isomorphism

$$\mathcal{R}(\Gamma_n) \cong \mathcal{R}_K(\mathfrak{X}^\times_{\Gamma_n}).$$

Moreover, for $m \geqslant 0$, the commutative diagram

$$\begin{tikzcd}
o_L \arrow[r, "\ell_{n+m}^{-1}"] \arrow[d, "\pi_L^m"'] & \Gamma_{n+m} \arrow[d, hook, "\iota_{n+m,n}"'] \\
o_L \arrow[r, "\ell_n^{-1}"] & \Gamma_n
\end{tikzcd}$$

induces the commutative diagrams

$$\begin{tikzcd} \mathcal{O}_K(\mathfrak{X}) \arrow[r, equal] \arrow[d, "\varphi_L^m"'] & D(o_L,K) \arrow[r, "\ell_{n+m,*}^{-1}"] \arrow[d, "(\pi_L^m)_*"'] & D(\Gamma_{n+m},K) \arrow[d, hook, "(\iota_{n+m,n})_*"'] \\ \mathcal{O}_K(\mathfrak{X}) \arrow[r, equal] & D(o_L,K) \arrow[r, "\ell_{n,*}^{-1}"] & D(\Gamma_n,K) \end{tikzcd} \tag{47}$$

and

$$\begin{tikzcd} \mathcal{R}(\Gamma_{n+m}) \arrow[r, "\ell_{n+m,*}", "\cong"'] \arrow[d, hook, "(\iota_{n+m,n})_*"'] & \mathcal{R} \arrow[d, hook, "\varphi_L^m"] \\ \mathcal{R}(\Gamma_n) \arrow[r, "\ell_{n,*}", "\cong"'] & \mathcal{R} \end{tikzcd} \tag{48}$$

as well as

$$\begin{tikzcd} \mathcal{R}^{I^{q^m}}(\Gamma_{n+m}) \arrow[r, "\ell_{n+m,*}", "\cong"'] \arrow[d, hook, "(\iota_{n+m,n})_*"'] & \mathcal{R}^{I^{q^m}} \arrow[d, hook, "\varphi_L^m"] \\ \mathcal{R}^I(\Gamma_n) \arrow[r, "\ell_{n,*}", "\cong"'] & \mathcal{R}^I. \end{tikzcd} \tag{49}$$

For

- a commutative ring R,
- an abelian group G,
- a subgroup $H \subseteq G$ of finite index with
- a given inclusion $H \subseteq R^\times$

we define the commutative ring

$$R \rtimes_H G := R \otimes_{\mathbb{Z}[H]} \mathbb{Z}[G].$$

It is a special case of a crossed ring product (with trivial action) of R with respect to G/H. As R-modules we have a decomposition

$$R \rtimes_H G \cong \bigoplus_{\bar{g} \in G/H} Rs(\bar{g}), \tag{50}$$

depending on the choice of a set theoretic section $s : G/H \to G$ of the natural projection $G \twoheadrightarrow G/H$ satisfying $s(e) = e$. We have natural inclusions

$$R \hookrightarrow R \rtimes_H G, r \mapsto r \otimes 1, \quad \text{and} \quad G \hookrightarrow (R \rtimes_H G)^\times, g \mapsto 1 \otimes g.$$

The former one defines an R-linear trace map

$$Tr_{R,G,H} : R \rtimes_H G \to R,$$

which corresponds to $\#G/H$ times the projection $pr_{G,H}$ onto the trivial component ($\bar{g} = e$) with respect to the decomposition (50). In particular, $pr_{G,H}$ is independent of the choice of the section s.

It follows from (48), (36) and (49), (35), (39) that the inclusions $\mathcal{R}(\Gamma_{n+m}) \xrightarrow{\iota_{n+m,n,*}} \mathcal{R}(\Gamma_n)$ and $\Gamma_n \hookrightarrow \mathcal{R}(\Gamma_n)^\times$ induce topological isomorphisms

$$\mathcal{R}(\Gamma_{n+m}) \rtimes_{\Gamma_{n+m}} \Gamma_n \cong \mathcal{R}(\Gamma_n) \tag{51}$$

and

$$\mathcal{R}^{I^{q^m}}(\Gamma_{n+m}) \rtimes_{\Gamma_{n+m}} \Gamma_n \cong \mathcal{R}^I(\Gamma_n), \tag{52}$$

when we endow the left hand sides with the product topology and maximum norm, respectively. If we extend our definitions to $1 \leqslant n \leqslant n_0$ by setting

$$\mathcal{R}^I(\Gamma_n) := \mathcal{R}^{I^{q^{n_0-n}}}(\Gamma_{n_0}) \rtimes_{\Gamma_{n_0}} \Gamma_n$$

and

$$\mathcal{R}(\Gamma_n) := \mathcal{R}(\Gamma_{n_0}) \rtimes_{\Gamma_{n_0}} \Gamma_n,$$

then we get commutative diagrams as above for all $n \geqslant 1, m \geqslant 0$. Finally, we define

$$\mathcal{R}(\Gamma_L) := \mathcal{R}(\Gamma_n) \rtimes_{\Gamma_n} \Gamma_L. \tag{53}$$

2.2.3 Tobias Schmidt's results adopted

The distribution algebra of locally L-analytic distributions of o_L - being of L-analytic dimension one - should morally be a ring of certain power series in one variable. This is not literally true, but we recall in this subsection that it holds for a certain completion.

Now we fix a $\mathbb{Z}_p$-basis $h_1 = 1, \ldots, h_d$ of o_L and set $b_i := h_i - 1$ and, for any multiindex $\mathbf{k} = (k_1, \ldots, k_d) \in \mathbb{N}_0^d$, $\mathbf{b}^\alpha := \prod_{i=1}^d b_i^{\alpha_i} \in \mathbb{Z}_p[o_L]$. We write $D_{\mathbb{Q}_p}(G, K)$ for the algebra of K-valued locally $\mathbb{Q}_p$-analytic distributions on a $\mathbb{Q}_p$-Lie group G. Any $\lambda \in D_{\mathbb{Q}_p}(o_L, K)$ has a unique convergent expansion $\lambda = \sum_{\mathbf{k}\in\mathbb{N}_0^d} \alpha_{\mathbf{k}} \mathbf{b}^{\mathbf{k}}$ with $\alpha_{\mathbf{k}} \in K$ such that, for any $0 < r < 1$, the set $\{\alpha_{\mathbf{k}} r^{|\mathbf{k}|}_{\mathbf{k}\in\mathbb{N}_0^d}\}$ is bounded. The completion with respect to the norm

$$||\lambda||_r := \sup_{\mathbf{k}\in\mathbb{N}_0^d} |\alpha_{\mathbf{k}}| r^{|\mathbf{k}|}$$

for $0 < r < 1$ is denoted by

$$D_{\mathbb{Q}_p,r}(o_L, K) = \{\sum_{\mathbf{k}\in\mathbb{N}_0^d} \alpha_{\mathbf{k}} \mathbf{b}^{\mathbf{k}} | \alpha_{\mathbf{k}} \in K \text{ and } |\alpha_{\mathbf{k}}| r^{|\mathbf{k}|} \to 0 \text{ as } |\mathbf{k}| \to \infty\}.$$

Similarly, we write $D_r(o_L, K)$ for the completion of $D(o_L, K)$ with respect to the quotient norm of $||\lambda||_r$. We use analogous notation for groups isomorphic to o_L.

If not otherwise specified, we denote by $V \otimes_K W$ the projective tensor product of locally convex K-vector spaces V, W.

Lemma 2.21. *Let*

$$0 \longrightarrow V \longrightarrow W \longrightarrow X \longrightarrow 0$$

be a strict exact sequence of locally convex topological K-vector spaces with W metrizable and X Hausdorff, then

(i) *the sequence of the associated Hausdorff completed spaces*

$$0 \longrightarrow \hat{V} \longrightarrow \hat{W} \longrightarrow \hat{X} \longrightarrow 0$$

is again strict exact,

(ii) *for a complete valued field extension F of K the associated sequence of completed base extension*

$$0 \longrightarrow F\hat{\otimes}_K V \longrightarrow F\hat{\otimes}_K W \longrightarrow F\hat{\otimes}_K X \longrightarrow 0$$

is again strict exact.

(iii) *If W is a K-Banach space, V a closed subspace with induced norm and $X = W/V$ endowed with the quotient norm, then in (ii) the quotient norm coincides with the tensor product norm on $F\hat{\otimes}_K X$.*

Proof. By [B-TVS, I.17 §2] with W also V, X and all their completions are metrizable. Hence the first statement follows from [B-TG, IX.26 Prop. 5]. For the second statement we first obtain the exact sequence

$$0 \longrightarrow F\otimes_K V \longrightarrow F\otimes_K W \longrightarrow F\otimes_K X \longrightarrow 0$$

of metrizable locally convex spaces ([PGS, Thm. 10.3.13]). The first non-trivial map is strict by Thm. 10.3.8 in (loc. cit.). Regarding the strictness of the second map one easily checks that $F\otimes_K W/F\otimes_K V$ endowed with the quotient topology satisfies the universal property of the projective tensor product $F\otimes_K X$. Now apply (i). The third item is contained in [G, §3, n° 2, Thm. 1], see also [vR, Thm. 4.28]. $\square$

As a consequence we have a strict exact sequence

$$0 \longrightarrow \hat{\mathfrak{a}} \longrightarrow D_{\mathbb{Q}_p,r}(o_L, K) \longrightarrow D_r(o_L, K) \longrightarrow 0, \tag{54}$$

where $\hat{\mathfrak{a}} = \bar{\mathfrak{a}}$ is the closure in $D_{\mathbb{Q}_p,r}(o_L, K)$ of $\mathfrak{a} := \ker\left(D_{\mathbb{Q}_p}(o_L, K) \twoheadrightarrow D(o_L, K)\right)$.

We set $\mathfrak{r}_0 := \begin{cases} p^{-1}, & p \neq 2; \\ 1/4, & p = 2. \end{cases}$ and $\mathfrak{r}_m := \begin{cases} p^{-\frac{1}{p^m}}, & p \neq 2; \\ 4^{-\frac{1}{p^m}}, & p = 2. \end{cases}$ for $m \geqslant 1$.

Lemma 2.22. *There is an isometric identification of the K-Banach spaces*

$$D_{\mathfrak{r}_0}(o_L, K) = \{\lambda = \sum_{k \geqslant 0} \alpha_k(\delta_1 - 1)^k | \alpha_k \in K, |\alpha_k|\mathfrak{r}_0^k \to 0 \text{ for } k \to \infty\}, \tag{55}$$

where on the right hand side λ has norm $\sup_k |a_k|\mathfrak{r}_0^k$.

Proof. For $K = L$ this follows from [Sc, Lem. 5.15] (applied first with $G = o_L$ and $m = 1$ and then using the group isomorphism $o_L \cong po_L$ in order to transport the description from $D_{\mathfrak{r}_0}(po_L, L)$ to $D_{\mathfrak{r}_0}(o_L, L)$ In (loc. cit.) the author assumes for simplicity that $p \neq 2$. But the cited result is based on [Sc1, 5.6/9] which allows $p = 2$ upon using the radius $\mathfrak{r}_0 = 1/4 < 2^{-\frac{1}{2-1}}$). For arbitrary K, we recall from the proof of Lemma 2.2 that $D(o_L, K) = K\hat{\otimes}_L D(o_L, L)$ and similarly $D_{\mathbb{Q}_p}(o_L, K) = K\hat{\otimes}_L D_{\mathbb{Q}_p}(o_L, L)$. Moreover, by [BGR, 9.3.6] we also have $D_{\mathbb{Q}_p,\mathfrak{r}_0}(o_L, K) = K\hat{\otimes}_L D_{\mathbb{Q}_p,\mathfrak{r}_0}(o_L, L)$ with $|| - ||_{\mathfrak{r}_0}\hat{\otimes}_L | - |_K = || - ||_{\mathfrak{r}_0}$. Indeed, since

$\mathfrak{r}_0$ belongs to the value group, $\{\sum_{\mathbf{k}\in\mathbb{N}_0^d} \alpha_{\mathbf{k}}\mathbf{b}^{\mathbf{k}} | \alpha_{\mathbf{k}} \in L \text{ and } |\alpha_{\mathbf{k}}|\mathfrak{r}_0^{|\mathbf{k}|} \to 0 \text{ as } |\mathbf{k}| \to \infty\}$ is visibly isometric to the d-dimensional Tate algebra $Z_n(L)$ over L (similarly for K) and Corollary 8 in §6.1.1 tells that we have an isometric isomorphism $K\hat{\otimes}_L Z_n(L) \cong Z_n(K)$.

We conclude that $K\hat{\otimes}_L D_{\mathbb{Q}_p,\mathfrak{r}_0}(o_L, L)$ is the completion of $K\hat{\otimes}_L D_{\mathbb{Q}_p}(o_L, L)$ with respect to $||-||_{\mathfrak{r}_0}$ and that $K\hat{\otimes}_L D(o_L, L)$ is a quotient of the latter. It follows from Lemma 2.21 (and (54)) that $D_{\mathfrak{r}_0}(o_L, K) = K\hat{\otimes}_L D_{\mathfrak{r}_0}(o_L, L)$ holds isometrically, too. Since also the right hand side of the claim is compatible with this base extension by the same reasoning [BGR, 9.3.6, Cor. 8 in §6.1.1] as above, the Lemma follows.

An alternative geometric argument is the following. The Fourier isomorphism (24) induces, for any $r \in (0,\infty) \cap p^{\mathbb{Q}}$, an isomorphism between $D_r(o_L, K)$ and the ring of rigid analytic functions $\mathcal{O}_K(\mathfrak{X}(r))$ on the affinoid subdomain $\mathfrak{X}(r)$ in the character variety $\mathfrak{X}$ (cf. [BSX, §1.2]). It sends the Dirac distribution δ_1 to the function ev_1 (cf. §2.1.1). Suppose now that $r \in (0, p^{-\frac{1}{p-1}}) \cap p^{\mathbb{Q}}$. Then [BSX, Lem. 1.16] tells us that

$$\begin{aligned} \mathbb{B}_{[0,r]} &\xrightarrow{\cong} \mathfrak{X}(r) \\ y &\longmapsto \text{character } \chi_y(g) := \exp(gy) \end{aligned}$$

is an isomorphism of affinoid varieties. The function ev_1 corresponds to the function exp on $\mathbb{B}_{[0,r]}$. But under our condition on r the function $\exp -1$ is an automorphism of the disk $\mathbb{B}_{[0,r]}$. It follows that $\mathrm{ev}_1 - 1$ is a global coordinate function on $\mathfrak{X}(r)$. This means that

$$\mathcal{O}_K(\mathfrak{X}(r)) = \{f = \sum_{k\geqslant 0} \alpha_k(\mathrm{ev}_1 - 1)^k | \alpha_k \in K, |\alpha_k| r^k \to 0 \text{ for } k \to \infty\} \tag{56}$$

with the spectral norm of f being given by $\sup_k |a_k| r^k$. By the Fourier transform this translates into the assertion of our lemma. □

According to [Sc, Cor. 5.13, Lem. 5.15] one has

$$D(o_L, K) = \operatorname*{proj\,lim}_m D_{\mathfrak{r}_m}(o_L, K)$$

with $D_{\mathfrak{r}_m}(o_L, K) = D_{\mathfrak{r}_0}(p^m o_L, K) \rtimes_{p^m o_L} o_L$. Again this description transports to any Γ_n. Note that $\Gamma_n^{p^m} = \Gamma_{n+me}$.

2.2.4 (φ, Γ)-modules

Any (φ_L, Γ_L)-module M over the usual Robba ring $\mathcal{R}$ is, by definition, in particular an $\mathcal{R}$-module with a semilinear action of the group Γ_L. Our aim in this section is to show that these two structures on M give rise to a module structure on M under the 'group' Robba ring $\mathcal{R}(\Gamma_L)$. We keep the assumptions and conventions of the previous subsection.

Definition 2.23. *A (φ_L, Γ_L)-module M over $\mathcal{R}$ is a φ_L-module M (see Definition 2.19) equipped with a semilinear continuous action of Γ_L which commutes with the endomorphism φ_M. We shall write $\mathcal{M}(\mathcal{R})$ for the category of (φ_L, Γ_L)-modules over $\mathcal{R}$.*

The continuity condition for the Γ_L-action on M, of course, refers to the product topology on $M \cong \mathcal{R}^d$. Technically important is the following fact (cf. [BSX] Prop. 2.4 for a more general case).

Proposition 2.24. *Let M be a (φ_L, Γ_L)-module over $\mathcal{R}$. Then there exists a model (M_0, r_0) as in Proposition 2.20 equipped with a semilinear continuous action of Γ_L such that*

$$\mathcal{R} \otimes_{\mathcal{R}^{[r_0,1)}} M_0 = M$$

respects the the Γ_L-actions (acting diagonally on the left hand side).

In the following we fix a (φ, Γ)-module M over $\mathcal{R}$ and a pair (r_0, M_0) as in Prop. 2.24. For any $r_0 \leqslant r \leqslant s < 1$, the finitely generated free modules $M^{[r,1)}$, and $M^{[r,s]}$ are each equipped with a semilinear continuous Γ_L-action, compatible with the identities

$$M^{[r,1)} = \varprojlim_{s>r} M^{[r,s]} \qquad \text{and} \qquad M = \varinjlim_{r} M^{[r,1)} .$$

Moreover, the Γ_L-actions commutes with the $\psi^?$-operators and the decompositions (45) and (44) are Γ_L-equivariant.

Definition 2.25. *A (φ_L, Γ_L)-module M over $\mathcal{R}$ is called L-analytic if the induced action* $\mathrm{Lie}(\Gamma_L) \to \mathrm{End}(M)$ *of the Lie algebra* $\mathrm{Lie}(\Gamma_L)$ *of Γ_L is L-linear (and not just $\mathbb{Q}_p$-linear). We shall write $\mathcal{M}^{an}(\mathcal{R})$ for the category of L-analytic (φ_L, Γ_L)-modules over $\mathcal{R}$.*

Assume henceforth that M is an *L-analytic* (φ_L, Γ_L)-module over $\mathcal{R}$.

Proposition 2.26. *The Γ_L-action on M extends uniquely to a separately continuous action of the locally L-analytic distribution algebra $D(\Gamma_L, K)$ of Γ_L with coefficients in K. If $M \xrightarrow{f} N$ is a homomorphism of L-analytic (φ_L, Γ_L)-modules, then f is $D(\Gamma_L, K)$-equivariant with regard to this action.*

Proof. First of all we observe that the Dirac distributions generate a dense L-subspace in $D(\Gamma_L, L)$ by [ST1] Lemma 3.1. Since $\Gamma_L \cong o_L^\times$ we have seen in the proof of Lemma 2.2 that $D(\Gamma_L, K) = K \widehat{\otimes}_L D(\Gamma_L, L)$. Hence the Dirac distributions also generate a dense K-linear subspace of $D(\Gamma_L, K)$. Therefore the extended action is unique provided it exists.

Our assertion is easily reduced to the analogous statement concerning the Banach spaces M^I for a closed interval $I = [r, s]$. From [BSX] Prop. 2.16 and Prop. 2.17 we know that the Γ_L-action on M^I is locally $\mathbb{Q}_p$-analytic. But since we assume M to be L-analytic it is actually locally L-analytic (cf. [BSX] Addendum to Prop. 2.25 and the argument at the end of the proof of Prop. 2.17).

For our purpose we show more generally the existence, for any K-Banach space W, of a continuous K-linear map

$$I : \mathcal{C}^{an}(\Gamma_L, W) \to \mathcal{L}_b(D(\Gamma_L, K), W)$$

satisfying $I(f)(\delta_g) = f(g)$. Note that this map, if it exists is unique by our initial observation. Recall (cf. [pLG] §12) that the locally convex vector space $\mathcal{C}^{an}(\Gamma_L, W)$ is the locally convex inductive limit of finite products of Banach spaces of the form $B \widehat{\otimes}_K W$ with a Banach space B, and that its strong dual $D(\Gamma_L, K)$ is the corresponding projective limit of the finite sums of dual Banach spaces B'. We therefore may construct the map I as the inductive limit of finite products of maps of the form

$$\begin{aligned} B \widehat{\otimes}_K W &\longrightarrow \mathcal{L}_b(B', W) \\ x \otimes y &\longmapsto [\ell \mapsto \ell(x) y] . \end{aligned}$$

Since B as a Banach space is barrelled this map is easily seen to be continuous (cf. the argument in the proof of [NFA] Lemma 9.9).

Now suppose that W carries a locally L-analytic Γ_L-action (e.g., $W = M^I$). For $y \in W$ let $\rho_y(g) := gy$ denote the orbit map in $\mathcal{C}^{an}(\Gamma_L, W)$. We then define

$$\begin{aligned} D(\Gamma_L, K) \times W &\longrightarrow W \\ (\mu, y) &\longmapsto I(\rho_y)(\mu) \ . \end{aligned}$$

Due to our initial observation the proof of [ST1] Prop. 3.2, that the above is a separately continuous module structure, remains valid even so K is not assumed to be spherically complete. $\square$

Recall that each M^I bears a natural Γ_L-action. Now, for each $n \geqslant 1$, we will define a different action of Γ_n on $M^{[r,s]}$, which is motivated by Lemma 2.27 below and which is crucial for analysing the structure of $M^{\psi_M=0}$ in the next subsection. To this end consider for each $\gamma \in \Gamma_n$ the operator $H_n(\gamma)$ on $M^{[r,s]}$ defined by

$$H_n(\gamma)(m) = \eta(\frac{\chi_{LT}(\gamma) - 1}{\pi_L^n}, Z)\gamma m.$$

For simplicity we identify $\Gamma_L = o_L^\times$. For $n \geqslant 1$ let $\Gamma_n := 1 + \pi_L^n o_L$. Note that, since Γ_n acts on $\mathcal{R}^+$, we may form the skew group ring $\mathcal{R}^+[\Gamma_n]$, which due to the semi-linear action of Γ_L on M maps into the K-Banach algebra $\mathcal{E}nd_K(M^I)$ of continuous K-linear endomorphisms of M^I, endowed with the operator norm. Hence we obtain the ring homomorphism

$$\begin{aligned} H_n : K[\Gamma_n] \longrightarrow \mathcal{R}^+[\Gamma_n] &\longrightarrow \mathcal{E}nd_K(M^I) \\ \gamma &\longmapsto \eta(\pi_L^{-n}(\gamma - 1), Z)\gamma, \end{aligned}$$

which we now shall extend to $D(\Gamma_n, K)$ below after some preparation.

Lemma 2.27. (i) *We have*

$$\sigma\eta(1, Z)\varphi_L^n(m) = \eta(1, T)\varphi_L^n(H_n(\sigma)(m)),$$

i.e., the isomorphisms

$$\begin{aligned} M &\xrightarrow{\cong} \eta(1, Z)\varphi_L^n(M), \\ M^{[r,s]} &\xrightarrow{\cong} \eta(1, T)\varphi_L^n(M^{[r,s]}) \\ m &\mapsto \eta(1, Z)\varphi_L^n(m) \end{aligned}$$

are Γ_n-equivariant with respect to the natural action on the right hand side and the action via H_n on the left hand side.

(ii) *The map*

$$\begin{aligned} \mathbb{Z}[\Gamma_1] \otimes_{\mathbb{Z}[\Gamma_n], H_n} M^{[r,s]} &\to M^{[r^{1/q^{n-1}}, s^{1/q^{n-1}}]} \\ \gamma \otimes y &\mapsto \eta(\frac{\gamma - 1}{\pi_L}, Z)\varphi_M^{n-1}(\gamma y) \end{aligned}$$

is a bijection of Banach-spaces, where the left hand side is equipped with the maximum norm, and which is Γ_1-equivariant with respect to the H_1-action on the right hand side.

(iii) *If there is an $R^I(\Gamma_n)$-action on M^I induced by H_n, such that the corresponding map $R^I(\Gamma_n) \to \mathcal{E}nd_K(M^I)$ is continuous, then the H_1-action induces an $R^{I^{1/q^{n-1}}}(\Gamma_1)$-action on $M^{I^{1/q^{n-1}}}$ such that we obtain again a continuous map $R^{I^{1/q^{n-1}}}(\Gamma_1) \to \mathcal{E}nd_K(M^{I^{1/q^{n-1}}})$.*

Proof. (i) Setting $b := \frac{\sigma-1}{\pi_L^n}$ we calculate

$$\begin{aligned}\sigma\eta(1,Z)\varphi_L^n(m) &= \sigma\eta(1,Z)\varphi_L^n(\sigma m)\\ &= \eta(1+\pi_L^n b, Z)\varphi_L^n(\sigma m)\\ &= \eta(1,Z)\eta(\pi^n b, Z)\varphi_L^n(\sigma m)\\ &= \eta(1,Z)\varphi_L^n\left(\eta(b,Z)\sigma m\right).\end{aligned}$$

where we used the multiplicativity of η in the first variable in the third and (40) in the last equality.

(ii) follows from (44) using the bijection $1+\pi_L o_L/1+\pi_L^n o_L \xrightarrow{\cong} o_L/\pi_L^{n-1}o_L$, $\gamma \mapsto \frac{\gamma-1}{\pi_L}$.

(iii) Base change induces the $R^{I^{1/q^{n-1}}}(\Gamma_1)$-action on

$$\begin{aligned}R^{I^{1/q^{n-1}}}(\Gamma_1)\otimes_{R^I(\Gamma_n),H_n} M^I &\cong \mathbb{Z}[\Gamma_1]\otimes_{\mathbb{Z}[\Gamma_n]} R^I(\Gamma_n)\otimes_{R^I(\Gamma_n),H_n} M^I\\ &\cong \mathbb{Z}[\Gamma_1]\otimes_{\mathbb{Z}[\Gamma_n],H_n}\otimes M^I\\ &\cong M^{I^{1/q^{n-1}}},\end{aligned} \tag{57}$$

where we used (52) and (ii). The continuity is easily checked by considering 'matrix entries' which are build by composites of the original continuous map by other continuous transformations. Here we use that the identifications (52) and (57) are homeomorphisms when we endow the left hand side with the maximum norm. □

There is a natural ring homomorphism $\mathcal{R}^I \to \mathcal{E}nd_K(M^I)$ by assigning to $f \in \mathcal{R}^I$ the multiplication- with-f-operator, which we denote by the same symbol f.

Remark 2.28. (i) *We have* $\sup_{x\in o_L}|\eta(x,Z)-1|_I < 1$ *and* $|\eta(x,Z)|_I = 1$ *for all* $x \in o_L$.

(ii) $|\eta(px,Z)-1|_I \leqslant \max\{|\eta(x,Z)-1|_I^p, \frac{1}{p}|\eta(x,Z)-1|_I\}(= |\eta(x,Z)-1|_I^p,$ *if* $|\eta(x,Z)-1|_I < p^{-\frac{1}{p-1}}$).

(iii) $|f|_I = ||f||_I$ *for all* $f \in \mathcal{R}^I$.

Proof. It is known ([ST2]) that $\eta(x,Z) = \eta(1,[x](Z))$ belongs to $1+Zo_{\mathbb{C}_p}[[Z]]$, whence we have, for any $x \in o_L$, that $|\eta(x,Z)-1|_I < 1$ from the definition of $|-|_I$, and (i) follows from the fact that the map $o_L \to \mathbb{R}$, $x \mapsto |\eta(x,Z)-1|_I$ is continuous with compact source. Affirmation (ii) is a consequence of the expansion

$$\begin{aligned}\eta(px,Z)-1 &= (\eta(x,Z)-1+1)^p - 1\\ &= (\eta(x,Z)-1)^p + \sum_{k=1}^{p-1}\binom{p}{k}(\eta(x,Z)-1)^k\end{aligned}$$

and $|\binom{p}{k}| = p^{-1}$ for $k = 1,\dots,p-1$. (iii) follows from the submultiplicativity of $|-|_I$ plus the fact that $1 \in \mathcal{R}^I$, which implies the statement on $M \cong (\mathcal{R}^I)^m$. □

The above Remark allows us to fix a natural number $m_0 = m_0(r_0)$ such that for all $m \geqslant m_0$ we have that

(58) $$|\eta(x, Z) - 1|_I < \mathfrak{r}_m \text{ for all } x \in o_L \text{ and } |\eta(x, Z) - 1|_I \leqslant \mathfrak{r}_0 \text{ for all } x \in p^m o_L,$$

(59) $$r_0^{1/q} < \mathfrak{r}_m,$$

for any of the intervals $I = [r_0, r_0]$, $[r_0, r_0^{1/q}]$ and $[r_0^{1/q}, r_0^{1/q}]$. In the following let I always denote one of those intervals.

Lemma 2.29. *Let $\epsilon > 0$ arbitrary. Then there exists $n_1 \gg 0$ such that, for any $n \geqslant n_1$, the operator norm $|| - ||_I$ on M^I satisfies*

(60) $$||\gamma - 1||_I \leqslant \epsilon \text{ for all } \gamma \in \Gamma_n.$$

Proof. We first proof the statement for the module $M = \mathcal{R}$. For the convenience of the reader we adopt the proof of [Ked, Lem. 5.2]. First note that for any fixed $f \in R^I$ by continuity of the action of Γ_L there exists an open normal subgroup H of Γ_L such that

(61) $$|(\gamma - 1)f|_I < \epsilon |f|_I$$

holds for all $\gamma \in H$. So me may assume that the latter inequality holds for Z and Z^{-1} simultaneously. Using the twisted Leibniz rule

$$(\gamma - 1)(gf) = (\gamma - 1)(g)f + \gamma(g)(\gamma - 1)(f)$$

and induction we get (61) for all powers $Z^{\mathbb{Z}}$; Since the latter form an orthogonal basis, the claim follows using that $|\gamma(g)|_I = |g|_I$ for any $\gamma \in H$ by (61) for the induction hypothesis. If $M \cong \bigoplus_{i=1}^d \mathcal{R}e_i$ and $m = \sum f_i e_i$, we may assume that

(62) $$|(\gamma - 1)e_i|_{M^I} < \epsilon |e_i|_{M^I}$$

holds for $1 \leqslant i \leqslant d$, and apply the same Leibniz rule to $f_i e_i$ instead of gf, whence the result follows, noting that $|e_i|_{M^I} = 1$ by the definition of the maximum norm and that $|\gamma(e_i)|_{M^I} = |e_i|_{M^I} = 1$ for any $\gamma \in H$ and $1 \leqslant i \leqslant d$ as a consequence of (62). □

We fix $n_1 = n_1(r_0) \geqslant n_0$ such that the Lemma holds for $\epsilon = \mathfrak{r}_0$. Then, for any $n \geqslant n_1, m \geqslant m_0$, the above H_n extends to a continuous ring homomorphism

$$\tilde{H}_n : D_{\mathbb{Q}_p, \mathfrak{r}_m}(\Gamma_n, K) \to \mathcal{E}nd_K(M^I),$$
$$\sum_{\mathbf{k} \in \mathbb{N}_0^d} \alpha_{\mathbf{k}} \ell_{n,*}^{-1}(\mathbf{b})^{\mathbf{k}} \mapsto \sum_{k \geqslant 0} \alpha_{\mathbf{k}} \prod_{i=1}^d H_n(\ell_{n,*}^{-1}(b_i))^{k_i},$$

Indeed, we have

$$H_n(\ell_{n,*}^{-1}(b_i)) = \eta(\frac{\ell_n^{-1}(h_i) - 1}{\pi_L^n}, Z) - 1 + \eta(\frac{\ell_n^{-1}(h_i) - 1}{\pi_L^n}, Z)(\ell_n^{-1}(h_i) - 1)$$

and since

$$||\eta(\frac{\ell_n^{-1}(h_i) - 1}{\pi_L^n}, Z) - 1 + \eta(\frac{\ell_n^{-1}(h_i) - 1}{\pi_L^n}, Z)(\ell_n^{-1}(h_i) - 1)||_I \leqslant$$
$$\max\{||\eta(\frac{\ell_n^{-1}(h_i) - 1}{\pi_L^n}, Z) - 1||_I, ||\eta(\frac{\ell_n^{-1}(h_i) - 1}{\pi_L^n}, Z)(\ell_n^{-1}(h_i) - 1)||_I\} \leqslant \mathfrak{r}_m$$

by (60),(58) and Remark 2.28 (i), the above defining sum converges with respect to the operator norm. Since M is assumed to be L-analytic, $\tilde{H}_n$ factorises over the desired ring homomorphism

$$H_n : \Big(D(\Gamma_n, K) \subseteq \Big) D_{\mathfrak{r}_m}(\Gamma_n, K) \to \mathcal{E}nd_K(M^I)$$

by (54).

Theorem 2.30.

(i) *Let I be any of the intervals*

$$[r_0, r_0]^{1/q^n} \text{ or } [r_0, r_0^{1/q}]^{1/q^n} \text{ for } n \geqslant n_1.$$

Then the Γ_1-action on M^I via H_1 extends uniquely to a continuous $\mathcal{R}^I(\Gamma_1)$-module structure. Moreover, $M^I$is a finitely generated free $\mathcal{R}^I(\Gamma_1)$-module; any $\mathcal{R}^{[r_0,1)}$-basis of M_0 is also a $\mathcal{R}^I(\Gamma_1)$-basis of M^I. If $M \xrightarrow{f} N$ is a homomorphism of L-analytic (φ_L, Γ_L)-modules, then $f^I : M^I \to N^I$ is $\mathcal{R}^I(\Gamma_1)$-equivariant with regard to this action.

(ii) *The Γ_1-action on M via H_1 extends uniquely to a separately continuous $\mathcal{R}(\Gamma_1)$-module structure. Moreover, M is a finitely generated free $\mathcal{R}(\Gamma_1)$-module; any $\mathcal{R}^{[r_0,1)}$-basis of M_0 is also a $\mathcal{R}(\Gamma_1)$-basis of M. If $M \xrightarrow{f} N$ is a homomorphism of L-analytic (φ_L, Γ_L)-modules, then f is $\mathcal{R}(\Gamma_1)$-equivariant with regard to this action.*

Proof. Part (i) will be proved in the next subsection, see Lemma 2.31. Since the intervals in (i) cover $[r_0^{1/q^{n_0}}, 1)$ the $\mathcal{R}^I(\Gamma_1)$-module structures glue together to $\mathcal{R}^{[r_0^{1/q^n},1)}(\Gamma_1)$-module structures on $M^{[r_0^{1/q^n},1)}$ for $n \geqslant n_1$: Indeed, the operations are uniquely determined by the H_1-rule everywhere and thus coincide on the intervals $[r_0, r_0]^{1/q^n}$ for all $n \geqslant n_1$. By the same reason these structures are compatible when varying the left boundary, whence taking direct limits gives the desired results (ii). □

2.2.5 The proof of Theorem 2.30

Let I be either $[r_0, r_0]$ or $[r_0, r_0^{1/q}]$ and let $e_1, \ldots, e_d$ be a basis of our chosen model M_0. By Lemma 2.27 (iii) for the proof of Theorem 2.30(i) it suffices to show that for all $n \geqslant n_1$ there is a $\mathcal{R}^I(\Gamma_n)$-module structure on M^I induced by H_n such that the e_i form a basis.

To achieve the module structure we will actually first show the existence of an a continuous ring homomorphism

$$\mathcal{R}^{I^{q^{m_0 e}}}(\Gamma_{n+m_0 e}) \to \mathcal{E}nd_K(M^I),$$

which extends then by the universal property of cross products to a continuous ring homomorphism

$$\mathcal{R}^I(\Gamma_n) \cong \mathcal{R}^{I^{q^{m_0 e}}}(\Gamma_{n+m_0 e}) \rtimes_{\Gamma_{n+m_0 e}} \Gamma_n \to \mathcal{E}nd_K(M^I),$$

as claimed. Here we use again the topological ring isomorphism (52). Therefore we have reduced the proof of the Theorem to the following

Lemma 2.31.

(i) *The map* $\Theta_n : D(\Gamma_{n+m_0e}, K) \subseteq D(\Gamma_n, K) \xrightarrow{H_n} \mathcal{E}nd_K(M^I)$ *extends to a continuous ring homomorphism*

$$\mathcal{R}^{I^{q^{m_0e}}}(\Gamma_{n+m_0e}) \to \mathcal{E}nd_K(M^I).$$

If $M \xrightarrow{f} N$ *is a homomorphism of* L*-analytic* (φ_L, Γ_L)*-modules, then* $f^I : M^I \to N^I$ *is* $\mathcal{R}^{I^{q^{m_0e}}}(\Gamma_{n+m_0e})$*-equivariant with regard to this action.*

(ii) M^I *is a free* $\mathcal{R}^I(\Gamma_n)$*-module of rank* $\mathrm{rk}_{\mathcal{R}} M$.

The diagram 47 extends to the commutative diagram

$$\begin{tikzcd}
\mathcal{O}_K(\mathfrak{X}) \arrow[r, hook] \arrow[d, "\varphi_L^{m_0e}"'] & D_{\mathfrak{r}_0}(o_L, K) \arrow[r, "\ell_{n+m_0e,*}^{-1}"] \arrow[d, "(\pi_L^{m_0e})_*"'] & D_{\mathfrak{r}_0}(\Gamma_{n+m_0e}, K) \arrow[d, hook] & \\
\mathcal{O}_K(\mathfrak{X}) \arrow[r, hook] & D_{\mathfrak{r}_{m_0}}(o_L, K) \arrow[r, "\ell_{n,*}^{-1}"] & D_{\mathfrak{r}_{m_0}}(\Gamma_n, K) \arrow[r, "H_n"] & \mathcal{E}nd_K(M^I).
\end{tikzcd} \tag{63}$$

Now we set $c_n := \ell_{n+m_0e}^{-1}(1) - 1 \in D(\Gamma_{n+m_0e}, K)$ and

$$\begin{aligned}
\mu_n :=& (\ell_{n+m_0e,*})^{-1}(Z) = (\ell_{n,*})^{-1}(\varphi_L^{m_0e}(Z)) \quad \text{in} \quad D(\Gamma_{n+m_0e}).\\
\lambda_n :=& H_n(\mu_n) \quad \text{in} \quad \mathcal{E}nd_K(M^I)
\end{aligned}$$

μ_n will have the meaning of a variable in $\mathcal{R}^{I^{q^{m_0e}}}(\Gamma_{n+m_0e})$, it will act via H_n on M^I. On the other hand, defining u_n by $\beta_n = \frac{\ell_{n+m_0e}^{-1}(1)-1}{\pi_L^n} =: \pi_L^{m_0e} u_n \in \pi_L^{m_0e} o_L$ we set

$$Z_n := [\pi_L^{m_0e} u_n]^*(Z) = [u_n]^*(\varphi_L^{m_0e}(Z)) \in \mathcal{O}_K(\mathfrak{X}),$$

which will have the meaning of a variable of $\varphi_L^{m_0e}(\mathcal{R}_K^{I^{q^{m_0e}}})$. It acts as element of R^I on M^I. The strategy now consists of comparing the actions of μ_n and Z_n.

Note that u induces an isometry $z \mapsto [u_n](z)$ on the open unit ball, in particular it induces isometries $[u_n]^*$ on $\mathcal{O}_K(\mathfrak{X})$ with respect to all norms which are induced from $|-|_I$ for some interval I as above. E.g. we have

$$|Z_n|_I = |\varphi_L^{m_0e}(Z)|_I \tag{64}$$

for all intervals I. The element Z_n has an inverse $Z_n^{-1} = [\beta_n]^*(Z^{-1})$ in $\mathcal{R}_K \subseteq \mathcal{R}_K^I$.

Lemma 2.32. (i) *There exist* $n_2 \geqslant n_1$ *such that for all* $n \geqslant n_2$ *we have that*

$$||Z_n - \lambda_n||_I < |Z_n^{-1}|_I^{-1} \leqslant |\varphi_L^{m_0e}(Z)|_I$$

for any of the intervals $I = [r_0, r_0]$, $[\frac{r_0}{q}, r_0]$ *and* $[\frac{r_0}{q}, \frac{r_0}{q}]$.

(ii) λ_n *has a (left and right) inverse* λ_n^{-1} *in* $\mathcal{E}nd_K(M^I)$ *and it holds* $||Z_n^{-1} - \lambda_n^{-1}||_I < |Z_n^{-1}|_I = |\varphi_L^{m_0e}(Z)^{-1}|_I$.

Proof. Since $Z \in \mathcal{O}_K(\mathfrak{X}) \subseteq D_{\mathfrak{r}_0}(o_L, K)$ we can express it by Lemma 2.22 as

$$Z = \sum_{k \geqslant 0} \alpha_k (\delta_1 - 1)^k$$

with finite norm $\epsilon := ||Z||_{\mathfrak{r}_0} = \sup_{k \geqslant 0} |\alpha_k| \mathfrak{r}_0^k > 0$ with respect to the variable $\eta(1, Z) - 1$. Since $D_{\mathfrak{r}_{m_0 e}}(o_L, K) \subseteq \mathcal{R}^I$ by assumption (59), using the composite $D_{\mathfrak{r}_0}(o_L, K) \xrightarrow{(\pi_L^{m_0 e})_*} D_{\mathfrak{r}_{m_0 e}}(o_L, K) \subseteq \mathcal{R}^I \xrightarrow{[u_n]^*} \mathcal{R}^I$ we have

$$\begin{aligned} Z_n &= \sum_{k \geqslant 0} \alpha_k (\eta(\beta_n, Z) - 1)^k \text{ in } \mathcal{R}^I, \\ \mu_n &= \sum_{k \geqslant 0} \alpha_k c_n^k \text{ in } D_{\mathfrak{r}_0}(\Gamma_{n+m_0 e}, K) \subseteq D_{\mathfrak{r}_{m_0}}(\Gamma_n, K) \end{aligned}$$

and

$$\begin{aligned} ||Z_n - \lambda_n||_I &= || \sum_{k \geqslant 0} \alpha_k \left\{ (\eta(\beta_n, Z) - 1)^k - (\eta(\beta_n, Z) - 1 + \eta(\beta_n, Z) c_n)^k \right\} ||_I \\ &\leqslant \sup_{k \geqslant 1} |\alpha_k| || (\eta(\beta_n, Z) - 1)^k - (\eta(\beta_n, Z) - 1 + \eta(\beta_n, Z) c_n)^k ||_I \\ &\leqslant \frac{||c_n||_I}{\mathfrak{r}_0} \sup_{k \geqslant 1} |\alpha_k| || \mathfrak{r}_0^k \\ &\leqslant \frac{||c_n||_I}{\mathfrak{r}_0} \sup_{k \geqslant 0} |\alpha_k| || \mathfrak{r}_0^k = \frac{||c_n||_I}{\mathfrak{r}_0} \epsilon, \end{aligned}$$

because for $k \geqslant 1$ one easily checks

$$\begin{aligned} ||(\eta(\beta_n, Z) - 1)^k - (\eta(\beta_n, Z) - 1 + \eta(\beta_n, Z) c_n)^k ||_I &\leqslant \max_{1 \leqslant l \leqslant k} ||\eta(\beta_n, Z) - 1||_I^{k-l} ||c_n||_I^l \\ &\leqslant ||c_n||_I \mathfrak{r}_0^{k-1} \end{aligned}$$

using that the operator norm is submultiplicative as well as (58) and remark 2.28(i).

There is some $0 < r < \mathfrak{r}_0$ such that $\frac{r}{\mathfrak{r}_0} \epsilon < |\varphi_L^{m_0 e}(Z)^{-1}|_I^{-1} = |Z_n^{-1}|_I^{-1}$ for all I (note that the latter norm is independent of n). Now choose $n_2 \geqslant n_1$ sufficiently huge such that $||c_n||_I < r$ for all I and $n \geqslant n_2$, which is possible Lemma 2.29. Then,

$$||Z_n - \lambda_n||_I \leqslant \frac{||c_n||_I}{\mathfrak{r}_0} \epsilon \leqslant \frac{r}{\mathfrak{r}_0} \epsilon < |Z_n^{-1}|_I^{-1}$$

for all I and $n \geqslant n_2$. Using (64) this proves (i) because $1 = |Z_n^{-1} Z_n|_I \leqslant |Z_n^{-1}|_I |Z_n|_I$.

By Remark 2.28 (iii) and from (i) we have $||1 - Z_n^{-1} \lambda_n||_I = ||Z_n^{-1}(Z_n - \lambda_n)||_I < 1$, whence $\sum_{k \geqslant 0} (1 - Z_n^{-1} \lambda_n)^k$ converges in $\mathcal{E}nd_K(M^I)$ and $\lambda_n^{-1} := \left(\sum_{k \geqslant 0} (1 - Z_n^{-1} \lambda_n)^k \right) Z_n^{-1}$ is the (left and right) inverse of $\lambda_n = Z_n(1 - (1 - Z_n^{-1} \lambda_n))$. Furthermore,

$$\begin{aligned} ||\lambda_n^{-1} - Z_n^{-1}||_I &= || \left(\sum_{k \geqslant 1} (1 - Z_n^{-1} \lambda_n)^k \right) Z_n^{-1} ||_I \\ &\leqslant \sup_{k \geqslant 1} ||1 - Z_n^{-1} \lambda_n||_I^k |Z_n^{-1}|_I < |Z_n^{-1}|_I. \end{aligned}$$

This proves (ii). $\square$

Proof of Lemma 2.31. Inductively, we obtain - by expressing $(\lambda_n^\pm)^k-(Z_n^\pm)^k$ as $\sum_{l=1}^k \binom{k}{l} (\lambda_n^\pm - Z_n^\pm)^l (Z_n^\pm)^{k-l}$ - from (i) and (ii) of Lemma 2.32 that

$$||\lambda_n^k - Z_n^k||_I < \begin{cases} |Z_n|_I^k, & \text{for } k \geqslant 0; \\ \leqslant |Z_n^{-1}|_I^{-k} \leqslant |Z_n|_I^k \leqslant |Z_n^k|_I, & \text{for } k < 0. \end{cases} \tag{65}$$

for all $k \in \mathbb{Z}$. It follows that, if $\sum_{k\in\mathbb{Z}} a_k \varphi_L^{m_0 e}(Z)^k \in \mathcal{R}_K^I(\mathfrak{X})$ with $a_i \in K$, then $\sum_{k\in\mathbb{Z}} a_k \lambda_n^k$ converges in $\mathcal{E}nd_K(M^I)$, because

$$||a_k\lambda_n^k||_I \leqslant \max\{||a_k(\lambda_n^k - Z_n^k)||_I, ||a_k Z_n^k||_I\} \leqslant \begin{cases} |a_k||\varphi_L^{m_0 e}(Z)|_I^k, & \text{for } k \geqslant 0; \\ |a_k||\varphi_L^{m_0 e}(Z)^{-1}|_I^{-k}, & \text{for } k < 0. \end{cases}$$

goes to zero for k going to $\pm\infty$. In other words, we have extended the continuous ring homomorphism Θ_n to a continuous ring homomorphism

$$\mathcal{R}_K^{q^{m_0 e}I} \to \mathcal{E}nd_K(M^I), \quad Z \mapsto \lambda_n.$$

As by definition $\ell_{n+m_0 e,*}$ extends to an continuous ring isomorphism $\mathcal{R}_K^{q^{m_0 e}I}(\Gamma_{n+m_0 e}) \cong \mathcal{R}_K^{q^{m_0 e}I}$ we have constructed a continuous ring homomorphism

$$\mathcal{R}_K^{q^{m_0 e}I}(\Gamma_{n+m_0 e}) \to \mathcal{E}nd_K(M^I)$$

as claimed. Concerning functoriality observe that the maps f and f^I are automatically continuous by [BSX, Rem. 2.20] (with respect to the canonical topologies). Without loss of generality we may assume that the estimates of Lemma 2.32 hold for M and N simultaneously. By the invariance under the distribution algebra and $\mathcal{R}$-linearity of f, the map f^I is compatible with respect to the operators $\lambda_n^\pm$ of M^I and N^I. By continuity this extends to arbitrary elements of $\mathcal{R}^{I^{q^{m_0 e}}}(\Gamma_{n+m_0 e})$. (ii) follows similarly as in [KPX]: Recall that (e_k) denotes a $\mathcal{R}_K^I$-basis of M^I and consider the maps

$$\Phi : \bigoplus_{k=1}^m \mathcal{R}_K^I = \bigoplus_{k=1}^m \varphi_L^{m_0 e}(\mathcal{R}_K^{q^{m_0 e}I}) \otimes_{\mathbb{Z}_p[p^m o_L]} \mathbb{Z}_p[o_L] \cong M^I, (f_k) \mapsto \sum_{k=1}^m f_k e_k,$$

$$\Phi' : \bigoplus_{k=1}^m \mathcal{R}_K^I(\Gamma_n) = \bigoplus_{k=1}^m \mathcal{R}_K^{q^{m_0 e}I}(\Gamma_{n+m_0 e}) \otimes_{\mathbb{Z}_p[\Gamma_{n+m_0 e}]} \mathbb{Z}_p[\Gamma_n] \to M^I, (f_k) \mapsto \sum_{k=1}^m f_k(e_k),$$

and

$$\Upsilon : \bigoplus_{k=1}^m \mathcal{R}_K^I \cong \bigoplus_{k=1}^m \mathcal{R}_K^I(\Gamma_n),$$

which in each component is induced by

$$\varphi_L^{m_0 e}(\mathcal{R}_K^{q^{m_0 e}I}) \cong \mathcal{R}_K^{q^{m_0 e}I}(\Gamma_{n+m_0 e}), \quad Z_n \mapsto \lambda_n$$

on the first tensor factor and

$$\ell_{n,*}^{-1} : \mathbb{Z}_p[o_L] \cong \mathbb{Z}_p[\Gamma_n]$$

on the second tensor factor. Then we have from (65) that

$$|\Phi' \circ \Upsilon \circ \Phi^{-1}(m) - m|_I < |m|_I,$$

i.e.,

$$||\Phi' \circ \Upsilon \circ \Phi^{-1} - \mathrm{id}\,||_I < 1,$$

whence with Φ and Υ also Φ' is an isomorphism because $\Phi' \circ \Upsilon \circ \Phi^{-1}$ is invertible by the usual argument using the geometric series. $\square$

2.2.6 The structure of $M^{\psi_M=0}$

Now let M be a L-analytic (φ_L,Γ)-module over $\mathcal{R}$. We want to show that $M^{\psi=0}$ carries a natural $\mathcal{R}(\Gamma_L)$-action extending the action of $D(\Gamma_L,K)$.

From (45) and using formula (42) and (41) we have

$$M^{\psi_L=0} = \bigoplus_{b\in(o_L/\pi_L)^\times} \eta(b,Z)\varphi_M(M) = \mathbb{Z}[\Gamma_L]\otimes_{\mathbb{Z}[\Gamma_1]}(\eta(1,Z)\varphi_M(M))\,. \tag{66}$$

Theorem 2.33. *The Γ_L action on M extends to a unique separately continuous $\mathcal{R}(\Gamma_L)$-action on $M^{\psi_L=0}$ (with respect to the LF-topology on $\mathcal{R}(\Gamma_L)$ and the subspace topology on $M^{\psi_L=0}$); moreover the latter is a free $\mathcal{R}(\Gamma_L)$-module of rank $\mathrm{rk}_{\mathcal{R}}M$. If $e_1,\ldots,e_r$ is a basis of M over $\mathcal{R}$, a $\mathcal{R}(\Gamma_L)$-basis of $M^{\psi_L=0}$ is given by $\eta(1,Z)\varphi_M(e_1),\ldots,\eta(1,Z)\varphi_M(e_r)$. If $M\xrightarrow{f}N$ is a homomorphism of L-analytic (φ_L,Γ_L)-modules, then $M\xrightarrow{f^{\psi_L=0}}N$ is $\mathcal{R}(\Gamma_L)$-equivariant with regard to this action.*

Proof. By Lemma 2.27 (i) we transfer the $\mathcal{R}(\Gamma_1)$-action on M from Theorem 2.30(ii) to $\eta(1,Z)\varphi_M(M)$. Note that the resulting action is separately continuous for the subspace topology of $\eta(1,Z)\varphi_M(M)$, because the map $\varphi_L: M\to M$ is a homeomorphism onto its image. The latter is a consequence of the existence of the continuous operator ψ_L and the relation $\psi_L\circ\varphi_L=\frac{q}{\pi_L}\mathrm{id}_M$. Finally, because of (53) and (66) the $\mathcal{R}(\Gamma_1)$-action extends to the asserted $\mathcal{R}(\Gamma_L)$-action. $\square$

2.2.7 The Mellin transform and twists

We define the isomorphism

$$\mathfrak{M}:\mathcal{R}(\Gamma_L)\cong\mathcal{R}^{\psi_L=0},\quad \lambda\mapsto\lambda(\eta(1,Z))$$

induced from Theorem 2.33. Moreover, let $\sigma_{-1}\in\Gamma_L$ be the element with $\chi_{LT}(\sigma_{-1})=-1$.

Recall the twist operators Tw_χ from section 2.1.3.

Lemma 2.34. *The diagram*

$$\begin{tikzcd} \mathcal{R}_K(\Gamma_L) \arrow[r, "\mathfrak{M}"] \arrow[d, "Tw_{\chi_{LT}}"'] & \mathcal{R}_K^{\psi_L=0} \arrow[d, "\frac{1}{\Omega}\partial_{\mathrm{inv}}", "\cong"'] \\ \mathcal{R}_K(\Gamma_L) \arrow[r, "\mathfrak{M}"] & \mathcal{R}_K^{\psi_L=0} \end{tikzcd} \tag{67}$$

is commutative; in particular, the right hand vertical map is an isomorphism.

Proof. The corresponding result for $\mathcal{R}_K(\Gamma_L)$ replaced by $D(\Gamma_L,K)$ is implicitly given in sections 2.1.3 and 2.1.4, see also [Co2, §1.2.4] for the relation $\partial_{\mathrm{inv}}\circ\gamma=\chi_{LT}(\gamma)\gamma\circ\partial_{\mathrm{inv}}$ as operators on $\mathcal{R}_K$. It follows by K-linearity and continuity that the relation of operators $\partial_{\mathrm{inv}}\circ\lambda = Tw_{\chi_{LT}}(\lambda)\circ\partial_{\mathrm{inv}}$ for all $\lambda\in D(\Gamma_L,K)$. By continuity of the action of $\mathcal{R}_K(\Gamma_L)=\mathcal{R}_K(\Gamma_n)\rtimes_{\Gamma_n}\Gamma_L$ on $\mathcal{R}_K^{\psi_L=0}$ it suffices to check the compatibility for the element

Y_n^{-1}, where $Y_n \in D(\Gamma_n, K)$ has been defined at the beginning of section 2.2.2. Using that $Tw_{\chi_{LT}}$ is multiplicative the claim follows from the relation

$$\begin{aligned} Tw_{\chi_{LT}}(Y_n^{-1})\eta(1,Z) &= Tw_{\chi_{LT}}(Y_n)^{-1}\frac{1}{\Omega}\partial_{\mathrm{inv}}\left(Y_nY_n^{-1}\eta(1,Z)\right) \\ &= Tw_{\chi_{LT}}(Y_n)^{-1}Tw_{\chi_{LT}}(Y_n)\frac{1}{\Omega}\partial_{\mathrm{inv}}\left(Y_n^{-1}\eta(1,Z)\right) \\ &= \frac{1}{\Omega}\partial_{\mathrm{inv}}\left(Y_n^{-1}\eta(1,Z)\right). \end{aligned}$$

$\square$

Moreover, we define $\iota_* : \mathcal{R}(\Gamma_L) \to \mathcal{R}(\Gamma_L)$ to be the map which is induced by sending $\gamma \in \Gamma_L$ to its inverse γ^{-1}, i.e., more precisely it is defined to be the usual involution on $\mathbb{Z}[\Gamma_L]$ and the inversion on $\mathcal{R}(\Gamma_{n_0})$ satisfying

(68)
$$\begin{tikzcd} \mathcal{R}(\Gamma_{n_0}) \arrow[r, "\ell_{n_0,*}", "\cong"'] \arrow[d, hook, "\iota"'] & \mathcal{R} \arrow[d, hook, "\iota=\sigma_{-1}"] \\ \mathcal{R}(\Gamma_{n_0}) \arrow[r, "\ell_{n_0,*}", "\cong"'] & \mathcal{R}. \end{tikzcd}$$

where the involution on $\mathcal{R}$ sends $\eta(x,Z)$ to $\eta(-x,Z)$.

Lemma 2.35. *Let n_1 be as in Lemma 2.29. Then, for $n \geqslant n_1$, the map $\mathfrak{M}$ induces isomorphisms*

$$\mathcal{R}(\Gamma_n) \cong \varphi_L^n(\mathcal{R})\eta(1,Z) \ \ (\subseteq R^{\psi_L=0})$$

of $\mathcal{R}(\Gamma_n)$-modules and

$$D(\Gamma_n, K) \cong \varphi_L^n(\mathcal{R}^+)\eta(1,Z) \ \ (\subseteq (R^+)^{\psi_L=0})$$

of $D(\Gamma_n, K)$-modules.

Proof. The first isomorphism follows from Lemma 2.31 (ii) by taking limits in combination with Lemma 2.27 (i). Since all involved constructions respect the +-structures (in the sense that $D(\Gamma_n, K)$ corresponds to the +-version of $\mathcal{R}(\Gamma_n)$) it induces the second isomorphism. $\square$

In the next Remark we identify Γ with an open subgroup of $o_L^\times$ for simplicity.

Remark 2.36. *If we assume that Ω belongs to K as in the next subsection, we can - via the LT-isomorphism and $\frac{\log}{\pi_L^n}$ - identify $\mathcal{R}_L(\mathfrak{X}_{\Gamma_n}^\times)$ with the Robba ring $\mathcal{R}$ as in section 2.2. The image $\widetilde{\Xi_b}$ of Ξ_b is then*

$$\widetilde{\Xi_b} = \frac{(\frac{\Omega}{\pi_L^n}\log_{LT}(Z))^{d-1}}{\prod_j(\exp(\log(b_j)\frac{\Omega}{\pi_L^n}\log_{LT}(Z)) - 1)}$$

and it follows from the proof of Proposition 2.12 that

$$Z\widetilde{\Xi_b}$$

belongs to $\mathcal{R}_K^+$ with constant term $(\frac{\Omega}{\pi_L^n}\prod_j \log(b_j))^{-1}$, whence

$$\widetilde{\Xi_b} \equiv \frac{\pi_L^n}{\Omega\prod_j \log(b_j)Z} \mod \mathcal{R}_K^+.$$

Proof. One checks that the isomorphism

$$D(\Gamma_n, K) \xrightarrow{(\frac{\log}{\pi_L^n})_*} D(o_L, K) \cong \mathcal{O}_K(\mathfrak{X}) \cong \mathcal{O}_K(\mathbb{B})$$

sends a distribution μ to the map $g_\mu(z) = \mu(\exp(\Omega\frac{\log(-)}{\pi_L^n}\log_{LT}(z)))$. In particular, a Dirac distribution δ_a is sent to $\exp(\Omega\frac{\log(a)}{\pi_L^n}\log_{LT}(z))$. Recall that the action of ∇ as distribution sends a locally L-analytic function f to $-\left(\frac{d}{dt}f(\exp(-t))\right)_{|t=0}$, whence ∇ is sent to

$$\nabla\left(\exp(\Omega\frac{\log(-)}{\pi_L^n}\log_{LT}(z))\right) = -\left(\frac{d}{dt}\exp(\Omega\frac{\log(\exp(-t))}{\pi_L^n}\log_{LT}(z))\right)_{|t=0} = \frac{\Omega}{\pi_L^n}\log_{LT}(z).$$

$\square$

Now we are going to discuss a variant of Remark 2.36 setting $\Theta_b := \nabla\Xi_b$ and $t_{LT} := \log_{LT}(Z)$.

Remark 2.37. *If we assume $\Gamma = \Gamma_n$ and that Ω belongs to K as in the next subsection, we obtain for sufficiently large n*

$$\mathfrak{M}(\Theta_b) = \varphi_L^n(\xi_b)\eta(1, Z)$$

with

$$\xi_b \equiv \frac{t_{LT}}{Z\prod_j \log(b_j)} \mod t_{LT}\mathcal{R}_K^+.$$

Proof. Consider the the element

$$F(X) = \frac{X}{\exp(X) - 1} = 1 + XQ(X)$$

with $Q(X) \in \mathbb{Q}_p[[X]]$ and let $r > 0$ be such that $Q(X)$ converges on $|X| \leqslant r$. We shall proof the claim within the Banach-algebra $\mathcal{R}_K^I$ for $I = [0, r]$ (which contains $\mathcal{R}_K^+$ and using that the actions on both rings are compatible). We assume for the operator norm that $||\delta_{b_i} - 1||_I < \min(p^{-\frac{1}{p-1}}, r)$ for all i (otherwise we enlarge n according to Lemma 2.29). From [BSX, Cor. 2.3.2, proof of Lem. 2.3.1] it follows that $\nabla = \frac{\log(\delta_{b_i})}{\log(b_i)}$ as operators in the Banach algebra A of continuous liniear endomorphisms of $\mathcal{R}_K^I$ and

$$\exp(\log(b_i)\nabla) = \exp(\log(\delta_{b_i})) = \delta_{b_i} \tag{69}$$

in A. Moreover,

$$|| \log(\delta_{b_i})||_I < \min(p^{-\frac{1}{p-1}}, r) \tag{70}$$

for all i, whence $||\nabla||_I < \min(p^{-\frac{1}{p-1}}, r)|\log(b_i)|$. Then, as operators in A we have

$$\log(b_i)^{-1} + \nabla Q(\log(b_i)\nabla) = \log(b_i)^{-1}F(\log(b_i)\nabla) = \frac{\nabla}{\exp(\log(b_i)\nabla) - 1} = \frac{\nabla}{\delta_{b_i} - 1}. \tag{71}$$

Hence

$$\Theta_b = \frac{\nabla^d}{\prod(\exp(\log(b_j)\nabla) - 1)} = \left(\prod_j \log(b_j)\right)^{-1} + \nabla g(\log(b_j)\nabla)$$

for some power series $g \in \mathcal{R}_K^I$. It follows that

$$\mathfrak{M}(\Theta_b) = \left(\left(\prod_j \log(b_j)\right)^{-1} + \Omega t_{LT} f(Z)\right)\eta(1,Z). \tag{72}$$

for some $f(Z) \in \mathcal{R}_K^I$. Indeed, concerning the derived action we have

$$\nabla\left(\eta(1,Z)\right) = \left(\frac{d}{dt}\exp(\Omega\exp(t)t_{LT})\right)_{|t=0} = \Omega t_{LT}\eta(1,Z)$$

(cf. also [BSX, end of §2.3] for the fact that

$$\nabla \text{ acts as } t_{LT}\partial_{\mathrm{inv}} \text{ on } \mathcal{R}_K^+ \tag{73}$$

) and

$$\nabla(\Omega t_{LT}) = \Omega t_{LT}.$$

Furthermore, we obtain inductively that

$$\nabla^i\eta(1,Z) = \left(\prod_{k=0}^{i-1}(\Omega t_{LT} + k)\right)\eta(1,Z)$$

for all $i \geqslant 0$. The convergence of $f(Z)$ can be deduced using the operator norm (70).

On the other hand, according to [BF, Lem. 2.4.2] we have

$$\Theta_b\eta(1,Z) = \frac{t_{LT}}{\varphi_L^n(Z)}g(Z)$$

for some $g(Z) \in \mathcal{R}_K^+$. Since the element $\Theta_b\eta(1,Z)$ lies in the $(\mathcal{R}^+)^{\psi_L=0}$, we conclude from

$$0 = \psi_L\left(\frac{\pi_L^{-1}\varphi_L(t_{LT})}{\varphi_L^n(Z)}g(Z)\right) = \frac{\pi_L^{-1}t_{LT}}{\varphi_L^{n-1}(Z)}\psi_L(g(Z))$$

that $g(Z)$ belongs to $(\mathcal{R}_K^+)^{\psi_L=0}$, whence it is of the form $\sum_{a\in(o_L/\pi_L)^\times}\varphi_L(g_a(Z))\eta(a,Z)$ for some $g_a \in \mathcal{R}_K^+$ by the analogue of (66) for $\mathcal{R}_K^+$. From Lemma 2.35 we derive that, for some $a(Z) \in \mathcal{R}_K^+$, we have

$$\Theta_b\eta(1,Z) = \varphi_L^n(a(Z))\eta(1,Z).$$

Since the decomposition in (66) is direct, we conclude that $g(Z) = \varphi_L(g_1(Z))\eta(1,Z)$ and $\frac{t_{LT}}{\varphi_L^n(Z)}\varphi_L(g_1(Z)) = \varphi_L^n(a(Z))$, whence t_{LT} divides $\varphi_L^n(a(Z)Z)$. Since φ_L^n sends the zeroes of t_{LT}, i.e., the points in $LT(\pi_L) = \bigcup_k LT[\pi_L^k]$, surjectively onto itself, we conclude by Lema 2.9 that t_{LT} divides also $a(Z)Z$ in $\mathcal{R}_K^+$ and that there exists $c(Z) \in \mathcal{R}_K^+$ such that

$$\mathfrak{M}(\Theta_b) = \varphi_L^n\left(\frac{t_{LT}}{Z}c(Z)\right)\eta(1,Z). \tag{74}$$

Comparing (74) with the first description (72) gives the claim as $c(0) = \left(\prod_j \log(b_j)\right)^{-1}$ because evaluation at 0 is compatible with the embedding $\mathcal{R}_K^+ \subseteq \mathcal{R}_K^I$ and $\frac{t_{LT}}{Z}(0) = 1$ by (1). $\square$

2.3 Pairings

In this section we discuss various pairings. The starting point is Serre-Duality on the open unit disk which induces a (residue) pairing

$$\{\ ,\ \}' : \mathcal{R} \times \mathcal{R} \to K,$$

see Proposition 2.40. This induces a pairing

$$<\ ,\ >: \mathcal{R}(\Gamma_L) \times \mathcal{R}(\Gamma_L) \to K \tag{75}$$

for the Robba ring of Γ_L, which is actually already characterized by its restriction to $\mathcal{R}(\Gamma_n)$ for any n and which can be seen to stem from Serre-duality on the character variety $\mathfrak{X}_{\Gamma_L}$. It is related to the additive pairing $\{\ ,\ \}'$ in two ways, firstly by using the 'logarithm' $\mathcal{R}(\Gamma_n) \xrightarrow{(\ell_n)_*} \mathcal{R}$ and secondly by a topological isomorphism (which is not a ring homomorphism) $\Upsilon'' : \mathcal{R}(\Gamma_n) \to \mathcal{R}$ by requiring $\varphi_L^n(\Upsilon''(\lambda))\eta(1,Z) = \lambda \cdot \eta(1,Z)$ using Theorem 2.33 and Lemma 2.35, cf. (91). While we take the first way as (ad hoc)-definition, the second interpretation stems from an identity of certain residues, see Theorem 2.54, which forms one main ingredient in the proof of the abstract reciprocity formula 2.74 below.

Based on the (generalized) residue pairings (77)

$$\{\ ,\ \} : \check{M} \times M \to K,$$

with $\check{M} := Hom_{\mathcal{R}}(M, \Omega^1_{\mathcal{R}})$ the pairing (75) induces an Iwasawa pairing (92)

$$\{\ ,\ \}_{Iw} : \check{M}^{\psi_L = \frac{q}{\pi_L}} \times M^{\psi_L=1} \to D(\Gamma_L, K).$$

for any (analytic) (φ_L, Γ_L)-module M, which behaves well with twisting (cf. Lemma 2.65).

Since, by construction and the comparison isomorphisms - the second main ingredient -

$$M[\frac{1}{t_{LT}}] \cong \mathcal{R}[\frac{1}{t_{LT}}] \otimes_L D_{cris,L}(V(\tau^{-1})) \text{ and } \check{M}[\frac{1}{t_{LT}}] \cong \mathcal{R}[\frac{1}{t_{LT}}] \otimes_L D_{cris,L}(V^*(1)),$$

the pairing $\{\ ,\ \}_{Iw}$ is closely related to a pairing

$$[\ ,\ ] : \mathcal{R}^{\psi_L=0} \otimes_L D_{cris,L}(V^*(1)) \times \mathcal{R}^{\psi_L=0} \otimes_L D_{cris,L}(V(\tau^{-1})) \to \mathcal{R}(\Gamma_L)$$

induced from the natural pairing for $D_{cris,L}$, one gets an abstract form of a reciprocity formula almost formally, see Theorem 2.74. As a consequence we shall later derive the adjointness of Berger's and Fourquaux' big exponential map with our regulator map, see Theorem 3.2.

2.3.1 The residuum pairing of the Robba ring

Let K be a complete field which contains Ω and L. We consider the residue pairing

$$\{\ ,\ \} : \mathcal{R} \times \Omega^1_{\mathcal{R}} \to K, \quad (g, \omega) \mapsto Res(g\omega) \tag{76}$$

where

$$Res : \Omega^1_{\mathcal{R}} \to K, \quad \sum_i a_i Z^i dZ \mapsto a_{-1}$$

is the residuum map at Z.

Lemma 2.38. *The pairing* $\{\ ,\ \}$ *identifies* $\mathcal{R}$*, resp.* $\Omega^1_{\mathcal{R}}$*, with the (strong) topological dual of* $\Omega^1_{\mathcal{R}}$*, resp.* $\mathcal{R}$.

Proof. This is a consequence of Serre duality. □

Setting $\check{M} := \mathrm{Hom}_{\mathcal{R}}(M, \Omega^1_{\mathcal{R}}) \cong \mathrm{Hom}_{\mathcal{R}}(M, \mathcal{R})(\chi_{LT})$, for any finitely generated projective $\mathcal{R}$-module M, we obtain more generally the pairing

$$\{\ ,\ \} := \{\ ,\ \}_M : \check{M} \times M \to K, \quad (g, f) \mapsto Res(g(f)), \tag{77}$$

which satisfies the following properties:

Lemma 2.39. (i) $\{\ ,\ \}$ *identifies* M *and* $\check{M}$ *with the (strong) topological duals of* $\check{M}$ *and* M*, respectively.*

(ii) $\{\varphi_L(g), \varphi_L(f)\} = \frac{q}{\pi_L}\{g, f\}$ *for all* $g \in \check{M}$ *and* $f \in M$,

(iii) $\{\sigma(g), \sigma(f)\} = \{g, f\}$ *for all* $g \in \check{M}$*,* $f \in M$*, and* $\sigma \in \Gamma_L$,

(iv) $\{\psi_L(g), f\} = \{g, \varphi_L(f)\}$ *and* $\{\varphi_L(g), f\} = \{g, \psi_L(f)\}$ *for all* $g \in \check{M}$ *and* $f \in M$,

Proof. i. Lemma 2.38. ii. - iv. [SV15, prop. 3.17, cor. 3.18, prop. 3.19]. □

Proposition 2.40. *The pairing* $\{\ ,\ \}' : \mathcal{R} \times \mathcal{R} \to K$*,* $(f, g) \mapsto \{f, gd\log_{LT}\}$*, induces topological isomorphisms*

$$\mathrm{Hom}_{K,cts}(\mathcal{R}, K) \cong \mathcal{R} \qquad \text{and} \qquad \mathrm{Hom}_{K,cts}(\mathcal{R}/\mathcal{R}^+, K) \cong \mathcal{R}^+.$$

Proof. This is a consequence of Serre duality. □

Assume henceforth that M is an analytic (φ_L, Γ_L)-module over $\mathcal{R}$.

Corollary 2.41. *i. The* Γ_L*-action on* M *extends continuously to a* $D(\Gamma_L, K)$*-module structure*

ii. The isomorphism $\check{M} \cong \mathrm{Hom}_{K,cts}(M, K)$ *(induced by* $\{\ ,\ \}$*) is* $D(\Gamma_L, K)$*-linear.*

Proof. i. See proof of Prop. 6.5 in "mult-char-variety". ii. This follows from Lemma 2.39(iii) since Γ_L generates a dense subspcae of $D(\Gamma_L, K)$. □

Since $\frac{\pi_L}{q}\psi_L \circ \varphi_L = \mathrm{id}_M$ we have a canonical decomposition $M = \varphi_L(M) \oplus M^{\psi_L=0}$. By Lemma 2.39 we see that $M^{\psi_L=0}$ is the exact orthogonal complement of $\varphi_L(\check{M})$, i.e., we obtain a canonical isomorphism

$$\check{M}^{\psi_L=0} \cong \mathrm{Hom}_{K,cts}(M^{\psi=0}, K). \tag{78}$$

Lemma 2.42. *The isomorphism* (78) *is* $\mathcal{R}(\Gamma_L)$*-equivariant, i.e., we have for all* $\check{m} \in \check{M}^{\psi_L=0}$*,* $m \in M^{\psi_L=0}$*, and* $\lambda \in \mathcal{R}(\Gamma_L)$ *that*

$$\{\lambda\check{m}, m\} = \{\check{m}, \iota(\lambda)m\}\ .$$

Proof. This is clear for $D(\Gamma_L, K)$ by Cor. 2.41.ii. It then follows for the localization $D(\Gamma_L, K)[\mu_{n_1}^{-1}]$, where we use the notation and considerations from subsection 2.2.5, especially Lemma 2.31 and its proof. Since $D(\Gamma_L, K)[\mu_{n_1}^{-1}]$ is dense in $\mathcal{R}(\Gamma_L)$ the assertion now is a consequence of the continuity property in Theorem 2.33. □

2.3.2 The duality pairing $<,>$ for the group Robba ring

We define $res : \mathcal{R} \to K$ by sending λ to $Res(\lambda d\log_{LT})$, and we write $pr_{n,m} = pr_{\Gamma_n,\Gamma_m}$ (and similarly $pr_{L,m} = pr_{\Gamma_L,\Gamma_m}$) for the projection maps induced by (50). Consider the following pairing

$$< \, , \, >: \mathcal{R}(\Gamma_L) \times \mathcal{R}(\Gamma_L) \xrightarrow{mult} \mathcal{R}(\Gamma_L) \xrightarrow{\varrho} K, \tag{79}$$

where

$$\varrho : \mathcal{R}(\Gamma_L) = \mathcal{R}(\Gamma_{n_0}) \rtimes_{\Gamma_{n_0}} \Gamma_L \xrightarrow{(\frac{q}{\Pi_L})^{n_0} res\circ\ell_{n_0,*}\circ pr_{L,n_0}} K$$

with n_0 as defined at the beginning of subsection 2.2.2.

Lemma 2.43. *This definition does not depend on the choice of n_0.*

Proof. For $m \geqslant n \geqslant n_0$ and $k := m - n$, we have commutative diagrams

$$\begin{tikzcd}
\mathcal{R}(\Gamma_n) \rtimes_{\Gamma_n} \Gamma_L \arrow[r, "\cong"] \arrow[d, "pr_{L,n}"'] & (\mathcal{R}(\Gamma_m) \rtimes_{\Gamma_m} \Gamma_n) \rtimes_{\Gamma_n} \Gamma_L \arrow[rr, "\cong"] \arrow[d, "pr_{L,n}"'] & & \mathcal{R}(\Gamma_m) \rtimes_{\Gamma_m} \Gamma_L \arrow[d, "pr_{L,m}"'] \\
\mathcal{R}(\Gamma_n) \arrow[r, "\cong"] \arrow[d, "\ell_{n,*}"'] & \mathcal{R}(\Gamma_m) \rtimes_{\Gamma_m} \Gamma_n \arrow[r, "pr_{n,m}"] \arrow[d, "(\ell_{n,*}\circ\iota_{m,n})\rtimes\ell_n"'] & \mathcal{R}(\Gamma_m) \arrow[r, equal] \arrow[d, "\ell_{n,*}\circ\iota_{m,n}"'] & \mathcal{R}(\Gamma_m) \arrow[d, "\ell_{m,*}"'] \\
\mathcal{R} \arrow[r, "\cong"] \arrow[dd, "(\frac{q}{\pi_L})^n res"'] & \bigoplus_{i\in o_L/\pi_L^k o_L} \eta(i,Z)\varphi_L^k(\mathcal{R}) \arrow[r, "pr_0^k"] & \varphi_L^k(\mathcal{R}) \arrow[d, hook] & \mathcal{R} \arrow[l, "\varphi_L^k"'] \arrow[dd, "(\frac{q}{\pi_L})^m res"] \\
 & & \mathcal{R} \arrow[d, "(\frac{q}{\pi_L})^n res"] & \\
K \arrow[rr, equal] & & K \arrow[r, equal] & K,
\end{tikzcd} \tag{80}$$

where pr_0^k denotes the natural projection onto the component corresponding to the trivial coset $i = 0$. Note that $pr_0^k(\lambda) = \left(\frac{\pi_L}{q}\right)^k \varphi_L^k\psi_L^k(\lambda)$ by formula (37). While the upper/middle rectangles commute obviously by construction (using subsection §2.2.2) , the left lower one does, because

$$res\left((\frac{\pi_L}{q})^k\varphi_L^k\psi_L^k(\lambda)\right) = \left(\frac{\pi_L}{q}\right)^k res(\varphi_L^k\psi_L^k(\lambda)) = \left(\frac{\pi_L}{q}\right)^k \left(\frac{q}{\pi_L}\right)^k res(\psi_L^k(\lambda)) = res(\lambda)$$

by Lemma 2.39 (ii) (with $g = 1$, $f = \lambda$) for the second and (iv) (with $g = \lambda$ and $f = 1$) for the third equation. By the same argument as for the last equation also the right lower rectangle commutes. $\square$

Remark 2.44. *We rather can start with the identification*

$$\mathcal{R}(\Gamma_n) \cong \varphi_L^n(\mathcal{R}),$$

extending the isomorphism $D(\Gamma_n, K) \xrightarrow{\ell_*} D(\pi_L^n o_L, K)$*. Then, for* $m \geqslant n \geqslant n_0$ *and* $k := m-n$*, we have the commutative diagram*

$$\begin{tikzcd} \mathcal{R}(\Gamma_m) \arrow[r, "\ell_*", "\cong"'] \arrow[d, hook, "\iota'_{m,n}"'] & \varphi_L^m(\mathcal{R}) \arrow[d, hook] \\ \mathcal{R}(\Gamma_n) \arrow[r, "\ell_*", "\cong"'] & \varphi_L^n(\mathcal{R}) \end{tikzcd}$$

and we may also describe the pairing as

$$<\ ,\ >: \mathcal{R}(\Gamma_L) \times \mathcal{R}(\Gamma_L) \xrightarrow{mult} \mathcal{R}(\Gamma_L) = \mathcal{R}(\Gamma_{n_0}) \rtimes_{\Gamma_{n_0}} \Gamma_L \xrightarrow{pr_{L,n_0}} \mathcal{R}(\Gamma_{n_0}) \xrightarrow{\ell} \varphi_L^{n_0}(\mathcal{R}) \subseteq \mathcal{R} \xrightarrow{res} K,$$

which obviously coincides with the one in the previous subsection (and, in particular, is independent of the choice of n_0*).*

The following properties follow immediately from the definition:

Lemma 2.45. *We have for all* $f, \lambda, \mu \in \mathcal{R}(\Gamma_L)$ *that*

(i) $< \lambda, f\mu > = < f\lambda, \mu >$,

(ii) $< \lambda, \mu > = < \mu, \lambda >$.

Remark 2.46. *For an open subgroup* $U \subseteq \Gamma_L$ *denote by* $<\ ,\ >_U$ *the restriction of* $<\ ,\ >$ *to* $\mathcal{R}(\Gamma_L) \times \mathcal{R}(\Gamma_L)$*. Then, for a pair* $U \subseteq U'$ *of open subgroups of* Γ_L*, one immediately checks the projection formula* $pr_{U',U}(\iota_{U,U'}(x)y) = x pr_{U',U}(y)$*, whence*

$$< \iota_{U,U'}(x), y >_{U'} = < x, pr_{U',U}(y) >_U \tag{81}$$

for $x \in \mathcal{R}(U)$*,* $y \in \mathcal{R}(U')$ *and the canonical inclusion* $\mathcal{R}(U) \xrightarrow{\iota_{U,U'}} \mathcal{R}(U')$*.*

The following proposition follows easily from Proposition 2.40, equation (51) and its analogue for the distribution algebra.

Proposition 2.47. *The pairing* $<\ ,\ >: \mathcal{R}(\Gamma_L) \times \mathcal{R}(\Gamma_L) \to K$ *induces topological isomorphisms*

$$\mathrm{Hom}_{K,cts}(\mathcal{R}(\Gamma_L), K) \cong \mathcal{R}(\Gamma_L) \text{ and } \mathrm{Hom}_{K,cts}(\mathcal{R}(\Gamma_L)/D(\Gamma_L, K), K) \cong D(\Gamma_L, K).$$

Proposition 2.48. *The map*

$$\mathrm{Hom}_{\mathcal{R}(\Gamma_L)}(M^{\psi=0}, \mathcal{R}(\Gamma_L))^\iota \xrightarrow{\cong} \mathrm{Hom}_{K,cts}(M^{\psi=0}, K) \xrightarrow[(78)]{\cong} \check{M}^{\psi_L=0} \tag{82}$$
$$F \longmapsto \rho \circ F$$

is an isomorphism of $\mathcal{R}(\Gamma_L)$*-modules, where the superscript "*ι*" on the left hand side indicates that* $\mathcal{R}(\Gamma_L)$ *acts through the involution* ι*.*

Proof. According to Thm. 2.33 the $\mathcal{R}(\Gamma_L)$-module $M^{\psi=0}$ is finitely generated free. Hence it suffices to show that the map

$$\mathrm{Hom}_{\mathcal{R}(\Gamma_L)}(\mathcal{R}(\Gamma_L), \mathcal{R}(\Gamma_L)) \longrightarrow \mathrm{Hom}_{K,cts}(\mathcal{R}(\Gamma_L), K)$$
$$F \longmapsto \rho \circ F$$

is bijective. But this map is nothing else than the duality isomorphism in Prop. 2.47. □

Remark 2.49 (Frobenius reciprocity). *For an open subgroup U of Γ_L the projection map $pr_{\Gamma_L,U}: \mathcal{R}(\Gamma_L) \to \mathcal{R}(U)$ induces an isomorphism*

$$\mathrm{Hom}_{\mathcal{R}(\Gamma_L)}(N, \mathcal{R}(\Gamma_L)) \cong \mathrm{Hom}_{\mathcal{R}(U)}(N, \mathcal{R}(U))$$

for any $\mathcal{R}(\Gamma_L)$-module N; the inverse sends f to the homomorphism $x \mapsto \sum_{g \in \Gamma_L/U} gf(g^{-1}x)$.

Twists One checks that the isomorphism

$$D(\Gamma_n, K) \xrightarrow{(\ell_n)_*} D(o_L, K) \cong \mathcal{O}_K(\mathfrak{X}) \cong \mathcal{O}_K(\mathbb{B})$$

sends a distribution μ to the map $g_\mu(z) = \mu(\exp(\Omega\ell_n(-)\log_{LT}(z)))$. In particular, a Dirac distribution δ_γ is sent to $\exp(\Omega\ell_n(\gamma)\log_{LT}(z))$.

Let $z_\chi \in \mathbb{B}(K)$ be the point which corresponds via the LT-isomorphism to the character $\chi: o_L \to L^\times$, $a \mapsto \exp(\pi_L^n a)$, i.e., satisfying

$$\exp(\pi_L^n a) = \exp(a\Omega\log_{LT}(z_\chi))$$

or

$$\pi_L^n = \Omega\log_{LT}(z_\chi).$$

Lemma 2.50. *We have the commutative diagram*

$$\begin{tikzcd} D(\Gamma_n, K) \arrow[r, "LT\circ\text{Fourier}\circ(\ell_n)_*"] \arrow[d, "Tw_{\chi_{LT}}"'] & \mathcal{O}_K(\mathbb{B}) \arrow[d, "(z_\chi +_{LT} Z)^*"] \\ D(\Gamma_n, K) \arrow[r, "LT\circ\text{Fourier}\circ(\ell_n)_*"] & \mathcal{O}_K(\mathbb{B}) \end{tikzcd}$$

Proof. This follows from properties 1. and 3. in section 2.1.3 together with the fact that the LT-isomorphism $\kappa: \mathfrak{X}_K \xrightarrow{\cong} \mathbb{B}_K$ is an isomorphism of group varieties. □

Since, with respect to the maximal ideal $\mathfrak{m}_K$ of o_K,

$$z_\chi +_{LT} Z \equiv 0 +_{LT} Z = Z \mod \mathfrak{m}_K$$

we can write

$$Z' := z_\chi +_{LT} Z = \alpha Z(1 + \frac{\beta}{Z}) \tag{83}$$

with $\alpha \in 1 + \mathfrak{m}_K$, $\beta \in z_\chi o_K[[Z]]$.

Sending Z to $z_\chi +_{LT} Z$ defines a continous K-linear ring automorphism $\eta: \mathcal{R} \to \mathcal{R}$, which in turn induces an automorphism of $\Omega^1_{\mathcal{R}}$ sending $f(Z)dZ$ to $\eta(F)d\eta(Z)$.

Lemma 2.51. *For all $\omega \in \Omega^1_{\mathcal{R}}$ we have*

$$Res(\eta(\omega)) = Res(\omega).$$

Proof. By the same reasoning as in the proof of [KPX, Lem. 2.1.19] this is reduced to the statement and proof of [SV, Rem. 3.4 ii.]. □

Lemma 2.52. $\eta(d\log_{LT}) = d\log_{LT}$

Proof.

$$\begin{aligned} d\eta(\log_{LT}(Z)) &= d\log_{LT}(z_\chi +_{LT} Z) \\ &= d\log_{LT}(z_\chi) + d\log_{LT}(Z) \\ &= d\log_{LT}(Z) \end{aligned}$$

□

From the definitions and Lemmata 2.50, 2.51 and 2.52 we conclude the following

Corollary 2.53. *For all $\lambda, \mu \in \mathcal{R}(\Gamma_n)$ we have*

$$< Tw_{\chi_{LT}}(\mu), Tw_{\chi_{LT}}(\lambda) >_{\Gamma_n} = < \mu, \lambda >_{\Gamma_n} .$$

This extends to $\mathcal{R}(\Gamma_L)$ using the projection formula (81).

2.3.3 A residuum identity and an alternative description of $<\,,\,>$

Consider the continuous map

$$\begin{aligned} \varsigma : \mathcal{R}(\Gamma_L) &\to K, \\ \lambda &\mapsto Res(\mathfrak{M}(\sigma_{-1})\mathfrak{M}_{\chi_{LT}}(\lambda)) \end{aligned}$$

where $\mathfrak{M}_{\chi_{LT}} : \mathcal{R}(\Gamma_L) \to \Omega^1_{\mathcal{R}}$ sends λ to

$$\lambda \cdot (\eta(1,Z) \otimes d\log_{LT}) = (Tw_{\chi_{LT}}(\lambda)\eta(1,Z)) \otimes d\log_{LT} . \tag{84}$$

Recall the definition from ϱ from (79).

Theorem 2.54. *We have*

$$\varsigma = \varrho,$$

i.e., the following identity for the residue map holds

$$(\frac{q}{\pi_L})^{n_0} Res\Big(\ell_{n_0,*} \circ pr_{L,n_0}(\lambda) d\log_{LT}\Big) = Res\Big(\eta(-1,Z)\lambda \cdot \big(\eta(1,Z) d\log_{LT}\big)\Big)$$

for all $\lambda \in \mathcal{R}(\Gamma_L)$.

Remark 2.55. *Compare with [Ben, Prop. 2.2.1, 3.2.1] where also residue identities play a crucial role in the proof of his reciprocity formula.*

Proof. Due to Proposition 2.47 there exists $g \in D(\Gamma_L, K)$ such that $\varsigma(\lambda) = < g, \lambda >$ for all $\lambda \in \mathcal{R}(\Gamma_L)$, because ς sends $D(\Gamma_L, K)$ to zero. We claim that

$$Tw_{\chi_{LT}^j}(g) = g \tag{85}$$

for all $j \in \mathbb{Z}$: By Corollary 2.53 and Lemma 2.56 below we have

$$\begin{aligned} < Tw_{\chi_{LT}^j}(g), f > &= < g, Tw_{\chi_{LT}^{-j}}(f) > \\ &= \varsigma(Tw_{\chi_{LT}^{-j}}(f)) \\ &= \varsigma(f) \\ &= < g, f > \end{aligned}$$

for all $f \in \mathcal{R}(\Gamma_L)$.

Now it follows from (85) combined with Lemma 2.57 below that g is constant (and equal to $ev_{\chi_{LT}^0}(g)$), i.e., $\varsigma(-) = g < 1, - >$. Finally, it follows from (87) below that $g = 1$. $\square$

Lemma 2.56. *For all* $\lambda \in \mathcal{R}(\Gamma_L)$ *we have*

$$\varrho(Tw_{\chi_{LT}^j}(\lambda)) = \varrho(\lambda).$$

Proof. Using

$$\begin{aligned}
\partial_{\mathrm{inv}}(\mathfrak{M}(\sigma_{-1})\mathfrak{M}(Tw_{\chi_{LT}}(\lambda))) &= \partial_{\mathrm{inv}}(\mathfrak{M}(\sigma_{-1}))\mathfrak{M}(Tw_{\chi_{LT}}(\lambda)) + \mathfrak{M}(\sigma_{-1})\partial_{\mathrm{inv}}(\mathfrak{M}(Tw_{\chi_{LT}}(\lambda)))\\
&= \mathfrak{M}(Tw_{\chi_{LT}}(\sigma_{-1}))\mathfrak{M}(Tw_{\chi_{LT}}(\lambda)) + \mathfrak{M}(\sigma_{-1})\mathfrak{M}(Tw_{\chi_{LT}}(Tw_{\chi_{LT}}(\lambda)))\\
&= -\mathfrak{M}(\sigma_{-1})\mathfrak{M}(Tw_{\chi_{LT}}(\lambda)) + \mathfrak{M}(\sigma_{-1})\mathfrak{M}(Tw_{\chi_{LT}}(Tw_{\chi_{LT}}(\lambda)))
\end{aligned}$$

and the fact that $res \circ \partial_{\mathrm{inv}} = 0$ by [FX, Prop. 2.12] the case $j = 1$ follows directly from the relations (89) with $\mu = 1$, from which the general case is immediate. $\square$

Lemma 2.57. *Let* $\lambda \in D(\Gamma_L, \mathbb{C}_p)$ *with* $ev_{\chi_{LT}^j}(\lambda) = 0$ *for infinitely many* j*, then* $\lambda = 0$.

Proof. On the character variety the characters χ_{LT}^j corresponds to points which converge to the trivial character. It follows that λ corresponds to the trivial function, since otherwise its divisor of zeroes would have only finitely many zeroes in any disk with fixed radius strictly smaller than 1 by (27), which would contradict the assumptions. $\square$

Now fix a $\mathbb{Z}_p$-basis $b = (b_1, \ldots, b_d)$ of $\Gamma := \Gamma_{n_0}$ and set $\ell^*(b) := \ell^*_\Gamma(b) := q^{-n_0}\prod_{i=1}^d \ell(b_i) \in o_L^\times$ According to subsection 2.1.2 we may define the operators

$$\begin{aligned}
\Theta_b :=& \ell^*(b)\prod_{i=1}^d \frac{\nabla}{b_i - 1}\\
\widehat{\Xi_b} :=& \ell^*(b)\Xi_b = \frac{\Theta_b}{\nabla}
\end{aligned}$$

in $\mathcal{R}(\Gamma)$. Let $\mathrm{aug} : D(\Gamma, K) \to K$ denote the augmentation map, induced by the trivial map $\Gamma \to \{1\}$.

Lemma 2.58. *The element* $\Omega\widehat{\Xi_b}$ *induces the augmentation map*

$$< \Omega\widehat{\Xi_b}, - >_\Gamma = \mathrm{aug} : D(\Gamma, K) \to K. \tag{86}$$

Moreover, we have

$$\varsigma(\Omega\widehat{\Xi_b}) = 1 = \varrho(\Omega\widehat{\Xi_b}). \tag{87}$$

Proof. Since $\ell_{n_0,*}(\widehat{\Xi_b}) \equiv \frac{\pi_L^n}{q^{n_0}\Omega}\frac{1}{Z} \mod \mathcal{R}_K^+$ by Remark 2.36, one has for every $\lambda \in D(\Gamma, K)$ by definition (79)

$$\begin{aligned}
< \widehat{\Xi_b}, \lambda >_\Gamma &= (\frac{q}{\pi_L})^{n_0} Res(\ell_{n_0,*}(\widehat{\Xi_b}\lambda)g_{LT}dZ)\\
&= \frac{1}{\Omega}Res(\frac{1}{Z}\ell_{n_0,*}(\lambda)g_{LT}dZ)\\
&= \frac{1}{\Omega}\mathrm{aug}(\lambda),
\end{aligned}$$

where we use for the last equation that $g_{LT}(Z)$ has constant term 1 and the fact that the augmentation map corresponds via Fourier theory and the LT-isomorphism to the 'evaluation at $Z = 0$' map.

For the second claim one has by definition of ς

$$\begin{aligned}
\varsigma(\widehat{\Omega\Xi_b}) &= = \Omega\ell^*(b)Res(\sigma_{-1}(\mathfrak{M}(\iota_*(\Xi_b)))\mathfrak{M}(1)d\log_{LT}) \\
&= \Omega\ell^*(b)Res(\mathfrak{M}(\sigma_{-1})\mathfrak{M}(Tw_{\chi_{LT}}(\Xi_b))d\log_{LT}) \\
&= \ell^*(b)Res(\mathfrak{M}(\sigma_{-1})\log_{LT}(Z)\partial_{\mathrm{inv}}\mathfrak{M}(\Xi_b)\frac{d\log_{LT}}{\log_{LT}(Z)}) \\
&= \ell^*(b)Res(\mathfrak{M}(\sigma_{-1})\mathfrak{M}(\nabla\Xi_b)\frac{d\log_{LT}}{\log_{LT}(Z)}) \\
&= \ell^*(b)Res(\mathfrak{M}(\sigma_{-1})\frac{\pi_L^{n_0}\log_{LT}(Z)}{\varphi_L^{n_0}(Z\prod_j \ell(b_j))}\eta(1,Z)\frac{d\log_{LT}}{\log_{LT}(Z)}) \\
&= \ell^*(b)\pi_L^{n_0}Res(\eta(-1,Z)\frac{1}{\varphi_L^{n_0}(Z\prod_j \ell(b_j))}\eta(1,Z)d\log_{LT}) \\
&= \ell^*(b)\pi_L^{n_0}Res(\eta(1-1,Z)\frac{1}{\varphi_L^{n_0}(Z\prod_j \ell(b_j))}d\log_{LT}) \\
&= \ell^*(b)\pi_L^{n_0}Res(\varphi_L^{n_0}\left(\eta(0,Z)\frac{1}{Z\prod_j \ell(b_j)}d\log_{LT}\right)) \\
&= \frac{\ell^*(b)}{\prod_j \ell(b_j)}\pi_L^{n_0}(\frac{q}{\pi_L})^{n_0}Res(\frac{1}{Z}d\log_{LT}) \\
&= 1,
\end{aligned}$$

where we use (67) in the third equation, the fact that ∇ acts on $\mathcal{R}$ as $\log_{LT}(Z)\partial_{\mathrm{inv}}$ in the fourth equation, Remark 2.37 for the fifth equation and finally for the last equation that $g_{LT}(Z)$ has constant term 1. The result follows because the delta distributions span a dense subspace in $D(\Gamma_{n_0}, K)$. ∎

Corollary 2.59. *The pairing $<,>$ makes the following diagram commutative*

(88)
$$\begin{tikzcd}
\mathcal{R}^{\psi_L=0} & \times & (\Omega^1_{\mathcal{R}})^{\psi_L=0} \arrow[r, "mult"] & \Omega^1_{\mathcal{R}} \arrow[r, "Res"] & K \\
<,>\colon \mathcal{R}(\Gamma_L) \arrow[u, "\sigma_{-1}\mathfrak{M}\circ\iota_*"'] & \times & \mathcal{R}(\Gamma_L) \arrow[u, "\mathfrak{M}_{\chi_{LT}}"'] \arrow[rr, dashed] & & K, \arrow[u, equal]
\end{tikzcd}$$

i.e., we have

(89)
$$\begin{aligned}
<\mu,\lambda> &= \{\mathfrak{M}(\sigma_{-1}\iota(\mu)), \mathfrak{M}_{\chi_{LT}}(\lambda)\} \\
&= Res(\sigma_{-1}\mathfrak{M}(\iota(\mu))\mathfrak{M}_{\chi_{LT}}(\lambda)) \\
&= Res(\mathfrak{M}(\sigma_{-1}\iota(\mu))\mathfrak{M}(Tw_{\chi_{LT}}(\lambda))d\log_{LT}) \\
&= Res(\mathfrak{M}(\iota(\mu))\mathfrak{M}(Tw_{\chi_{LT}}(\sigma_{-1}\lambda))d\log_{LT}).
\end{aligned}$$

Proof. By Theorem 2.54, the definition of ς and of $<\,,\,>$ we have

(90)
$$\begin{aligned}
<\mu,\lambda> &=< 1, \mu\lambda > \\
&= \{\mathfrak{M}(\sigma_{-1}), \mathfrak{M}_{\chi_{LT}}(\mu\lambda)\} \\
&= \{\mathfrak{M}(\sigma_{-1}\iota(\mu)), \mathfrak{M}_{\chi_{LT}}(\lambda)\}
\end{aligned}$$

where we use Lemma 2.42 for the last equation. $\square$

Remark 2.60. *For every n the pairing $<,>_{\Gamma_n}$ induces topological isomorphisms*

$$\mathrm{Hom}_{K,cts}(\mathcal{R}(\Gamma_n),K)\cong\mathcal{R}(\Gamma_n)\text{ and }\mathrm{Hom}_{K,cts}(\mathcal{R}(\Gamma_n)/D(\Gamma_n,K),K)\cong D(\Gamma_n,K).$$

Proof. We define topological isomorphisms $\Upsilon',\Upsilon:\mathcal{R}(\Gamma_n)\to\mathcal{R}$ by requiring

$$\varphi_L^n(\Upsilon'(\lambda))\eta(-1,Z)=\iota_*(\lambda)\eta(-1,Z)\text{ and }\varphi_L^n(\Upsilon(\lambda))\eta(1,Z)=Tw_{\chi_{LT}}(\lambda)\eta(1,Z)$$

and we observe that they restrict to topological isomorphisms between $D(\Gamma_n,K)$ and $\mathcal{R}_K^+$ by Lemmata 2.35 and 2.27. Then, by construction and (89), we obtain a commutative diagram

(91)
$$\begin{tikzcd}
\mathcal{R} & \times & \mathcal{R} \arrow[r, "mult"] & \mathcal{R} \arrow[r, "res"] & K \arrow[d, "(\frac{q}{\pi_L})^n"] \\
<,>_{\Gamma_n}:\mathcal{R}(\Gamma_n) \arrow[u, "\Upsilon'"] & \times & \mathcal{R}(\Gamma_n) \arrow[u, "\Upsilon"] \arrow[rr, dashed] & & K
\end{tikzcd}$$

because $res(\varphi_L(f))=\frac{q}{\pi_L}res(f)$ by Lemma 2.39 (ii) (with $g=1$, $f=\lambda$). Hence, the claim follows from Proposition 2.40. $\square$

Lemma 2.61. *We have for all $f,\lambda,\mu\in\mathcal{R}(\Gamma_L)$ that $<\lambda,\mu>=-<\iota(\lambda),\iota(\mu)>$.*

Proof. Using (89), Lemma 2.39 and Lemma 2.56 we see that

$$\begin{aligned}
<\mu,\lambda> &= Res\big(\mathfrak{M}(Tw_{\chi_{LT}}(\sigma_{-1}\lambda))\mathfrak{M}(\iota(\mu))d\log_{LT}\big)\\
&= -Res\big(\mathfrak{M}(\sigma_{-1}\iota_*(Tw_{\chi_{LT}^{-1}}(\iota_*(\lambda))))\mathfrak{M}(\iota(\mu))d\log_{LT}\big)\\
&= -<Tw_{\chi_{LT}^{-1}}(\iota(\lambda)),Tw_{\chi_{LT}^{-1}}(\iota(\mu))>\\
&= -<\iota(\lambda),\iota(\mu)>
\end{aligned}$$

$\square$

2.3.4 The Iwasawa pairing for (φ_L,Γ_L)-modules over the Robba ring

Using Proposition 2.47 we define an $\mathcal{R}(\Gamma_L)$-ι_*-sesquilinear pairing

$$\{\ ,\ \}'_{Iw}:=\{\ ,\ \}'_{M,Iw}:\check{M}^{\psi_L=0}\times M^{\psi_L=0}\to\mathcal{R}(\Gamma_L)$$

requiring the commutativity of the diagram

$$\begin{tikzcd}
\mathcal{R}(\Gamma_L) \arrow[d, equal] & \times & \check{M}^{\psi_L=0}\times M^{\psi_L=0} \arrow[r] \arrow[d, dashed, "\{\,,\,\}'_{Iw}"] & K \arrow[d, equal] \\
\mathcal{R}(\Gamma_L) & \times & \mathcal{R}(\Gamma_L) \arrow[r, "<,>"] & K,
\end{tikzcd}$$

in which the upper line sends (f,x,y) to $\{f(x),y\}$. Indeed, the property

$$\{\lambda\check{m},m\}'_{Iw}=\{\check{m},\iota_*(\lambda)m\}'_{Iw}$$

follows from the corresponding property for $\{,\}$ by Lemma 2.42, while with regard to the second one

$$\lambda\{\check{m},m\}'_{Iw} = \{\lambda\check{m},m\}'_{Iw}$$

we have for all $f \in \mathcal{R}(\Gamma_L)$

$$\begin{aligned} < f, \{\lambda\check{m},m\}'_{Iw} > &= \{f\cdot\lambda\check{m},m\} \\ &=< \lambda f, \{\check{m},m\}'_{Iw} > \\ &=< f, \lambda\{\check{m},m\}'_{Iw} > \end{aligned}$$

by Lemma 2.45. Note that the pairing $\{\ ,\ \}'_{Iw}$ induces the isomorphism (82).

We set

$$\mathcal{C} := (\frac{\pi_L}{q}\varphi_L - 1)M^{\psi_L=1} \quad \text{and } \check{\mathcal{C}} := (\varphi_L - 1)\check{M}^{\psi_L=\frac{q}{\pi_L}}$$

and we shall need the following

Lemma 2.62. *For $f \in D(\Gamma_L,K)$ we have $\{f\cdot(\varphi_L-1)x, (\frac{\pi_L}{q}\varphi_L-1)y\} = 0$ for all $x \in \check{M}^{\psi_L=\frac{q}{\pi_L}}$ and $y \in M^{\psi_L=1}$.*

Proof. Straightforward calculation using Lemma 2.39 above, cp. [KPX, Lem. 4.2.7]. □

This Lemma combined with the second statement of Proposition 2.47 implies that the restriction of $\{\ ,\ \}'_{Iw}$ to $\check{\mathcal{C}}\times\mathcal{C}$, which by abuse of notation we denote by the same symbol, is characterized by the commutativity of the diagram

$$\begin{array}{ccccc} \check{\mathcal{C}} \times \mathcal{C} & \times & \mathcal{R}(\Gamma_L)/D(\Gamma_L,K) & \longrightarrow & K \\ \Big\downarrow{\scriptstyle \{\,,\,\}'_{Iw}} & & \Big\| & & \Big\| \\ D(\Gamma_L,K) & \times & \mathcal{R}(\Gamma_L)/D(\Gamma_L,K) & \xrightarrow{<,>} & K, \end{array}$$

in which the upper line sends (x,y,f) to $\{f(x),y\}$. In particular, it takes values in $D(\Gamma_L,K)$.

Finally, we obtain a $D(\Gamma_L,K)$-ι_*-sesquilinear pairing $\{\ ,\ \}_{Iw} := \{\ ,\ \}_{M,Iw}$ which by definition fits into the following commutative diagram

$$\begin{array}{ccccc} \{\ ,\ \}_{M,Iw} : \check{M}^{\psi_L=\frac{q}{\pi_L}} & \times & M^{\psi_L=1} & \dashrightarrow & D(\Gamma_L,K) \\ \Big\downarrow{\scriptstyle \varphi_L-1} & & \Big\downarrow{\scriptstyle \frac{\pi_L}{q}\varphi_L-1} & & \Big\| \\ \{\ ,\ \}'_{M,Iw} : \check{\mathcal{C}} & \times & \mathcal{C} & \longrightarrow & D(\Gamma_L,K). \end{array}$$

Altogether we obtain the following

Theorem 2.63. *There is a $D(\Gamma_L,K)$–ι_*-sesquilinear pairing*

(92) $$\{\ ,\ \}_{Iw} : \check{M}^{\psi_L=\frac{q}{\pi_L}} \times M^{\psi_L=1} \to D(\Gamma_L,K).$$

It is characterized by the following property

(93) $$< f, \{\check{m},m\}_{Iw} >= \{f\cdot(\varphi_L-1)\check{m}, (\frac{\pi_L}{q}\varphi_L-1)m\} \text{ for all } f \in \mathcal{R}(\Gamma_L),\ \check{m}\in\check{M},\ m\in M.$$

Remark 2.64. *For any open subgroup U of Γ_L, we obtain similarly as in* (92) $D(U,K)$-ι_*-*sesquilinear pairings*

$$\{\ ,\ \}_{Iw,U} : \check{M}^{\psi_L=\frac{q}{\pi_L}} \times M^{\psi_L=1} \to D(U,K).$$

It follows immediately from the definitions, the projection formulae (81)*and Frobenius reciprocity 2.49 that*

$$\{\ ,\ \}_{Iw,U} := pr_{\Gamma_L,U} \circ \{\ ,\ \}_{Iw}.$$

If $\chi : \Gamma_L \longrightarrow o_L^\times$ is any continuous character with representation module $W_\chi = o_L t_\chi$ then, for any (φ_L,Γ_L)-module M over $\mathcal{R}$, we have the twisted (φ_L,Γ_L)-module $M(\chi)$ where $M(\chi) := M \otimes_{o_L} W_\chi$ as $\mathcal{R}$-module, $\varphi_{M(\chi)}(m\otimes w) := \varphi_M(m)\otimes w$, and $\gamma|M(\chi)(m\otimes w) := \gamma|M(m)\otimes\gamma|W_\chi(w) = \chi(\gamma)\cdot\gamma|M(m)\otimes w$ for $\gamma\in\Gamma_L$. It follows that $\psi_{M(\chi)}(m\otimes w) = \psi_M(m)\otimes w$. For the character χ_{LT} we take $W_{\chi_{LT}} = T = o_L\eta$ and $W_{\chi_{LT}^{-1}} = T^* = o_L\eta^*$ as representation module, where T^* denotes the o_L-dual with dual basis η^* of η.

Consider the $\mathcal{R}_K$-linear (but of course not $\mathcal{R}_K(\Gamma_L)$-linear) map

$$tw_\chi : M \to M(\chi),\ m \mapsto m\otimes t_\chi.$$

Lemma 2.65. *There is a commutative diagram*

$$\begin{array}{ccccc}
\check{M}(\chi_{LT}^{-j})^{\psi_L=\frac{q}{\pi_L}} & \times & M(\chi_{LT}^{j})^{\psi_L=1} & \xrightarrow{\{,\}_{Iw}} & D(\Gamma_L,\mathbb{C}_p) \\
\uparrow{\scriptstyle tw_{\chi_{LT}^{-j}}} & & \uparrow{\scriptstyle tw_{\chi_{LT}^{j}}} & & \uparrow{\scriptstyle Tw_{\chi_{LT}^{j}}} \\
\check{M}^{\psi_L=\frac{q}{\pi_L}} & \times & M^{\psi_L=1} & \xrightarrow{\{,\}_{Iw}} & D(\Gamma_L,\mathbb{C}_p).
\end{array}$$

Proof. We have for all $f\in\mathcal{R}_K(\Gamma_L)$,

$$\begin{aligned}
< f, \{tw_{\chi_{LT}^{-j}}(\check{m}), tw_{\chi_{LT}^{j}}(m)\}_{Iw} > &= \{f\cdot\left((\varphi_L-1)\check{m}\otimes\eta^{\otimes -j}\right), (\frac{\pi_L}{q}\varphi_L-1)m\otimes\eta^{\otimes j}\} \\
&= \{\left(Tw_{\chi_{LT}^{-j}}(f)\cdot(\varphi_L-1)\check{m}\right)\otimes\eta^{\otimes -j}, (\frac{\pi_L}{q}\varphi_L-1)m\otimes\eta^{\otimes j}\} \\
&= \{\left(Tw_{\chi_{LT}^{-j}}(f)\cdot(\varphi_L-1)\check{m}\right), (\frac{\pi_L}{q}\varphi_L-1)m\} \\
&=< Tw_{\chi_{LT}^{-j}}(f), \{\check{m},m\}_{Iw} > \\
&=< f, Tw_{\chi_{LT}^{j}}(\{\check{m},m\}_{Iw}) >
\end{aligned}$$

where we used Corollary 2.53 for the last equation. The second equation is clear for δ-distributions and hence extends by the uniqueness result of Theorem 2.33, cf. the proof of Theorem 2.30. □

2.3.5 The abstract reciprocity formula

Compatibility of the Iwasawa pairing under comparison isomorphisms Let M,N be (not necessarily étale) L-analytic (φ_L,Γ_L)-modules over $\mathcal{R}$. We extend the action of Γ_L,

φ_L and ψ_L to the $\mathcal{R}[\frac{1}{t_{LT}}]$-module $M[\frac{1}{t_{LT}}]$ (and in the same way to $N[\frac{1}{t_{LT}}]$) as follows:

$$\gamma\frac{m}{t_{LT}^k} := \frac{\gamma m}{\gamma t_{LT}^k} = \frac{\frac{\gamma m}{\chi_{LT}^k(\gamma)}}{t_{LT}^k},$$
$$\varphi_L(\frac{m}{t_{LT}^k}) := \frac{\varphi_L(m)}{\varphi_L(t_{LT}^k)} = \frac{\frac{\varphi_L(m)}{\pi_L^k}}{t_{LT}^k} \text{ and}$$
$$\psi_L(\frac{m}{t_{LT}^k}) := \frac{\pi_L^k\psi_L(m)}{t_{LT}^k}.$$

Now we assume that there is an isomorphism

$$c : \mathcal{R}[\frac{1}{t_{LT}}] \otimes_{\mathcal{R}} M \xrightarrow{\cong} \mathcal{R}[\frac{1}{t_{LT}}] \otimes_{\mathcal{R}} N$$

of (φ_L, Γ_L)-modules over $\mathcal{R}[\frac{1}{t_{LT}}]$.

Lemma 2.66. (i) $(M[\frac{1}{t_{LT}}])^{\psi_L=0} = (M^{\psi_L=0})[\frac{1}{t_{LT}}] := \{\frac{m}{t_{LT}^k} | m \in M^{\psi_L=0},\ k \geqslant 0\}$.

(ii) *The (continuous) $\mathcal{R}(\Gamma_L)$-action on $M^{\psi_L=0}$ extends to a (continuous with respect to direct limit topology) action of $\mathcal{R}(\Gamma_L)$ on $(M[\frac{1}{t_{LT}}])^{\psi_L=0}$.*

Proof. For (i) note that $0 = \psi_L(\frac{m}{t_{LT}^k}) = \frac{\pi_L^k\psi_L(m)}{t_{LT}^k}$ if and only if $\psi_L(m) = 0$. For (ii) take for any $f \in \mathcal{R}(\Gamma_L)$ the direct limit of the following commutative diagram

$$\begin{array}{ccccccccc}
M^{\psi_L=0} & \xrightarrow{t_{LT}} & M^{\psi_L=0} & \xrightarrow{t_{LT}} & \cdots & \xrightarrow{t_{LT}} & M^{\psi_L=0} & \xrightarrow{t_{LT}} & \cdots \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle Tw_{\chi_{LT}^{-1}}(f)} & & & & \downarrow{\scriptstyle Tw_{\chi_{LT}^{-i}}(f)} & & \\
M^{\psi_L=0} & \xrightarrow{t_{LT}} & M^{\psi_L=0} & \xrightarrow{t_{LT}} & \cdots & \xrightarrow{t_{LT}} & M^{\psi_L=0} & \xrightarrow{t_{LT}} & \cdots
\end{array}$$

This defines a (separatedly continuous) action. □

Consider the composite map

$$\begin{aligned}
\check{c} : \mathcal{R}[\frac{1}{t_{LT}}] \otimes_{\mathcal{R}} \check{M} &\cong \mathrm{Hom}_{\mathcal{R}[\frac{1}{t_{LT}}]}(\mathcal{R}[\frac{1}{t_{LT}}] \otimes_{\mathcal{R}} M, \mathcal{R}[\frac{1}{t_{LT}}] \otimes_{\mathcal{R}} \Omega^1_{\mathcal{R}}) \\
&\cong \mathrm{Hom}_{\mathcal{R}[\frac{1}{t_{LT}}]}(\mathcal{R}[\frac{1}{t_{LT}}] \otimes_{\mathcal{R}} N, \mathcal{R}[\frac{1}{t_{LT}}] \otimes_{\mathcal{R}} \Omega^1_{\mathcal{R}}) \\
&\cong \mathcal{R}[\frac{1}{t_{LT}}] \otimes_{\mathcal{R}} \check{N}
\end{aligned}$$

where the second isomorphism is $(c^{-1})^*$.

Lemma 2.67. *$c^{\psi_L=0}$ and $\check{c}^{\psi_L=0}$ are $\mathcal{R}(\Gamma_L)$-equivariant.*

Lemma 2.68. *The following diagram commutes on the vertical intersections*

$$
\begin{tikzcd}
\check{M}^{\psi_L=0} \arrow[d, hook] & \times & M^{\psi_L=0} \arrow[d, hook] \arrow[r, "\{,\}'_{M,Iw}"] & \mathcal{R}(\Gamma_L) \arrow[ddd, equal] \\
(\mathcal{R}[\frac{1}{t_{LT}}] \otimes_{\mathcal{R}} \check{M})^{\psi_L=0} \arrow[d, "\check{c}"', "\cong"] & \times & (\mathcal{R}[\frac{1}{t_{LT}}] \otimes_{\mathcal{R}} M)^{\psi_L=0} \arrow[d, "c"', "\cong"] & \\
(\mathcal{R}[\frac{1}{t_{LT}}] \otimes_{\mathcal{R}} \check{N})^{\psi_L=0} & \times & (\mathcal{R}[\frac{1}{t_{LT}}] \otimes_{\mathcal{R}} N)^{\psi_L=0} & \\
\check{N}^{\psi_L=0} \arrow[u, hook] & \times & N^{\psi_L=0} \arrow[u, hook] \arrow[r, "\{,\}'_{N,Iw}"] & \mathcal{R}(\Gamma_L),
\end{tikzcd}
$$

i.e., if $\check{m} \in \check{M}, m \in M, \check{n} \in \check{N}, n \in N$ with $\check{c}(\check{m}) = \check{n}$ and $c(m) = n$, then

$$\{\check{m}, m\}'_{M,Iw} = \{\check{n}, n\}'_{N,Iw}.$$

Proof. By definition of the Iwasawa pairings we have for all $f \in \mathcal{R}(\Gamma_L)$

$$
\begin{aligned}
< f, \{\check{n}, n\}'_{N,Iw} > &= \{f \cdot \check{n}, n\}_N \\
&= \{f \cdot \check{c}(\check{m}), c(m)\}_N \\
&= \{\check{c}(f \cdot \check{m}), c(m)\}_N \\
&= Res(\check{c}(f \cdot \check{m})(c(m)) \\
&= Res(((f \cdot \check{m}) \circ c^{-1})(c(m)) \\
&= Res((f \cdot \check{m})(m)) \\
&= \{f \cdot \check{m}, m\}_M \\
&=< f, \{\check{m}, m\}'_{M,Iw} >
\end{aligned}
$$

whence the claim. Here we use the $\mathcal{R}(\Gamma_L)$-equivariance of $\check{c}$ in the third equality. $\square$

Now let D be any φ_L-module over L of finite dimension, say d, (with trivial Γ_L-action) and consider the (φ_L, Γ_L)-module $N := \mathcal{R} \otimes_L D$ over $\mathcal{R}$ (with diagonal actions) Since $N \cong \mathcal{R}^d$ as Γ_L-module, it is L-analytic. Moreover, we have $\check{N} \cong \Omega^1_{\mathcal{R}} \otimes D^*$ with $D^* = \mathrm{Hom}_L(D, L)$ being the dual φ_L-module.

Lemma 2.69. *There is a commutative diagram*

$$
\begin{tikzcd}
(\Omega^1_{\mathcal{R}} \otimes D^*)^{\psi_L=0} & \times & (\mathcal{R} \otimes_L D)^{\psi_L=0} \arrow[r, "\{,\}'_{N,Iw}"] & \mathcal{R}(\Gamma_L) \arrow[d, equal] \\
\mathcal{R}(\Gamma_L) \otimes_L D^* \arrow[u, "\mathfrak{M}_{\chi_{LT}} \otimes \mathrm{id}", "\cong"'] & \times & \mathcal{R}(\Gamma_L) \otimes_L D \arrow[u, "\sigma_{-1}\mathfrak{M} \circ \iota_* \otimes \mathrm{id}", "\cong"'] \arrow[r] & \mathcal{R}(\Gamma_L),
\end{tikzcd}
$$

where the bottom line is the $\mathcal{R}(\Gamma_L)$-linear extension of the canonical pairing between D^* and D, i.e., it maps $(\lambda \otimes l, \mu \otimes d)$ to $\lambda \mu l(d)$.

Proof. Let $\check{d}_j$ and d_i be a basis of D^* and D, respectively, and $x = \sum_j \lambda_j \cdot \check{d}_j$ and $y = \sum_i \mu_i \cdot d_i$. Then, by definition of $\{,\}'_{Iw}$ we have for all $\lambda \in \mathcal{R}(\Gamma_L)$

$$\begin{aligned}
&< \lambda, \{(\mathfrak{M}_{\chi_{LT}} \otimes \mathrm{id})(x), (\sigma_{-1}\mathfrak{M} \circ \iota_* \otimes \mathrm{id})(y)\}'_{Iw} > \\
&\qquad = \{(\lambda\mathfrak{M}_{\chi_{LT}} \otimes \mathrm{id})(x), (\sigma_{-1}\mathfrak{M} \circ \iota_* \otimes \mathrm{id})(y)\} \\
&\qquad = \{\sum_j (\lambda\lambda_j) \cdot (\eta(1,Z) d\log_{LT} \otimes \check{d}_j), \sum_i \iota_*(\mu_i) \cdot \eta(-1,Z) \otimes d_i\} \\
&\qquad = \sum_{i,j} \{(\lambda\lambda_j\mu_i) \cdot (\eta(1,Z) d\log_{LT}) \otimes \check{d}_j, \eta(-1,Z) \otimes d_i\} \\
&\qquad = \sum_{i,j} Res\Big(\check{d}_j(d_i)\eta(-1,Z)(\lambda\lambda_j\mu_i) \cdot (\eta(1,Z) d\log_{LT})\Big). \\
&\qquad = \sum_{i,j} \check{d}_j(d_i) Res\Big(\eta(-1,Z)(\lambda\lambda_j\mu_i) \cdot (\eta(1,Z) d\log_{LT})\Big).
\end{aligned}$$

Here, for the third equation we used property (iii) in Lemma 2.39. On the other hand we can pair the image $\sum_{i,j} \lambda_j\mu_i\check{d}_j(d_i))$ of (x,y) under the bottom pairing with λ using the description (90)

$$\begin{aligned}
< \lambda, \sum_{i,j} \lambda_j\mu_i\check{d}_j(d_i)) > &= \sum_{i,j} \check{d}_j(d_i)\{\mathfrak{M}(\sigma_{-1}), \mathfrak{M}_{\chi_{LT}}(\lambda\lambda_j\mu_i)\} \\
&= \sum_{i,j} \check{d}_j(d_i) Res\Big(\eta(-1,Z)(\lambda\lambda_j\mu_i) \cdot (\eta(1,Z) d\log_{LT})\Big).
\end{aligned}$$

whence comparing with the above gives the result using Proposition 2.47. □

Definition 2.70. *An L-analytic (φ_L, Γ_L)-module M over $\mathcal{R}_L$ is called étale, if it is semistable and of slope* 0*. We write $\mathfrak{M}^{an,ét}(\mathcal{R}_L)$ for the category of étale, analytic (φ_L, Γ_L)-modules over $\mathcal{R}_L$.*

Crucial is the following

Theorem 2.71. *There is an equivalence of categories*

$$\begin{aligned}
Rep_L^{an}(G_L) &\longleftrightarrow \mathfrak{M}^{an,ét}(\mathcal{R}_L) \\
V &\mapsto D^\dagger_{\mathrm{rig}}(V).
\end{aligned}$$

Proof. Theorem D in [Be16]. □

Recall the paragraph before Remark A.24 in the Appendix for the definition of the subring $\mathbf{B}_L^\dagger$ of $\mathcal{R}$. It follows from the proof of [Be16, Thm. 10.1] that for $V \in Rep_L^{an}(G_L)$ we have $D^\dagger_{\mathrm{rig}}(V) = \mathcal{R} \otimes_{\mathbf{B}_L^\dagger} D^\dagger(V)$, where $D^\dagger(V)$ belongs to $\mathfrak{M}^{ét}(\mathbf{B}_L^\dagger)$. From the theory of Wach modules we actually know that $D_{LT}(V)$ is even of finite height, if V is crystallin in addition:

$$D^\dagger(V) = \mathbf{B}_L^\dagger \otimes_{\mathbf{A}_L^+} N(T) = \mathbf{B}_L^\dagger \otimes_{\mathbf{B}_L^+} N(V)$$

for any Galois stable o_L-lattice $T \subseteq V$. From the big diagram in section 1.1 we thus obtain the following diagram, in which the horizontal maps are equivalences of categories.

$$\begin{tikzcd}
\mathrm{Mod}^{\varphi_L,\Gamma_L,an}_{\mathbf{B}_L^+} \arrow[rr, "\mathbf{B}_L\otimes_{\mathbf{B}_L^+}-", "\simeq"'] \arrow[d, "\mathcal{O}\otimes_{\mathbf{B}_L^+}-"'] & & \mathfrak{M}^{et,cris}(\mathbf{B}_L) \\
\mathrm{Mod}^{\varphi_L,\Gamma_L,0}_{\mathcal{O}} \arrow[rr, shift left, "V_L\circ D", "\simeq"'] \arrow[d, "\mathcal{R}\otimes_{\mathcal{O}}-"'] & & \mathrm{Rep}_L^{cris,an}(G_L) \arrow[ll, shift left, "\mathcal{M}\circ D_{cris,L}"] \arrow[u, "\simeq", "D_{LT}(-)"'] \arrow[ull, "N(-)"'] \arrow[d, "\subseteq"'] \\
\mathfrak{M}(\mathcal{R})^{an,ét} & & \mathrm{Rep}_L^{an}(G_L) \arrow[ll, "D^\dagger(V)"', "\simeq"]
\end{tikzcd}$$

Here $\mathfrak{M}^{et,cris}(\mathbf{B}_L)$ denotes the essential image of $\mathrm{Rep}_L^{cris,an}(G_L)$ under $D_{LT}(-)$ in $\mathfrak{M}^{et}(\mathbf{B}_L)$ with $\mathbf{B}_L := \mathbf{A}_L[\frac{1}{\pi_L}]$.

Now let T be an o_L-lattice in an L-linear continuous representation of G_L such that $V^*(1)$ (and hence $V(\tau^{-1})$) is L-analytic and crystalline: Then it follows from [KR] and the discussion above that

$$M := D^\dagger_{\mathrm{rig}}(V(\tau^{-1})) = \mathcal{R}\otimes_{\mathcal{R}^+}\mathcal{M}(D_{cris,L}(V(\tau^{-1}))) = \mathcal{R}\otimes_{\mathbb{A}_L^+} N(T(\tau^{-1}))$$

as well as

$$\check{M} = D^\dagger_{\mathrm{rig}}(V^*(1)) = \mathcal{R}\otimes_{\mathcal{R}^+}\mathcal{M}(D_{cris,L}(V^*(1))) = \mathcal{R}\otimes_{\mathbb{A}_L^+} N(T^*(1))$$

and the comparison isomorphism (20) induces isomorphisms

$$\mathrm{comp}_M : M[\frac{1}{t_{LT}}] \cong \mathcal{R}[\frac{1}{t_{LT}}]\otimes_L D_{cris,L}(V(\tau^{-1}))$$

and

$$\mathrm{comp}_{\check{M}} : \check{M}[\frac{1}{t_{LT}}] \cong \mathcal{R}[\frac{1}{t_{LT}}]\otimes_L D_{cris,L}(V^*(1)).$$

Note that for $c = \mathrm{comp}_M$ and $D = D_{cris,L}(V(\tau^{-1}))$ we have

(94) $$\mathrm{comp}_{\check{M}} = (\mathrm{comp}_{\Omega^1_{\mathcal{R}}}\otimes_L \mathrm{id}_{D^*})\circ \check{c}$$

using the identifications $\Omega^1_{\mathcal{R}} \cong \mathcal{R}(\chi_{LT})$ and

$$D_{cris,L}(V^*(1)) \cong D^*\otimes D_{cris,L}(L(\chi_{LT})).$$

We set $b := \mathrm{comp}_{\Omega^1_{\mathcal{R}}}(t_{LT}^{-1} d\log_{LT}) = \frac{1}{t_{LT}}\otimes\eta \in D_0 := D_{cris,L}(L(\chi_{LT}))$.

Lemma 2.72. *The following diagram commutes*

$$\begin{tikzcd}
\Omega^1_{\mathcal{R}}[\frac{1}{t_{LT}}]\otimes D^* \arrow[r, "\mathrm{comp}_{\Omega^1_{\mathcal{R}}}\otimes_L \mathrm{id}_{D^*}"] & \mathcal{R}[\frac{1}{t_{LT}}]\otimes D^*\otimes D_0 \\
(\Omega^1_{\mathcal{R}}\otimes_L D^*)^{\psi_L=0} \arrow[u, hook] & \mathcal{R}^{\psi_L=0}\otimes D^*\otimes D_0 \arrow[u, hook, "\frac{\nabla}{\Omega}"] \\
\mathcal{R}(\Gamma_L)\otimes_L D^* \arrow[u, "\mathfrak{M}_{\chi_{LT}}\otimes \mathrm{id}_{D^*}", "\cong"'] \arrow[r, "\mathrm{id}_{\mathcal{R}(\Gamma_L)\otimes_L D^*}\otimes b"] & \mathcal{R}(\Gamma_L)\otimes_L D^*\otimes D_0 \arrow[u, "\mathfrak{M}\otimes \mathrm{id}_{D^*\otimes D_0}", "\cong"']
\end{tikzcd}$$

Proof. Observe, since on D^* we have the identity throughout, that the commutativity of the above diagram follows from the commutativity of

$$\begin{tikzcd}
\mathcal{R}(\Gamma_L) \arrow[r, "\mathfrak{M}_{\chi_{LT}}"] \arrow[dd, equal] & (\Omega^1_{\mathcal{R}})^{\psi_L=0} \arrow[r, hook] & \Omega^1_{\mathcal{R}}[\frac{1}{t_{LT}}]^{\psi_L=0} \arrow[ddd, "\cong"', "\mathrm{comp}_{\Omega^1_{\mathcal{R}}}"] \\
 & \mathcal{R}(\chi_{LT})^{\psi_L=0} \arrow[u, "\cong"'] & \\
\mathcal{R}(\Gamma_L) \arrow[r, "\mathfrak{M}\otimes b"] \arrow[d, "\frac{\nabla}{\Omega}"'] & \mathcal{R}^{\psi_L=0}\otimes_L D_{cris,L}(L(\chi_{LT})) \arrow[u, "\frac{\partial_{\mathrm{inv}}}{\Omega}\otimes t_{LT}"] \arrow[d, "t_{LT}\frac{\partial_{\mathrm{inv}}}{\Omega}\otimes \mathrm{id}"'] & \\
\mathcal{R}(\Gamma_L) \arrow[r, "\mathfrak{M}\otimes b"] & \mathcal{R}^{\psi_L=0}\otimes_L D_{cris,L}(L(\chi_{LT})) \arrow[r, hook] & (\mathcal{R}[\frac{1}{t_{LT}}]\otimes_L D_{cris,L}(L(\chi_{LT})))^{\psi_L=0}
\end{tikzcd} \tag{95}$$

where the map $\frac{\partial_{\mathrm{inv}}}{\Omega}\otimes t_{LT} : \mathcal{R}\otimes_L D_{cris,L}(L(\chi_{LT})) \to \mathcal{R}(\chi_{LT})$ sends $f\otimes\frac{1}{t_{LT}}\otimes\eta$ to $\frac{\partial_{\mathrm{inv}}}{\Omega}f\otimes\eta$ and the composite with the natural identification $\mathcal{R}(\chi_{LT})\cong\Omega^1$, which sends η to $d\log_{LT}$, is the map $\frac{d}{\Omega}:\mathcal{R}\to\Omega^1_{\mathcal{R}}$ upon identifying $\mathcal{R}\otimes_L D_{cris,L}(L(\chi_{LT}))$ with $\mathcal{R}$ by sending $f\otimes\frac{1}{t_{LT}}\otimes\eta$ to f. The fact (73) implies the commutativity of the left lower corner while for the upper left corner it follows from (67) and (84)

$$\begin{aligned}
\mathfrak{M}_{\chi_{LT}}(\lambda) &= Tw_{\chi_{LT}}(\lambda)\cdot\eta(1,Z)d\log_{LT}\\
&= Tw_{\chi_{LT}}(\lambda)\cdot\frac{\partial_{\mathrm{inv}}}{\Omega}\eta(1,Z)d\log_{LT}\\
&= \frac{\partial_{\mathrm{inv}}}{\Omega}(\lambda\cdot\eta(1,Z))d\log_{LT}
\end{aligned}$$

Finally, since $\eta(1,Z)\otimes b\in\mathcal{R}^{\psi_L=0}\otimes_L D_{cris,L}(L(\chi_{LT}))$ is send up to $\eta(1,Z)d\log_{LT}$ and down to $t_{LT}\eta(1,Z)\otimes b$, the compatibility with $\mathrm{comp}_{\Omega^1_{\mathcal{R}}}$ is easily checked. $\square$

Now we introduce a pairing

$$[\ ,\ ] := [\ ,\ ]_{D_{cris,L}(V(\tau^{-1}))} : \mathcal{R}^{\psi_L=0}\otimes_L D_{cris,L}(V^*(1)) \times \mathcal{R}^{\psi_L=0}\otimes_L D_{cris,L}(V(\tau^{-1})) \to \mathcal{R}(\Gamma_L)$$

by requiring that the following diagram becomes commutative

$$\begin{tikzcd}
\mathcal{R}^{\psi_L=0}\otimes D^*\otimes D_0 & \times & \mathcal{R}^{\psi_L=0}\otimes_L D \arrow[r, "{[\,,\,]}"] & \mathcal{R}(\Gamma_L) \arrow[d, equal] \\
\mathcal{R}(\Gamma_L)\otimes_L D^*\otimes D_0 \arrow[u, "\mathfrak{M}\otimes\mathrm{id}_{D^*\otimes D_0}", "\cong"'] & \times & \mathcal{R}(\Gamma_L)\otimes_L D \arrow[u, "\sigma_{-1}\mathfrak{M}\circ\iota_*\otimes\mathrm{id}", "\cong"'] \arrow[r] & \mathcal{R}(\Gamma_L),
\end{tikzcd} \tag{96}$$

where the bottom line sends $(\lambda\otimes l\otimes\beta b, \mu\otimes d)$ to $\lambda\mu\beta l(d)$.

Combining the Lemmata 2.69 and 2.72 we obtain for $N=\mathcal{R}\otimes_L D_{cris,L}(V(\tau^{-1}))$

Lemma 2.73. $[-,-]_{D_{cris,L}(V(\tau^{-1}))} = \{\frac{\nabla}{\Omega}(\mathrm{comp}_{\Omega^1_{\mathcal{R}}}\otimes_L \mathrm{id}_{D^*})^{-1}(-),-\}'_{N,Iw}$.

Setting $M' := \mathrm{comp}^{-1}(\mathcal{R}^{\psi_L=0}\otimes_L D_{cris,L}(V(\tau^{-1})))$ and $\check{M}' := \mathrm{comp}^{-1}(\mathcal{R}^{\psi_L=0}\otimes_L D_{cris,L}(V^*(1)))$ we obtain

Theorem 2.74. *For all $x \in \check{M}' \cap (\check{M}^{\psi_L=0})$ and $y \in M' \cap (M^{\psi_L=0})$ it holds*

$$\{\frac{\nabla}{\Omega}x, y\}'_{Iw} = [x, y],$$

i.e., the following diagram commutes on the vertical intersections

$$\begin{tikzcd}
\check{M}^{\psi_L=0} \arrow[d, hook] & \times & M^{\psi_L=0} \arrow[r, "\frac{\nabla}{\Omega}\{,\}'_{M,Iw}"] \arrow[d, hook] & \mathcal{R}(\Gamma_L) \arrow[ddd, equal] \\
(\mathcal{R}[\frac{1}{t_{LT}}] \otimes_{\mathcal{R}} \check{M})^{\psi_L=0} \arrow[d, "\mathrm{comp}_{\check{M}}"', "\cong"] & \times & (\mathcal{R}[\frac{1}{t_{LT}}] \otimes_{\mathcal{R}} M)^{\psi_L=0} \arrow[d, "\mathrm{comp}_{M}"', "\cong"] & \\
(\mathcal{R}[\frac{1}{t_{LT}}] \otimes_L D_{cris,L}(V^*(1)))^{\psi_L=0} & \times & (\mathcal{R}[\frac{1}{t_{LT}}] \otimes_L D_{cris,L}(V(\tau^{-1})))^{\psi_L=0} & \\
\mathcal{R}^{\psi_L=0} \otimes_L D_{cris,L}(V^*(1)) \arrow[u, hook] & \times & \mathcal{R}^{\psi_L=0} \otimes_L D_{cris,L}(V(\tau^{-1})) \arrow[u, hook] \arrow[r, "{[,]_{D_{cris,L}(V(\tau^{-1}))}}"] & \mathcal{R}(\Gamma_L).
\end{tikzcd}$$

Proof. Combine Lemmata 2.73 and 2.68 using (94). □

Interpretation of the abstract reciprocity formula in terms of the $D_{cris,L}$-pairing The canonical pairing $D_{cris,L}(V^*(1)) \times D_{cris,L}(V(\tau^{-1})) \to D_{cris,L}(L(\chi_{LT}))$ extends to a pairing

$$\mathcal{R}^{\psi_L=0} \otimes_L D_{cris,L}(V^*(1)) \quad \times \quad \mathcal{R}^{\psi_L=0} \otimes_L D_{cris,L}(V(\tau^{-1})) \xrightarrow{[,]_{cris}} \mathcal{R}^{\psi_L=0} \otimes_L D_{cris,L}(L(\chi_{LT}))$$

by requiring that the following diagram is commutative (in which the lower one is induced by multiplication within $\mathcal{R}(\Gamma_L)$ and the natural duality paring on $D_{cris,L}$)
(97)

$$\begin{tikzcd}
\mathcal{R}^{\psi_L=0} \otimes_L D_{cris,L}(V^*(1)) & \times & \mathcal{R}^{\psi_L=0} \otimes_L D_{cris,L}(V(\tau^{-1})) \arrow[r, "{[,]_{cris}}"] & \mathcal{R}^{\psi_L=0} \otimes_L D_{cris,L}(L(\chi_{LT})) \\
\mathcal{R}(\Gamma_L) \otimes_L D_{cris,L}(V^*(1)) \arrow[u, "\mathfrak{M}\otimes\mathrm{id}"'] & \times & \mathcal{R}(\Gamma_L) \otimes_L D_{cris,L}(V(\tau^{-1})) \arrow[u, "\sigma_{-1}\mathfrak{M}\circ\iota_*\otimes\mathrm{id}"'] \arrow[r] & \mathcal{R}(\Gamma_L) \otimes_L D_{cris,L}(L(\chi_{LT})) \arrow[u, "\mathfrak{M}\otimes\mathrm{id}"']
\end{tikzcd}$$

Note that

$$\mathrm{comp}\left([x,y] \cdot \eta(1,Z) \otimes (t_{LT}^{-1} \otimes \eta)\right) = [x,y]_{cris}.$$

Hence using the diagram (95) Theorem 2.74 is also equivalent to

$$\mathrm{comp} \circ \mathfrak{M}_{\chi_{LT}} \circ \{x,y\}'_{Iw} = [\mathrm{comp}(x), \mathrm{comp}(y)]_{cris},$$

i.e., the 'commutativity' (whenever it makes sense) of the following diagram
(98)

$$\begin{tikzcd}
D^\dagger_{\mathrm{rig}}(V^*(1))^{\psi_L=\frac{q}{\pi_L}}[\frac{1}{t_{LT}}] \arrow[d, "1-\varphi_L"'] & \times & D^\dagger_{\mathrm{rig}}(V(\tau^{-1}))^{\psi_L=1}[\frac{1}{t_{LT}}] \arrow[r, dashed, "{\{,\}_{Iw}}"] \arrow[d, "1-\frac{\pi_L}{q}\varphi_L"'] & \mathcal{R}(\Gamma_L) \arrow[d, equal] & \\
D^\dagger_{\mathrm{rig}}(V^*(1))^{\psi_L=0}[\frac{1}{t_{LT}}] \arrow[d, "\mathrm{comp}"', "\cong"] & \times & D^\dagger_{\mathrm{rig}}(V(\tau^{-1}))^{\psi_L=0}[\frac{1}{t_{LT}}] \arrow[r, dashed, "{\{,\}'_{Iw}}"] \arrow[d, "\mathrm{comp}"', "\cong"] & \mathcal{R}(\Gamma_L) \arrow[r, "\mathfrak{M}_{\chi_{LT}}"] & \mathcal{R}(\chi_{LT})[\frac{1}{t_{LT}}]^{\psi_L=0} \arrow[d, "\cong"', "\mathrm{comp}"] \\
\mathcal{R}^{\psi_L=0}[\frac{1}{t_{LT}}] \otimes_L D_{cris,L}(V^*(1)) & \times & \mathcal{R}^{\psi_L=0}[\frac{1}{t_{LT}}] \otimes_L D_{cris,L}(V(\tau^{-1})) \arrow[rr, dashed, "{[,]_{cris}}"] & & \mathcal{R}^{\psi_L=0}[\frac{1}{t_{LT}}] \otimes_L D_{cris,L}(L(\chi_{LT}))
\end{tikzcd}$$

Question: Can one extend the definition of [,] and { , } to $(\check{M}[\frac{1}{t_{LT}}])^{\psi_L=0} \times (M[\frac{1}{t_{LT}}])^{\psi_L=0}$ by perhaps enlarging the target $\mathcal{R}(\Gamma_L)$ by an appropriate localisation, which reflects the inversion of t_{LT} somehow?

3 Application

3.1 The regulator map

Recall that we write $\tau^{-1} = \chi_{LT}\chi_{cyc}^{-1}$. Let T be in $\mathrm{Rep}^{cris}_{o_L,f}(G_L)$ such that $T(\tau^{-1})$ belongs to $\mathrm{Rep}^{cris,an}_{o_L,f}(G_L)$ with all Hodge-Tate weights in $[0,r]$, and such that $V := L \otimes_{o_L} T$ does not have any quotient isomorphic to $L(\tau)$. Then we define the regulator maps

$$\begin{aligned}
\mathbf{L}_V : & H^1_{Iw}(L_\infty/L, T) \to D(\Gamma_L, \mathbb{C}_p) \otimes_L D_{cris,L}(V(\tau^{-1})),\\
\mathcal{L}^0_V : & H^1_{Iw}(L_\infty/L, T) \to (\mathcal{R}^+_L)^{\psi_L=0} \otimes_L D_{cris,L}(V(\tau^{-1})),\\
\mathcal{L}_V : & H^1_{Iw}(L_\infty/L, T) \to D(\Gamma_L, \mathbb{C}_p) \otimes_L D_{cris,L}(V)
\end{aligned}$$

as (part of) the composite

$$\begin{aligned}
H^1_{Iw}(L_\infty/L,T) &\cong D_{LT}(T(\tau^{-1}))^{\psi_L=1} = N(T(\tau^{-1}))^{\psi_{D_{LT}(T(\tau^{-1}))}=1} \xrightarrow{(1-\frac{\pi_L}{q}\varphi_L)} \varphi_L^*(N(V(\tau^{-1})))^{\psi_L=0}\\
&\hookrightarrow \mathcal{O}^{\psi_L=0} \otimes_L D_{cris,L}(V(\tau^{-1})) \subseteq (\mathcal{R}^+_{\mathbb{C}_p})^{\psi_L=0} \otimes_L D_{cris,L}(V(\tau^{-1})) \qquad (99)\\
&\xrightarrow{\mathfrak{M}^{-1}\otimes\mathrm{id}} D(\Gamma_L, \mathbb{C}_p) \otimes_L D_{cris,L}(V(\tau^{-1})) \to D(\Gamma_L, \mathbb{C}_p) \otimes_L D_{cris,L}(V)
\end{aligned}$$

using [SV15, Thm. 5.13], Lemma 1.30, the inclusion (22) and where the last map sends $\mu\otimes d \in D(\Gamma_L,\mathbb{C}_p)\otimes_L D_{cris,L}(V(\tau^{-1}))$ to $\mu\otimes d\otimes\mathbf{d}_1 \in D(\Gamma_L,\mathbb{C}_p)\otimes_L D_{cris,L}(V(\tau^{-1}))\otimes_L D_{cris,L}(L(\tau)) \cong D(\Gamma_L,\mathbb{C}_p) \otimes_L D_{cris,L}(V)$. Note that $D := D_{cris,L}(L(\tau)) = D^0_{dR,L}(L(\tau)) = L\mathbf{d}_1$ with $\mathbf{d}_1 = t_{LT}t_{\mathbb{Q}_p}^{-1} \otimes (\eta^{\otimes -1} \otimes \eta^{cyc})$.

Alternatively, in order to stress that the regulator is essentially the map $1-\varphi_L$, one can rewrite this as

(100)

$$\begin{aligned}
H^1_{Iw}(L_\infty/L,T) &\cong D_{LT}(V(\tau^{-1}))^{\psi_L=1} = N(T(\tau^{-1}))^{\psi_{D_{LT}(T(\tau^{-1}))}=1} \hookrightarrow N(V(\tau^{-1}))^{\psi_{D_{LT}(V(\tau^{-1}))}=1} \otimes_L D\\
&\xrightarrow{1-\varphi_L} \varphi_L^*(N(V(\tau^{-1})))^{\psi_L=0} \otimes_L D \hookrightarrow \mathcal{O}^{\psi_L=0} \otimes_L D_{cris,L}(V(\tau^{-1})) \otimes_L D \subseteq (\mathcal{R}^+_{\mathbb{C}_p})^{\psi_L=0} \otimes_L D_{cris,L}(V)\\
&\xrightarrow{\mathfrak{M}^{-1}\otimes\mathrm{id}} D(\Gamma_L,\mathbb{C}_p) \otimes_L D_{cris,L}(V)
\end{aligned}$$

where the $\hookrightarrow$ in the first line sends n to $n\otimes\mathbf{d}_1$ and the φ_L now acts diagonally. By construction, this regulator map $\mathcal{L}_V$ takes values in $D(\Gamma_L,K)^{G_L} \otimes_L D_{cris,L}(V)$.

One significance of regulator maps is that it should interpolate (dual) Bloch-Kato exponential maps. We shall prove such interpolation formulae in subsection 3.2.4 by means of a reciprocity formula.

3.1.1 The basic example

We are looking for a map

$$\mathcal{L} : U \otimes_{\mathbb{Z}} T^*_\pi \to D(\Gamma_L, \mathbb{C}_p) \otimes_L D_{cris,L}(L(\tau))$$

such that

$$\frac{\Omega^r}{r!}\frac{1-\pi_L^{-r}}{1-\frac{\pi_L^r}{q}}\mathcal{L}(u\otimes a\eta^*)(\chi_{LT}^r)\otimes(t_{LT}^{r-1}\otimes\eta^{\otimes -r+1}) = CW(u\otimes a\eta^{\otimes -r})$$

for all r, u, a. where CW denotes the diagonal map in

Theorem 3.1 (A special case of Kato's explicit reciprocity law). *For $r\geqslant 1$ the diagram*

$$\begin{tikzcd}
\varprojlim_n o_{L_n}^\times \otimes_{\mathbb{Z}} T_\pi^{\otimes -r} \arrow[d, "-\kappa\otimes\mathrm{id}"'] \arrow[ddr, "''(1-\pi_L^{-r})r\psi_{CW}^r(-)\mathbf{d}_r''"] & \\
H^1_{Iw}(L_\infty/L, T_\pi^{\otimes -r}(1)) \arrow[d, "\mathrm{cor}"'] & \\
H^1(L, T_\pi^{\otimes -r}(1)) \arrow[r, "\exp^*"] & D^0_{dR,L}(V_\pi^{\otimes -r}(1)) = L\mathbf{d}_r,
\end{tikzcd}$$

commutes, i.e., the diagonal map sends $u\otimes a\eta^{\otimes -r}$ to

$$a(1-\pi_L^{-r})r\psi_{CW}^r(u)\mathbf{d}_r = a\frac{1-\pi_L^{-r}}{(r-1)!}\partial_{\mathrm{inv}}^r\log g_{u,\eta}(Z)_{|Z=0}\mathbf{d}_r$$

with $\mathbf{d}_r := t_{LT}^r t_{\mathbb{Q}_p}^{-1}\otimes(\eta^{\otimes -r}\otimes\eta^{cyc})$.

We set $\mathcal{L} = L\otimes\mathbf{d}_1$ with L given as follows

$$L: U\otimes T_\pi^* \xrightarrow{\nabla} o_L[[\omega_{LT}]]^{\psi_L=1} \xrightarrow{(1-\frac{\pi_L}{q}\varphi)} (\mathcal{R}^+_{\mathbb{C}_p})^{\psi_L=0} \xrightarrow{\log_{LT}\cdot} (\mathcal{R}^+_{\mathbb{C}_p})^{\psi_L=0} \xrightarrow{\mathfrak{M}^{-1}} D(\Gamma_L,\mathbb{C}_p)$$

Using Lemmata 2.15, 2.14 we obtain

$$\begin{aligned}
L(u\otimes a\eta^*)(\chi_{LT}^r) &= a\mathfrak{M}^{-1}(\log_{LT}(1-\frac{\pi_L}{q}\varphi)\partial_{\mathrm{inv}}\log g_u)(\chi_{LT}^r)\\
&= a\Omega^{-1}r\mathfrak{M}^{-1}(1-\frac{\pi_L}{q}\varphi)\partial_{\mathrm{inv}}\log g_u)(\chi_{LT}^{r-1})\\
&= ar\Omega^{-r}(1-\frac{\pi_L}{q}\pi_L^{r-1})(\partial_{\mathrm{inv}}^{r-1}\partial_{\mathrm{inv}}\log g_u)_{|\omega=0}\\
&= ar\Omega^{-r}(1-\frac{\pi_L^r}{q})(\partial_{\mathrm{inv}}^{r-1}\partial_{\mathrm{inv}}\log g_u)_{|\omega=0}.
\end{aligned}$$

By construction and Propostion 2.18 the image of $\mathcal{L}$ actually lies in the G_L-invariants:

$$\mathcal{L}: U\otimes_{\mathbb{Z}} T_\pi^* \to D(\Gamma_L, K)^{G_L}\otimes_L D_{cris,L}(L(\tau)).$$

We claim that

$$(\varprojlim_n o_{L_n}^\times)\otimes_{\mathbb{Z}} T_\pi^* \xrightarrow{-\kappa\otimes T_\pi^*} H^1_{Iw}(L_\infty/L, o_L(\tau)) \xrightarrow{\mathcal{L}_{L(\tau)}} D(\Gamma_L,\mathbb{C}_p)\otimes_L D_{cris,L}(L(\tau))$$

coincides with

$$\mathcal{L} : U \otimes_{\mathbb{Z}} T_\pi^* \to D(\Gamma_L, \mathbb{C}_p) \otimes_L D_{cris,L}(L(\tau)).$$

More generally, we have the following commutative diagram (cp. with [LVZ15, Appendix C] for $L = \mathbb{Q}_p$), setting $e_r := t_{LT}^{-r} \otimes \eta^{\otimes r} \in D_{cris,L}(L(\chi_{LT}^r))$, $r \geqslant 1$:

$$\begin{tikzcd}
U \otimes T_\pi^{\otimes(r-1)} \arrow[rr, "-\kappa \otimes T_\pi^{\otimes(r-1)}"] \arrow[d, "\nabla \otimes \eta^{\otimes r}"'] & & \mathrm{H}^1_{Iw}(L_\infty/L, o_L(\tau\chi_{LT}^r)) \arrow[d, "\cong"] \arrow[dddd, bend left=60, "\mathcal{L}_{L(\tau\chi_{LT}^r)} \otimes \mathbf{d}_1^{\otimes(-1)}"] \\
(o_L[[\omega_{LT}]] \otimes \eta^{\otimes r})^{\psi=1} \arrow[r, hook] \arrow[d, "(1-\frac{\pi_L}{q}\varphi_L)\otimes \mathrm{id}"'] & (\omega_{LT}^{-r} o_L[[\omega_{LT}]] \otimes \eta^{\otimes r})^{\psi=1} \arrow[r, equal] \arrow[d, "(1-\frac{\pi_L}{q}\varphi_L)\otimes \mathrm{id}"] & N(o_L(\chi_{LT}^r))^{\psi=1} \arrow[d, "1-\frac{\pi_L}{q}\varphi_L"] \\
o_L[[\omega_{LT}]][\frac{1}{p}]^{\psi=0} \otimes \eta^{\otimes r} \arrow[r, hook] \arrow[d, "\partial_{\mathrm{inv}}^{-r} \otimes t_{LT}^{-r}"'] & \varphi(\omega_{LT})^{-r} o_L[[\omega_{LT}]][\frac{1}{p}]^{\psi=0} \otimes \eta^{\otimes r} \arrow[r, equal] \arrow[d, "t_{LT}^r \otimes t_{LT}^{-r}"] & \varphi^*(N(L(\chi_{LT}^r)))^{\psi=0} \arrow[d, "comp"] \\
\mathcal{O}^{\psi=0} \otimes e_r \arrow[r, "\mathfrak{l}_0 \cdots \mathfrak{l}_{r-1} \otimes \mathrm{id}", "= t_{LT}^r \partial_{\mathrm{inv}}^r \otimes \mathrm{id}"'] \arrow[dd, "\mathfrak{M}^{-1} \otimes \mathrm{id}"'] & \mathcal{O}^{\psi=0} \otimes e_r \arrow[r, equal] & \mathcal{O}^{\psi=0} \otimes_L D_{cris,L}(L(\chi_{LT}^r)) \arrow[d, "\mathfrak{M}^{-1} \otimes \mathrm{id}"] \\
 & & D(\Gamma_L, K)^{G_L} \otimes_L D_{cris,L}(L(\chi_{LT}^r)) \arrow[d, "\mathfrak{l}_{L(\chi_{LT}^r)}^{-1}"'] \\
D(\Gamma_L, K)^{G_L} \otimes e_r \arrow[rr, equal] \arrow[d, "\mathrm{id} \otimes t_{LT}^r"'] & & D(\Gamma_L, K)^{G_L} \otimes_L D_{cris,L}(L(\chi_{LT}^r)) \arrow[d, "\mathrm{id} \otimes t_{LT}^r"'] \\
D(\Gamma_L, K)^{G_L} \otimes_L \eta^r \arrow[rr, equal] & & D(\Gamma_L, K)^{G_L} \otimes_L o_L(\chi_{LT}^r)
\end{tikzcd}$$

where $\mathfrak{l}_i := t_{LT}\partial_{\mathrm{inv}} - i$, $\partial_{\mathrm{inv}} = \frac{d}{dt_{LT}}$. We write $\nabla_{\mathrm{Lie}} \in \mathrm{Lie}(\Gamma_L)$ for the element in the Lie algebra of Γ_L corresponding to 1 under the identification $\mathrm{Lie}(\Gamma_L) = L$. Note that we have

$$\mathfrak{M}^{-1}(\mathfrak{l}_0 f) = \lim_{\gamma \to 1} \frac{\delta_\gamma(\mathfrak{M}^{-1}(f)) - \mathfrak{M}^{-1}(f)}{\ell(\gamma))} = \nabla_{\mathrm{Lie}} \mathfrak{M}^{-1}(f), \tag{101}$$

see [KR, Lem. 2.1.4] for the fact that $\nabla_{\mathrm{Lie}} = t_{LT}\partial_{\mathrm{inv}}$ as operators on $\mathcal{O}$. By abuse of notation we thus also write $\mathfrak{l}_i = \nabla_{\mathrm{Lie}} - i$ for the corresponding element in $D(\Gamma_L, K)$, compare [ST1, §2.3] for the action of $\mathrm{Lie}(\Gamma_L)$ on and its embedding into $D(\Gamma_L, K)$. Moreover we set $\mathfrak{l}_{L(\chi_{LT}^r)} = \prod_{i=0}^{r-1} \mathfrak{l}_i$. Note that ∂_{inv} is invertible on $\mathcal{O}^{\psi=0}$ by [FX, Prop. 3.12]. Finally the map

$$comp : \varphi^*(N(o_L(\chi_{LT}^r)))^{\psi=0} \to \mathcal{O}^{\psi=0} \otimes_L D_{cris,L}(L(\chi_{LT}^r))$$

is (22).

3.2 Relation to Berger's and Fourquaux' big exponential map

Let V denote a L-analytic representation of G_L and take an integer $h \geqslant 1$ such that $\mathrm{Fil}^{-h} D_{cris,L}(V) = D_{cris,L}(V)$ and such that $D_{cris,L}(V)^{\varphi_L = \pi_L^{-h}} = 0$ holds. Under these conditions in [BF] a big exponential map à la Perrin-Riou

$$\Omega_{V,h} : \left(\mathcal{O}^{\psi_L=0} \otimes_L D_{cris,L}(V)\right)^{\Delta=0} \to D_{\mathrm{rig}}^\dagger(V)^{\psi_L = \frac{q}{\pi_L}}$$

is constructed as follows: According to [BF, Lem. 3.5.1] there is an exact sequence

$$0 \to \bigoplus_{k=0}^{h} t_L^k D_{cris,L}(V)^{\varphi_L=\pi_L^{-k}} \to (\mathcal{O} \otimes_{o_L} D_{cris,L}(V))^{\psi_L=\frac{q}{\pi_L}} \xrightarrow{1-\varphi_L}$$

$$\mathcal{O}^{\psi_L=0} \otimes_L D_{cris,L}(V) \xrightarrow{\Delta} \bigoplus_{k=0}^{h} D_{cris,L}(V)/(1-\pi_L^k\varphi_L)D_{cris,L}(V) \to 0,$$

where, for $f \in \mathcal{O} \otimes_L D_{cris,L}(V)$, $\Delta(f)$ denotes the image of $\bigoplus_{k=0}^{h}(\partial_{\mathrm{inv}}^k \otimes \mathrm{id}_{D_{cris,L}(V)})(f)(0)$ in $\bigoplus_{k=0}^{h} D_{cris,L}(V)/(1-\pi_L^k\varphi_L)D_{cris,L}(V)$. Hence, if $f \in \left(\mathcal{O}^{\psi_L=0} \otimes_L D_{cris,L}(V)\right)^{\Delta=0}$ there exists $y \in (\mathcal{O} \otimes_{o_L} D_{cris,L}(V))^{\psi_L=\frac{q}{\pi_L}}$ such that $f = (1-\varphi_L)y$. Setting $\nabla_i := \nabla - i$ for any integer i, one observes that $\nabla_{h-1} \circ \ldots \circ \nabla_0$ annihilates $\bigoplus_{k=0}^{h-1} t_L^k D_{cris,L}(V)^{\varphi_L=\pi_L^{-k}}$ whence $\Omega_{V,h}(f) := \nabla_{h-1} \circ \ldots \circ \nabla_0(y)$ is well-defined and belongs under the comparison isomorphism (20) to $D_{\mathrm{rig}}^{\dagger}(V)^{\psi_L=\frac{q}{\pi_L}}$ by Proposition 1.13.

Note that $\left(\mathcal{O}^{\psi_L=0} \otimes_L D_{cris,L}(V)\right)^{\Delta=0} = \mathcal{O}^{\psi_L=0} \otimes_L D_{cris,L}(V)$ if $D_{cris,L}(V)^{\varphi_L=\pi_L^{-k}} = 0$ for all $0 \leqslant k \leqslant h$. If this does not hold for V itself, it does hold for $V(\chi_{LT}^{-r})$ for r sufficiently large (with respect to the same h).

In the case $L = \mathbb{Q}_p$ the above map specialises to the exponential map due to Perrin-Riou and satisfies the following adjointness property with Loeffer's and Zerbes' regulator map, see [LVZ15, A.2.2], where the upper pairing and notation are introduced:

$$\begin{array}{ccccc}
D(\Gamma,\mathbb{Q}_p) \otimes_{\Lambda_L} H_{Iw}(\mathbb{Q}_p, V^*(1)) & \times & D(\Gamma,\mathbb{Q}_p) \otimes_{\Lambda_{\mathbb{Q}_p}} H_{Iw}(\mathbb{Q}_p, V) & \longrightarrow & D(\Gamma,\mathbb{Q}_p) \\
\uparrow {\scriptstyle \Omega_{V^*(1),1}} & & {\scriptstyle \gamma_{-1}\mathcal{L}_V} \downarrow & & \| \\
D(\Gamma,\mathbb{Q}_p) \otimes_{\mathbb{Q}_p} D_{cris,\mathbb{Q}_p}(V^*(1)) & \times & D(\Gamma,\mathbb{Q}_p) \otimes_{\mathbb{Q}_p} D_{cris,\mathbb{Q}_p}(V) & \longrightarrow & D(\Gamma,\mathbb{Q}_p)
\end{array}$$

In fact this is a variant of Perrin-Riou's reciprocity law comparing $\Omega_{V,h}$ with $\Omega_{V^*(1),1-h}$.

For $L \neq \mathbb{Q}_p$ the issue of L-analyticity requires that $V^*(1)$ is L-analytic for the construction of $\Omega_{V^*(1),1-h}$, which then implies that V is not L-analytic. Instead our regulator map is available and the purpose of this subsection is to prove an analogue of the above adjointness for arbitrary L.

Theorem 3.2 (Reciprocity formula/Adjointness of Big exponential and regulator map). *Assume that $V^*(1)$ is L-analytic with $\mathrm{Fil}^{-1}D_{cris,L}(V^*(1)) = D_{cris,L}(V^*(1))$ and $D_{cris,L}(V^*(1))^{\varphi_L=\pi_L^{-1}} = D_{cris,L}(V^*(1))^{\varphi_L=1} = 0$. Then the following diagram commutes:*

$$\begin{array}{ccccc}
D_{\mathrm{rig}}^{\dagger}(V^*(1))^{\psi_L=\frac{q}{\pi_L}} & \times & D(V(\tau^{-1}))^{\psi_L=1} & \xrightarrow{\{,\}_{Iw}} & D(\Gamma_L,\mathbb{C}_p) \\
\uparrow {\scriptstyle \Omega_{V^*(1),1}} & & {\scriptstyle \Omega\mathcal{L}_V^0} \downarrow & & \| \\
\mathcal{O}^{\psi_L=0} \otimes_L D_{cris,L}(V^*(1)) & \times & \mathcal{O}^{\psi_L=0} \otimes_L D_{cris,L}(V(\tau^{-1})) & \xrightarrow{[,]} & D(\Gamma_L,\mathbb{C}_p).
\end{array} \tag{102}$$

Note that the terms on the right hand side of the pairings are all defined over L!

Proof. This follows from the abstract reciprocity formula 2.74 (with $M := D_{\mathrm{rig}}^{\dagger}(V(\tau^{-1}))$ as before) by construction. Indeed, assuming that $z \in \mathcal{O}^{\psi_L=0} \otimes_L D_{cris,L}(V^*(1))$ and $y \in D(V(\tau^{-1}))^{\psi_L=1}$ we have that $(1-\frac{\pi_L}{q}\varphi_L)y \in M' \cap (M^{\psi_L=0})$ (see (99)) and $\mathrm{comp}^{-1}((1-\varphi_L)x) \in$

$\check{M}'$ for $x \in (\mathcal{O} \otimes_L D_{cris,L}(V^*(1)))^{\psi_L=\frac{q}{\pi_L}}$ such that $z = (1-\varphi_L)x$. Moreover, $\mathrm{comp}^{-1}((1-\varphi_L)x) \in \check{M}^{\psi_L=0}$ by Proposition 1.13 as $V^*(1)$ is positive by assumption. Recall that $\mathrm{comp}^{-1}(\nabla x)$ is an element in $D^\dagger_{\mathrm{rig}}(V^*(1))^{\psi_L=\frac{q}{\pi_L}}$ again by Proposition 1.13. We thus obtain

$$\begin{aligned}\{\mathrm{comp}^{-1}(\nabla x), y\}_{Iw} &= \{\nabla \mathrm{comp}^{-1}((1-\varphi_L)x), (1-\frac{\pi_L}{q}\varphi_L)y\}'_{Iw}\\ &= \Omega[(1-\varphi_L)x, \mathrm{comp}((1-\frac{\pi_L}{q}\varphi_L)y)].\end{aligned}$$

By definition of the big exponential and regulator map the latter is equivalent to

$$\{\Omega_{V^*(1),1}(z), y\}_{Iw} = [z, \Omega\mathcal{L}^0_V(y)].$$

□

We also could consider the following variant of the big exponential map (under the assumptions of the theorem)

$$\mathbf{\Omega}_{V,h} : D(\Gamma_L, \mathbb{C}_p) \otimes_L D_{cris,L}(V^*(1)) \to D^\dagger_{\mathrm{rig}}(V)^{\psi_L=\frac{q}{\pi_L}}$$

by extending scalars from L to $\mathbb{C}_p$ and composing the original one with Ω^{-h} times

$$D(\Gamma_L, \mathbb{C}_p) \otimes_L D_{cris,L}(V^*(1)) \xrightarrow{\mathfrak{M}\otimes\mathrm{id}} (\mathcal{R}^+)^{\psi_L=0} \otimes_L D_{cris,L}(V^*(1)).$$

Corollary 3.3 (Reciprocity formula/Adjointness of Big exponential and regulator map)**.** *Under the assumptions of the theorem the following diagram commutes:*

$$\begin{array}{ccccc} D^\dagger_{\mathrm{rig}}(V^*(1))^{\psi_L=\frac{q}{\pi_L}} & \times & D(V(\tau^{-1}))^{\psi_L=1} & \xrightarrow{\{,\}_{Iw}} & D(\Gamma_L, \mathbb{C}_p) \\ \uparrow{\scriptstyle \mathbf{\Omega}_{V^*(1),1}} & & {\scriptstyle \sigma_{-1}\mathbf{L}_V}\downarrow & & \| \\ D(\Gamma_L, \mathbb{C}_p) \otimes_L D_{cris,L}(V^*(1)) & \times & D(\Gamma_L, \mathbb{C}_p) \otimes_L D_{cris,L}(V(\tau^{-1})) & \xrightarrow{[,]^0} & D(\Gamma_L, \mathbb{C}_p), \end{array} \tag{103}$$

where $[-,-]^0 = [\mathfrak{M} \otimes \mathrm{id}(-), \sigma_{-1}\mathfrak{M} \otimes \mathrm{id}(-)]$, *i.e.,*

$$[\lambda \otimes \check{d}, \mu \otimes d]^0 \cdot \eta(1,Z) \otimes (t_{LT}^{-1} \otimes \eta) = \lambda\iota_*(\mu) \cdot \eta(1,Z) \otimes [\check{d}, d]_{cris}, \tag{104}$$

where $D_{cris,L}(V^*(1)) \times D_{cris,L}(V(\tau^{-1})) \xrightarrow{[\,,\,]_{cris}} D_{cris,L}(L(\chi_{LT}))$ *is the canonical pairing.*

Remark 3.4. *By [BF, Cor. 3.5.4] we have* $\Omega_{V,h}(x) \otimes \eta^{\otimes j} = \Omega_{V(\chi^j_{LT}),h+j}(\partial_{\mathrm{inv}}^{-j} x \otimes t_{LT}^{-j}\eta^{\otimes j})$ *and* $\nabla_h \circ \Omega_{V,h} = \Omega_{V,h+1}$, *whence we obtain* $\mathbf{\Omega}_{V,h}(x) \otimes \eta^{\otimes j} = \mathbf{\Omega}_{V(\chi^j_{LT}),h+j}(Tw_{\chi_{LT}^{-j}}(x) \otimes t_{LT}^{-j}\eta^{\otimes j})$ *and* $\nabla_h \circ \mathbf{\Omega}_{V,h} = \mathbf{\Omega}_{V,h+1}$.

3.2.1 Some homological algebra

Let $X \xrightarrow{f} Y$ be a morphism of cochain complexes. Its mapping cone $\mathrm{cone}(f)$ is defined as $X[1] \bigoplus Y$ with differential $d^i_{\mathrm{cone}(f)} := \begin{pmatrix} d^i_{X[1]} & 0 \\ f[1]^i & d^i_Y \end{pmatrix}$ (using column notation) and we define the mapping fibre of f as $\mathrm{Fib}(f) := \mathrm{cone}(f)[-1]$. Here the translation $X[n]$ of a complex X is given by $X[n]^i := X^{i+n}$ and $d^i_{X[n]} := (-1)^n d_X^{i+n}$. Alternatively, we may consider f as a double cochain complex concentrated horizontally in degree 0 and 1 and form the total complex (as in [SP, Def. 18.3/tag 012Z]). Then the associated total complex coincides with $\mathrm{Fib}(-f)$.

For a complex $(X^\bullet, d_X)$ of topological L-vector spaces we define its L-dual $((X^*)^\bullet, d_{X^*})$ to be the complex with

$$(X^*)^i := \mathrm{Hom}_{L,cts}(X^{-i}, L)$$

and

$$d_{X^*}(f) := (-1)^{\deg(f)-1} f \circ d_X.$$

By $((X^{*,\mathrm{naive}})^\bullet, d_{X^{*,\mathrm{naive}}})$ we call the naive version by $d_{X^{*,\mathrm{naive}}}(f) := f \circ d_X$.

More generally, for two complexes $(X^\bullet, d_X)$ and $(Y^\bullet, d_Y)$ of topological L-vector spaces we define the complex $\mathrm{Hom}^\bullet_{L,cts}(X^\bullet, Y^\bullet)$ by

$$\mathrm{Hom}^n_{L,cts}(X^\bullet, Y^\bullet) = \prod_{i \in \mathbb{Z}} \mathrm{Hom}_{L,cts}(X^i, Y^{i+n})$$

with differentials $df = d \circ f + (-1)^{\deg(f)-1} f \circ d$. Note that the canonical isomorphism

$$\mathrm{Hom}^\bullet(X^\bullet, Y^\bullet)[n] \xrightarrow{\cong} \mathrm{Hom}^\bullet(X^\bullet, Y^\bullet[n])$$

does not involve any sign, i.e., it is given by the identity map in all degrees.

Also we recall that the tensor product of two complexes $X^\bullet$ and $Y^\bullet$ is given by

$$(X^\bullet \otimes_L Y^\bullet)^i := \bigoplus_n X^n \otimes_L Y^{i-n}$$

and

$$d(x \otimes y) = dx \otimes y + (-1)^{\deg(x)} x \otimes dy.$$

The adjunction morphism on the level of complexes

$$\mathrm{adj} : \mathrm{Hom}^\bullet_{L,cts}(X^\bullet \otimes_L Y^\bullet, Z^\bullet) \to \mathrm{Hom}^\bullet_{L,cts}(Y^\bullet, \mathrm{Hom}^\bullet_{L,cts}(X^\bullet, Z^\bullet))$$

sends u to $(y \mapsto (x \mapsto (-1)^{\deg(x)\deg(y)} u(x \otimes y)))$. It is well-defined and continuous with respect to the projective tensor product topology and the strong topology for the Homs. Furthermore, by definition we have the following commutative diagram

$$\begin{tikzcd} X^\bullet \otimes_L Y^\bullet \arrow[r, "u"] \arrow[d, "\mathrm{id} \otimes \mathrm{adj}(u)"'] & L[-2] \arrow[d, equal] \\ X^\bullet \otimes_L \mathrm{Hom}^\bullet_{L,cts}(X^\bullet, L[-2]) \arrow[r, "ev_2"] & L[-2] \end{tikzcd} \tag{105}$$

where ev_2 sends (x, f) to $(-1)^{\deg(x)\deg(f)} f(x)$.

Lemma 3.5. *Let $(\mathcal{C}^\bullet, d^\bullet)$ be a complex in the category of locally convex topological L-vector spaces.*

(i) *If $\mathcal{C}$ consists of Fréchet spaces and $h^i(\mathcal{C}^\bullet)$ is finite-dimensional over L, then d^{i-1} is strict and has closed image.*

(ii) *If d^i is strict, then $h^{-i}(\mathcal{C}^*) \cong h^i(\mathcal{C})^*$.*

Proof. (i) Apply the argument from [BW, § IX, Lem. 3.4] and use the open mapping theorem [NFA, Prop. 8.8]. (ii) If

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C$$

forms part of the complex with B in degree i, one immediately obtains a map

$$\ker(\alpha^*)/\mathrm{im}(\beta^*) \to (\ker(\beta)/\mathrm{im}(\alpha))^*,$$

where $\ker(\beta)$ carries the subspace topology and $\ker(\beta)/\mathrm{im}(\alpha)$ the quotient topology. Now use the Hahn-Banach theorem [NFA, Cor. 9.4] for the strict maps $B/\ker(\beta) \hookrightarrow C$ (induced from β) and $\ker(\beta) \hookrightarrow B$ in order to show that this map is an isomorphism. □

Definition 3.6. *A locally convex topological vector space is called an LF-space, if it is the direct limit of a countable family of Fréchet spaces, the limit being formed in the category of locally convex vector spaces.*

Remark 3.7. (i) *If $V \xrightarrow{\alpha} W$ is a continuous linear map of Hausdorff LF-spaces with finite dimensional cokernel, then α is strict and has closed image by the same argument used in (i) of the previous lemma. However, since a closed subspace of an LF-space need not be an LF-space, we cannot achieve the same conclusion for complexes by this argument as $\ker(d^i)$ may fail to be an LF-space, whence one cannot apply the open mapping theorem, in general. But consider the following special situation. Assume that the complex $\mathcal{C}^\bullet$ consists of LF-spaces and $h^i(\mathcal{C}^\bullet)$ is finite-dimensional. If moreover $\mathcal{C}^{i+1} = 0$, i.e., $\mathcal{C}^i = \ker(d^i)$, then d^{i-1} is strict and $h^{1-i}(\mathcal{C}^*) \cong h^{i-1}(\mathcal{C})^*$.*

(ii) *If d^i is not strict, the above proof still shows that we obtain a surjection $h^{-i}(\mathcal{C}^*) \twoheadrightarrow h^i(\mathcal{C})^*$.*

However, for a special class of LF-spaces and under certain conditions we can say more about how forming duals and cohomology interacts.

Lemma 3.8. *Let $(\mathcal{C}^\bullet, d^\bullet) = \varinjlim_r (\mathcal{C}_r^\bullet, d_r^\bullet)$ be a complex in the category of locally convex topological L-vector spaces arising as regular inductive limit of complexes of Fréchet spaces, i.e., in each degree i the transition maps in the countable sequence $(\mathcal{C}_r^i)_r$ are injective and for each bounded subset $B \subseteq \mathcal{C}^i$ there exists an $r \geqslant 1$ such that B is contained in $\mathcal{C}_r^i$ and is bounded as a subset of the Fréchet space $\mathcal{C}_r^i$. Then,*

(i) *we have topological isomorphisms $(\mathcal{C}^\bullet)^* \cong \varprojlim_r (\mathcal{C}_r^\bullet)^*$,*

(ii) *if, in addition, $\varprojlim^1_{r\geqslant 0} h^i((\mathcal{C}_r^\bullet)^*) = 0$ for all i, we have a long exact sequence*

$$\dots \longrightarrow h^i((\mathcal{C}^\bullet)^*) \longrightarrow \varprojlim_{r\geqslant 0} h^i((\mathcal{C}_r^\bullet)^*) \longrightarrow h^{i-1}(\varprojlim^1_{r\geqslant 0}(\mathcal{C}_r^\bullet)^*) \longrightarrow h^{i+1}((\mathcal{C}^\bullet)^*) \longrightarrow \dots,$$

(iii) *if, in addition to (ii), the differentials $d_r^\bullet$ are strict, e.g., if all $h^i(\mathcal{C}_r^\bullet)$ have finite dimension over L, and $\varprojlim^1_{r\geqslant 0}(\mathcal{C}_r^\bullet)^* = 0$, we have isomorphisms*

$$h^i((\mathcal{C}^\bullet)^*) \cong \varprojlim_{r\geqslant 0} h^{-i}(\mathcal{C}_r^\bullet)^*.$$

Proof. (i) is [PGS, Thm: 11.1.13] while (ii), (iii) follows from (i) and [1, Ch. 3, Prop. 1] applied to the inverse system $((\mathcal{C}_r^\bullet)^*)_r$ combined with Lemma 3.5. . □

3.2.2 Koszul complexes

In this paragraph we restrict to the situation $U \cong \mathbb{Z}_p^d$ and fix topological generators $\gamma_1, \dots \gamma_d$ of U and we set $\Lambda := \Lambda(U)$. Furthermore, let M be any complete linearly topologized o_L-module with a continuous U-action. Then by [Laz2, Thm. II.2.2.6] this actions extends to continuous Λ-action and one has $\mathrm{Hom}_{\Lambda,cts}(\Lambda, M) = \mathrm{Hom}_\Lambda(\Lambda, M)$.

Consider the (homological) complexes $K_\bullet(\gamma_i) := [\Lambda \xrightarrow{\gamma_i - 1} \Lambda]$ concentrated in degrees 1 and 0 and define

$$\begin{aligned}
K_\bullet :=& K_\bullet^U := K_\bullet(\gamma) := \bigotimes_{i=1}^{d}{}_{\Lambda} K_\bullet(\gamma_i),\\
K^\bullet(M) :=& K_U^\bullet(M) := \mathrm{Hom}_\Lambda^\bullet(K_\bullet, M) \cong \mathrm{Hom}_\Lambda^\bullet(K_\bullet, \Lambda) \otimes_\Lambda M = K^\bullet(\Lambda) \otimes_\Lambda M,\\
K_\bullet(M) :=& K_\bullet \otimes_\Lambda M \text{ (homological complex)},\\
K_\bullet(M)^\bullet :=& (K_\bullet \otimes_\Lambda M)^\bullet \text{ (the associated cohomological complex)}.
\end{aligned}$$

If we want to indicate the dependence on $\gamma = (\gamma_1, \dots \gamma_d)$ we also write $K^\bullet(\gamma, M)$ instead of $K^\bullet(M)$ and similarly for other notation; moreover, we shall use the notation $\gamma^{-1} = (\gamma_1^{-1}, \dots, \gamma_d^{-1})$ and $\gamma^{p^n} = (\gamma_1^{p^n}, \dots \gamma_d^{p^n})$. Note that in each degree these complexes consists of a direct sum of finitely many copies of M and will be equipped with the corresponding direct product topology.

The complex $K_\bullet$ will be identified with the exterior algebra complex $\bigwedge_\Lambda^\bullet \Lambda^d$ of the free Λ-module with basis $e_1, \dots, e_d$, for which the differentials $d_q : \bigwedge_\Lambda^q \Lambda^d \to \bigwedge_\Lambda^{q-1} \Lambda^d$ with respect to the standard basis $e_{i_1,\dots,i_q} = e_{i_1} \wedge \dots \wedge e_{i_q}$, $1 \leqslant i_1 < \dots < i_q \leqslant d$, is given by the formula

$$d_q(a_{i_1,\dots,i_q}) = \sum_{k=1}^{q} (-1)^{k+1} (\gamma_{i_k} - 1) a_{i_1,\dots,\widehat{i_k},\dots,i_q}.$$

Then the well-known selfduality (compare [Ei, Prop. 17.15] although the claim there is not precisely the same) of the Koszul complex, i.e., the isomorphism of complexes

$$K_\bullet(\Lambda)^\bullet \cong K^\bullet(\Lambda)[d] \tag{106}$$

can be explicitly described in degree $-q$ as follows (by identifying $\bigwedge_\Lambda^d \Lambda^d = \Lambda e_1 \wedge \dots \wedge e_d = \Lambda$):

$$\begin{aligned}
\bigwedge\nolimits_\Lambda^q \Lambda^d &\xrightarrow{\alpha_{-q}} \mathrm{Hom}_\Lambda(\bigwedge\nolimits_\Lambda^{d-q} \Lambda^d, \Lambda)\\
e_{i_1,\dots,i_q} &\mapsto \quad \mathrm{sign}(I, J) e^*_{j_1,\dots,j_{d-q}},
\end{aligned}$$

where $e_1^*, \dots e_d^*$ denotes the dual basis of $e_1, \dots, e_d$, the elements $e^*_{j_1,\dots,j_{d-q}} = e^*_{j_1} \wedge \dots \wedge e^*_{j_{d-q}}$, $1 \leqslant j_1 < \dots < j_{d-q} \leqslant d$, form a (dual) basis of $\mathrm{Hom}_\Lambda(\bigwedge_\Lambda^{d-q} \Lambda^d, \Lambda)$, the indices $J = (j_k)_k$ are complementary to $I = (i_n)_n$ in the following sense $\{i_1, \dots, i_q\} \cup \{j_1, \dots, j_{d-q}\} = \{1, \dots, d\}$ and $\mathrm{sign}(I, J)$ denotes the sign of the permutation $[i_1, \dots, i_q, j_1, \dots, j_{d-q}]$. Indeed, the verification that the induced diagram involving the differentials from cohomological degree $-q$ to $-q+1$

$$\begin{tikzcd}
\bigwedge_\Lambda^q \Lambda^d \arrow[r, "\alpha_{-q}"] \arrow[d, "d_q"'] & \mathrm{Hom}_\Lambda(\bigwedge_\Lambda^{d-q} \Lambda^d, \Lambda) \arrow[d, "(-1)^d (-1)^{d-q-1} d^*_{d-q+1}"] \\
\bigwedge_\Lambda^{q-1} \Lambda^d \arrow[r, "\alpha_{-q+1}"] & \mathrm{Hom}_\Lambda(\bigwedge_\Lambda^{d-q+1} \Lambda^d, \Lambda)
\end{tikzcd}$$

commutes, relies on the observation that

$$\operatorname{sign}(I,J)\operatorname{sign}(I_{\widehat{k}},J_k)^{-1} = (-1)^{q-k+l-1},$$

where $I_{\widehat{k}} := (i_1,\dots,\widehat{i_k},\dots,i_q)$ denotes the sequence which results from I by omitting i_k while $J_k = (j_1,\dots,j_{l-1},i_k,j_l,\dots i_{d-q})$ denotes the sequence which arises from J by inserting i_k at position l with regard to the strict increasing ordering: The permutations $[i_1,\dots,i_q,j_1,\dots,j_{d-q}]$ and $[i_1,\dots,\widehat{i_k},\dots,i_q,j_1,\dots,j_{l-1},i_k,j_l,\dots,j_{d-q}]$ differ visibly by $q-k+l-1$ transpositions.

Now we assume that M is any complete locally convex L-vector space with continuous U-action such that its strong dual is again complete with continuous U-action. Then we obtain isomorphisms of complexes

$$\begin{aligned} K^\bullet(\gamma,M)^* &= \operatorname{Hom}^\bullet_{L,cts}(\operatorname{Hom}^\bullet_\Lambda(K_\bullet(\gamma),\Lambda)\otimes_\Lambda M, L)\\ &\cong \operatorname{Hom}^\bullet_\Lambda(\operatorname{Hom}^\bullet_\Lambda(K_\bullet(\gamma^{-1}),\Lambda),\operatorname{Hom}_{L,cts}(M,L))\\ &\cong \operatorname{Hom}^\bullet_\Lambda(\operatorname{Hom}^\bullet_\Lambda(K_\bullet(\gamma^{-1}),\Lambda),\Lambda)\otimes_\Lambda \operatorname{Hom}_{L,cts}(M,L)\\ &\cong K_\bullet(\gamma^{-1},\Lambda)^\bullet\otimes_\Lambda \operatorname{Hom}_{L,cts}(M,L)\\ &\cong K^\bullet(\gamma^{-1},\Lambda)[d]\otimes_\Lambda M^*\\ &\cong K^\bullet(\gamma^{-1},M^*)[d], \end{aligned}\tag{107}$$

where in the second line we use the adjunction morphism; the isomorphism in the fourth line being the biduality morphism (according to [Ne, (1.2.8)])

$$\begin{aligned} K_\bullet(\Lambda)^\bullet &\xrightarrow{\cong} \operatorname{Hom}^\bullet_\Lambda(\operatorname{Hom}^\bullet_\Lambda(K_\bullet,\Lambda),\Lambda)\\ x &\mapsto (-1)^i x^{**} \end{aligned}$$

with the usual biduality of modules

$$\begin{aligned} K_\bullet(\Lambda)^i &\xrightarrow{\cong} \operatorname{Hom}_\Lambda(\operatorname{Hom}_\Lambda(K_{-i},\Lambda),\Lambda)\\ x &\mapsto (x^{**}: f\mapsto f(x)) \end{aligned}$$

involves a sign, while the isomorphism in the third last line stems from (106) together with Lemma 2.39 (i). Note that the isomorphism in the second last line does not involve any further signs by [Ne, (1.2.15)].

We finish this subsection by introducing restriction and corestriction maps concerning the change of group for Koszul complexes. To this end let $U_1 \subseteq U$ be the open subgroup generated by $\gamma_1^{p^n},\dots\gamma_d^{p^n}$. Then $\operatorname{Hom}^\bullet_\Lambda(-,M)$ applied to the tensor product of the diagrams

$$\begin{tikzcd} \Lambda(U) \arrow[r, "\gamma_i^{p^n}-1"] & \Lambda(U) \\ \Lambda(U) \arrow[u, equal] \arrow[r, "\gamma_i-1"] & \Lambda(U) \arrow[u, "\sum_{k=0}^{p^n-1}\gamma_i^k"'] \end{tikzcd}$$

gives a map $cor_U^{U_1}: K^\bullet_{U_1}(\gamma^{p^n})(M)\to K^\bullet_U(\gamma)(M)$ which we call corestriction map and which is compatible under (121) below with the corestriction map on cocylces (for appropriate choices of representatives in the definition of the latter). Using the diagram

$$\begin{tikzcd} \Lambda(U) \arrow[r, "\gamma_i^{p^n}-1"] \arrow[d, "\sum_{k=0}^{p^n-1}\gamma_i^k"'] & \Lambda(U) \arrow[d, equal] \\ \Lambda(U) \arrow[r, "\gamma_i-1"] & \Lambda(U) \end{tikzcd}$$

instead, one obtains the restriction map $res_{U_1}^U : K_U^\bullet(\gamma)(M) \to K_{U_1}^\bullet(\gamma^{p^n})(M)$, again compatible under (121) with the restriction map on cocycles.

3.2.3 Continuous and analytic cohomology

For any profinite group G and topological abelian group M with continuous G-action we write $\mathcal{C}^\bullet := \mathcal{C}^\bullet(G,M)$ for the continuous (inhomogeneous) cochain complex of G with coefficients in M and $H^*(G,M) := h^*(\mathcal{C}^\bullet(G,M))$ for continuous group cohomology. Note that $\mathcal{C}^0(G,M) = M$.

If G is moreover a L-analytic group and $M = \varinjlim_s \varprojlim_r M^{[r,s]}$ with Banach spaces $M^{[r,s]}$ a LF space with a pro-L-analytic action of G, i.e., a locally analytic action on each $M^{[r,s]}$, which means that for all $m \in M^{[r,s]}$ there exist an open L-analytic subgroup $\Gamma_n \subseteq \Gamma$ in the notation of subsection 2.2.2 such that the orbit map of m restricted to Γ_n is a power series of the form $g(m) = \sum_{k\geqslant 0} \ell(g)^k m_k$ for a sequence m_k of elements in $M^{[r,s]}$ with $\pi_L^{nk} m_k$ converging to zero. Following [Co2, §5] we write $\mathcal{C}_{an}^\bullet := \mathcal{C}_{an}^\bullet(G,M)$ for the locally L-analytic cochain complex of G with coefficients in M and $H_{an}^*(G,M) := h^*(\mathcal{C}_{an}^\bullet(G,M))$ for locally L-analytic group cohomology. More precisely, if $\mathrm{Maps}_{locL-an}(G, M^{[r,s]})$ denotes the space of locally L-analytic maps from G to $M^{[r,s]}$, then

$$C_{an}^n(G,M) = \varinjlim_s \varprojlim_r \mathrm{Maps}_{locL-an}(G, M^{[r,s]})$$

is the space of locally L-analytic functions (locally with values in $\varprojlim_r M^{[r,s]}$ for some s and such that the composite with the projection onto $M^{[r,s]}$ is locally L-analytic for all r). Note that again $\mathcal{C}_{an}^0(G,M) = M$ and that there are canonical homomorphisms

$$\mathcal{C}_{an}^\bullet(G,M) \hookrightarrow \mathcal{C}^\bullet(G,M), \tag{108}$$

$$H_{an}^\bullet(G,M) \to H^\bullet(G,M). \tag{109}$$

Let f be any continuous endomorphism of M which commutes with the G-action. We define

$$H^0(f,M) := M^{f=1} \qquad \text{and} \qquad H^1(f,M) := M_{f=1} \tag{110}$$

as the kernel and cokernel of the map $M \xrightarrow{f-1} M$, respectively.

The endomorphism f induces an operator on $\mathcal{C}^\bullet$ or $\mathcal{C}_{an}^\bullet$ and we denote by $\mathcal{T} := \mathcal{T}_{f,G}(M)$ and $\mathcal{T}^{an} := \mathcal{T}_{f,G}^{an}(M)$ the mapping fibre of $\mathcal{C}^\bullet(G,f)$ and $\mathcal{C}_{an}^\bullet(G,f)$, respectively.

Again there are canonical homomorphisms

$$\mathcal{T}_{f,G}^{an}(M) \hookrightarrow \mathcal{T}_{f,G}(M), \tag{111}$$

$$h^\bullet(\mathcal{T}_{f,G}^{an}(M)) \to h^\bullet(\mathcal{T}_{f,G}(M)). \tag{112}$$

For ? either empty or an, one of the corresponding double complex spectral sequences is

$$ {}_IE_2^{i,j} = H^i(f, H^j_?(G,M)) \Longrightarrow h^{i+j}(\mathcal{T}^?) \tag{113}$$

It degenerates into the short exact sequences

$$0 \longrightarrow H^{i-1}_?(G,M)_{f=1} \longrightarrow h^i(\mathcal{T}^?_{f,G}(M)) \longrightarrow H^i_?(G,M)^{f=1} \longrightarrow 0.$$

In (loc. cit.) as well as in [BF] analytic cohomology is also defined for the semigroups $\Gamma_L \times \Phi$ and $\Gamma_L \times \Psi$ with $\Phi = \{\varphi_L^n | n \geqslant 0\}$ and $\Psi = \{(\frac{\pi}{q}\psi_L)^n | n \geqslant 0\}$, if M denotes an L-*analytic* (φ_L, Γ_L)-module over the Robba ring $\mathcal{R}$.

Remark 3.9. *Any L-analytic (φ_L, Γ_L)-module M over the Robba ring $\mathcal{R}$ is a pro-L-analytic Γ_L-module by the discussion at the end of the proof of [BSX, Prop. 2.25], whence it is also an L-analytic $\Gamma_L \times \Phi$- and $\Gamma_L \times \Psi$-module as Φ and Ψ possess the discrete structure as L-analytic manifolds.*

Proposition 3.10. *We have canonical isomorphisms*

$$h^i(\mathcal{T}^{an}_{\varphi_L,\Gamma_L}(M)) \cong H^i_{an}(\Gamma_L \times \Phi, M) \cong H^i_{an}(\Gamma_L \times \Psi, M) \cong h^i(\mathcal{T}^{an}_{\frac{\pi}{q}\psi_L,\Gamma_L}(M)).$$

and an exact sequence

(114)

$$0 \longrightarrow H^1_{an}(\Gamma_L, M^{\psi_L = \frac{q}{\pi}}) \longrightarrow h^i(\mathcal{T}^{an}_{\frac{\pi}{q}\psi_L,\Gamma_L}(M)) \longrightarrow (M_{\psi_L=\frac{q}{\pi}})^{\Gamma_L} \longrightarrow H^2_{an}(\Gamma_L, M^{\psi_L=\frac{q}{\pi}}) \longrightarrow h^2(\mathcal{T}^{an}_{\frac{\pi}{q}\psi_L,\Gamma_L}(M)) \; .$$

Proof. The isomorphism in the middle is [BF, cor. 2.2.3]. For the two outer isomorphism we refer the reader to [Th, 3.7.6]. The exact sequence is the extension [Th, Thm. 5.1.5] of [BF, Thm. 2.2.4]. □

Note that, for $U \subseteq U'$, the restriction and corestriction homomorphisms $\mathcal{C}^\bullet(U', M) \xrightarrow{\text{res}} \mathcal{C}^\bullet(U, M)$ and $\mathcal{C}^\bullet(U, M) \xrightarrow{\text{cor}} \mathcal{C}^\bullet(U', M)$ induce maps on $\mathcal{T}_{f,U'}(M) \xrightarrow{\text{res}} \mathcal{T}_{f,U}(M)$ and $\mathcal{T}_{f,U}(M) \xrightarrow{\text{cor}} \mathcal{T}_{f,U'}(M)$, respectively.

We write $\mathrm{Ext}^1_{\mathfrak{C}}(A, B)$ for isomorphism classes of extensions of B by A in any abelian category $\mathfrak{C}$. Furthermore, we denote by $\mathfrak{M}_U(R)$ ($\mathfrak{M}^{ét}_U(R)$, $\mathfrak{M}^\dagger_U(R)$) the category of all (étale, overconvergent) (φ_L, U)-modules over R, respectively, and by $\mathrm{Rep}^\dagger_L(G_{L^U_\infty})$ the category of overconvergent representations of $G_{L^U_\infty}$ consisting of those representations V of $G_{L^U_\infty}$ such that $D(V)$ is an overconvergent (φ_L, U)-module; see Definition A.25, where also the notation $D^\dagger(V)$) is introduced.

Theorem 3.11. *Let V be in $\mathrm{Rep}_L(G_L)$ and $U \subseteq \Gamma_L$ be any open subgroup.*

(i) *For $D(V)$ the corresponding (φ_L, Γ_L)-module over $\mathbf{B}_L$ we have canonical isomorphisms*

$$h^* = h^*_{U,V} : H^*(L^U_\infty, V) \xrightarrow{\cong} h^*(\mathcal{T}_{\varphi_L,U}(D(V))) \tag{115}$$

which are functorial in V and compatible with restriction and corestriction.

(ii) *If V is in addition overconvergent there are isomorphisms*

$$h^0(\mathcal{T}_{\varphi_L,U}(D^\dagger_{rig}(V))) \cong V^{G_{L^U_\infty}}, \tag{116}$$

$$h^1(\mathcal{T}_{\varphi_L,U}(D^\dagger_{rig}(V))) \cong H^1_\dagger(L^U_\infty, V), \tag{117}$$

which are functorial in V and compatible with restriction and corestriction and where by definition $H^1_\dagger(L^U_\infty, V) \subseteq H^1(L^U_\infty, V)$ classifies the overconvergent extensions of L by V. In particular, these L-vector spaces have finite dimension.

(iii) *If V is in addition L-analytic, then we have*

$$H^1_{an}(L^U_\infty, V) \xrightarrow{\cong} h^1(\mathcal{T}^{an}_{\varphi_L,U}(D^\dagger_{rig}(V))) \tag{118}$$

where by definition $H^1_{an}(L^U_\infty, V) \subseteq H^1_\dagger(L^U_\infty, V) \subseteq H^1(L^U_\infty, V)$ *classifies the L-analytic extensions of L by V.*

Proof. (i) is [Ku, Thm. 5.1.11.] or [KV, Thm. 5.1.11.]. The statement (iii) is [BF, prop. 2.2.1] combined with Proposition 3.10 while (ii) follows from [FX] (the reference literally only covers the case $U = \Gamma_L$, but the same arguments allow to extend the result to general U) as follows: Firstly, by Lemma 3.12 below one has an isomorphism $h^1(\mathcal{T}_{\varphi_L,U}(D^\dagger_{rig}(V))) \cong \mathrm{Ext}^1_{\mathfrak{M}_U(\mathcal{R}_L)}(\mathcal{R}_L, D^\dagger_{rig}(V))$. Then use the HN-filtration à la Kedlaya to see that any extension of étale (φ_L, U)-modules is étale again, whence

$$\mathrm{Ext}^1_{\mathfrak{M}_U(\mathcal{R}_L)}(\mathcal{R}_L, D^\dagger_{rig}(V)) = \mathrm{Ext}^1_{\mathfrak{M}^{ét}_U(\mathcal{R}_L)}(\mathcal{R}_L, D^\dagger_{rig}(V))$$

and the latter group equals

$$\mathrm{Ext}^1_{\mathfrak{M}^\dagger_U(\mathcal{R}_L)}(\mathcal{R}_L, D^\dagger_{rig}(V)) \cong \mathrm{Ext}^1_{\mathrm{Rep}^\dagger_L(G_{L^U_\infty})}(L, V) = H^1_\dagger(L^U_\infty, V)$$

by prop. 1.5 and 1.6 in (loc. cit.). For the claim in degree 0 one has to show that the inclusion $D^\dagger(V) \subseteq D^\dagger_{rig}(V)$ induces an isomorphism on φ_L-invariants, which follows from Lemma A.36. □

Lemma 3.12. *Let M be in $\mathfrak{M}_U(\mathcal{R})$. Then we have a canonical isomorphism*

$$h^1(\mathcal{T}_{\varphi_L,U}(M)) \cong \mathrm{Ext}^1_{\mathfrak{M}_U(\mathcal{R}_L)}(\mathcal{R}_L, M).$$

Proof. Starting with a class $z = [(c_1, -c_0)]$ in $h^1(\mathcal{T}_{\varphi_L,U}(M))$ with $c_1 \in C^1(M)$ and $c_0 \in C^0(M) = M$ (i.e., we work with *inhomogeneous* continuous cocycle) satisfying the cocycle property

$$c_1(\sigma\tau) = \sigma c_1(\tau) + c_1(\sigma) \text{ for all } \sigma, \tau \in U, \quad \text{and} \quad (\varphi_L - 1)c_1(\tau) = (\tau - 1)c_0 \text{ for all } \tau \in U, \tag{119}$$

we define an extension of (φ_L, U)-modules

$$0 \to M \longrightarrow E_c \longrightarrow \mathcal{R}_L \to 0$$

with $E_c := M \times \mathcal{R}_L$ as $\mathcal{R}_L$-module, $g(m, r) := (gm + gr \cdot c_1(g), gr)$ for $g \in U$ and $\varphi_{E_c}((m, r)) := (\varphi_M(m) + \varphi_L(r)c_0, \varphi_L(r))$; note that this defines a (continuous) group-action by the first identity in (119), while the U- and φ_L-action commute by the second identity in (119). If we change the representatives $(c_1, -c_0)$ by the coboundary induced by $m_0 \in M$, then sending $(0, 1)$ to $(-m_0, 1)$ induces an isomorphism of extensions from the first to the second one, whence our map is well-defined.

Conversely, if E is any such extension, choose a lift $e \in E$ of $1 \in \mathcal{R}_L$ and define

$$c_1(\tau) := (\tau - 1)e \in M, \quad c_0 := (\varphi_E - 1)e,$$

which evidently satisfy the cocycle conditions (119). Choosing another lift $\tilde{e}$ leads to a cocylce which differs from the previous one by the coboundary induced by $\tilde{e} - e \in M$, whence the inverse map is well-defined.

One easily verifies that these maps are mutually inverse to each other. □

Question 3.13. *Can one show that $h^2(\mathcal{T}_{\varphi_L,U}(D^\dagger_{rig}(V)))$ is finite-dimensional (and related to $H^2(L^U_\infty, V)$) and that the groups $h^i(\mathcal{T}_{\varphi_L,U}(D^\dagger_{rig}(V)))$ vanish for $i \geqslant 3$?*

Remark 3.14. *By [FX, Thm. 0.2, Rem. 5.21] it follows that the inclusions*

$$H^1_{an}(L^U_\infty, V) \subseteq H^1_\dagger(L^U_\infty, V) \subseteq H^1(L^U_\infty, V)$$

are in general strict. More precisely, the codimension for the left one equals $([L^U_\infty : \mathbb{Q}_p] - 1)\dim_L V^{G_{L^U_\infty}}$.

Let us recall Tate's local duality in this context.

Proposition 3.15 (Local Tate duality)**.** *Let V be an object in $\mathrm{Rep}_L(G_L)$, and K any finite extension of L. Then the cup product and the local invariant map induce perfect pairings of finite dimensional L-vector spaces*

$$H^i(K,V) \times H^{2-i}(K, \mathrm{Hom}_{\mathbb{Q}_p}(V, \mathbb{Q}_p(1))) \longrightarrow H^2(K, \mathbb{Q}_p(1)) = \mathbb{Q}_p$$

and

$$H^i(K,V) \times H^{2-i}(K, \mathrm{Hom}_L(V, L(1))) \longrightarrow H^2(K, L(1)) = L$$

where $-(1)$ denotes the Galois twist by the cyclotomic character. In other words, there are canonical isomorphisms

$$H^i(K,V) \cong H^{2-i}(K, V^*(1))^* \ .$$

Proof. This is well known. For lack of a reference (with proof) we sketch the second claim (the first being proved similarly). Choose a Galois stable o_L-lattice $T \subseteq V$ and denote by ${}_{\pi_L^n}A$ the kernel of multiplication by π_L^n on any o_L-module A. Observe that we have short exact sequences

$$0 \longrightarrow H^i(K,T)/\pi_L^n \longrightarrow H^i(K, T/\pi_L^n T) \longrightarrow {}_{\pi_L^n}H^{i+1}(K,T) \longrightarrow 0$$

for $i \geqslant 0$ and similarly for T replaced by $T^*(1) = \mathrm{Hom}_{o_L}(T, o_L(1))$. By [SV15, Prop. 5.7] (remember the normalisation given there!) the cup product induces isomorphism

$$H^i(K, T/\pi_L^n T) \cong \mathrm{Hom}_{o_L}(H^{2-i}(K, T^*(1)/\pi_L^n T^*(1)), o_L/\pi_L^n)$$

such that we obtain altogether canonical maps

$$H^i(K,T)/\pi_L^n \to \mathrm{Hom}_{o_L}(H^{2-i}(K,T^*(1))/\pi_L^n, o_L/\pi_L^n) \cong \mathrm{Hom}_{o_L}(H^{2-i}(K,T^*(1)), o_L)/\pi_L^n.$$

Using that the cohomology groups are finitely generated o_L-modules and isomorphic to the inverse limits of the corresponding cohomology groups with coefficients modulo π_L^n we see that the inverse limit of the above maps induces a surjective map

$$H^i(K,T) \twoheadrightarrow \mathrm{Hom}_{o_L}(H^{2-i}(K,T^*(1)), o_L)$$

with finite kernel, whence the claim after tensoring with L over o_L using the isomorphism $H^i(K,T) \otimes_{o_L} L \cong H^i(K,V)$ and analogously for $T^*(1)$. □

Now let W be a L-analytic representation of G_L and set

$$H^1_{/\dagger}(L^U_\infty, W^*(1)) := H^1_\dagger(L^U_\infty, W)^*,$$

which, by local Tate duality and Theorem 3.11, is a quotient of $H^1(L^U_\infty, W^*(1))$. By definition, the local Tate pairing induces a non-degenerate pairing

$$<,>_{Tate,L,\dagger}\colon H^1_\dagger(L^U_\infty, W) \quad \times H^1_{/\dagger}(L^U_\infty, W^*(1)) \longrightarrow H^2(L, L(1)) \cong L. \tag{120}$$

In order to compute this pairing more explicitly in certain situations we shall use Koszul-complexes. For this we have to assume first that U is torsionfree. Following [CoNi, §4.2] we obtain for any complete linearly topologised o_L-module M with continuous U-action a quasi-isomorphism

$$K^\bullet_U(M) \xrightarrow{\simeq} \mathcal{C}^\bullet(U, M) \tag{121}$$

which arises as follows: Let $X_\bullet := X_\bullet(U)$ and $Y_\bullet = Y_\bullet(U)$ denote the completed standard complex [Laz2, V.1.2.1], i.e., $X_n = \mathbb{Z}_p[[U]]^{\hat{\otimes}(n+1)}$, and the standard complex computing group cohomology, i.e., $Y_n = \mathbb{Z}_p[U]^{\otimes(n+1)}$. Then, by [Laz2, Lem. V.1.1.5.1] we obtain a diagram of complexes

$$\begin{array}{ccc} Y_\bullet(U) & \xrightarrow{\Delta} & Y_\bullet(U\times U) \cong Y_\bullet(U)\otimes_{\mathbb{Z}_p} Y_\bullet(U) \\ \downarrow & & \downarrow \\ X_\bullet(U) & \xrightarrow{\Delta} & X_\bullet(U\times U) \cong X_\bullet(U)\hat{\otimes}_{\mathbb{Z}_p} X_\bullet(U) \\ \downarrow & & \downarrow \\ K^U_\bullet & \xrightarrow{\Delta} & K^{U\times U}_\bullet \cong K^U_\bullet \hat{\otimes}_{\mathbb{Z}_p} K^U_\bullet, \end{array} \tag{122}$$

which commutes up to homotopy (of filtered Λ-modules) . Here the maps Δ are induced by the diagonal maps $U \to U \times U$, e.g., $\mathbb{Z}_p[[U]] \to \mathbb{Z}_p[[U\times U]] \cong \mathbb{Z}_p[[U]]\hat{\otimes}_{\mathbb{Z}_p}\mathbb{Z}_p[[U]]$. The first column induces a morphism

$$\mathrm{Hom}_\Lambda(K^U_\bullet, M) \to \mathrm{Hom}_{\Lambda,cts}(X_\bullet(U), M) \to \mathrm{Hom}_{\mathbb{Z}_p[U],cts}(Y_\bullet(U), M),$$

which is (121). The upper line induces as usual the cup product on continuous group cohomology

$$H^r(U,M) \times H^s(U,N) \xrightarrow{\cup_U} H^{r+s}(U, M\otimes N)$$

via

$$\begin{aligned} \mathrm{Hom}_{\mathbb{Z}_p[U],cts}(Y_\bullet(U), M) \times \mathrm{Hom}_{\mathbb{Z}_p[U],cts}(Y_\bullet(U), N) &\xrightarrow{\times} \mathrm{Hom}_{\mathbb{Z}_p[U]\otimes\mathbb{Z}_p[U],cts}(Y_\bullet(U)\otimes_{\mathbb{Z}_p} Y_\bullet(U), M\otimes N) \\ &\xrightarrow{\Delta^*} \mathrm{Hom}_{\mathbb{Z}_p[U],cts}(Y_\bullet(U), M\otimes N). \end{aligned}$$

The lower line induces analogously the Koszul-product

$$K^r_U(M) \times K^s_U(N) \xrightarrow{\cup_K} K^{r+s}_U(M\otimes N).$$

By diagram (122) both products are compatible with each other.

Let f be any continuous endomorphism of M which commutes with the U-action; it induces an operator on $K^\bullet(M)$ and we denote by $K_{f,U}(M) := \mathrm{cone}\left(K^\bullet(M) \xrightarrow{f-\mathrm{id}} K^\bullet(M)\right)[-1]$ the mapping fibre of $K^\bullet(f)$. Then the quasi-isomorphism (121) induces a quasi-isomorphism

$$K_{\varphi,U}(M) \xrightarrow{\simeq} \mathcal{T}_{\varphi,U}(M). \tag{123}$$

Remark 3.16. *By a standard procedure cup products can be extended to hyper-cohomology (defined via total complexes), we follow [Ne, (3.4.5.2)], but for the special case of a cone, see also [Ni, Prop. 3.1]. In particular, we obtain compatible cup products $\cup_K$ and $\cup_U$ for $K_{\varphi,U}(M)$ and $\mathcal{T}_{\varphi,U}(M)$, respectively.*

Now we allow some arbitrary open subgroup $U \subseteq \Gamma_L$ and let $L' = L_\infty^U$. Note that we obtain a decomposition $U \cong \Delta \times U'$ with a subgroup $U' \cong \mathbb{Z}_p^d$ of U and Δ the torsion subgroup of U. By Lemma A.44 we obtain a canonical isomorphism

$$K_{\varphi,U'}(M^\Delta) \xrightarrow{\simeq} \mathcal{T}_{\varphi,U}(M). \tag{124}$$

Now let M be a finitely generated projective $\mathcal{R}$-module M with continuous U-action. Then $M^* = \check{M}$ is again a finitely generated projective $\mathcal{R}$-module M with continuous U-action by Lemma 2.39 (i). Hence M as well as M^Δ satisfies the assumptions of (107) and we have isomorphisms

$$\begin{aligned} K_{\varphi,U}(M^\Delta)^* &\cong \mathrm{cone}\left(K^\bullet(M^\Delta)^* \xrightarrow{\varphi^*-1} K^\bullet(M^\Delta)^*\right) \\ &= \mathrm{cone}\left(K^\bullet((M^\Delta)^*)[d] \xrightarrow{\varphi^*-1} K^\bullet((M^\Delta)^*)[d]\right) \\ &= K_{\varphi^*,U}((M^*)_\Delta)[d+1] \\ &= K_{\psi,U}(\check{M}_\Delta)[d+1] \\ &= K_{\psi,U}(\check{M}^\Delta)[d+1]. \end{aligned} \tag{125}$$

The last isomorphism is induced by the canonical isomorphism $\check{M}^\Delta \cong \check{M}_\Delta$.

Now note that

$$D^\dagger_{\mathrm{rig}}(W)^\vee \cong D^\dagger_{\mathrm{rig}}(W^*(\chi_{LT})) \tag{126}$$

for any L-analytic representation W by the fact that the functor $D^\dagger_{\mathrm{rig}}$ respects inner homs, (cp. [SV, Remark 5.6] for the analogous case D_{LT}). Hence the tautological pairing ev_2 from (105) together with the above isomorphism (125) induces the following pairing (see also the lower pairing of diagram (172)):

(127)

$$\cup_{K,\psi} : h^1(K_{\varphi,U'}(D^\dagger_{rig}(W)^\Delta)) \times h^1(K_{\psi,U'}(D^\dagger_{rig}(W^*(\chi_{LT}))^\Delta)[d-1]) \longrightarrow L$$

Remark 3.17. *For $U = U'$ and $M = D^\dagger_{rig}(W)$, on the level of cochains this pairing is given as follows:*

$$\check{M} \oplus K^{d-1}(\check{M}) \times K^1(M) \oplus M \to L, ((x,y),(x',y')) \mapsto \{y',x\} - y(x'),$$

where we again use that $K^{d-1}(\check{M}) \cong K^1(M)^*$ and where $\{\ ,\ \}$ denotes the pairing (77). More generally, we have the following diagram

(128)

$$\begin{tikzcd}[column sep=small]
K_{\varphi,U}(M): & 0 \arrow[r] & M \arrow[r, "\begin{pmatrix} d_K^0 \\ 1-\varphi \end{pmatrix}"] & K^1(M)\oplus M \arrow[r, "\begin{pmatrix} d_K^1 & 0 \\ 1-\varphi & -d_K^0 \end{pmatrix}"] & K^2(M)\oplus K^1(M) \arrow[r] & {} \\
& & \times & \times & \times & \\
K_{\psi,U}(\check{M})[d-1]: & {} \arrow[r] & K^{d-1}(\check{M})\oplus K^{d-2}(\check{M}) \arrow[r, "\begin{pmatrix} d_K^{d-1} & 0 \\ 1-\psi & -d_K^{d-2} \end{pmatrix}"] \arrow[d] & \check{M}\oplus K^{d-1}(\check{M}) \arrow[r, "\begin{pmatrix} 1-\psi & -d_K^{d-1} \end{pmatrix}"] \arrow[d] & \check{M} \arrow[r] \arrow[d] & 0 \\
& & L & L & L & \\
\textit{in degrees:} & & 0 & 1 & 2 &
\end{tikzcd}$$

Recall that $W = V^*(1)$ is L-analytic and set $M = D^\dagger_{rig}(W)$ as well as $\check{M} = D^\dagger_{rig}(V(\tau^{-1})) = D^\dagger_{rig}(W^*(\chi_{LT}))$. We obtain a Fontaine-style, explicit map

$$pr_U : D^\dagger_{rig}(V(\tau^{-1}))^{\psi=1} \to h^1(K_{\psi,U'}(\check{M}^\Delta)[d-1]),\ m \mapsto [(\bar{m},0)], \tag{129}$$

where $\bar{m} = \frac{1}{\#\Delta}\sum_{\delta\in\Delta}\delta m$ denotes the image of m under the map $\check{M} \twoheadrightarrow \check{M}_\Delta \cong \check{M}^\Delta$.

Remark 3.18. *Let $U_1 \subseteq U$ an open subgroup with torsion subgroups Δ_1 and Δ, respectively. Assume that the torsionfree parts U_1' and U' are generated by $\gamma_1^{p^n},\dots\gamma_d^{p^n}$ and $\gamma_1,\dots\gamma_d$, respectively. Then, for M any complete locally convex L-vector space with continuous U-action, the restriction and corestriction maps of Koszul-complexes from section 3.2.2 extend by functoriality to the mapping fibre*

$$\begin{aligned}
cor_U^{U_1} :=& cor_{U'}^{U_1'} \circ K_{\varphi,U_1'}(N_{\Delta/\Delta_1}) : K_{\varphi,U_1'}(M^{\Delta_1}) \to K_{\varphi,U'}(M^\Delta)\\
res_{U_1}^U :=& K_{\varphi,U_1'}(\iota)\circ res_{U_1'}^{U'} : K_{\varphi,U'}(M^\Delta) \to K_{\varphi,U_1'}(M^{\Delta_1})
\end{aligned}$$

Here $N_{\Delta/\Delta_1} : M^{\Delta_1} \to M^\Delta$ denotes the norm/trace map sending m to $\sum_{\delta\in\Delta/\Delta_1}\delta m$ while $\iota : M^\Delta \to M^{\Delta_1}$ is the inclusion. Taking duals as in (125) we also obtain

$$\begin{aligned}
cor_U^{U_1} :=& (res_{U_1}^U)^*[1-d] : K_{\psi,U_1'}(M^{\Delta_1}) \to K_{\psi,U'}(M^\Delta)\\
res_{U_1}^U :=& (cor_U^{U_1})^*[1-d] : K_{\psi,U'}(M^\Delta) \to K_{\psi,U_1'}(M^{\Delta_1})
\end{aligned}$$

(co)restriction maps for the ψ-Herr complexes.

Since inflation is compatible with restriction and corestriction one checks that the above maps are compatible under the isomorphism (115) with the usual maps in Galois cohomology. Moreover, they define such maps on $H^1_\dagger$ and $H^1_{/\dagger}$ via (117) and $h^1(K_{\psi,U'}(D^\dagger_{rig}(W^*(\chi_{LT}))^\Delta[d-1]) \cong H^1_{/\dagger}(L', W^*(1))$.

By the discussion at the end of section 3.2.2 the restriction map $K_{\varphi,U'}(M^{\Delta}) \xrightarrow{\mathrm{res}^U_{U_1}} K_{\varphi,U_1'}(M^{\Delta_1})$ and corestriction map $K_{\varphi,U_1'}(M^{\Delta_1}) \xrightarrow{\mathrm{cor}^{U_1}_U} K_{\varphi,U'}(M^{\Delta})$ in degree 0 are given as inclusion $M^{\Delta} \hookrightarrow M^{\Delta_1}$ and norm $M^{\Delta_1} \xrightarrow{N_{U',U_1'}\circ N_{\Delta/\Delta_1}} M^{\Delta}$, respectively, where

$$N_{U',U_1'} := \prod_{i=1}^{d}\sum_{k=0}^{p^n-1}\gamma_i^k \in \Lambda(U').$$

Hence, by duality the restriction map $K_{\psi,U}(\check{M}^{\Delta})[d-1]^2 \xrightarrow{\mathrm{res}^U_{U_1}} K_{\psi,U_1}(\check{M}^{\Delta_1})[d-1]^2$ and corestriction map $K_{\psi,U_1}(\check{M}^{\Delta_1})[d-1]^2 \xrightarrow{\mathrm{cor}^{U_1}_U} K_{\psi,U}(\check{M}^{\Delta})[d-1]^2$ are given by the norm $\check{M}^{\Delta} \xrightarrow{(\Delta:\Delta_1)\iota(N_{U',U_1'})} \check{M}^{\Delta_1}$ and projection map $\check{M}^{\Delta_1} \xrightarrow{\frac{1}{(\Delta:\Delta_1)}N_{\Delta/\Delta_1}} \check{M}^{\Delta}$, respectively. Here ι denotes the involution of $\Lambda(U)$ sending u to u^{-1}. Note that the latter two descriptions also hold for the first components of $K_{\psi,U}(\check{M}^{\Delta})[d-1]^1 \xrightarrow{\mathrm{res}^U_{U_1}} K_{\psi,U_1}(\check{M}^{\Delta_1})[d-1]^1$ and $K_{\psi,U_1}(\check{M}^{\Delta_1})[d-1]^1 \xrightarrow{\mathrm{cor}^{U_1}_U} K_{\psi,U}(\check{M}^{\Delta})[d-1]^1$, respectively. Hence, we obtain

$$\mathrm{cor}^{U_1}_U \circ pr_{U_1} = pr_U \text{ and } \mathrm{res}^U_{U_1} \circ pr_U = pr_{U_1} \circ N_{\Delta/\Delta_1} \circ \iota(N_{U',U_1'}).$$

Berger and Fourquaux in contrast define a different Fontaine-style map in [BF, Thm. 2.5.8] for an L-analytic representation Z and $N = D^{\dagger}_{\mathrm{rig}}(Z)$

$$\begin{aligned}(130)\quad h^1_{L^U_\infty,Z} : D^{\dagger}_{\mathrm{rig}}(Z)^{\psi_L=\frac{q}{\pi_L}} &\to H^1_{an}(U, D^{\dagger}_{\mathrm{rig}}(Z)^{\psi_L=\frac{q}{\pi_L}}) \to h^1(\mathcal{T}_{\varphi_L,U}(N)) \cong h^1(K_{\varphi_L,U}(N^{\Delta})),\\ y &\mapsto \qquad\qquad [c_b(y)] \mapsto \qquad [(c_b(y),-m_c)] \mapsto [(\tilde{c}_b(y),-\tilde{m}_c)],\end{aligned}$$

in which the cocycle $h^1_{L^U_\infty,Z}(y)$ is given in terms of the pair $(c_b(y),-m_c)$ in the notation of Theorem 2.5.8 in (loc. cit.): m_c is the unique element in $D^{\dagger}_{\mathrm{rig}}(Z)^{\psi_L=0}$ such that

$$(\varphi_L - 1)c_b(y)(\gamma) = (\gamma-1)m_c \tag{131}$$

for all $\gamma \in U$ and this pair defines the extension class in the sense of Lemma 3.12. Here, the first map is implicity given by Proposition 2.5.1 in (loc. cit.), the second one is the composite from maps arising in Cor. 2.2.3, Thm. 2.2.4, of (loc. cit.) with the natural map from analytic to continuous cohomology

$$H^1_{an}(U, D^{\dagger}_{\mathrm{rig}}(Z)^{\psi_L=\frac{q}{\pi_L}}) \to H^1_{an}(U\times\Psi, D^{\dagger}_{\mathrm{rig}}(Z)) \cong H^1_{an}(U\times\Phi, D^{\dagger}_{\mathrm{rig}}(Z)) \to H^1(U\times\Phi, D^{\dagger}_{\mathrm{rig}}(Z))$$

combined with the interpretation of extension classes (see §1.4 in (loc. cit.) and Lemma 3.12), while the last one is (124) (the concrete image $(\tilde{c}_b(y),-\tilde{m}_c)$ will be of interest for us only in the situation where Δ is trivial, when $\tilde{m}_c = m_c$).

According to [BF, Prop. 2.5.6, Rem. 2.5.7] this map also satisfies

$$cor^U_{U'} \circ h^1_{L^{U'}_\infty,Z} = h^1_{L^U_\infty,Z}. \tag{132}$$

Since $D^{\dagger}_{\mathrm{rig}}(V(\tau^{-1})))^{\vee} \cong D^{\dagger}_{\mathrm{rig}}(V^*(1))$ by (126), concerning the Iwasawa-pairing we have the following

Proposition 3.19. *For a G_L-representation V such that $V^*(1)$ is L-analytic the following diagram is commutative*

$$\begin{tikzcd}
h^1(K_{\varphi,U'}(D^\dagger_{rig}(V^*(1)(\chi_{LT}^j))^\Delta)) & \times & h^1(K_{\psi,U'}(D^\dagger_{rig}(V(\chi_{LT}^{-j}))^\Delta)[d-1]) \arrow[r, "\cup_{K,\psi}"] & L \subseteq \mathbb{C}_p \\
D^\dagger_{\mathrm{rig}}(V^*(1))^{\psi_L=\frac{q}{\pi_L}} \arrow[u, "h^1_{L,V^*(1)(\chi_{LT}^j)}\circ tw_{\chi_{LT}^j}"] & \times & D^\dagger_{\mathrm{rig}}(V(\tau^{-1}))^{\psi_L=1} \arrow[u, "pr_U\circ tw_{\chi_{LT}^{-j}}"] \arrow[r, "\{,\}_{Iw}"] & D(\Gamma_L,\mathbb{C}_p). \arrow[u, "\frac{1}{\Omega}ev_{\chi_{LT}^{-j}}"]
\end{tikzcd}$$

Proof. By Lemma 2.65, it suffices to show the case $j=0$, i.e., the trivial character χ_{triv}. Furthermore, it suffices to show the statement for any subgroup of the form Γ_n without any p-torsion:

$$\frac{1}{\Omega}ev_{L_n,\chi_{triv}} \circ \{x,y\}_{Iw,\Gamma_n} = h^1_{L_n,V^*(1)}(x) \cup_{K,\psi} pr_{\Gamma_n}(y) \tag{133}$$

for $x \in D^\dagger_{\mathrm{rig}}(V^*(1))^{\psi_L=\frac{q}{\pi_L}}$, $y \in D^\dagger_{\mathrm{rig}}(V(\tau^{-1}))^{\psi_L=1}$.

Indeed, by Remark 3.18, for every such n, we have the commutative diagram

$$\begin{tikzcd}
h^1(K_{\varphi,\Gamma_n}(D^\dagger_{rig}(V^*(1)))) \arrow[d, "\mathrm{cor}"] & \times & h^1(K_{\psi,\Gamma_n}(D^\dagger_{rig}(V))[d-1]) \arrow[r, "\cup_{K,\psi}"] & L \arrow[d, equal] \\
h^1(K_{\varphi,U'}(D^\dagger_{rig}(V^*(1))^\Delta)) & \times & h^1(K_{\psi,U'}(D^\dagger_{rig}(V)^\Delta)[d-1]) \arrow[u, "\mathrm{res}"] \arrow[r, "\cup_{K,\psi}"] & L.
\end{tikzcd}$$

Hence we obtain using (132)

$$\begin{aligned}
h^1_{L',V^*(1)}(x) \cup_{K,\psi} pr_U(y) &= (\mathrm{cor} \circ h^1_{L_n,V^*(1)}(x)) \cup_{K,\psi} pr_U(y) \\
&= h^1_{L_n,V^*(1)}(x) \cup_{K,\psi} (\mathrm{res} \circ pr_U(y)) \\
&= h^1_{L_n,V^*(1)}(x) \cup_{K,\psi} (pr_{\Gamma_n}(N_\Delta \circ \iota(N_{U',\Gamma_n})y)),
\end{aligned}$$

where we use Remark 3.18 for the last equality. On the other hand one easily checks that

$$ev_{L_n,\chi_{triv}} \circ pr_{U,\Gamma_n} \circ N_\Delta \circ \iota(N_{U',\Gamma_n}) = ev_{L,\chi_{triv}} : D(U,\mathbb{C}_p) \to \mathbb{C}_p,$$

whence

$$\begin{aligned}
ev_{L,\chi_{triv}} \circ \{x,y\}_{Iw,U} &= ev_{L_n,\chi_{triv}} \circ pr_{U,\Gamma_n}(N_\Delta \circ \iota(N_{U',\Gamma_n})\{x,y\}_{Iw,U}) \\
&= ev_{L_n,\chi_{triv}} \circ pr_{U,\Gamma_n}(\{x, N_\Delta \circ \iota(N_{U',\Gamma_n})y\}_{Iw,U}) \\
&= ev_{L_n,\chi_{triv}} \circ \{x, N_\Delta \circ \iota(N_{U',\Gamma_n})y\}_{Iw,\Gamma_n}
\end{aligned}$$

where we have used Remark 2.64 for the last equation.

In order to prove (133) choose $n=n_0$ (see section 2.2.2). As recalled in (130)) the map

$$h^1_{L_{n_0},V^*(1)} : D^\dagger_{\mathrm{rig}}(V^*(1))^{\psi_L=\frac{q}{\pi_L}} \to h^1(K_{\varphi_L,\Gamma_{n_0}}(D^\dagger_{\mathrm{rig}}(V^*(1))))$$

is given by the cocycle $h^1_{L_{n_0},V^*(1)}(x)$ in terms of the pair $(\tilde{c}_b(x), -m_c)$. Note that we have

$$m_c = \widehat{\Xi}_b(\varphi_L - 1)x.$$

Indeed, by [BF, Thm. 2.5.8] we have $c_b(x)(b_j^k) = (b_j^k - 1)\widehat{\Xi_b}x$ for all $j, k \geqslant 0$, which together with (131) and the uniqueness of m_c (loc. cit.) implies the claim. On the other hand we have the map (129)

$$pr_{\Gamma_{n_0}} : D^\dagger_{rig}(V(\tau^{-1}))^{\psi_L=1} \to h^1(K_{\psi,\Gamma_{n_0}}(D^\dagger_{rig}(V(\tau^{-1})))[d-1]), \quad y \to \text{ class of } (y, 0).$$

Thus the pairing $\cup_{K,\psi}$ sends by construction (see diagram (128)) the above classes to

$$\begin{aligned} h^1_{L_n,V^*(1)}(x) \cup_{K,\psi} pr_{\Gamma_n}(y) &= 0(\tilde{c}_b(x)) + \{-\widehat{\Xi_b}(\varphi_L - 1)x, y\} \\ &= \{\widehat{\Xi_b}(\varphi_L - 1)x, (\frac{\pi_L}{q}\varphi_L - 1)y\} \\ &= < \widehat{\Xi_b}, \{x, y\}_{Iw,\Gamma_n} >_{\Gamma_n} \\ &= \frac{1}{\Omega}\mathrm{aug}(\{x, y\}_{Iw,\Gamma_n}). \end{aligned}$$

Here the second equality holds due to Lemma 2.39 because the left hand side belongs to $D^\dagger_{\mathrm{rig}}(V^*(1))^{\psi_L=0}$, the third one is (93) while the last one comes from (86). $\square$

Proposition 3.20. *For W an L-analytic representation we have a canonical commutative diagram*

$$\begin{tikzcd} \cup_{K,\psi} : h^1(K_{\varphi,U'}(D^\dagger_{rig}(W)^\Delta)) \arrow[d, "b"', "\cong"] & \times & h^1(K_{\psi,U'}(D^\dagger_{rig}(W^*(\chi_{LT}))^\Delta)[d-1]) \arrow[d, "a"', "\cong"] \arrow[rr] & & L \arrow[d, equal] \\ <,>_{Tate,L,\dagger}: H^1_\dagger(L', W) \arrow[d, hook] & \times & H^1_{/\dagger}(L', W^*(1)) \arrow[r] & H^2(L', L(1)) \arrow[r, "\cong"] \arrow[d, equal] & L \arrow[d, equal] \\ <,>_{Tate,L'}: H^1(L', W) & \times & H^1(L', W^*(1)) \arrow[r] \arrow[u, two heads, "pr"] & H^2(L', L(1)) \arrow[r, "\cong"] & L. \end{tikzcd}$$

Moreover, the isomorphism a is compatible with the middle maps of the diagrams (172) *and* (178).

Proof. The lower square of pairings comes from Tate duality as in Prop. 3.15 and (120). Its commutativity holds by definition. In the upper square of pairings the left upper vertical isomorphism b arises from (117) combined with (124), while the middle vertical isomorphism a is uniquely determined as adjoint of the latter because both pairings are non-degenerate: The middle one by definition of $H^1_{/\dagger}$ while the upper one due to Corollary A.48(ii) with $W = V^*(1)$. Therefore one immediately checks that $a^{-1} \circ pr$ is induced by the cohomology of the middle map going down in diagram (172) in Appendix A.2 again with $W = V^*(1)$. By the same reason, $a^{-1} \circ pr$ it is also induced by the cohomology of the middle map (going down) in diagram (178) (being the same as the middle map (going from right to left) of diagram (179) upon identifying $h^1\left(K^\bullet_{\psi,U'}(D(V(\tau^{-1}))^\Delta)[d-1]\right)$ and $H^1(L', V)$ by the isomorphism described there). $\square$

Combining the last two propositions we get the following result.

Corollary 3.21. *For a G_L-representation V such that $V^*(1)$ is L-analytic the following diagram is commutative*

$$\begin{tikzcd}
H^1_\dagger(L', V^*(1)(\chi_{LT}^j)) & \times & H^1_{/\dagger}(L', V(\chi_{LT}^{-j})) \arrow[r, "<{,}>_{Tate,L'}"] & H^2(L', L(1)) \cong L \subseteq \mathbb{C}_p \\
D^\dagger_{\mathrm{rig}}(V^*(1))^{\psi_L = \frac{q}{\pi_L}} \arrow[u, "h^1_{L',V^*(1)} \circ tw_{\chi_{LT}^j}"'] & \times & D^\dagger_{\mathrm{rig}}(V(\tau^{-1}))^{\psi_L=1} \arrow[u, "pr_{L'} \circ tw_{\chi_{LT}^{-j}}"'] \arrow[r, "\{{,}\}_{Iw}"] & D(\Gamma_L, \mathbb{C}_p). \arrow[u, "\frac{1}{\Omega} ev_{\chi_{LT}^{-j}}"']
\end{tikzcd}$$

With respect to evaluating at a character we have the following analogue of Corollary 3.21.

Proposition 3.22. *For a G_L-representation V such that $V^*(1)$ is L-analytic the following diagram is commutative*

$$\begin{tikzcd}
\mathbb{C}_p \otimes_L D_{cris,L}(V^*(1)(\chi_{LT}^j)) & \times & \mathbb{C}_p \otimes_L D_{cris,L}(V(\tau^{-1})(\chi_{LT}^{-j})) \arrow[r, "{[,]_{cris}}"] & \mathbb{C}_p \otimes_L D_{cris,L}(L(\chi_{LT})) \cong \mathbb{C}_p \\
D(\Gamma_L, \mathbb{C}_p) \otimes_L D_{cris,L}(V^*(1)) \arrow[u, "ev_{\chi_{LT}^{-j}} \otimes t_{LT}^{-j} \eta^{\otimes j}"'] & \times & D(\Gamma_L, \mathbb{C}_p) \otimes_L D_{cris,L}(V(\tau^{-1})) \arrow[u, "ev_{\chi_{LT}^{j}} \otimes t_{LT}^{j} \eta^{\otimes -j}"'] \arrow[r, "{[,]^0}"] & D(\Gamma_L, \mathbb{C}_p), \arrow[u, "ev_{\chi_{LT}^{-j}}"']
\end{tikzcd}$$

where, for the identification in the right upper corner we choose $t_{LT}^{-1} \otimes \eta$ as a basis.

Proof. Using Lemma 3.23 below the statement is reduced to $j = 0$. Evaluation of (104) implies the claim in this case. $\square$

Lemma 3.23. *There is a commutative diagram*

$$\begin{tikzcd}
D(\Gamma_L, \mathbb{C}_p) \otimes_L D_{cris,L}(V^*(1)(\chi_{LT}^j)) & \times & D(\Gamma_L, \mathbb{C}_p) \otimes_L D_{cris,L}(V(\tau^{-1})(\chi_{LT}^{-j})) \arrow[r, "{[,]^0}"] & D(\Gamma_L, \mathbb{C}_p) \\
D(\Gamma_L, \mathbb{C}_p) \otimes_L D_{cris,L}(V^*(1)) \arrow[u, "Tw_{\chi_{LT}^{-j}} \otimes t_{LT}^{-j} \eta^{\otimes j}"'] & \times & D(\Gamma_L, \mathbb{C}_p) \otimes_L D_{cris,L}(V(\tau^{-1})) \arrow[u, "Tw_{\chi_{LT}^{j}} \otimes t_{LT}^{j} \eta^{\otimes -j}"'] \arrow[r, "{[,]^0}"] & D(\Gamma_L, \mathbb{C}_p). \arrow[u, "Tw_{\chi_{LT}^{-j}}"']
\end{tikzcd}$$

Proof. The claim follows immediately from (104), the compatibility of the usual D_{cris}-pairing with twists and the fact that $Tw_{\chi_{LT}^j}(\lambda \iota_*(\mu)) = Tw_{\chi_{LT}^j}(\lambda)\iota_*(Tw_{\chi_{LT}^{-j}}(\mu))$ holds. $\square$

3.2.4 The interpolation formula for the regulator map

In this subsection we are going to prove the interpolation property for $\mathcal{L}_V$. First recall that we introduced in section 1.1 the notation $D_{dR,L'}(V) := (B_{dR} \otimes_{\mathbb{Q}_p} V)^{G_{L'}}$. Since B_{dR} contains the algebraic closure $\overline{L}$ of L we have the isomorphism

$$B_{dR} \otimes_{\mathbb{Q}_p} V = (B_{dR} \otimes_{\mathbb{Q}_p} L) \otimes_L V \xrightarrow{\cong} \prod_{\sigma \in G_{\mathbb{Q}_p}/G_L} B_{dR} \otimes_{\sigma,L} V$$

which sends $b \otimes v$ to $(b \otimes v)_\sigma$. The tensor product in the factor $B_{dR} \otimes_{\sigma,L} V$ is formed with respect to L acting on B_{dR} through σ. With respect to the G_L-action the right hand side decomposes according to the double cosets in $G_L \backslash G_{\mathbb{Q}_p}/G_L$. It follows, in particular, that $D^{\mathrm{id}}_{dR}(V) := (B_{dR} \otimes_L V)^{G_L}$ is a direct summand of $D_{dR,L}(V)$ and we denote by pr^{id} the corresponding projection. Similarly, $tan_{L,\mathrm{id}}(V) := (B_{dR}/B^+_{dR} \otimes_L V)^{G_L}$ is a direct summand of $tan_L(V) := (B_{dR} \otimes_L V)^{G_L}$. More generally, also the filtration $D^i_{dR,L}(V)$ decomposes into direct summands.

According to [SV15, Appendix A] the dual Bloch-Kato exponential map is uniquely determined by the commutativity of the following diagram, in which all pairings are perfect:

(134)
$$\begin{tikzcd}
H^1(L',W) \arrow[d, "\exp^*_{L',W}"] & \times & H^1(L',W^*(1)) \arrow[rr, "<{,}>_{Tate,L'}"] & & L \arrow[d, hook] \\
D^0_{dR,L'}(W) \arrow[d, hook] & \times & tan_{L'}(W^*(1)) \arrow[u, "\exp_{L',W^*(1)}"'] \arrow[r] & D_{dR,L'}(\mathbb{Q}_p(1)) \arrow[r, "\cong"] \arrow[d, equal] & L' \arrow[d, equal] \\
D_{dR,L'}(W) & \times & D_{dR,L'}(W^*(1)) \arrow[u, two heads, "pr"] \arrow[r] & D_{dR,L'}(\mathbb{Q}_p(1)) \arrow[r, "\cong"] & L'.
\end{tikzcd}$$

In the Lubin-Tate setting we can also consider the dual of the identity component $\exp_{L',W^*(1),\mathrm{id}}$ of $\exp_{L',W^*(1)}$:

(135)
$$\begin{tikzcd}
H^1(L',W) \arrow[d, "\widetilde{\exp}^*_{L',W,\mathrm{id}}"] & \times & H^1(L',W^*(1)) \arrow[rr, "<{,}>_{Tate,L'}"] & & L \arrow[d, hook] \\
D^{\mathrm{id},0}_{dR,L'}(W(\tau^{-1})) \arrow[d, hook] & \times & tan_{L',\mathrm{id}}(W^*(1)) \arrow[u, "\exp_{L',W^*(1),\mathrm{id}}"'] \arrow[r] & D^{\mathrm{id}}_{dR,L'}(L(\chi_{LT})) \arrow[r, "\cong"] \arrow[d, equal] & L' \arrow[d, equal] \\
D^{\mathrm{id}}_{dR,L'}(W(\tau^{-1})) & \times & D^{\mathrm{id}}_{dR,L'}(W^*(1)) \arrow[u, two heads, "pr"] \arrow[r] & D^{\mathrm{id}}_{dR,L'}(L(\chi_{LT})) \arrow[r, "\cong"] & L'.
\end{tikzcd}$$

Upon noting that under the identifications $D_{dR,L'}(\mathbb{Q}_p(1)) \cong L'$ and $D^{\mathrm{id}}_{dR,L'}(\mathbb{Q}_p(1)) \cong L'$ the elements $t_{\mathbb{Q}_p} \otimes \eta_{cyc}$ and $t_{LT} \otimes \eta$ are sent to 1, one easily checks that, if $W^*(1)$ is L-analytic, whence the inclusion $tan_{L',\mathrm{id}}(W^*(1)) \subseteq tan_{L'}(W^*(1))$ is an equality and $\exp_{L',W^*(1),\mathrm{id}} = \exp_{L',W^*(1)}$, it holds

$$\mathbb{T}_{\tau^{-1}} \circ \exp^*_{L',W} = \widetilde{\exp}^*_{L',W,\mathrm{id}}, \tag{136}$$

where $\mathbb{T}_{\tau^{-1}} : D^0_{dR,L'}(W) \to D^{\mathrm{id},0}_{dR,L'}(W(\tau^{-1}))$ is the isomorphism, which sends $b \otimes v$ to $b\frac{t_{\mathbb{Q}_p}}{t_{LT}} \otimes v \otimes \eta \otimes \eta_{cyc}^{\otimes -1}$; note that $\frac{t_{\mathbb{Q}_p}}{t_{LT}} \in (B^+_{dR})^\times$, whence the filtration is preserved.

Now let W be an L-analytic, crystalline L-linear representation of G_L. Recall that $\eta = (\eta_n)_n$ denotes a fixed generator of T_π and that the map $tw_{\chi^j_{LT}} : D^\dagger_{rig}(W) \to D^\dagger_{rig}(W(\chi^j_{LT}))$ has been defined before Lemma 2.65. For D_{cris} twisting $D_{cris,L}(W) \xrightarrow{-\otimes e_j} D_{cris,L}(W(\chi^j_{LT}))$ maps d to $d \otimes e_j$ with $e_j := t_{LT}^{-j} \otimes \eta^{\otimes j} \in D_{cris,L}(L(\chi^j_{LT}))$.

If we assume, in addition, that

(i) W has Hodge Tate weights $\leqslant 0$, whence $W^*(1)$ has Hodge Tate weights $\geqslant 1$ and $D^0_{dR,L}(W^*(1)) = 0$, and

(ii) $D_{cris,L}(W^*(\chi_{LT}))^{\varphi_L = \frac{q}{\pi_L}} = 0$, then $exp_{L,W^*(1)} : D_{dR,L}(W^*(1)) \hookrightarrow H^1(L, W^*(1))$ is injective with image $H^1_e(L, W^*(1)) = H^1_f(L, W^*(1))$ by our assumption (see [BK, Cor. 3.8.4]). We denote its inverse by

$$\log_{L,W^*(1)} : H^1_f(L, W^*(1)) \to D_{dR,L}(W^*(1))$$

and define

$$\widetilde{\log}_{L,W^*(1)} : H^1_f(L, W^*(1)) \to D_{dR,L}(W^*(1)) \xrightarrow{\mathbb{T}_{\tau^{-1}}} D_{cris,L}(W^*(\chi_{LT}))$$

where (by abuse of notation) we also write $\mathbb{T}_{\tau^{-1}} : D_{dR,L}(W^*(1)) \to D^{\mathrm{id}}_{dR,L}(W^*(\chi_{LT})) = D_{cris,L}(W^*(\chi_{LT}))$ for the isomorphism, which sends $b \otimes v$ to $b\frac{t_{\mathbb{Q}_p}}{t_{LT}} \otimes v \otimes \eta \otimes \eta_{cyc}^{\otimes -1}$. We obtain the following commutative diagram, which defines the dual map $\log^*_{L,W}$ being inverse to $\exp^*_{L,W}$ (up to factorisation over $H^1(L, W)/H^1_f(L, W)$):

(137)
$$\begin{tikzcd}
H^1(L,W)/H^1_f(L,W) & \times & H^1_f(L,W^*(1)) \arrow[d, "\log_{L,W^*(1)}"'] \arrow[rr, "<{,}>_{Tate,L}"] & & L \arrow[d, equal] \\
D_{dR,L}(W) \arrow[u, "\log^*_{L,W}"] & \times & D_{dR,L}(W^*(1)) \arrow[r] & D_{dR,L}(\mathbb{Q}_p(1)) \arrow[r, "\cong"] & L.
\end{tikzcd}$$

Similarly as above we obtain a commutative diagram more convenient for the Lubin-Tate setting:

(138)
$$\begin{tikzcd}
H^1(L,W)/H^1_f(L,W) & \times & H^1_f(L,W^*(1)) \arrow[d, "\widetilde{\log}_{L,W^*(1)}"'] \arrow[rr, "<{,}>_{Tate,L}"] & & L \arrow[d, equal] \\
D^{\mathrm{id}}_{dR,L}(W) \arrow[u, "\log^*_{L,W,\mathrm{id}}"] & \times & D^{\mathrm{id}}_{dR,L}(W^*(\chi_{LT})) \arrow[r] & D^{\mathrm{id}}_{dR,L}(L(\chi_{LT})) \arrow[r, "\cong"] & L.
\end{tikzcd}$$

We write $\mathrm{Ev}_{W,n} : \mathcal{O}_L \otimes_L D_{cris,L}(W) \to L_n \otimes_L D_{cris,L}(W)$ for the composite $\partial_{D_{cris,L}(W)} \circ \varphi_q^{-n}$ from the introduction of [BF], which actually sends $f(Z) \otimes d$ to $f(\eta_n) \otimes \varphi_L^{-n}(d)$. By abuse of notation we also use $\mathrm{Ev}_{W,0}$ for the analogous map $\mathcal{O}_K \otimes_L D_{cris,L}(W) \to K \otimes_L D_{cris,L}(W)$. For $x \in D(\Gamma_L, K) \otimes_L D_{cris,L}(W)$ we denote by $x(\chi^j_{LT})$ the image under the map $D(\Gamma_L, K) \otimes_L D_{cris,L}(W) \to K \otimes_L D_{cris,L}(W)$, $\lambda \otimes d \mapsto \lambda(\chi^j_{LT}) \otimes d$.

Lemma 3.24. *Assume that Ω is contained in K. Then there are commutative diagrams*

$$\begin{tikzcd}
D(\Gamma_L,K) \otimes_L D_{cris,L}(W) \arrow[r, "\mathfrak{M}\otimes\mathrm{id}"] \arrow[d, "ev_{\mathrm{triv}}"'] & \mathcal{O}_K \otimes_L D_{cris,L}(W) \arrow[d, "\mathrm{Ev}_{W,0}"] & \mathcal{O}_K \otimes_L D_{cris,L}(W) \arrow[l, "1-\varphi_L\otimes\varphi_L"'] \arrow[d, "\mathrm{Ev}_{W,0}"] \\
K \otimes_L D_{cris,L}(W) \arrow[r, equal] & K \otimes_L D_{cris,L}(W) & K \otimes_L D_{cris,L}(W) \arrow[l, "1-\mathrm{id}\otimes\varphi_L"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
D(\Gamma_L,K) \otimes_L D_{cris,L}(W) \arrow[r, "\mathfrak{M}\otimes\mathrm{id}"] \arrow[d, "Tw_{\chi_{LT}^{-j}}\otimes e_j"'] & \mathcal{O}_K \otimes_L D_{cris,L}(W) \arrow[d, "(\frac{\partial}{\Omega})^{-j}\otimes e_j"] & \mathcal{O}_K \otimes_L D_{cris,L}(W) \arrow[l, "1-\varphi_L\otimes\varphi_L"'] \arrow[d, "(\frac{\partial}{\Omega})^{-j}\otimes e_j"] \\
D(\Gamma_L,K) \otimes_L D_{cris,L}(W(\chi^j_{LT})) \arrow[r, "\mathfrak{M}\otimes\mathrm{id}"] & \mathcal{O}_K \otimes_L D_{cris,L}(W(\chi^j_{LT})) & \mathcal{O}_K \otimes_L D_{cris,L}(W(\chi^j_{LT})). \arrow[l, "1-\varphi_L\otimes\varphi_L"']
\end{tikzcd}$$

In the latter we follow the (for $j > 0$) abusive notation ∂^{-j} from [BF, Rem. 3.5.5.].

Proof. For the upper diagram note that $\eta_0 = 0$ and $(\delta_g \cdot \eta(1, Z))_{|Z=0} = 1$, from which the claim follows for Dirac distributions, whence in general. For the right square we observe that $\varphi_L(g(Z))_{|Z=0} = g(0)$. Regarding the lower diagram we use 2.34 and the relation $\partial_{\mathrm{inv}} \circ \varphi_L = \pi_L \varphi_L \circ \partial$. $\square$

With this notation Berger's and Fourquaux' interpolation property reads as follows:

Theorem 3.25 (Berger-Fourquaux [BF, Thm. 3.5.3]). *Let W be L-analytic and $h \geqslant 1$ such that $\mathrm{Fil}^{-h} D_{cris,L}(W) = D_{cris,L}(W)$, $f \in \left(\mathcal{O}^{\psi=0} \otimes_L D_{cris,L}(W)\right)^{\Delta=0}$ and $y \in (\mathcal{O} \otimes_L D_{cris,L}(W))^{\psi=\frac{q}{\pi_L}}$ with $f = (1-\varphi_L)y$. If $h + j \geqslant 1$, then*

$$h^1_{L_n,W(\chi_{LT}^j)}(tw_{\chi_{LT}^j}(\Omega_{W,h}(f))) =$$

(139)

$$(-1)^{h+j-1}(h+j-1)! \begin{cases} \exp_{L_n,W(\chi_{LT}^j)}\left(q^{-n}\mathrm{Ev}_{W(\chi_{LT}^j),n}(\partial_{\mathrm{inv}}^{-j} y \otimes e_j)\right) & \text{if } n \geqslant 1; \\ \exp_{L,W(\chi_{LT}^j)}\left((1-q^{-1}\varphi_L^{-1})\mathrm{Ev}_{W(\chi_{LT}^j),0}(\partial_{\mathrm{inv}}^{-j} y \otimes e_j)\right), & \text{if } n = 0. \end{cases}$$

If $h + j \leqslant 0$, then

$$\exp^*_{L_n,W(\chi_{LT}^j)}\left(h^1_{L_n,W(\chi_{LT}^j)}(tw_{\chi_{LT}^j}(\Omega_{W,h}(f)))\right) =$$

$$\frac{1}{(-h-j)!} \begin{cases} q^{-n}\mathrm{Ev}_{W(\chi_{LT}^j),n}(\partial_{\mathrm{inv}}^{-j} y \otimes e_j) & \text{if } n \geqslant 1; \\ (1-q^{-1}\varphi_L^{-1})\mathrm{Ev}_{W(\chi_{LT}^j),0}(\partial_{\mathrm{inv}}^{-j} y \otimes e_j), & \text{if } n = 0. \end{cases} \tag{140}$$

By abuse of notation we shall denote the base change $K \otimes_L -$ of the (dual) Bloch-Kato exponential map by the same expression. Using Lemma 3.24 we deduce the following interpolation property for the modified big exponential map with $x \in D(\Gamma_L, K) \otimes_L D_{cris,L}(W)$: If $j \geqslant 0$, then

$$h^1_{L,W(\chi_{LT}^j)}(tw_{\chi_{LT}^j}(\mathbf{\Omega}_{W,1}(x))) =$$

$$(-1)^j j! \Omega^{-j-1} \exp_{L,W(\chi_{LT}^j)}\left((1-q^{-1}\varphi_L^{-1})(1-\varphi_L)^{-1}\left(x(\chi_{LT}^{-j}) \otimes e_j\right)\right); \tag{141}$$

if $j < 0$, then

$$h^1_{L,W(\chi_{LT}^j)}(tw_{\chi_{LT}^j}(\mathbf{\Omega}_{W,1}(f))) =$$

$$\frac{1}{(-1-j)!}\Omega^{-j-1} \log^*_{L,W(\chi_{LT}^j)}\left((1-q^{-1}\varphi_L^{-1})(1-\varphi_L)^{-1}\left(x(\chi_{LT}^{-j}) \otimes e_j\right)\right), \tag{142}$$

assuming in both cases that the operator $1 - \varphi_L$ is invertible on $D_{cris,L}(W(\chi_{LT}^j))$ and for $j < 0$ also that the operator $1 - q^{-1}\varphi_L^{-1}$ is invertible on $D_{cris,L}(W(\chi_{LT}^j))$ (in order to grant the existence of $\log_{L,W(\chi_{LT}^j)}$).

Recall that the generalized Iwasawa cohomology of $T \in Rep_{o_L}(G_L)$ is defined by

$$H^*_{Iw}(L_\infty/L, T) := \varprojlim_K H^*(K, T)$$

where K runs through the finite Galois extensions of L contained in L_∞ and the transition maps in the projective system are the cohomological corestriction maps. For $V := T \otimes_{o_L} L \in Rep_L(G_L)$ we define

$$H^*_{Iw}(L_\infty/L, V) := H^*_{Iw}(L_\infty/L, T) \otimes_{o_L} L,$$

which is independent of the choice of T. As usual we denote by $cor : H^*_{Iw}(L_\infty/L,T) \to H^*(L',T)$ the projection map and analogously for rational coefficients. Similarly as in (129) we have a map

$$pr_U : D(V(\tau^{-1}))^{\psi=1} \to h^1(K_{\psi,U'}(D(V(\tau^{-1}))^{\Delta})[d-1]) \cong H^1(L',V),\ m \mapsto [(\bar{m},0)]. \tag{143}$$

m under the map $\check{M} \twoheadrightarrow \check{M}_\Delta \cong \check{M}^\Delta$. Note that under the assumptions of Lemma 1.30 for $V(\tau^{-1})$ there is a commutative diagram

$$\begin{tikzcd} H^1_{Iw}(L_\infty/L,T) \arrow[r, "\cong"] \arrow[d, "cor"'] & D_{LT}(T(\tau^{-1}))^{\psi=1} \arrow[r, hook] \arrow[d, "pr_U"'] & D^\dagger_{rig}(V(\tau^{-1}))^{\psi=1} \arrow[d, "pr_U"] \\ H^1(L',V) \arrow[r, equal] & H^1(L',V) \arrow[r, two heads] & H^1_{/\dagger}(L',V), \end{tikzcd} \tag{144}$$

where the right vertical map is induced by (129). Indeed, for the commutativity of the left rectangle and the right rectangle we refer the reader to (A.58) and (179), respectively. Let $y_{\chi_{LT}^{-j}}$ denote the image of y under the map

$$H^1_{Iw}(L_\infty/L,T) \xrightarrow{\cdot\otimes\eta^{\otimes -j}} H^1_{Iw}(L_\infty/L,T(\chi_{LT}^{-j})) \xrightarrow{\text{cor}} H^1(L,T(\chi_{LT}^{-j})) \to H^1(L,V(\chi_{LT}^{-j})).$$

The following result generalizes [LVZ15, Thm. A.2.3] and [LZ, Thm. B.5] from the cyclotomic case.

Theorem 3.26. *Assume that $V^*(1)$ is L-analytic with $\mathrm{Fil}^{-1}D_{cris,L}(V^*(1)) = D_{cris,L}(V^*(1))$ and $D_{cris,L}(V^*(1))^{\varphi_L=\pi_L^{-1}} = D_{cris,L}(V^*(1))^{\varphi_L=1} = 0$. Then it holds that for $j \geqslant 0$*

$$\begin{aligned}\Omega^j\mathbf{L}_V(y)(\chi_{LT}^j) &= j!\Big((1-\pi_L^{-1}\varphi_L^{-1})^{-1}(1-\frac{\pi_L}{q}\varphi_L)\widetilde{\exp}^*_{L,V(\chi_{LT}^{-j}),\mathrm{id}}(y_{\chi_{LT}^{-j}})\Big)\otimes e_j \\ &= j!(1-\pi_L^{-1-j}\varphi_L^{-1})^{-1}(1-\frac{\pi_L^{j+1}}{q}\varphi_L)\Big(\widetilde{\exp}^*_{L,V(\chi_{LT}^{-j}),\mathrm{id}}(y_{\chi_{LT}^{-j}})\otimes e_j\Big)\end{aligned}$$

and for $j \leqslant -1$:

$$\begin{aligned}\Omega^j\mathbf{L}_V(y)(\chi_{LT}^j) &= \frac{(-1)^j}{(-1-j)!}\Big((1-\pi_L^{-1}\varphi_L^{-1})^{-1}(1-\frac{\pi_L}{q}\varphi_L)\widetilde{\log}_{L,V(\chi_{LT}^{-j}),\mathrm{id}}(y_{\chi_{LT}^{-j}})\Big)\otimes e_j \\ &= \frac{(-1)^j}{(-1-j)!}(1-\pi_L^{-1-j}\varphi_L^{-1})^{-1}(1-\frac{\pi_L^{j+1}}{q}\varphi_L)\Big(\widetilde{\log}_{L,V(\chi_{LT}^{-j}),\mathrm{id}}(y_{\chi_{LT}^{-j}})\otimes e_j\Big),\end{aligned}$$

if the operators $1-\pi_L^{-1-j}\varphi_L^{-1}, 1-\frac{\pi_L^{j+1}}{q}\varphi_L$ or equivalently $1-\pi_L^{-1}\varphi_L^{-1}, 1-\frac{\pi_L}{q}\varphi_L$ are invertible on $D_{cris,L}(V(\tau^{-1}))$ and $D_{cris,L}(V(\tau^{-1}\chi_{LT}^j))$, respectively.

Proof. From the reciprocity formula in Corollary 3.3 and Propositions 3.21, 3.22 we obtain for $x \in D(\Gamma_L, \mathbb{C}_p) \otimes_L D_{cris,L}(V^*(1))$, $y \in D(V(\tau^{-1}))^{\psi_L=1}$ and $j \geqslant 0$ using (144)

$$\begin{aligned}
&[x(\chi_{LT}^{-j}) \otimes e_j, (-1)^j \mathbf{L}_V(y)(\chi_{LT}^{j}) \otimes e_{-j}]_{cris} \\
&\quad = [x, \sigma_{-1}\mathbf{L}_V(y)]^0(\chi_{LT}^{-j}) \\
&\quad = \{\mathbf{\Omega}_{V^*(1),1}(x), y\}_{Iw}(\chi_{LT}^{-j}) \\
&\quad = \Omega < h_L^1 \circ tw_{\chi_{LT}^{j}}\left(\mathbf{\Omega}_{V^*(1),1}(x)\right), y_{\chi_{LT}^{-j}} >_{Tate} \\
&\quad = \Omega^{-j} < (-1)^j j! \exp_{L,V^*(1)(\chi_{LT}^{j})}((1-q^{-1}\varphi_L^{-1})(1-\varphi_L)^{-1}(x(\chi_{LT}^{-j}) \otimes e_j), y_{\chi_{LT}^{-j}} >_{Tate} \\
&\quad = (-1)^j \Omega^{-j} j! [(1-q^{-1}\varphi_L^{-1})(1-\varphi_L)^{-1}(x(\chi_{LT}^{-j}) \otimes e_j), \widetilde{\exp}^*_{L,V(\chi_{LT}^{-j}),\mathrm{id}}(y_{\chi_{LT}^{-j}})]_{cris} \\
&\quad = [x(\chi_{LT}^{-j}) \otimes e_j, (-1)^j \Omega^{-j} j! (1-\pi_L^{-1}\varphi_L^{-1})^{-1}(1-\frac{\pi_L}{q}\varphi_L)\widetilde{\exp}^*_{L,V(\chi_{LT}^{-j}),\mathrm{id}}(y_{\chi_{LT}^{-j}})]_{cris}
\end{aligned}$$

Here we used (141) in the fourth equation for the interpolation property of $\mathbf{\Omega}_{V^*(1),1}$. The fifth equation is the defining equation for the dual exponential map resulting from (135). Furthermore, for the last equality we use that $\pi_L^{-1}\varphi_L^{-1}$ is adjoint to φ_L under the lower pairing. The claim follows since the evaluation map is surjective and $[\ ,\ ]_{cris}$ is non-degenerated. Now assume that $j < 0$:

$$\begin{aligned}
&[x(\chi_{LT}^{-j}) \otimes e_j, (-1)^j \mathbf{L}_V(y)(\chi_{LT}^{j}) \otimes e_{-j}]_{cris} \\
&\quad = [x, \sigma_{-1}\mathbf{L}_V(y)]^0(\chi_{LT}^{-j}) \\
&\quad = \{\mathbf{\Omega}_{V^*(1),1}(x), y\}_{Iw}(\chi_{LT}^{-j}) \\
&\quad = \Omega < h_L^1 \circ tw_{\chi_{LT}^{j}}\left(\mathbf{\Omega}_{V^*(1),1}(x)\right), y_{\chi_{LT}^{-j}} >_{Tate} \\
&\quad = \Omega^{-j} < \frac{1}{(-1-j)!} \log^*_{L,W(\chi_{LT}^{j})}\Big((1-q^{-1}\varphi_L^{-1})(1-\varphi_L)^{-1}\Big(x(\chi_{LT}^{-j}) \otimes e_j\Big)\Big), y_{\chi_{LT}^{-j}} >_{Tate} \\
&\quad = \frac{\Omega^{-j}}{(-1-j)!}[(1-q^{-1}\varphi_L^{-1})(1-\varphi_L)^{-1}(x(\chi_{LT}^{-j}) \otimes e_j), \widetilde{\log}_{L,V(\chi_{LT}^{-j}),\mathrm{id}}(y_{\chi_{LT}^{-j}})]_{cris} \\
&\quad = [x(\chi_{LT}^{-j}) \otimes e_j, \frac{\Omega^{-j}}{(-1-j)!}(1-\pi_L^{-1}\varphi_L^{-1})^{-1}(1-\frac{\pi_L}{q}\varphi_L)\widetilde{\log}_{L,V(\chi_{LT}^{-j}),\mathrm{id}}(y_{\chi_{LT}^{-j}})]_{cris}
\end{aligned}$$

□

A Appendix

A.1 Perfect and imperfect (φ_L, Γ_L)-modules and their cohomology

A.1.1 An analogue of Tate's result

The aim of this section is to prove an analogue of Tate's classical result [Ta, Prop. 10] for the tilt $\mathbb{C}_p^\flat$ instead of $\mathbb{C}_p$ itself and in the Lubin Tate situation instead of the cyclotomic one.

Proposition A.1. *$H^n(H, \mathbb{C}_p^\flat) = 0$ for all $n \geqslant 1$ and $H \subseteq H_L$ any closed subgroup.*

Since the proof is formally very similar to that of loc. cit. or [BC, Prop. 14.3.2.] we only sketch the main ingredients. To this aim we fix H and write sometimes W for $\mathbb{C}_p^\flat$ as well as $W_{\geqslant m} := \{x \in W \,||x|_\flat \leqslant \frac{1}{p^m}\}$.

Lemma A.2. *The Tate-Sen axiom* **(TS1)** *is satisfied for $\mathbb{C}_p^\flat$ with regard to H, i.e., there exists a real constant $c > 1$ such that for all open subgroups $H_1 \subseteq H_2$ in H there exists $\alpha \in (\mathbb{C}_p^\flat)^{H_1}$*

with $|\alpha|_\flat < c$ and $Tr_{H_2|H_1}(\alpha) := \sum_{\tau\in H_2|H_1}\tau(\alpha) = 1$. Moreover, for any sequence $(H_m)_m$ of open subgroups $H_{m+1} \subseteq H_m$ of H there exists a trace compatible system $(y_{H_m})_m$ of elements $y_{H_m} \in (\mathbb{C}_p^\flat)^{H_m}$ with $|y_{H_m}|_\flat < c$ and $Tr_{H|H_m}(y_{H_m}) = 1$.

Proof. Note that for a perfect field K (like $(\mathbb{C}_p^\flat)^H$) of characteristic p complete for a multiplicative norm with maximal ideal $\mathfrak{m}_K$ and a separable finite extension F one has $\mathrm{Tr}_{F/K}(\mathfrak{m}_F) = \mathfrak{m}_K$ because the trace pairing is non-degenerate . Fix some $x \in (\mathbb{C}_p^\flat)^H$ with $0 < |x|_\flat < 1$ and set $c := |x|_\flat^{-1} > 1$. Then we find $\tilde{\alpha}$ in the maximal ideal of $(\mathbb{C}_p^\flat)^{H_1}$ with $Tr_{H|H_1}(\tilde{\alpha}) = x$ and $\alpha := (Tr_{H_2|H_1}(\tilde{\alpha}))^{-1}\tilde{\alpha}$ satisfies the requirement as $|Tr_{H_2|H_1}(\tilde{\alpha})|_\flat^{-1} \leqslant |x|_\flat^{-1} = c$.

For the second claim we successively choose elements $\tilde{\alpha}_m$ in the maximal ideal of $(\mathbb{C}_p^\flat)^{H_m}$ such that $Tr_{H|H_1}(\tilde{\alpha}_1) = x$ and $Tr_{H_{m+1}|H_m}(\tilde{\alpha}_{m+1}) = \tilde{\alpha}_m$ for all $m \geqslant 1$. Renormalization $\alpha_m := x^{-1}\tilde{\alpha}_m$ gives the desired system. □

Remark A.3. *Since H is also a closed subgroup of the absolute Galois group G_L of L it possesses a countable fundamental system $(H_m)_m$ of open neighbourhoods of the identity, as for any $n > 0$ the local field L of characteristic 0 has only finitely many extensions of degree smaller than n.*

Proof. The latter statement reduces easily to finite Galois extensions L' of L, which are known to be solvable, i.e. L' has a series of at most n intermediate fields $L \subseteq L_1 \subseteq \ldots \subseteq L_n = L'$ such that each subextension is abelian. Now its known by class field theory that each local field in characteristic 0 only has finitely many abelian extensions of a given degree. □

We write $\mathcal{C}^n(G,V)$ for the abelian group of continuous n-cochains of a profinite group G with values in a topological abelian group V carrying a continuous G-action and ∂_{inv} for the usual differentials. In particular, we endow $\mathcal{C}^n(H,W)$ with the maximum norm $||-||$ and consider the subspace $\mathcal{C}^n(H,W)^\delta := \bigcup_{H' \trianglelefteq H \text{ open}} \mathcal{C}^n(H/H',W) \subseteq \mathcal{C}^n(H,W)$ of those cochains with are even continuous with respect to the discrete topology of W.

Lemma A.4. (i) *The completion of $\mathcal{C}^n(H,W)^\delta$ with respect to the maximum norm equals $\mathcal{C}^n(H,W)$.*

(ii) *There exist $(\mathbb{C}_p^\flat)^H$-linear continuous maps*

$$\sigma^n : \mathcal{C}^n(H,W) \to \mathcal{C}^{n-1}(H,W)$$

satisfying $||f - \partial_{\mathrm{inv}}\sigma^n f|| \leqslant c||\partial_{\mathrm{inv}} f||$.

Proof. Since the space $\mathcal{C}^n(H,W)$ is already complete we only have to show that an arbitrary cochain f in it can be approximated by a Cauchy sequence f_m in $\mathcal{C}^n(H,W)^\delta$. To this end we observe that, given any m, the induced cochain $H^n \xrightarrow{f} W \xrightarrow{pr_m} W/W_{\geqslant m}$ comes, for some open normal subgroup H_m, from a cochain in $\mathcal{C}^n(H/H_m, W/W_{\geqslant m})$, which in turn gives rise to $f_m \in \mathcal{C}^n(H,W)^\delta$ when composing with any set theoretical section $W/W_{\geqslant m} \xrightarrow{s_m} W$ of the canonical projection $W \xrightarrow{pr_m} W/W_{\geqslant m}$. Note that s_m is automatically continuous, since $W/W_{\geqslant m}$ is discrete. By construction we have $||f - f_m|| \leqslant \frac{1}{p^m}$ and $(f_m)_m$ obviously is a Cauchy sequence. This shows (i).

For (ii) recall from Lemma A.2 together with Remark A.3 the existence of a trace compatible system $(y_{H'})_{H'}$ of elements $y_{H'} \in (\mathbb{C}_p^\flat)^{H'}$ with $|y_{H'}|_\flat < c$ and $Tr_{H|H'}(y_{H'}) = 1$, where H' runs over the open normal subgroups of H. Now we first define $(\mathbb{C}_p^\flat)^H$-linear maps

$$\sigma^n : \mathcal{C}^n(H,W)^\delta \to \mathcal{C}^{n-1}(H,W)$$

satisfying $||f - \partial_{\rm inv}\sigma^n f|| \leqslant c||\partial_{\rm inv} f||$ and $||\sigma^n f|| \leqslant c||f||$ by setting for $f \in \mathcal{C}^n(H/H', W)$

$$\sigma^n(f) := y_{H'} \cup f$$

(by considering $y_{H'}$ as a -1-cochain), i.e.,

$$\sigma^n(f)(h_1, \ldots, h_{n-1}) = (-1)^n \sum_{\tau \in H/H'} (h_1 \ldots h_{n-1}\tau)(y_{H'}) f(h_1, \ldots, h_{n-1}, \tau).$$

The inequality $||y_{H'} \cup f|| \leqslant c||f||$ follows immediately from this description, see the proof of [BC, Lem. 14.3.1.]. Upon noting that $\partial_{\rm inv} y_{H'} = Tr_{H|H'}(y_{H'}) = 1$, the Leibniz rule for the differential $\partial_{\rm inv}$ with respect to the cup-product then implies that

$$f - \partial_{\rm inv}(y_{H'} \cup f) = y_{H'} \cup \partial_{\rm inv} f,$$

hence

$$||f - \partial_{\rm inv}(y_{H'} \cup f)|| \leqslant c||\partial_{\rm inv} f||$$

by the previous inequality, see again loc. cit. In order to check that this map σ^n is well defined we assume that f arises also from a cochain in $\mathcal{C}^n(H/H'', W)$. Since we may make the comparison within $\mathcal{C}^n(H/(H' \cap H''), W)$ we can assume without loss of generality that $H'' \subseteq H'$. Then

$$\begin{aligned}
(y_{H''} \cup f)(h_1, \ldots, h_{n-1}) &= (-1)^n \sum_{\tau \in H/H''} (h_1 \ldots h_{n-1}\tau)(y_{H''}) f(h_1, \ldots, h_{n-1}, \tau) \\
&= (-1)^n \sum_{\tau \in H/H'} \left(h_1 \ldots h_{n-1} \sum_{\tau' \in H'/H''} \tau' \right) (y_{H''}) f(h_1, \ldots, h_{n-1}, \tau) \\
&= (-1)^n \sum_{\tau \in H/H'} (h_1 \ldots h_{n-1}) \, (\sum_{\tau' \in H'/H''} \tau'(y_{H''})) f(h_1, \ldots, h_{n-1}, \tau) \\
&= (-1)^n \sum_{\tau \in H/H'} (h_1 \ldots h_{n-1}) \, (y_{H'}) f(h_1, \ldots, h_{n-1}, \tau) \\
&= (y_{H'} \cup f)(h_1, \ldots, h_{n-1})
\end{aligned}$$

using the trace compatibility in the fourth equality. Finally the inequality $||\sigma^n f|| \leqslant c||f||$ implies that σ^n is continuous on $\mathcal{C}^n(H, W)^\delta$ and therefore extends continuously to its completion $\mathcal{C}^n(H, W)$. $\square$

The proof of Proposition A.1 is now an immediate consequence of Lemma A.4(ii).

A.1.2 The functors D, $\tilde{D}$ and $\tilde{D}^\dagger$

Recall the definition of $\mathbf{A}$, $\mathbf{A}_L$, $\mathbf{A}_L^{nr}$, $\mathbf{E}_L^{sep}$ and $\mathrm{Rep}_{o_L,f}(G_L)$ from section 1.1. Let $\mathrm{Rep}_{o_L}(G_L)$ and $\mathrm{Rep}_L(G_L)$ denote the category of finitely generated o_L-modules and finite dimensional L-vector spaces, respectively, equipped with a continuous linear G_L-action. The following result is established in [KR] Thm. 1.6. (see also [GAL, Thm. 3.3.10]) and [SV15, Prop. 4.4 (ii)].

Theorem A.5. *The functors*

$$T \longmapsto D(T) := (\mathbf{A} \otimes_{o_L} T)^{H_L} \qquad \textit{and} \qquad M \longmapsto (\mathbf{A} \otimes_{\mathbf{A}_L} M)^{\varphi_q \otimes \varphi_M = 1}$$

are exact quasi-inverse equivalences of categories between $\mathrm{Rep}_{o_L}(G_L)$ *and* $\mathfrak{M}^{et}(\mathbf{A}_L)$*. Moreover, for any* T *in* $\mathrm{Rep}_{o_L}(G_L)$ *the natural map*

$$\mathbf{A} \otimes_{\mathbf{A}_L} D(T) \xrightarrow{\cong} \mathbf{A} \otimes_{o_L} T \tag{145}$$

is an isomorphism (compatible with the G_L*-action and the Frobenius on both sides).*

In the following we would like to establish a perfect version of the above and prove similar properties for it. In the classical situation such versions have been studied by Kedlaya et al using the unramified rings of Witt vectors $W(R)$. In our Lubin-Tate situation we have to work with ramified Witt vectors $W(R)_L$. Many results and their proofs transfer almost literally from the classical setting. Often we will try to at least sketch the proofs for the convenience of the reader, but when we just quote results from the classical situation, e.g. from [KLI], this usually means that the transfer is purely formal.

We start defining $\tilde{\mathbf{A}} := W(\mathbb{C}_p^\flat)_L$ and

$$\tilde{\mathbf{A}}^\dagger := \{x = \sum_{n \geqslant 0} \pi_L^n[x_n] \in \tilde{\mathbf{A}} : |\pi_L^n||x_n|_\flat^r \xrightarrow{n\to\infty} 0 \text{ for some } r > 0\}$$

as well as $\tilde{D}(T) := (\tilde{\mathbf{A}} \otimes_{o_L} T)^{H_L}$ and $\tilde{D}^\dagger(T) := (\tilde{\mathbf{A}}^\dagger \otimes_{o_L} T)^{H_L}$.

More generally, let K be any perfectoid field containing L. For $r > 0$ let $W^r(K^\flat)_L$ be the set of $x = \sum_{n=0}^\infty \pi_L^n[x_n] \in W(K^\flat)_L$ such that $|\pi_L|^n|x_n|_\flat^r$ tends to zero as n goes to ∞. This is a subring by [KLI, Prop. 5.1.2] on which the function

$$|x|_r := \sup_n\{|\pi_L^n||x_n|_\flat^r\} = \sup_n\{q^{-n}|x_n|_\flat^r\}$$

is a complete multiplicative norm; it extends multiplicatively to $W^r(K^\flat)_L[\frac{1}{\pi_L}]$. Furthermore, $W^\dagger(K^\flat)_L := \bigcup_{r>0} W^r(K^\flat)_L$ is a henselian discrete valuation ring by [Ked05, Lem. 2.1.12], whose π_L-adic completion equals $W(K^\flat)_L$ since they coincide modulo π_L^n. Then $\tilde{\mathbf{A}}^\dagger = W^\dagger(\mathbb{C}_p^\flat)_L$, and we write $\tilde{\mathbf{A}}_L$ and $\tilde{\mathbf{A}}_L^\dagger$ for $W(\hat{L}_\infty^\flat)_L$ and $W^\dagger(\hat{L}_\infty^\flat)_L$, respectively. We set $\tilde{\mathbf{B}}_L = \tilde{\mathbf{A}}_L[\frac{1}{\pi_L}]$, $\tilde{\mathbf{B}} = \tilde{\mathbf{A}}[\frac{1}{\pi_L}]$, $\tilde{\mathbf{B}}_L^\dagger = \tilde{\mathbf{A}}_L^\dagger[\frac{1}{\pi_L}]$ and $\tilde{\mathbf{B}}^\dagger = \tilde{\mathbf{A}}^\dagger[\frac{1}{\pi_L}]$ for the corresponding fields of fractions.

Remark A.6. *By the Ax-Tate-Sen theorem [Ax] and since* $\mathbb{C}_p^\flat$ *is the completion of an algebraic closure* $\overline{\hat{L}_\infty^\flat}$ *he have that* $(\mathbb{C}_p^\flat)^H = ((\overline{\hat{L}_\infty^\flat})^H)^\wedge$ *for any closed subgroup* $H \subseteq H_L$*, in particular* $(\mathbb{C}_p^\flat)^{H_L} = \hat{L}_\infty^\flat$*. As completion of an algebraic extension of the perfect field* $\hat{L}_\infty^\flat$ *the field* $(\mathbb{C}_p^\flat)^H$ *is perfect, too. Moreover, we have* $\tilde{\mathbf{A}}^{H_L} = \tilde{\mathbf{A}}_L$*,* $(\tilde{\mathbf{A}}^\dagger)^{H_L} = \tilde{\mathbf{A}}_L^\dagger$ *and analogously for the rings* $\tilde{\mathbf{B}}$ *and* $\tilde{\mathbf{B}}^\dagger$*. It also follows that* $\tilde{\mathbf{A}}$ *is the* π_L*-adic completion of a maximal unramified extension of* $\tilde{\mathbf{A}}_L$*.*

Lemma A.7. *The rings* $\mathbf{A}_L$ *and* $\mathbf{A}$ *embed into* $\tilde{\mathbf{A}}_L$ *and* $\tilde{\mathbf{A}}$*, respectively.*

Proof. The embedding $\mathbf{A}_L \hookrightarrow \tilde{\mathbf{A}}_L$ is explained in [GAL, p. 94]. Moreover, $\mathbf{A}$ is the π_L-adic completion of the maximal unramified extension of $\mathbf{A}_L$ inside $\tilde{\mathbf{A}} = W(\mathbb{C}_p^\flat)_L$ (cf. [GAL, §3.1]). □

On $\tilde{\mathbf{A}} = W(\mathbb{C}_p^\flat)_L$ the weak topology is defined to be the product topology of the valuation topologies on the components $\mathbb{C}_p^\flat$. The induced topology on any subring R of it is also called weak topology of R. If M is a finitely generated R-module, then we call the *canonical* topology of M *(with respect to the weak topology of R)* the quotient topology with respect to any surjection $R^n \twoheadrightarrow M$ where the free module carries the product topology; this is independent of any choices. We recall that a (φ_L, Γ_L)-module M over $R \in \{\mathbf{A}_L, \tilde{\mathbf{A}}_L, \tilde{\mathbf{A}}_L^\dagger\}$ is a finitely generated R-module M together with

- a Γ_L-action on M by semilinear automorphisms which is continuous for the weak topology and
- a φ_L-linear endomorphism φ_M of M which commutes with the Γ_L-action.

We let $\mathfrak{M}(R)$ denote the category of (φ_L, Γ_L)-modules M over R. Such a module M is called étale if the linearized map

$$\begin{aligned} \varphi_M^{lin} : R \otimes_{R,\varphi_L} M &\xrightarrow{\cong} M \\ f \otimes m &\longmapsto f\varphi_M(m) \end{aligned}$$

is bijective. We let $\mathfrak{M}^{\acute{e}t}(R)$ denote the full subcategory of étale (φ_L, Γ_L)-modules over R.

Definition A.8. *For $* = \mathbf{B}_L, \tilde{\mathbf{B}}_L, \tilde{\mathbf{B}}_L^\dagger$ we write $\mathfrak{M}^{\acute{e}t}(*) := \mathfrak{M}^{\acute{e}t}(*')\otimes_{o_L} L$ with $*' = \mathbf{A}_L, \tilde{\mathbf{A}}_L, \tilde{\mathbf{A}}_L^\dagger$, respectively, and call the objects étale (φ_L, Γ_L)-modules over $*$.*

Lemma A.9. *Let G be a profinite group and $R \to S$ be a topological monomorphism of topological o_L-algebras, for which there exists a system of open neighbourhoods of 0 consisting of o_L-submodules. Consider a finitely generated R-module M, for which the canonical map $M \to S \otimes_R M$ is injective (e.g. if S is faithfully flat over R or M is free, in addition), and endow it with the canonical topology with respect to R. Assume that G acts continuously, o_L-linearly and compatible on R and S as well as continuously and R-semilinearly on M. Then the diagonal G-action on $S \otimes_R M$ is continuous with regard to the canonical topology with respect to S.*

Proof. Imitate the proof of [GAL, Lem. 3.1.11]. □

Proposition A.10. *The canonical map*

$$\tilde{\mathbf{A}}_L \otimes_{\mathbf{A}_L} D(T) \xrightarrow{\cong} \tilde{D}(T) \tag{146}$$

is an isomorphism and the functor $\tilde{D}(-) : \mathrm{Rep}_{o_L}(G_L) \to \mathfrak{M}^{\acute{e}t}(\tilde{\mathbf{A}}_L)$ is exact. Moreover, we have a comparison isomorphism

$$\tilde{\mathbf{A}} \otimes_{\tilde{\mathbf{A}}_L} \tilde{D}(T) \xrightarrow{\cong} \tilde{\mathbf{A}} \otimes_{o_L} T. \tag{147}$$

Proof. The isomorphism (146) implies formally the isomorphism (147) after base change of the comparison isomorphism (145). Secondly, the isomorphism (146), resp. (147), implies easily that $\tilde{D}(T)$ is finitely generated, resp. étale. Thirdly, since the ring extension $\tilde{\mathbf{A}}_L/\mathbf{A}_L$ is faithfully flat as local extension of (discrete) valuation rings, the exactness of $\tilde{D}$ follows from that of D. Moreover, the isomorphism (146) implies by Lemma A.9 that Γ_L acts continuously on $\tilde{D}(T)$, i.e., the functor $\tilde{D}$ is well-defined. Thus we only have to prove that

$$\tilde{\mathbf{A}}_L \otimes_{\mathbf{A}_L} (\mathbf{A} \otimes_{o_L} T)^{H_L} \xrightarrow{\cong} (\tilde{\mathbf{A}} \otimes_{o_L} T)^{H_L}$$

s an isomorphism. To this aim let us *assume first that T is finite.* Then we find an open normal subgroup $H \trianglelefteq H_L$ which acts trivially on T. Application of the subsequent Lemma A.11 to $M =$

$(\mathbf{A}\otimes_{o_L} T)^H$ and $G = H_L/H$ interprets the left hand side as $\left(\tilde{\mathbf{A}}_L \otimes_{\mathbf{A}_L} (\mathbf{A} \otimes_{o_L} T)^H\right)^{H_L/H}$ while the right hand side equals $\left((\tilde{\mathbf{A}} \otimes_{o_L} T)^H\right)^{H_L/H}$. Hence it suffices to establish the isomorphism

$$\tilde{\mathbf{A}}_L \otimes_{\mathbf{A}_L} (\mathbf{A} \otimes_{o_L} T)^H \xrightarrow{\cong} (\tilde{\mathbf{A}} \otimes_{o_L} T)^H.$$

By Lemma A.12 below this is reduced to showing that the canonical map

$$\tilde{\mathbf{A}}_L \otimes_{\mathbf{A}_L} \mathbf{A}^H \otimes_{o_L} T \xrightarrow{\cong} \tilde{\mathbf{A}}^H \otimes_{o_L} T$$

is an isomorphism, which follows from Lemma A.13 below. *Finally let T be arbitrary.* Then we have isomorphisms

$$\begin{aligned}\tilde{\mathbf{A}}_L \otimes_{\mathbf{A}_L} D(T) &\cong \tilde{\mathbf{A}}_L \otimes_{\mathbf{A}_L} \varprojlim_n D(T/\pi_L^n T)\\ &\cong \tilde{\mathbf{A}}_L \otimes_{\mathbf{A}_L} \varprojlim_n D(T)/\pi_L^n D(T)\\ &\cong \varprojlim_n \tilde{\mathbf{A}}_L \otimes_{\mathbf{A}_L} D(T)/\pi_L^n D(T)\\ &\cong \varprojlim_n \tilde{\mathbf{A}}_L \otimes_{\mathbf{A}_L} D(T/\pi_L^n T)\\ &\cong \varprojlim_n \tilde{D}(T/\pi_L^n T)\\ &\cong \tilde{D}(T),\end{aligned}$$

where we use for the second and fourth equation exactness of D, for the second last one the case of finite T and for the first, third and last equation the elementary divisor theory for the discrete valuation rings o_L, $\mathbf{A}_L$ and $\tilde{\mathbf{A}}_L$, respectively. $\square$

Lemma A.11. *Let $A \to B$ be a flat extension of rings and M an A-module with an A-linear action by a finite group G. Then $B \otimes_A M$ carries a B-linear G-action and we have*

$$(B \otimes_A M)^G = B \otimes_A M^G.$$

Proof. Apply the exact functor $B \otimes_A -$ to the exact sequence

$$0 \longrightarrow M^G \longrightarrow M \xrightarrow{(g-1)_{g\in G}} \bigoplus_{g\in G} M,$$

which gives the desired description of $(B \otimes_A M)^G$. $\square$

Lemma A.12. *Let A be $\mathbf{A}$, $\mathbf{A}_L^{nr}$, $\tilde{\mathbf{A}}^\dagger$ or $\tilde{\mathbf{A}}$ and T be a finitely generated o_L-module with trivial action by an open subgroup $H \subseteq H_L$. Then $(A \otimes_{o_L} T)^H = A^H \otimes_{o_L} T$. Moreover, $\mathbf{A}^H$ and $\tilde{\mathbf{A}}^H$ are free $\mathbf{A}_L$- and $\tilde{\mathbf{A}}_L$-modules of finite rank, respectively.*

Proof. Since $T \cong \bigoplus_{i=1}^r o_L/\pi_L^{n_i} o_L$ with $n_i \in \mathbb{N} \cup \{\infty\}$ we may assume that $T = o_L/\pi_L^n o_L$ for some $n \in \mathbb{N} \cup \{\infty\}$. We then we have to show that

$$(A/\pi_L^n A)^H = A^H/\pi_L^n A^H \tag{148}$$

For $n = \infty$ there is nothing to prove.

The case $n = 1$: First of all we have $\mathbf{A}/\pi_L\mathbf{A} = \mathbf{A}_L^{nr}/\pi_L\mathbf{A}_L^{nr} = \mathbf{E}_L^{sep}$. On the other hand, by the Galois correspondence between unramified extensions and their residue extensions, we have that $(\mathbf{E}_L^{sep})^H$ is the residue field of $(\mathbf{A}_L^{nr})^H$. Hence the case $n = 1$ holds true for $A = \mathbf{A}_L^{nr}$. After having finished all cases for $A = \mathbf{A}_L^{nr}$ we will see at the end of the proof that $(\mathbf{A}_L^{nr})^H = \mathbf{A}^H$. Therefore the case $n = 1$ for $A = \mathbf{A}$ will be settled, too.

For $A = \tilde{\mathbf{A}}$ we only need to observe that $\tilde{\mathbf{A}}/\pi_L\tilde{\mathbf{A}} = W(\mathbb{C}_p^\flat)_L/\pi_L W(\mathbb{C}_p^\flat)_L = \mathbb{C}_p^\flat$ and that $(\mathbb{C}_p^\flat)^H$ is the residue field of $(W(\mathbb{C}_p^\flat)_L)^H = W((\mathbb{C}_p^\flat)^H)_L$.

For $A = \tilde{\mathbf{A}}^\dagger$ we argue by the following commutative diagram

$$\begin{tikzcd}
(\mathbf{C}_p^\flat)^H \arrow[r, "\cong"] \arrow[dr, "\cong"'] & W^\dagger((\mathbf{C}_p^\flat)^H)_L/\pi_L W^\dagger((\mathbf{C}_p^\flat)^H)_L \arrow[rr, "\cong"] & & (\tilde{\mathbf{A}}^\dagger)^H/\pi_L(\tilde{\mathbf{A}}^\dagger)^H \arrow[d] \\
& \tilde{\mathbf{A}}^H/\pi_L\tilde{\mathbf{A}}^H \arrow[r, "\cong"] & (\tilde{\mathbf{A}}/\pi_L\tilde{\mathbf{A}})^H \arrow[r, "\cong"] & (\tilde{\mathbf{A}}^\dagger/\pi_L\tilde{\mathbf{A}}^\dagger)^H.
\end{tikzcd}$$

The case $1 < n < \infty$: This follows by induction using the commutative diagram with exact lines

$$\begin{tikzcd}
0 \arrow[r] & A^H/\pi_L^n A^H \arrow[r, "\pi_L\cdot"] \arrow[d, "\cong"'] & A^H/\pi_L^{n+1}A^H \arrow[r] \arrow[d] & A^H/\pi_L A^H \arrow[r] \arrow[d, "\cong"'] & 0 \\
0 \arrow[r] & (A/\pi_L^n A)^H \arrow[r, "\pi_L\cdot"] & (A/\pi_L^{n+1}A)^H \arrow[r] & (A/\pi_L A)^H,
\end{tikzcd}$$

in which the outer vertical arrows are isomorphism by the case $n = 1$ and the induction hypothesis.

Finally we can check, using the above equality (148) for $A = \mathbf{A}_L^{nr}$ in the third equation:

$$\begin{aligned}
\mathbf{A}^H &= \left(\varprojlim_n \mathbf{A}_L^{nr}/\pi_L^n\mathbf{A}_L^{nr}\right)^H \\
&= \varprojlim_n (\mathbf{A}_L^{nr}/\pi_L^n\mathbf{A}_L^{nr})^H \\
&= \varprojlim_n \left(\mathbf{A}_L^{nr}\right)^H/\pi_L^n\left(\mathbf{A}_L^{nr}\right)^H \\
&= (\mathbf{A}_L^{nr})^H.
\end{aligned}$$

Note that $(\mathbf{A}_L^{nr})^H$ is a finite unramified extension of $\mathbf{A}_L$ and therefore is π_L-adically complete. We also see that $\mathbf{A}^H$ is a free $\mathbf{A}_L$-module of finite rank. Similarly, $W(\mathbb{C}_p^\flat)_L^H \cong (W(\hat{L}_\infty^\flat)_L^{nr})^H$ is a free $W(\hat{L}_\infty^\flat)_L$-module of finite rank. $\square$

Lemma A.13. *For any open subgroup H of H_L the canonical maps*

$$W(\hat{L}_\infty^\flat)_L \otimes_{\mathbf{A}_L} \mathbf{A}^H \xrightarrow{\cong} W((\mathbb{C}_p^\flat)^H)_L,$$
$$W(\hat{L}_\infty^\flat)_L \otimes_{\tilde{\mathbf{A}}_L^\dagger} (\tilde{\mathbf{A}}^\dagger)^H \xrightarrow{\cong} W((\mathbb{C}_p^\flat)^H)_L$$

are isomorphisms.

Proof. We begin with the first isomorphism. Since $\mathbf{A}^H$ is finitely generated free over $\mathbf{A}_L$ by Lemma A.12, we have

$$W(\hat{L}_\infty^\flat)_L \otimes_{\mathbf{A}_L} \mathbf{A}^H \cong \left(\varprojlim_n W_n(\hat{L}_\infty^\flat)_L\right) \otimes_{\mathbf{A}_L} \mathbf{A}^H \cong \varprojlim_n \left(W_n(\hat{L}_\infty^\flat)_L \otimes_{\mathbf{A}_L} \mathbf{A}^H\right).$$

It therefore suffices to show the corresponding assertion for Witt vectors of finite length:

$$W_n(\hat{L}_\infty^\flat)_L \otimes_{\mathbf{A}_L} \mathbf{A}^H/\pi_L^n\mathbf{A}^H = W_n(\hat{L}_\infty^\flat)_L \otimes_{\mathbf{A}_L} \mathbf{A}^H \xrightarrow{\cong} W_n((\mathbb{C}_p^\flat)^H)_L.$$

To this aim we first consider the case $n = 1$. From (148) we know that $\mathbf{A}^H/\pi_L^n\mathbf{A}^H = (\mathbf{E}_L^{sep})^H$. Hence we need to check that

$$\hat{L}_\infty^\flat \otimes_{\mathbf{E}_L} (\mathbf{E}_L^{sep})^H \xrightarrow{\cong} (\mathbb{C}_p^\flat)^H$$

is an isomorphism. Since $\mathbf{E}_L^{perf}$ (being purely inseparable and normal) and $(\mathbf{E}_L^{sep})^H$ (being separable) are linear disjoint extensions of $\mathbf{E}_L$ their tensor product is equal to the composite of fields $\mathbf{E}_L^{perf}(\mathbf{E}_L^{sep})^H$ (cf. [Coh, Thm. 5.5, p. 188]), which moreover has to have degree $[H_L : H]$ over $\mathbf{E}_L^{perf}$. Since the completion of the tensor product is $\hat{L}_\infty^\flat \otimes_{\mathbf{E}_L} (\mathbf{E}_L^{sep})^H$, we see that the completion of the field $\mathbf{E}_L^{perf}(\mathbf{E}_L^{sep})^H$ is the composite of fields $\hat{L}_\infty^\flat(\mathbf{E}_L^{sep})^H$, which has degree $[H_L : H]$ over $\hat{L}_\infty^\flat$. But $\hat{L}_\infty^\flat(\mathbf{E}_L^{sep})^H \subseteq (\mathbb{C}_p^\flat)^H$. By the Ax-Tate-Sen theorem $(\mathbb{C}_p^\flat)^H$ has also degree $[H_L : H]$ over $\hat{L}_\infty^\flat$. Hence the two fields coincide, which establishes the case $n = 1$.

The commutative diagram

$$\begin{tikzcd}
\hat{L}_\infty^\flat \otimes_{\mathbf{A}_L} \mathbf{A}^H \arrow[r, "\cong"] \arrow[d, "\varphi_q^m\otimes\mathrm{id}"', "\cong"] & (\mathbb{C}_p^\flat)^H \arrow[d, "\cong"', "\varphi_q^m"] \\
\hat{L}_\infty^\flat \otimes_{\varphi_q^m, \mathbf{A}_L} \mathbf{A}^H \arrow[r, "\mathrm{id}\,\varphi_q^m"] & (\mathbb{C}_p^\flat)^H
\end{tikzcd}$$

shows that also the lower map is an isomorphism. Using that Verschiebung V on $W_n((\mathbb{C}_p^\flat)^H)_L$ and $W_n(\hat{L}_\infty^\flat)_L$ is additive and satisfies the projection formula $V^m(x) \cdot y = V^m(x \cdot \varphi_q^m(y))$ we see that we obtain a commutative exact diagram

$$\begin{tikzcd}
0 \arrow[r] & \hat{L}_\infty^\flat \otimes_{\varphi_q^n, \mathbf{A}_L} \mathbf{A}^H \arrow[r, "V^n\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\,\varphi_q^n"'] & W_{n+1}(\hat{L}_\infty^\flat)_L \otimes_{\mathbf{A}_L} \mathbf{A}^H \arrow[r] \arrow[d, "can"'] & W_n(\hat{L}_\infty^\flat)_L \otimes_{\mathbf{A}_L} \mathbf{A}^H \arrow[r] \arrow[d, "\cong"'] & 0 \\
0 \arrow[r] & (\mathbb{C}_p^\flat)^H \arrow[r, "V^n"] & W_{n+1}((\mathbb{C}_p^\flat)^H)_L \arrow[r] & W_n((\mathbb{C}_p^\flat)^H)_L, &
\end{tikzcd}$$

from which the claim follows by induction because the outer vertical maps are isomorphisms by the above and the induction hypothesis. Here the first non-trivial horizontal morphisms map onto the highest Witt vector component.

The second isomorphism is established as follows: We choose a subgroup $N \subseteq H \subseteq H_L$ which is open normal in H_L and obtain the extensions of henselian discrete valuation rings

$$\tilde{\mathbf{A}}_L^\dagger \subseteq (\tilde{\mathbf{A}}^\dagger)^H = W^\dagger((\mathbb{C}_p^\flat)^H)_L \subseteq (\tilde{\mathbf{A}}^\dagger)^N = W^\dagger((\mathbb{C}_p^\flat)^N)_L.$$

The corresponding extensions of their field of fractions

$$\tilde{\mathbf{B}}_L^\dagger \subseteq E := (\tilde{\mathbf{A}}^\dagger)^H[\tfrac{1}{\pi_L}] \subseteq F := (\tilde{\mathbf{A}}^\dagger)^N[\tfrac{1}{\pi_L}]$$

satisfy $F^{H/N} = E$ and $F^{H_L/N} = \tilde{\mathbf{B}}_L^\dagger$. Hence F/E and $F/\tilde{\mathbf{B}}_L^\dagger$ are Galois extensions of degree $[H : N]$ and $[H_L : N]$, respectively. It follows that $E/\tilde{\mathbf{B}}_L^\dagger$ is a finite extension of degree $[H_L : H]$. The henselian condition then implies that $(\tilde{\mathbf{A}}^\dagger)^H = W^\dagger((\mathbb{C}_p^\flat)^H)_L$ is free of rank

$[H_L : H]$ over $\tilde{\mathbf{A}}_L^\dagger = W^\dagger(\hat{L}_\infty^\flat)_L$. The π_L-adic completion $(-)\hat{}$ of the two rings therefore can be obtained by the tensor product with $\tilde{\mathbf{A}}_L = W(\hat{L}_\infty^\flat)_L$. This gives the wanted

$$W(\hat{L}_\infty^\flat)_L \otimes_{\tilde{\mathbf{A}}_L^\dagger} (\tilde{\mathbf{A}}^\dagger)^H = W^\dagger(\hat{L}_\infty^\flat)\hat{}_L \otimes_{\tilde{\mathbf{A}}_L^\dagger} (\tilde{\mathbf{A}}^\dagger)^H = W^\dagger((\mathbb{C}_p^\flat)^H)\hat{}_L = W((\mathbb{C}_p^\flat)^H)_L.$$

□

Proposition A.14. *The sequences*

$$0 \to o_L \to \mathbf{A} \xrightarrow{\varphi_q - 1} \mathbf{A} \to 0, \tag{149}$$

$$0 \to o_L \to \tilde{\mathbf{A}} \xrightarrow{\varphi_q - 1} \tilde{\mathbf{A}} \to 0, \tag{150}$$

$$0 \to o_L \to \tilde{\mathbf{A}}^\dagger \xrightarrow{\varphi_q - 1} \tilde{\mathbf{A}}^\dagger \to 0. \tag{151}$$

are exact.

Proof. The first sequence is [SV15, (26), Rem. 5.1]. For the second sequence one proves by induction the statement for finite length Witt vectors using that the Artin-Schreier equation has a solution in $\mathbb{C}_p^\flat$. Taking projective limits then gives the claim. For the third sequence only the surjectivity has to be shown. This can be achieved by the same calculation as in the proof of [KLII, Lem. 4.5.3] with $R = \mathbb{C}_p^\flat$. □

Lemma A.15. *For any finite T in $Rep_{o_L}(G_L)$ the map $\tilde{\mathbf{A}} \otimes_{o_L} T \xrightarrow{\varphi_q \otimes \mathrm{id} - 1} \tilde{\mathbf{A}} \otimes_{o_L} T$ has a continuous set theoretical section.*

Proof. Since $T \cong \bigoplus_{i=1}^r o_L/\pi_L^{n_i} o_L$ for some natural numbers r, n_i we may assume that $T = o_L/\pi_L^n o_L$ for some n and then we have to show that the surjective map $W_n(\mathbb{C}_p^\flat)_L \xrightarrow{\varphi_q - \mathrm{id}} W_n(\mathbb{C}_p^\flat)_L$ has a continuous set theoretical section. Thus me may neglect the additive structure and identify source and target with $X = (\mathbb{C}_p^\flat)^n$. In order to determine the components of the map $\varphi_q - \mathrm{id} =: f = (f_0, \ldots, f_{n-1}) : X \to X$ with respect to these coordinates we recall that the addition in Witt rings is given by polynomials

$$S_j(X_0, \ldots X_j, Y_0, \ldots, Y_j) = X_j + Y_j + \text{ terms in } X_0, \ldots, X_{j-1}, Y_0, \ldots, Y_{j-1}$$

while the additive inverse is given by

$$I_j(X_0, \ldots X_j) = -X_j + \text{ terms in } X_0, \ldots, X_{j-1}.$$

Indeed, the polynomials I_j are defined by the property that $\Phi_j(I_0, \ldots, I_j) = -\Phi_j(X_0, \ldots, X_j)$ where the Witt polynomials have the form $\Phi_j(X_0, \ldots, X_j) = X_0^{q^j} + \pi_L X_1^{q^{j-1}} + \ldots + \pi_L^j X_j$. Modulo $(X_0, \ldots, X_{j-1})$ we derive that $\pi_L^j I_j(X_0, \ldots, X_j) \equiv -\pi_L^j X_j$ and the claim follows. Since φ_q acts componentwise rising the entries to their qth power, we conclude that

$$f_j = S_j(X_0^q, \ldots X_j^q, I_0(X_0), \ldots, I_j(X_0, \ldots X_j)).$$

Hence the Jacobi matrix of f at a point $x \in X$ looks like

$$D_x(f) = \begin{pmatrix} -1 & & 0 \\ & \ddots & \\ * & & -1 \end{pmatrix},$$

i.e., is invertible in every point. As a polynomial map f is locally analytic. It therefore follows from the inverse function theorem [pLG, Prop. 6.4] that f restricts to a homeomorphism $f|U_0 : U_0 \xrightarrow{\cong} U_1$ of open neighbourhoods of x and $f(x)$, respectively. By the surjectivity of f every $x \in X$ has an open neighbourhood U_x and a continuous map $s_x : U_x \to X$ with $f \circ s_x = \mathrm{id}_{|U_x}$. But X is strictly paracompact by Remark 8.6 (i) in (loc. cit.), i.e., the covering $(U_x)_x$ has a disjoint refinement. There the restrictions of the s_x glue to a continuous section of f. □

Corollary A.16. *For T in $Rep_{o_L}(G_L)$, the nth cohomology groups of the complexes concentrated in degrees 0 and 1*

$$0 \longrightarrow \tilde{D}(T) \xrightarrow{\varphi-1} \tilde{D}(T) \longrightarrow 0 \quad \textit{and} \tag{152}$$

$$0 \longrightarrow D(T) \xrightarrow{\varphi-1} D(T) \longrightarrow 0 \tag{153}$$

are isomorphic to $H^n(H_L, T)$ for any $n \geqslant 0$.

Proof. Assume first that T is finite. For (153) see [SV15, Lemma 5.2]. For (152) we use Lemma A.15, which says that the right hand map in the exact sequence

$$0 \longrightarrow T \longrightarrow \tilde{\mathbf{A}} \otimes_{o_L} T \xrightarrow{\varphi_q \otimes \mathrm{id} - 1} \tilde{\mathbf{A}} \otimes_{o_L} T \longrightarrow 0$$

has a continuous set theoretical section and thus gives rise to the long exact sequence of continuous cohomology groups

$$0 \to H^0(H_L, T) \to \tilde{D}(T) \xrightarrow{\varphi-1} \tilde{D}(T) \to H^1(H_L, T) \to H^1(H_L, \tilde{\mathbf{A}} \otimes_{o_L} T) \to \dots \tag{154}$$

Using the comparison isomorphism (147) and the subsequent Proposition A.17 we see that all terms from the fifth on vanish.

For the general case (for $\tilde{D}(T)$ as well as $D(T)$) we take inverse limits in the exact sequences for the $(T/\pi_L^m T)$ and observe that $H^n(H_L, T) \cong \varprojlim_m H^n(H_L, T/\pi_L^m T)$. This follows for $n \neq 2$ from [NSW, Cor. 2.7.6]. For $n = 2$ we use [NSW, Thm. 2.7.5] and have to show that the projective system $(H^1(H_L, T/\pi_L^m T))_m$ is Mittag-Leffler. Since it is a quotient of the projective system $(D(T/\pi_L^m T))_m$, it suffices for this to check that the latter system is Mittag-Leffler. But due to the exactness of the functor D this latter system is equal to the projective system of artinian $\mathbf{A}_L$-modules $(D(T)/\pi_L^m D(T))_m$ and hence is Mittag-Leffler. We conclude by observing that taking inverse limits of the system of sequences (154) remains exact. The reasoning being the same for $\tilde{D}(T)$ and $D(T)$ we consider only the former. Indeed, we split the 4-term exact sequences into two short exact sequences of projective systems

$$0 \to H^0(H_L, V/\pi_L^m T) \to \tilde{D}(T/\pi_L^m T) \to (\varphi - 1)\tilde{D}(T/\pi_L^m T) \to 0$$

and

$$0 \to (\varphi - 1)\tilde{D}(T/\pi_L^m T) \to \tilde{D}(T/\pi_L^m T) \to H^1(H_L, T/\pi_L^m T) \to 0.$$

Passing to the projective limits remains exact provided the left most projective systems have vanishing $\varprojlim^1$. For the system $H^0(H_L, T/\pi_L^m T)$ this is the case since it is Mittag-Leffler. The system $(\varphi - 1)\tilde{D}(T/\pi_L^m T)$ even has surjective transition maps since the system $\tilde{D}(T/\pi_L^m T)$ has this property by the exactness of the functor $\tilde{D}$ (cf. Prop. A.10). □

Proposition A.17. *$H^n(H, \tilde{\mathbf{A}}/\pi_L^m\tilde{\mathbf{A}}) = 0$ for all $n, m \geqslant 1$ and $H \subseteq H_L$ any closed subgroup.*

Proof. For $j < i$ the canonical projection $W_i(\mathbb{C}_p^\flat) \cong \tilde{\mathbf{A}}/\pi_L^i\tilde{\mathbf{A}} \twoheadrightarrow \tilde{\mathbf{A}}/\pi_L^j\tilde{\mathbf{A}} \cong W_j(\mathbb{C}_p^\flat)$ corresponds to the projection $(\mathbb{C}_p^\flat)^i \twoheadrightarrow (\mathbb{C}_p^\flat)^j$ and hence have set theoretical continuous sections. Using the associated long exact cohomology sequence (after adding the kernel) allows to reduce the statement to Proposition A.1. □

For any commutative ring R with endomorphism φ we write $\Phi(R)$ for the category of φ-modules consisting of R-modules equipped with a semi-linear φ-action. We write $\Phi^{ét}(R)$ for the subcategory of étale φ-modules, i.e., such that M is finitely generated over R and φ induces an R-linear isomorphism $\varphi^*M \xrightarrow{\cong} M$. Finally, we denote by $\Phi_f^{ét}(R)$ the subcategory consisting of finitely generated free R-modules.

For $M_1, M_2 \in \Phi(R)$ the R-module $\mathrm{Hom}_R(M_1, M_2)$ has a natural structure as a φ-module satisfying

$$\varphi_{\mathrm{Hom}_R(M_1,M_2)}(\alpha)(\varphi_{M_1}(m)) = \varphi_{M_2}(\alpha(m)) \;, \tag{155}$$

hence in particular

$$\mathrm{Hom}_R(M_1, M_2)^{\varphi=\mathrm{id}} = \mathrm{Hom}_{\Phi(R)}(M_1, M_2). \tag{156}$$

Note that with M_1, M_2 also $\mathrm{Hom}_R(M_1, M_2)$ is étale.

Remark A.18. *We recall from [KLI, §1.5] that the cohomology groups $H^i_\varphi(M)$ of the complex $M \xrightarrow{\varphi-1} M$ can be identified with the Yoneda extension groups $\mathrm{Ext}^i_{\Phi(R)}(R, M)$. Indeed, if $S := R[X;\varphi]$ denotes the twisted polynomial ring satisfying $Xr = \varphi(r)X$ for all $r \in R$, then we can identify $\Phi(R)$ with the category S-Mod of (left) S-modules by letting X act via φ_M on X. Using the free resolution*

$$0 \longrightarrow S \xrightarrow{\cdot(X-1)} S \longrightarrow R \longrightarrow 0$$

the result follows. Compare also with Lemma 3.12.

Remark A.19. *Note that $\tilde{\mathbf{A}}_L^\dagger \subseteq \tilde{\mathbf{A}}_L$ is a faithfully flat ring extension as both rings are discrete valuation rings and the bigger one is the completion of the previous one.*

Proposition A.20. *Base extension induces*

(i) *an equivalence of categories*
$$\Phi_f^{ét}(\tilde{\mathbf{A}}_L^\dagger) \leftrightarrow \Phi_f^{ét}(\tilde{\mathbf{A}}_L)$$

(ii) *and an isomorphism of Yoneda extension groups*

$$\mathrm{Ext}^1_{\Phi(\tilde{\mathbf{A}}_L^\dagger)}(\tilde{\mathbf{A}}_L^\dagger, M) \cong \mathrm{Ext}^1_{\Phi(\tilde{\mathbf{A}}_L)}(\tilde{\mathbf{A}}_L, \tilde{\mathbf{A}}_L \otimes_{\tilde{\mathbf{A}}_L^\dagger} M)$$

for all $M \in \Phi_f^{ét}(\tilde{\mathbf{A}}_L^\dagger)$.

Proof. For the first item we imitate the proof of [KLI, Thm. 8.5.3], see also [Ked15, Lem. 2.4.2,Thm. 2.4.5]: First we will show that for every $M \in \Phi_f^{\acute{e}t}(\tilde{\mathbf{A}}_L^\dagger)$ it holds that $(\tilde{\mathbf{A}}_L \otimes M)^{\varphi=\mathrm{id}} \subseteq M^{\varphi=\mathrm{id}}$ and hence equality. Applied to $M := \mathrm{Hom}_{\tilde{\mathbf{A}}_L^\dagger}(M_1, M_2)$ this implies that the base change is fully faithful by the equation (156). We observe that the analogue of [KLI, Lem. 3.2.6] holds in our setting and that S in loc. cit. can be chosen to be a finite separable field extension of the perfect field $R = \hat{L}_\infty^\flat$. Thus we may choose S in the analogue of [KLI, Prop. 7.3.6] (with $a = 1$, $c = 0$ and M_0 being our M) as completion of a (possibly infinite) separable field extension of R. This means in our situation that there exists a closed subgroup $H \subseteq H_L$ such that $(\tilde{\mathbf{A}}^\dagger)^H \otimes_{\tilde{\mathbf{A}}_L^\dagger} M = \bigoplus (\tilde{\mathbf{A}}^\dagger)^H e_i$ for a basis e_i invariant under φ. Now let $v = \sum x_i e_i$ be an arbitrary element in

$$\tilde{\mathbf{A}}_L \otimes_{\tilde{\mathbf{A}}_L^\dagger} M \subseteq \tilde{\mathbf{A}}^H \otimes_{\tilde{\mathbf{A}}_L^\dagger} M = \tilde{\mathbf{A}}^H \otimes_{(\tilde{\mathbf{A}}^\dagger)^H} (\tilde{\mathbf{A}}^\dagger)^H \otimes_{\tilde{\mathbf{A}}_L^\dagger} M = \bigoplus \tilde{\mathbf{A}}^H e_i$$

with $x_i \in \tilde{\mathbf{A}}^H$ and such that $\varphi(v) = v$. The latter condition implies that $x_i \in \tilde{\mathbf{A}}^{H,\varphi_q=\mathrm{id}} = o_L$, i.e., v belongs to $(M \otimes_{\tilde{\mathbf{A}}_L^\dagger} (\tilde{\mathbf{A}}^\dagger)^H) \cap (M \otimes_{\tilde{\mathbf{A}}_L^\dagger} \tilde{\mathbf{A}}_L) = M$, because M is free and one has $\tilde{\mathbf{A}}_L \cap (\tilde{\mathbf{A}}^\dagger)^H = (\tilde{\mathbf{A}}^\dagger)^{H_L} = \tilde{\mathbf{A}}_L^\dagger$. To show essential surjectivity one proceeds literally as in the proof of [KLI, Thm. 8.5.3] adapted to ramified Witt vectors.

For the second statement choose a quasi-inverse functor $F : \Phi_f^{\acute{e}t}(\tilde{\mathbf{A}}_L) \to \Phi_f^{\acute{e}t}(\tilde{\mathbf{A}}_L^\dagger)$ with $F(\tilde{\mathbf{A}}_L) = \tilde{\mathbf{A}}_L^\dagger$. Given an extension $0 \longrightarrow M \longrightarrow E \longrightarrow \tilde{\mathbf{A}}_L \longrightarrow 0$ over $\Phi(\tilde{\mathbf{A}}_L)$ with $M \in \Phi_f^{\acute{e}t}(\tilde{\mathbf{A}}_L)$ first observe that $E \in \Phi_f^{\acute{e}t}(\tilde{\mathbf{A}}_L)$, too. Indeed, $\tilde{\mathbf{A}}_L \xrightarrow{\varphi_q} \tilde{\mathbf{A}}_L$ is a flat ring extension, whence $\varphi^* E \to E$ is an isomorphism, if the corresponding outer maps are. The analogous statement holds over $\tilde{\mathbf{A}}_L^\dagger$. Therefore the sequence $0 \longrightarrow F(M) \longrightarrow F(E) \longrightarrow \tilde{\mathbf{A}}_L^\dagger \longrightarrow 0$ is exact by Remark A.19, because its base extension - being isomorphic to the original extension - is, by assumption. $\square$

We denote by $\mathfrak{M}_f^{\acute{e}t}(\tilde{\mathbf{A}}_L^\dagger)$ and $\mathfrak{M}_f^{\acute{e}t}(\tilde{\mathbf{A}}_L)$ the full subcategories of $\mathfrak{M}^{\acute{e}t}(\tilde{\mathbf{A}}_L^\dagger)$ and $\mathfrak{M}^{\acute{e}t}(\tilde{\mathbf{A}}_L)$, respectively, consisting of finitely generated free modules over the base ring.

Remark A.21. *Let M be in $\mathfrak{M}_f^{\acute{e}t}(\tilde{\mathbf{A}}_L)$ and endow $N := \tilde{\mathbf{A}}_L \otimes_{\tilde{\mathbf{A}}_L^\dagger} M$ with the canonical topology with respect to the weak topology of $\tilde{\mathbf{A}}_L$. Then the induced subspace topology of $M \subseteq N$ coincides with the canonical topology with respect to the weak topology of $\tilde{\mathbf{A}}_L^\dagger$. Indeed for free modules this is obvious while for torsion modules this can be reduced by the elementary divisor theory to the case $M = \tilde{\mathbf{A}}_L^\dagger/\pi_L^n \tilde{\mathbf{A}}_L^\dagger \cong \tilde{\mathbf{A}}_L/\pi_L^n \tilde{\mathbf{A}}_L$. But the latter spaces are direct product factors of $\tilde{\mathbf{A}}_L^\dagger$ and $\tilde{\mathbf{A}}_L$, respectively, as topological spaces, from wich the claim easily follows.*

Proposition A.22. *For $T \in \mathrm{Rep}_{o_L}(G_L)$ and $V \in \mathrm{Rep}_L(G_L)$ we have natural isomorphisms*

$$\tilde{\mathbf{A}}_L \otimes_{\tilde{\mathbf{A}}_L^\dagger} \tilde{D}^\dagger(T) \cong \tilde{D}(T) \text{ and} \tag{157}$$

$$\tilde{\mathbf{B}}_L \otimes_{\tilde{\mathbf{B}}_L^\dagger} \tilde{D}^\dagger(V) \cong \tilde{D}(V), \tag{158}$$

as well as

$$\tilde{\mathbf{A}}^\dagger \otimes_{\tilde{\mathbf{A}}_L^\dagger} \tilde{D}^\dagger(T) \cong \tilde{\mathbf{A}}^\dagger \otimes_{o_L} T \text{ and} \tag{159}$$

$$\tilde{\mathbf{B}}^\dagger \otimes_{\tilde{\mathbf{B}}_L^\dagger} \tilde{D}^\dagger(V) \cong \tilde{\mathbf{B}}^\dagger \otimes_L V, \tag{160}$$

respectively. In particular, the functor $\tilde{D}^\dagger(-) : \mathrm{Rep}_{o_L}(G_L) \to \mathfrak{M}^{ét}(\tilde{\mathbf{A}}_L^\dagger)$ is exact.

Moreover, base extension induces equivalences of categories

$$\mathfrak{M}_f^{ét}(\tilde{\mathbf{A}}_L^\dagger) \leftrightarrow \mathfrak{M}_f^{ét}(\tilde{\mathbf{A}}_L),$$

and hence also an equivalence of categories

$$\mathfrak{M}^{ét}(\tilde{\mathbf{B}}_L^\dagger) \leftrightarrow \mathfrak{M}^{ét}(\tilde{\mathbf{B}}_L).$$

Proof. Note that the base change functor is well-defined - regarding the continuity of the Γ_L-action - by Lemma A.9 and Remark A.19 while $\tilde{D}^\dagger$ is well-defined by Remark A.21, once (157) will have been shown. We first show the equivalence of categories for free modules: By Proposition A.20 we already have, for $M_1, M_2 \in \mathfrak{M}_f^{ét}(\tilde{\mathbf{A}}_L^\dagger)$, an isomorphism

$$\mathrm{Hom}_{\Phi(\tilde{\mathbf{A}}_L^\dagger)}(M_1, M_2) \cong \mathrm{Hom}_{\Phi(\tilde{\mathbf{A}}_L)}(\tilde{\mathbf{A}}_L \otimes_{\tilde{\mathbf{A}}_L^\dagger} M_1, \tilde{\mathbf{A}}_L \otimes_{\tilde{\mathbf{A}}_L^\dagger} M_2).$$

Taking Γ_L-invariants gives that the base change functor in question is fully faithful.

In order to show that this base change functor is also essentially surjective, consider an arbitrary $N \in \mathfrak{M}_f^{ét}(\tilde{\mathbf{A}}_L)$. Again by A.20 we know that there is a free étale φ-module M over $\tilde{\mathbf{A}}_L^\dagger$ whose base change is isomorphic to N. By the fully faithfulness the Γ_L-action descends to M. Since the weak topology of M is compatible with that of N by Remark A.21, this action is again continuous.

To prepare for the proof of the isomorphism (157) we first observe the following fact. The isomorphism (147) implies that T and $\tilde{D}(T)$ have the same elementary divisors, i.e.: If $T \cong \oplus_{i=1}^r o_L/\pi_L^{n_i} o_L$ as o_L-module (with $n_i \in \mathbb{N} \cup \{\infty\}$) then $\tilde{D}(T) \cong \oplus_{i=1}^r \tilde{\mathbf{A}}_L/\pi_L^{n_i}\tilde{\mathbf{A}}_L$ as $\tilde{\mathbf{A}}_L$-module.

We shall prove (157) in several steps: First assume that T is *finite*. Then T is annihilated by some π_L^n. We have $\tilde{D}^\dagger(T) = \tilde{D}(T)$ and $\tilde{\mathbf{A}}_L^\dagger/\pi_L^n\tilde{\mathbf{A}}_L^\dagger = \tilde{\mathbf{A}}_L/\pi_L^n\tilde{\mathbf{A}}_L$ so that there is nothing to prove. Secondly we suppose that T is *free and* that $\tilde{D}^\dagger(T)$ is free over $\tilde{\mathbf{A}}_L^\dagger$ of the same rank $r := \mathrm{rk}_{o_L} T$. On the other hand, as the functor $\tilde{D}^\dagger$ is always left exact, we obtain the injective maps

$$\tilde{D}^\dagger(T)/\pi_L^n\tilde{D}^\dagger(T) \to \tilde{D}^\dagger(T/\pi_L^n T) = \tilde{D}(T/\pi_L^n T).$$

for any $n \geqslant 1$. We observe that both sides are isomorphic to $(\tilde{\mathbf{A}}_L^\dagger/\pi_L^n\tilde{\mathbf{A}}_L^\dagger)^r = (\tilde{\mathbf{A}}_L/\pi_L^n\tilde{\mathbf{A}}_L)^r$. Hence the above injective maps are bijections. We deduce that

$$\begin{aligned}\tilde{\mathbf{A}}_L \otimes_{\mathbf{A}_L^\dagger} \tilde{D}^\dagger(T) &\cong \varprojlim_n \tilde{D}^\dagger(T)/\pi_L^n\tilde{D}^\dagger(T)\\ &\cong \varprojlim_n \tilde{D}(T/\pi_L^n T)\\ &\cong \varprojlim_n \tilde{D}(T)/\pi_L^n\tilde{D}(T)\\ &\cong \tilde{D}(T)\end{aligned}$$

using that the above tensor product means π_L-adic completion for finitely generated $\tilde{\mathbf{A}}_L^\dagger$-modules.

Thirdly let $T \in \mathrm{Rep}_{o_L,f}(G_L)$ be arbitrary and $M \in \mathfrak{M}_f^{ét}(\tilde{\mathbf{A}}_L^\dagger)$ such that $\tilde{\mathbf{A}}_L \otimes_{\tilde{\mathbf{A}}_L^\dagger} M \cong \tilde{D}(T)$ according the equivalence of categories. Without loss of generality we may treat this isomorphism as an equality. Similarly as in the proof of Proposition A.20 and with the same notation one shows that $(\tilde{\mathbf{A}}^\dagger \otimes_{\tilde{\mathbf{A}}_L^\dagger} M)^{\varphi=1} = \bigoplus_{i=1}^r o_L e_i$ for some appropriate φ-invariant basis $e_1, \ldots, e_r$ of $\tilde{\mathbf{A}}^\dagger \otimes_{\tilde{\mathbf{A}}_L^\dagger} M$. Note that $r = \mathrm{rk}_{o_L} T$. Using (147), it follows that

$$\begin{aligned} T = (\tilde{\mathbf{A}} \otimes_{o_L} T)^{\varphi=1} &\cong (\tilde{\mathbf{A}} \otimes_{\tilde{\mathbf{A}}_L} \tilde{D}(T))^{\varphi=1} = (\tilde{\mathbf{A}} \otimes_{\tilde{\mathbf{A}}_L^\dagger} M)^{\varphi=1} \\ &= \bigoplus_{i=1}^r \tilde{\mathbf{A}}^{\varphi_q=1} e_i = \bigoplus_{i=1}^r o_L e_i = (\tilde{\mathbf{A}}^\dagger \otimes_{\tilde{\mathbf{A}}_L^\dagger} M)^{\varphi=1}. \end{aligned}$$

It shows that the comparison isomorphism (147) restricts to an injective map $T \hookrightarrow \tilde{\mathbf{A}}^\dagger \otimes_{\tilde{\mathbf{A}}_L^\dagger} M$, which extends to a homomorphism $\tilde{\mathbf{A}}^\dagger \otimes_{o_L} T \xrightarrow{\alpha} \tilde{\mathbf{A}}^\dagger \otimes_{\tilde{\mathbf{A}}_L^\dagger} M$ of free $\tilde{\mathbf{A}}^\dagger$-modules of the same rank r. Further base extension by $\tilde{\mathbf{A}}$ gives back the isomorphism (147). Since $\tilde{\mathbf{A}}$ is faithfully flat over $\tilde{\mathbf{A}}^\dagger$ the map α was an isomorphism already. By passing to H_L-invariants we obtain an isomorphism $\tilde{D}^\dagger(T) \cong M$ and see that $\tilde{D}^\dagger(T)$ is free of the same rank as T. Hence the second case applies and gives (157) for free T and (158). Finally, let T be just finitely generated over o_L. Write $0 \to T_{\mathrm{fin}} \to T \to T_{\mathrm{free}} \to 0$ with finite T_{fin} and free T_{free}. We then have the commutative exact diagram

$$\begin{tikzcd} 0 \arrow[r] & \tilde{\mathbf{A}}_L \otimes_{\tilde{\mathbf{A}}_L^\dagger} \tilde{D}^\dagger(T_{\mathrm{fin}}) \arrow[r] \arrow[d, "\cong"'] & \tilde{\mathbf{A}}_L \otimes_{\tilde{\mathbf{A}}_L^\dagger} \tilde{D}^\dagger(T) \arrow[r] \arrow[d] & \tilde{\mathbf{A}}_L \otimes_{\tilde{\mathbf{A}}_L^\dagger} \tilde{D}^\dagger(T_{\mathrm{free}}) \arrow[r] \arrow[d, "\cong"'] & \tilde{\mathbf{A}}_L \otimes_{\tilde{\mathbf{A}}_L^\dagger} H^1(H_L, \tilde{\mathbf{A}}^\dagger \otimes_{o_L} T_{\mathrm{fin}}) \\ 0 \arrow[r] & \tilde{D}(T_{\mathrm{fin}}) \arrow[r] & \tilde{D}(T) \arrow[r] & \tilde{D}(T_{\mathrm{free}}) \arrow[r] & 0, \end{tikzcd}$$

in which we use the first and third step for the vertical isomorphisms. In order to show that the middle perpendicular arrow is an isomorphism it suffices to prove that $H^1(H_L, \tilde{\mathbf{A}}^\dagger \otimes_{o_L} T_{\mathrm{fin}}) = 0$. But since T_{fin} is annihilated by some π_L^n we have

$$\tilde{\mathbf{A}}^\dagger \otimes_{o_L} T_{\mathrm{fin}} \cong \tilde{\mathbf{A}}/\pi_L^n \tilde{\mathbf{A}} \otimes_{o_L} T_{\mathrm{fin}} \cong \tilde{\mathbf{A}}/\pi_L^n \tilde{\mathbf{A}} \otimes_{\tilde{\mathbf{A}}_L} \tilde{D}(T_{\mathrm{fin}}),$$

the last isomorphism by (147). Thus it suffices to prove the vanishing of $H^1(H_L, \tilde{\mathbf{A}}/\pi_L^n \tilde{\mathbf{A}})$, which is established in Proposition A.17 and finishes the proof of the isomorphism (157).

Note that this base change isomorphism implies the exactness of $\tilde{D}^\dagger$ as $\tilde{D}$ is exact by Proposition A.10 and using that the base extension is faithfully flat by Remark A.19.

For free T the statement (159) (and hence (160)) is already implicit in the above arguments while for finite T the statement coincides with (147). The general case follows from the previous ones by exactness of $\tilde{D}^\dagger$ and the five lemma as above.

□

Corollary A.23. *For a T in $\mathrm{Rep}_{o_L,f}(G_L)$ and V in $\mathrm{Rep}_L(G_L)$, the nth cohomology group, for any $n \geqslant 0$, of the complexes concentrated in degrees* 0 *and* 1

(161) $$0 \longrightarrow \tilde{D}^\dagger(T) \xrightarrow{\varphi-1} \tilde{D}^\dagger(T) \longrightarrow 0 \quad \textit{and}$$

(162) $$0 \longrightarrow \tilde{D}^\dagger(V) \xrightarrow{\varphi-1} \tilde{D}^\dagger(V) \longrightarrow 0 \quad \textit{and}$$

is isomorphic to $H^n(H_L, T)$ and $H^n(H_L, V)$, respectively.

Proof. The integral result reduces, by (157), Remark A.18, and Proposition A.20, to Corollary A.16. Since inverting π_L is exact and commutes with taking cohomology [NSW, Prop. 2.7.11], the second statement follows. $\square$

Set $\mathbf{A}^\dagger := \tilde{\mathbf{A}}^\dagger \cap \mathbf{A}$ and $\mathbf{B}^\dagger := \mathbf{A}^\dagger[\frac{1}{\pi_L}]$ as well as $\mathbf{A}_L^\dagger := (\mathbf{A}^\dagger)^{H_L}$. Note that $\mathbf{B}_L^\dagger := (\mathbf{B}^\dagger)^{H_L} \subseteq \mathbf{B}^\dagger \subseteq \tilde{\mathbf{B}}^\dagger$. For $V \in \mathrm{Rep}_L(G_L)$ we define $D^\dagger(V) := (\mathbf{B}^\dagger \otimes_L V)^{H_L}$. The categories $\mathfrak{M}^{ét}(\mathbf{A}_L^\dagger)$ and $\mathfrak{M}^{ét}(\mathbf{B}_L^\dagger)$ are defined analogously as in Definition A.8.

Remark A.24. *There is also the following more concrete description for $\mathbf{A}_L^\dagger$ in terms of Laurent series in ω_{LT} :*

$$\mathbf{A}_L^\dagger = \{F(\omega_{LT}) \in \mathbf{A}_L | F(Z) \textit{ converges on } \rho \leqslant |Z| < 1 \textit{ for some } \rho \in (0,1)\} \subseteq \mathbf{A}_L.$$

Indeed this follows from the analogue of [ChCo1, Lem. II.2.2] upon noting that the latter holds with and without the integrality condition: "$rv_p(a_n) + n \geqslant 0$ for all $n \in \mathbb{Z}$" (for $r \in \overline{\mathbf{R}} \backslash \mathbf{R}$) in the notation of that article. In particular we obtain canonical embeddings $\mathbf{A}_L^\dagger \subseteq \mathbf{B}_L^\dagger \hookrightarrow \mathcal{R}_L$ of rings.

Definition A.25. *V in $\mathrm{Rep}_L(G_L)$ is called overconvergent, if $\dim_{\mathbf{B}_L^\dagger} D^\dagger(V) = \dim_L V$. We denote by $\mathrm{Rep}_L^\dagger(G_L) \subseteq \mathrm{Rep}_L(G_L)$ the full subcategory of overconvergent representations.*

Remark A.26. *We always have $\dim_{\mathbf{B}_L^\dagger} D^\dagger(V) \leqslant \dim_L V$. If $V \in \mathrm{Rep}_L(G_L)$ is overconvergent then we have the natural isomorphism*

$$\mathbf{B}_L \otimes_{\mathbf{B}_L^\dagger} D^\dagger(V) \xrightarrow{\cong} D(V). \tag{163}$$

Proof. Since $\mathbf{B}_L$ and $\mathbf{B}_L^\dagger$ are fields this is immediate from [FO, Thm. 2.13]. $\square$

Remark A.27. *In [Be16, §10] Berger uses the following condition to define overconvergence of V: There exists a $\mathbf{B}_L$-basis $x_1, \ldots, x_n$ of $D(V)$ such that $M := \bigoplus_{i=1}^n \mathbf{B}_L^\dagger x_i$ is a (φ_L, Γ_L)-module over $\mathbf{B}_L^\dagger$. This then implies a natural isomorphism*

$$\mathbf{B}_L \otimes_{\mathbf{B}_L^\dagger} M \cong D(V). \tag{164}$$

Lemma A.28. *V in $\mathrm{Rep}_L(G_L)$ is overconvergent if and only if V satisfies the above condition of Berger. In this case $M = D^\dagger(V)$.*

Proof. If V is overconvergent, we can take a basis within $M := D^\dagger(V)$. Conversely let V satisfy Berger's condition, i.e. we have the isomorphism (164). One easily checks by faithfully flat descent that with $D(V)$ also M is étale. By [FX, Prop. 1.5 (a)] we obtain the identity $V = \left(\mathbf{B}^\dagger \otimes_{\mathbf{B}_L^\dagger} M\right)^{\varphi=1}$ induced from the comparison isomorphism

$$\mathbf{B} \otimes_L V \cong \mathbf{B} \otimes_{\mathbf{B}_L} D(V) \cong \mathbf{B} \otimes_{\mathbf{B}_L^\dagger} M. \tag{165}$$

We shall prove that $M \subseteq D^\dagger(V) = (\mathbf{B}^\dagger \otimes_L V)^{H_L}$ as then $M = D^\dagger(V)$ by dimension reasons. To this aim we may write a basis $v_1, \ldots, v_n$ of V over L as $v_i = \sum c_{ij} x_j$ with $c_{ij} \in \mathbf{B}^\dagger$. Then (165) implies that the matrix $C = (c_{ij})$ belongs to $M_n(\mathbf{B}^\dagger) \cap GL_n(\mathbf{B}) = GL_n(\mathbf{B}^\dagger)$. Thus M is contained in $\mathbf{B}^\dagger \otimes_L V$ and - as subspace of $D(V)$ - also H_L-invariant, whence the claim. $\square$

Remark A.29. *Note that the imperfect version of Proposition A.22 is not true: base change $\mathfrak{M}^{ét}(\mathbf{B}_L^\dagger) \to \mathfrak{M}^{ét}(\mathbf{B}_L)$ is not essentially surjective in general, whence not an equivalence of categories, by [FX]. By definition, its essential image consists of overconvergent (φ_L, Γ_L)-modules, i.e., whose corresponding Galois representations are overconvergent.*

Lemma A.30. *Assume that $V \in \mathrm{Rep}_L(G_L)$ is overconvergent. Then there is natural isomorphism*

$$\tilde{\mathbf{B}}_L^\dagger \otimes_{\tilde{\mathbf{B}}_L^\dagger} D^\dagger(V) \cong \tilde{D}^\dagger(V).$$

Proof. By construction we have a natural map $\tilde{\mathbf{B}}_L^\dagger \otimes_{\tilde{\mathbf{B}}_L^\dagger} D^\dagger(V) \to \tilde{D}^\dagger(V)$, whose base change to $\tilde{\mathbf{B}}_L$

$$\tilde{\mathbf{B}}_L \otimes_{\tilde{\mathbf{B}}_L^\dagger} D^\dagger(V) \to \tilde{\mathbf{B}}_L \otimes_{\tilde{\mathbf{B}}_L^\dagger} \tilde{D}^\dagger(V) \cong \tilde{D}(V)$$

arises also as the base change of the isomorphism (163), whence is an isomorphism itself. Here we have used the (base change of the) isomorphisms (158), (146). By faithfully flatness the original map is an isomorphism, too. □

A.1.3 The perfect Robba ring

Again let K be any perfectoid field containing L and $r > 0$. For $0 < s \leqslant r$, let $\tilde{\mathcal{R}}^{[s,r]}(K)$ be the completion of $W^r(K^\flat)_L[\frac{1}{\pi_L}]$ with respect to the norm $\max\{|\ |_s, |\ |_r\}$, and put

$$\tilde{\mathcal{R}}^r(K) = \varprojlim_{s\in(0,r]} \tilde{\mathcal{R}}^{[s,r]}(K)$$

equipped with the Fréchet topology. Let $\tilde{\mathcal{R}}(K) = \varinjlim_{r>0} \tilde{\mathcal{R}}^r(K)$, equipped with the locally convex direct limit topology (LF topology). We set $\tilde{\mathcal{R}} = \tilde{\mathcal{R}}(\mathbb{C}_p)$ and $\tilde{\mathcal{R}}_L := \tilde{\mathcal{R}}(\hat{L}_\infty)$. As in [KLI, Thm. 9.2.15] we have

$$\tilde{\mathcal{R}}^{H_L} = \tilde{\mathcal{R}}_L.$$

Similarly as in [KLI, Def. 4.3.1] for the cyclotomic situation one shows that the embedding $o_L[[Z]] \to W(\hat{L}_\infty^\flat)_L$ from subsection 1.1 extends to a Γ_L- and φ_L-equivariant topological monomorphism $\mathcal{R}_L \to \tilde{\mathcal{R}}_L$, see also [W, Konstruktion 1.3.27] in the Lubin-Tate setting.

Let R be either $\mathcal{R}_L$ or $\tilde{\mathcal{R}}_L$. A (φ_L, Γ_L)-module over R is a finitely generated free R-module M equipped with commuting semilinear actions of φ_M and Γ_L, such that the action is continuous for the LF topology and such that the semi-linear map $\varphi_M : M \to M$ induces an isomorphism $\varphi_M^{lin} : R \otimes_{R,\varphi_R} M \xrightarrow{\cong} M$. Such M is called étale, if there exists an étale (φ_L, Γ_L)-module N over $\mathbf{A}_L^\dagger$ and $\tilde{\mathbf{A}}_L^\dagger$ (see before Definition A.8), such that $\mathcal{R}_L \otimes_{\mathbf{A}_L^\dagger} N \cong M$ and $\tilde{\mathcal{R}}_L \otimes_{\tilde{\mathbf{A}}_L^\dagger} N \cong M$, respectively.

By $\mathfrak{M}(R)$ and $\mathfrak{M}^{ét}(R)$ we denote the category of (φ_L, Γ_L)-modules and étale (φ_L, Γ_L)-modules over R, respectively.

We call the topologies on $\tilde{\mathbf{A}}_L^\dagger$ and $\tilde{\mathbf{A}}^\dagger$, which make the inclusions $\tilde{\mathbf{A}}_L^\dagger \subseteq \tilde{\mathbf{A}}^\dagger \subseteq \tilde{\mathcal{R}}$ topological embeddings, the LF-topologies.

Lemma A.31. *For $M \in \mathfrak{M}_f^{ét}(\tilde{\mathbf{A}}_L^\dagger)$ the Γ_L-action is also continuous with respect to the canonical topology with respect to the LF-topology of $\tilde{\mathbf{A}}_L^\dagger$.*

Proof. The proof in fact works in the following generality: Suppose that $\tilde{\mathbf{A}}^\dagger$ is equipped with an o_L-linear ring topology which induces the π_L-adic topology on o_L. Consider on $\tilde{\mathbf{A}}_L^\dagger$ the corresponding induced topology. We claim that then the Γ_L-action on M is continuous with respect to the corresponding canonical topology. By Proposition A.40 we may choose $T \in \mathrm{Rep}_{o_L,f}(G_L)$ such that $M \cong \tilde{D}^\dagger(T)$. Then we have a homeomorphism $\tilde{\mathbf{A}}^\dagger \otimes_{o_L} T \cong \tilde{\mathbf{A}}^\dagger \otimes_{\tilde{\mathbf{A}}_L^\dagger} M$ with respect to the canonical topology by (159) (as any R-module homomorphism of finitely generated modules is continuous with respect to the canonical topology with regard to any topological ring R). Since $o_L \subseteq \tilde{\mathbf{A}}^\dagger$ is a topological embedding with respect to the π_L-adic and the given topology, respectively, Lemma A.9 implies that G_L is acting continuously on $\tilde{\mathbf{A}}^\dagger \otimes_{\tilde{\mathbf{A}}_L^\dagger} M$, whence Γ_L acts continuously on $M = \left(\tilde{\mathbf{A}}^\dagger \otimes_{\tilde{\mathbf{A}}_L^\dagger} M\right)^{H_L}$ with respect to the induced topology as subspace of the previous module. Since all involved modules are free and hence carry the product topologies and since $\tilde{\mathbf{A}}_L^\dagger \subseteq \tilde{\mathbf{A}}^\dagger$ is a topological embedding, it is clear that the latter topology of M coincides with its canonical topology. □

We define the functor

$$\begin{aligned} \tilde{D}^\dagger_{rig}(-) : \mathrm{Rep}_L(G_L) &\longrightarrow \mathfrak{M}(\tilde{\mathcal{R}}_L) \\ V &\longmapsto (\tilde{\mathcal{R}} \otimes_L V)^{H_L}, \end{aligned}$$

where the fact, that Γ_L acts continuously on the image with respect to the LF-topology can be seen as follows, once we have shown the next lemma. Indeed, (166) implies that for any G_L-stable o_L-lattice T of V we also have an isomorphism $\tilde{\mathcal{R}}_L \otimes_{\tilde{\mathbf{A}}_L^\dagger} \tilde{D}^\dagger(T) \xrightarrow{\cong} \tilde{D}^\dagger_{rig}$. Now again Lemma A.9 applies to conclude the claim.

Lemma A.32. *The canonical map*

$$\tilde{\mathcal{R}}_L \otimes_{\tilde{\mathbf{B}}_L^\dagger} \tilde{D}^\dagger(V) \xrightarrow{\cong} \tilde{D}^\dagger_{rig}(V) \tag{166}$$

is an isomorphism and the functor $\tilde{D}^\dagger_{rig}(-) : \mathrm{Rep}_L(G_L) \to \mathfrak{M}(\tilde{\mathcal{R}}_L)$ is exact. Moreover, we have a comparison isomorphism

$$\tilde{\mathcal{R}} \otimes_{\tilde{\mathcal{R}}_L} \tilde{D}^\dagger_{rig}(V) \xrightarrow{\cong} \tilde{\mathcal{R}} \otimes_{o_L} V. \tag{167}$$

Proof. The comparison isomorphism in the proof of (an analogue of) [KP, Thm. 2.13] implies the comparison isomorphism

$$\tilde{\mathcal{R}} \otimes_{\tilde{\mathcal{R}}_L} \tilde{D}^\dagger_{rig}(V) \cong \tilde{\mathcal{R}} \otimes_{o_L} V$$

together with the identity $V = (\tilde{\mathcal{R}} \otimes_{\tilde{\mathcal{R}}_L} \tilde{D}^\dagger_{rig}(V))^{\varphi_L=1}$. On the other hand the comparison isomorphism (160) induces by base change an isomorphism

$$\tilde{\mathcal{R}} \otimes_{\tilde{\mathbf{B}}_L^\dagger} \tilde{D}^\dagger(V) \xrightarrow{\cong} \tilde{\mathcal{R}} \otimes_{o_L} V.$$

Taking H_L-invariants gives the first claim. The exactness of the functor $\tilde{D}^\dagger_{rig}(-)$ follows from the exactness of the functor $\tilde{D}^\dagger(-)$ by Proposition A.10. □

Let R be $\mathbf{B}_L$, $\mathbf{B}_L^\dagger$, $\mathcal{R}_L$, $\tilde{\mathbf{B}}_L$, $\tilde{\mathbf{B}}_L^\dagger$, $\tilde{\mathcal{R}}_L$ and let correspondingly R^{int} be $\mathbf{A}_L$, $\mathbf{A}_L^\dagger$, $\mathbf{A}_L^\dagger$, $\tilde{\mathbf{A}}_L$, $\tilde{\mathbf{A}}_L^\dagger$, $\tilde{\mathbf{A}}_L^\dagger$. We denote by $\Phi(R)^{ét}$ the essential image of the base change functor $R \otimes_{R^{int}} - : \Phi^{ét,f}(R^{int}) \to \Phi^{ét,f}(R)$ (sic!).

Proposition A.33. *Base change induces an equivalence of categories*

$$\Phi(\tilde{\mathbf{B}}_L^\dagger)^{\acute{e}t} \leftrightarrow \Phi(\tilde{\mathcal{R}}_L)^{\acute{e}t}$$

and an isomorphism of Yoneda extension groups

$$\mathrm{Ext}^1_{\Phi(\tilde{\mathbf{B}}_L^\dagger)}(\tilde{\mathbf{B}}_L^\dagger, M) \cong \mathrm{Ext}^1_{\Phi(\tilde{\mathcal{R}}_L)}(\tilde{\mathcal{R}}_L, \tilde{\mathcal{R}}_L \otimes_{\tilde{\mathbf{B}}_L^\dagger} M)$$

for all $M \in \Phi(\tilde{\mathbf{B}}_L^\dagger)^{\acute{e}t}$.

Proof. The first claim is an analogue of [KLI, Thm. 8.5.6]. The second claim follows as in the proof of Proposition (A.20) using the fact that by Lemma 8.6.3 in loc. cit. any extension of étale φ-modules over $\tilde{R}_L$ is again étale. Note that $\tilde{\mathcal{R}}_L/\tilde{\mathbf{B}}_L^\dagger$ is a faithfully flat ring extension, $\tilde{\mathbf{B}}_L^\dagger$ being a field. □

Corollary A.34. *If* V *belongs to* $Rep_L(G_L)$*, the following complex concentrated in degrees* 0 *and* 1 *is acyclic*

$$0 \longrightarrow \tilde{D}^\dagger_{rig}(V)/\tilde{D}^\dagger(V) \xrightarrow{\varphi-1} \tilde{D}^\dagger_{rig}(V)/\tilde{D}^\dagger(V) \longrightarrow 0. \tag{168}$$

In particular, we have that the nth cohomology groups of the complex concentrated in degrees 0 *and* 1

$$0 \longrightarrow \tilde{D}^\dagger_{rig}(V) \xrightarrow{\varphi-1} \tilde{D}^\dagger_{rig}(V) \longrightarrow 0$$

are isomorphic to $H^n(H_L, V)$ *for* $n \geqslant 0$.

Proof. Compare with [KLI, Thm. 8.6.4] and its proof (Note that the authors meant to cite Theorem 8.5.12 (taking c=0, d=1) instead of Theorem 6.2.9 - a reference which just does not exist within that book). Using the interpretation of the H^i_φ as Hom- and Ext^1-groups, respectively, the assertion is immediate from Proposition A.33. The last statement now follows from Corollary A.23. □

Proposition A.35. *Base extension gives rise to an equivalence of categories*

$$\mathfrak{M}^{\acute{e}t}(\mathbf{B}_L^\dagger) \leftrightarrow \mathfrak{M}^{\acute{e}t}(\mathcal{R}_L).$$

Proof. [FX, Prop. 1.6]. □

Lemma A.36. (i) $\mathbf{B}_L^\dagger \subseteq \mathcal{R}_L$ *are Bézout domains and the strong hypothesis in the sense of [Ked08, Hypothesis 1.4.1] holds, i.e., for any* $n \times n$ *matrix* A *over* $\mathbf{A}_L^\dagger$ *the map* $(\mathcal{R}_L/\mathbf{B}_L^\dagger)^n \xrightarrow{1-A\varphi_L} (\mathcal{R}_L/\mathbf{B}_L^\dagger)^n$ *is bijective.*

Proof. [Ked08, Prop. 1.2.6]. □

Proposition A.37. *If* V *belongs to* $Rep_L^\dagger(G_L)$*, the following complex concentrated in degrees* 0 *and* 1 *is acyclic*

$$0 \longrightarrow D^\dagger_{rig}(V)/D^\dagger(V) \xrightarrow{\varphi-1} D^\dagger_{rig}(V)/D^\dagger(V) \longrightarrow 0, \tag{169}$$

where $D^{\dagger}_{rig}(V) := \mathcal{R}_L \otimes_{\mathbf{B}^{\dagger}_L} D^{\dagger}(V)$*. In particular, we have that the* n*th cohomology groups of the complex concentrated in degrees* 0 *and* 1

$$0 \longrightarrow D^{\dagger}_{rig}(V) \xrightarrow{\varphi-1} D^{\dagger}_{rig}(V) \longrightarrow 0$$

are isomorphic to $H^n_{\dagger}(H_L, V) := H^n(\varphi_L, D^{\dagger}(V))$ *for* $n \geqslant 0$*, see* (110)*.*

Proof. This follows from the strong hypothesis in Lemma A.36 as the Frobenius endomorphism on $M \in \mathfrak{M}^{\acute{e}t}(\mathbf{B}^{\dagger}_L)$ is of the form $A\varphi_L$ by definition. $\square$

Lemma A.38. *Base change induces a fully faithful embeddings* $\Phi(\mathbf{A}^{\dagger}_L)^{\acute{e}t} \subseteq \Phi(\mathbf{A}_L)^{\acute{e}t}$ *and* $\Phi(\mathbf{B}^{\dagger}_L)^{\acute{e}t} \subseteq \Phi(\mathbf{B}_L)^{\acute{e}t}$*.*

Proof. As in the proof of Proposition A.20 this reduces to checking that $\left(\mathbf{A}_L \otimes_{\mathbf{A}^{\dagger}_L} M\right)^{\varphi=\mathrm{id}} \subseteq M$. By that proposition we know that

$$\left(\mathbf{A}_L \otimes_{\mathbf{A}^{\dagger}_L} M\right)^{\varphi=\mathrm{id}} \subseteq \left(\tilde{\mathbf{A}}_L \otimes_{\mathbf{A}^{\dagger}_L} M\right)^{\varphi=\mathrm{id}} \subseteq \tilde{\mathbf{A}}^{\dagger}_L \otimes_{\mathbf{A}^{\dagger}_L} M.$$

Since $\mathbf{A}_L \cap \tilde{\mathbf{A}}^{\dagger}_L = \mathbf{A}^{\dagger}_L$ within $\tilde{\mathbf{A}}_L$ by definition, the claim follows for the integral version, whence also for the other one my tensoring the integral embedding with L over o_L. $\square$

Remark A.39. *Note that* $H^0_{\dagger}(H_L, V) = H^0(H_L, V)$ *and* $H^1_{\dagger}(H_L, V) \subseteq H^1(H_L, V)$*. For the latter relation use the previous lemma, which implies that an extension which splits after base change already splits itself, together with Corollary A.16 and Remark A.18. In general the inclusion for* H^1 *is strict as follows indirectly from [FX]. Indeed, otherwise the complex*

$$0 \longrightarrow D(V)/D^{\dagger}(V) \xrightarrow{\varphi-1} D(V)/D^{\dagger}(V) \longrightarrow 0, \tag{170}$$

would be always acyclic, which would imply by the same observation as in Proposition A.43 below together with Theorem 3.11 (ii) that $H^1_{\dagger}(G_L, V) = H^1(G_L, V)$ *in contrast to Remark 3.14.*

A.1.4 The complete picture

Although formally we do not need them in this article we would also mention the following equivalences of categories, for which we only sketch proofs or indicate analogue results whose proofs can be transferred to our setting.

Proposition A.40. *The following categories are equivalent:*

(i) $Rep_{o_L}(G_L)$,

(ii) $\mathfrak{M}^{\acute{e}t}(\mathbf{A}_L)$,

(iii) $\mathfrak{M}^{\acute{e}t}(\tilde{\mathbf{A}}_L)$ *and*

(iv) $\mathfrak{M}^{\acute{e}t}(\tilde{\mathbf{A}}^{\dagger}_L)$.

The equivalences from (ii) and (iv) to (iii) are induced by base change.

Proof. This can be proved in the same way as in [Ked15, Thm. 2.3.5], although it seems to be only a sketch. Another way is to check that the very detailed proof for the equivalence between (i) and (ii) in [GAL] almost literally carries over to a proof for the equivalence between (i) and (iii). Alternatively, this is a consequence of Proposition B.2 by [KLII, Thm. 5.4.6]. See also [Kl]. For the equivalence between (iii) and (iv) consider the 2-commutative diagram

$$\begin{tikzcd} \mathfrak{M}^{ét}(\tilde{\mathbf{A}}_L^\dagger) \arrow[rr, "\text{faithfully flat base change}"] & & \mathfrak{M}^{ét}(\tilde{\mathbf{A}}_L) \arrow[dl] \\ & \mathrm{Rep}_{o_L}(G_L) \arrow[ul] \arrow[ur] & \end{tikzcd} ,$$

which is induced by the isomorphism (157) and immediately implies (essential) surjectivity on objects and morphisms while the faithfulness follows from faithfully flat base change. □

Corollary A.41. *The following categories are equivalent:*

(i) $Rep_L(G_L)$,

(ii) $\mathfrak{M}^{ét}(\mathbf{B}_L)$,

(iii) $\mathfrak{M}^{ét}(\tilde{\mathbf{B}}_L)$ *and*

(iv) $\mathfrak{M}^{ét}(\tilde{\mathbf{B}}_L^\dagger)$.

The equivalences from (ii) and (iv) to (iii) are induced by base change.

Proof. This follows from Propositions A.22 and A.40 by inverting π_L. □

Proposition A.42. *The categories in Corollary A.41 are - via base change from (iv) - also equivalent to*

(v) $\mathfrak{M}^{ét}(\tilde{\mathcal{R}}_L)$.

Proof. By definition base change is essentially surjective and it is well-defined - regarding the continuity of the Γ_L-action - by Lemma A.31 and Lemma A.9. Since for étale φ_L-modules we know fully faithfulness already, taking Γ_L-invariants gives fully faithfulness for (φ_L, Γ_L)-modules, too. □

Altogether we may visualize the relations between the various categories by the following diagram:

$$
\begin{tikzcd}
\mathfrak{M}^{\acute{e}t}(\tilde{\mathcal{R}}_L) & & \mathfrak{M}^{\acute{e}t}(\tilde{\mathbf{B}}_L^\dagger) \arrow[ll, Rightarrow] \arrow[rr, Rightarrow] & & \mathfrak{M}^{\acute{e}t}(\tilde{\mathbf{B}}_L) \\
\mathfrak{M}^{\acute{e}t}(\mathcal{R}_L) \arrow[u] & & \mathfrak{M}^{\acute{e}t}(\mathbf{B}_L^\dagger) \arrow[ll, Rightarrow] \arrow[rr] \arrow[u] & & \mathfrak{M}^{\acute{e}t}(\mathbf{B}_L) \arrow[u, Rightarrow] \\
\mathrm{Rep}_L^{an}(G_L) \arrow[r] & \mathrm{Rep}_L^\dagger(G_L) \arrow[rr] \arrow[ul, Leftrightarrow, "D^\dagger_{rig}", "V^\dagger_{rig}"'] \arrow[ur, Leftrightarrow, "D^\dagger", "V^\dagger"'] & & \mathrm{Rep}_L(G_L) \arrow[uulll, Leftrightarrow, "\tilde{D}^\dagger_{rig}", "\tilde{V}^\dagger_{rig}"'] \arrow[uul, Leftrightarrow, "\tilde{D}^\dagger", "\tilde{V}^\dagger"'] \arrow[uur, Leftrightarrow, "\tilde{D}", "\tilde{V}"'] \arrow[ur, Leftrightarrow, "D", "V"'] &
\end{tikzcd}
$$

Here all arrows represent functors which are fully faithful, i.e., embeddings of categories. Arrows without label denote base change functors. Under them the functors $D, \tilde{D}, D^\dagger, \tilde{D}^\dagger, D^\dagger_{rig}$, and $\tilde{D}^\dagger_{rig}$ are compatible. The arrows => represent equivalences of categories, while the arrows −> represent embeddings which are not essentially surjective in general. We recall that the quasi-inverse functors are given as follows

$$
\begin{aligned}
V(M) =& (\mathbf{B} \otimes_{\mathbf{B}_L} M)^{\varphi_L=1}, \;\; \tilde{V}(M) = (\tilde{\mathbf{B}} \otimes_{\tilde{\mathbf{B}}_L} M)^{\varphi_L=1}, \;\; V^\dagger(M) = (\mathbf{B}^\dagger \otimes_{\mathbf{B}_L^\dagger} M)^{\varphi_L=1}, \\
\tilde{V}^\dagger(M) =& (\tilde{\mathbf{B}}^\dagger \otimes_{\tilde{\mathbf{B}}_L^\dagger} M)^{\varphi_L=1}, \;\; \tilde{V}^\dagger_{rig}(M) = (\tilde{\mathcal{R}} \otimes_{\tilde{\mathcal{R}}_L} M)^{\varphi_L=1} \text{ and } V^\dagger_{rig}(M) = (\tilde{\mathcal{R}} \otimes_{\mathcal{R}_L} M)^{\varphi_L=1}.
\end{aligned}
$$

A.2 Cup products and local Tate duality

The aim of this subsection is to discuss cup products and to prove Proposition 3.20. We fix some open subgroup $U \subseteq \Gamma_L$ and let $L' = L_\infty^U$. Note that we obtain a decomposition $U \cong \Delta \times U'$ with a subgroup $U' \cong \mathbb{Z}_p^d$ of U and Δ the torsion subgroup of U.

Proposition A.43. *If V belongs to $Rep_L(G_L)$, the canonical inclusions of Herr complexes*

$$
\begin{aligned}
K^\bullet_{\varphi,U'}(\tilde{D}^\dagger(V)^\Delta) &\subseteq K^\bullet_{\varphi,U'}(\tilde{D}^\dagger_{rig}(V)^\Delta), \\
K^\bullet_{\varphi,U'}(\tilde{D}^\dagger(V)^\Delta) &\subseteq K^\bullet_{\varphi,U'}(\tilde{D}(V)^\Delta) \text{ and} \\
K^\bullet_{\varphi,U'}(D(V)^\Delta) &\subseteq K^\bullet_{\varphi,U'}(\tilde{D}(V)^\Delta)
\end{aligned}
$$

are quasi-isomorphisms and their cohomology groups are canonically isomorphic to $H^i(L', V)$ for all $i \geqslant 0$.

Proof. Forming Koszul complexes with regard to U' we obtain the following diagram of (double) complexes with exact columns

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
K^\bullet(D(V)^\Delta) \arrow[r, "\varphi-1"] \arrow[d] & K^\bullet(D(V)^\Delta) \arrow[d] \\
K^\bullet(\tilde{D}(V)^\Delta) \arrow[r, "\varphi-1"] \arrow[d] & K^\bullet(\tilde{D}(V)^\Delta) \arrow[d] \\
K^\bullet((\tilde{D}(V)/D(V))^\Delta) \arrow[r, "\varphi-1", "\cong"'] \arrow[d] & K^\bullet((\tilde{D}(V)/D(V))^\Delta) \arrow[d] \\
0 & 0
\end{tikzcd}$$

in which the bottom line is an isomorphism of complexes by A.16, as the action of Δ commutes with φ. Hence, going over to total complexes gives an exact sequence

$$0 \to K^\bullet_{\varphi,U}(D(V)^\Delta) \to K^\bullet_{\varphi,U}(\tilde{D}(V)^\Delta) \to K^\bullet_{\varphi,U}((\tilde{D}(V)/D(V))^\Delta) \to 0,$$

in which $K^\bullet_{\varphi,U}((\tilde{D}(V)/D(V))^\Delta)$ is acyclic. Thus we have shown the statement regarding the last inclusion. The other two cases are dealt with similarly, now using (168) and A.23 combined with (152). It follows in particular that all six Koszul complexes in the statement are quasi-isomorphic. Therefore it suffices for the second part of the assertion to show that the cohomology groups of $K^\bullet_{\varphi,U'}(D(V)^\Delta)$ are isomorphic to $H^i(L',V)$.

To this aim let T be a G_L-stable lattice of V. In [Ku, Thm. 5.1.11.], [KV, Thm. 5.1.11.] it is shown that the cohomology groups of $\mathcal{T}_{\varphi,U}(D(T))$ are canonically isomorphic to $H^i(L',T)$ for all $i \geqslant 0$, whence the cohomology groups of $\mathcal{T}_{\varphi,U}(D(T))[\frac{1}{\pi_L}]$ are canonically isomorphic to $H^i(L',V)$ for all $i \geqslant 0$. For $M_0 = D(T)$ Lemma A.44 below establishes a quasi-isomorphism

$$\mathcal{T}_{\varphi,U}(D(T))[\frac{1}{\pi_L}] \simeq K^\bullet_{\varphi,U'}(D(V)^\Delta)$$

which gives the last statement. □

Lemma A.44. *Let M_0 be a complete linearly topologised o_L-module M with continuous U-action. Then there is a canonical quasi-isomorphism*

$$\mathcal{T}_{\varphi,U}(M_0)[\frac{1}{\pi_L}] \simeq K^\bullet_{\varphi,U'}(M_0[\frac{1}{\pi_L}]^\Delta).$$

If M_0 is an L-vector space, the inversion of π_L can be omitted on both sides.

Proof. Let $\mathcal{C}^\bullet_n(U,M_0) \subseteq \mathcal{C}^\bullet(U,M_0)$ denote the subcomplex of normalized cochains. Since Δ is finite, [Th, Thm. 3.7.6] gives a canonical quasi-isomorphism:

$$\mathcal{C}^\bullet_n(U,M_0) = \mathcal{C}^\bullet_n(\Delta \times U', M_0) \xrightarrow{\simeq} \mathcal{C}^\bullet_n(\Delta, \mathcal{C}^\bullet_n(U',M_0)).$$

Here we understand the above objects in the sense of hypercohomology as total complexes of the obvious double complexes involved. After inverting π_L we may compute the right hand side further as

$$\mathcal{C}_n^\bullet(\Delta, \mathcal{C}_n^\bullet(U', M_0))[\frac{1}{\pi_L}] = \mathcal{C}_n^\bullet(\Delta, \mathcal{C}_n^\bullet(U', M_0)[\frac{1}{\pi_L}]) \xleftarrow{\simeq} \mathcal{C}_n^\bullet(U', M_0)[\frac{1}{\pi_L}]^\Delta = \mathcal{C}_n^\bullet(U', M_0^\Delta)[\frac{1}{\pi_L}] \ .$$

Here the middle quasi-isomorphism comes from the fact that a finite group has no cohomology in characteristic zero. The right hand equality is due to the fact that Δ acts on the cochains through its action on M_0. Altogether we obtain a natural quasi-isomorphism

$$\mathcal{C}_n^\bullet(U, M_0)[\frac{1}{\pi_L}] \cong \mathcal{C}_n^\bullet(U', M_0^\Delta)[\frac{1}{\pi_L}] \ .$$

By using [Th, Prop. 3.3.3] we may replace the normalized cochains again by general cochains obtaining the left hand quasi-isomorphism in

$$\mathcal{C}^\bullet(U, M_0)[\frac{1}{\pi_L}] \cong \mathcal{C}^\bullet(U', M_0^\Delta)[\frac{1}{\pi_L}] \cong K_{U'}^\bullet(M_0^\Delta)[\frac{1}{\pi_L}] = K_{U'}^\bullet(M_0[\frac{1}{\pi_L}]^\Delta) \ .$$

The middle quasi-isomorphism is (121). The claim follows by taking mapping fibres of the attached map $\varphi_L - 1$ of complexes. $\square$

Recall from [Ne, (5.2.1)] the quasi isomorphism $L[-2] \xrightarrow{\iota} \tau_{\geqslant 2}\mathcal{C}^\bullet(G_{L'}, L(1))$ which allows to define a trace map

$$\mathrm{tr}_{\mathcal{C}} : \mathcal{C}^\bullet(G_{L'}, L(1)) \to L[-2]$$

in the derived category $D(L\text{−Mod})$ as

$$\mathcal{C}^\bullet(G_{L'}, L(1)) \to \tau_{\geqslant 2}\mathcal{C}^\bullet(G_{L'}, L(1)) \xleftarrow{\iota} L[-2].$$

Then local Tate duality is induced by the following pairing on cocycles

$$\mathcal{C}^\bullet(G_{L'}, V^*(1)) \ \times \ \mathcal{C}^\bullet(G_{L'}, V) \xrightarrow{\cup_{G_{L'}}} \mathcal{C}^\bullet(G_{L'}, L(1)) \xrightarrow{\mathrm{tr}_{\mathcal{C}}} L[-2]. \tag{171}$$

As before let T be a G_L-stable lattice of V. Setting $M_0 = D(T^*(1))$, $M = D(V^*(1))$, $\tilde{M}_{rig}^\dagger = \tilde{D}_{rig}^\dagger(V^*(1))$, $N_0 = D(T)$, $N = D(V)$, $\tilde{N}_{rig}^\dagger = \tilde{D}_{rig}^\dagger(V)$ etc. we obtain the following commutative diagram, in which we require for the two last lines that $V^*(1)$ is L-analytic,

(172)

$$
\begin{tikzcd}[column sep=small]
\mathcal{C}^\bullet(G_{L'},V^*(1)) \arrow[d, dashed, "\simeq"] & \times & \mathcal{C}^\bullet(G_{L'},V) \arrow[r, "\cup_{G_{L'}}"] \arrow[d, dashed, "\simeq"] & \mathcal{C}^\bullet(L',L(1)) \arrow[r, dashed, "\mathrm{tr}_{\mathcal{C}}"] \arrow[d, dashed, "\simeq"] & L[-2] \arrow[d, equal] \\
\mathcal{T}^\bullet_{\varphi,U}(M_0)[\frac{1}{\pi_L}] & \times & \mathcal{T}^\bullet_{\varphi,U}(N_0)[\frac{1}{\pi_L}] \arrow[r, "\cup_U"] & \mathcal{T}^\bullet_{\varphi,U}(D(o_L(1)))[\frac{1}{\pi_L}] \arrow[r, dashed, "\mathrm{tr}_{\mathcal{T}}"] & L[-2] \arrow[d, equal] \\
K^\bullet_{\varphi,U'}(M^\Delta) \arrow[u, dashed, "\simeq"] \arrow[d, dashed, "\simeq"] & \times & K^\bullet_{\varphi,U'}(N^\Delta) \arrow[r, "\cup_K"] \arrow[u, dashed, "\simeq"] \arrow[d, dashed, "\simeq"] & K^\bullet_{\varphi,U'}(D(L(1))^\Delta) \arrow[r, dashed, "\mathrm{tr}_K"] \arrow[u, dashed, "\simeq"] \arrow[d, dashed, "\simeq"] & L[-2] \arrow[d, equal] \\
K^\bullet_{\varphi,U'}((\tilde{M}^\dagger_{rig})^\Delta) \arrow[d, equal] & \times & K^\bullet_{\varphi,U'}((\tilde{N}^\dagger_{rig})^\Delta) \arrow[r, "\cup_K"] \arrow[d, "\mathrm{adj}"] & K^\bullet_{\varphi,U'}(\tilde{D}^\dagger_{rig}(L(1))^\Delta) \arrow[r, dashed, "\mathrm{tr}_{\tilde{K}^\dagger_{rig}}"] & L[-2] \arrow[d, equal] \\
K^\bullet_{\varphi,U'}((\tilde{M}^\dagger_{rig})^\Delta) & \times & K^\bullet_{\varphi,U'}((\tilde{M}^\dagger_{rig})^\Delta)^*[-2] \arrow[rr, "ev_2"] \arrow[d, "c"] & & L[-2] \arrow[d, equal] \\
K^\bullet_{\varphi,U'}((M^\dagger_{rig})^\Delta) \arrow[u, hook, "b"] \arrow[d, equal] & \times & K^\bullet_{\varphi,U'}((M^\dagger_{rig})^\Delta)^*[-2] \arrow[rr, "ev_2"] \arrow[d, "g", "\cong"'] & & L[-2] \arrow[d, equal] \\
K^\bullet_{\varphi,U'}((M^\dagger_{rig})^\Delta) \arrow[d, equal] & \times & K^\bullet_{\psi,U'}(((M^\dagger_{rig})^\vee)^\Delta)[d-1] \arrow[rr, "\cup_{K,\psi}"] \arrow[d, "\cong"] & & L[-2] \arrow[d, equal] \\
K^\bullet_{\varphi,U'}(D^\dagger_{rig}(V^*(1))^\Delta) & \times & K^\bullet_{\psi,U'}(D^\dagger_{rig}(V(\tau^{-1}))^\Delta)[d-1] \arrow[rr, "\cup_{K,\psi}"] & & L[-2]
\end{tikzcd}
$$

in which trace maps $\mathrm{tr}_{\mathcal{T}}, \mathrm{tr}_K, \mathrm{tr}_{\tilde{K}^\dagger_{rig}}$ are defined by requiring that the first three rectangles on the right hand side become commutative in the derived category. Moreover, $\dashrightarrow$ indicates a map in the derived category while the usual arrows denote morphisms in the category of complexes; in both cases $\simeq$ indicates quasi-isomorphisms. The pairings in the 2nd, 3rd and 4th line arise as in indicated in Remark 3.16. The pairings in line 5 and 6 are the evaluation ones ev_2 according to (105). The pairing $\cup_{K,\psi}$ in line 7 is induced from the one in line 6 by requiring commutativity. Since Δ interchanges well with respect to $*$ and $\check{}$ as its action is semi-simple, the isomorphism (125) restricts to an isomorphism

$$g : K^\bullet_{\varphi,U'}((M^\dagger_{rig})^\Delta)^*[-2] \cong K^\bullet_{\psi,U'}(((M^\dagger_{rig})^\vee)^\Delta)[d-1].$$

Now we explain the remaining vertical maps. From line 1 to line 2 we have the quasi-isomorphisms from [KV, Thm. 5.1.11.]. They are compatible with cup products by Lemma A.45 below. The quasi-isomorphism between the 2nd and 3rd as well as 3rd and 4th lines are those induced from Proposition A.43 and its proof. They involve (123), which is compatible with products by Remark 3.16 as well as the quasi-isomorphism [Th, Thm. 3.7.6] which is compatible with cup products by [Th1, Theorem 11.16.] while the inclusion from normalized to all cocycles is obviously compatible. The map adj is given as

$$K^\bullet_{\varphi,U'}(\tilde{N}^\dagger_{rig}) \xrightarrow{\mathrm{adj}(\mathrm{tr}_{\tilde{\mathrm{K}}^\dagger_{\mathrm{rig}}} \circ \cup_{\mathrm{K}})} \mathrm{Hom}_{L,cts}(K^\bullet_{\varphi,U'}(\tilde{M}^\dagger_{rig}), L[-2]) \cong K^\bullet_{\varphi,U'}(\tilde{M}^\dagger_{rig})^*[-2],$$

where the first map uses the notation of (105) while the second map is the canonical isomorphism which pulls the twist outside: by our sign conventions this is the identity in all degrees. So commutativity between the 4th and 5th line follows by the same lemma. Moreover the map b arises by base change, while $c = b^*[-2]$ is the continuous L-dual of the corresponding base change map, whence the commutativity between the 5th and 6th line is clear. Finally, we use the identification $(M^{\dagger}_{rig})^{\vee} \cong D^{\dagger}_{rig}(V(\tau^{-1}))$ because the functor $D^{\dagger}_{rig}(-)$ is compatible with forming inner homomorphisms. Altogether we have established the commutativity of the diagram, which implies Proposition 3.20.

Lemma A.45. (i) *Cup products are compatible with inflation maps.*

(ii) *Let G be any profinite group and*

$$0 \longrightarrow A \xrightarrow{\iota} B \xrightarrow{f} C \longrightarrow 0$$

as well as

$$0 \longrightarrow A' \xrightarrow{\iota'} B' \xrightarrow{f'} C' \longrightarrow 0$$

two exact sequences of continuous G-modules (with continuous set theoretic section). Then there is a commutative diagram

$$\begin{tikzcd}
\mathcal{C}^\bullet(G,A) \arrow[d, "\iota_*"'] & \times & \mathcal{C}^\bullet(G,A') \arrow[r, "\cup"] \arrow[d, "\iota'_*"'] & \mathcal{C}^\bullet(G,A\otimes A') \arrow[d, "(\iota\otimes\iota')_*"] \\
\mathcal{C}^\bullet(G,B) \arrow[d, hook] & \times & \mathcal{C}^\bullet(G,B') \arrow[r, "\cup"] \arrow[d, hook] & \mathcal{C}^\bullet(G,B\otimes B') \arrow[d, hook] \\
\mathcal{C}^\bullet(G,B\to C) & \times & \mathcal{C}^\bullet(G,B'\to C') \arrow[r, "\cup"] & \mathcal{C}^\bullet(G,(B\to C)\otimes(B'\to C')),
\end{tikzcd}$$

in which the vertical composites are the canonical maps $\mathcal{C}^\bullet(G,A) \to \mathcal{C}^\bullet(G,B \to C)$, $\mathcal{C}^\bullet(G,A') \to \mathcal{C}^\bullet(G,B' \to C')$ and $\mathcal{C}^\bullet(G,A\otimes A') \to \mathcal{C}^\bullet(G,(B \to C)\otimes(B' \to C'))$ in the sense of hyper-cohomology of [Ne, (3.4.5.2)], e.g. $B \to C$ is considered as complex in degrees 0 and 1 and $\mathcal{C}^\bullet(G,B \to C) = \mathrm{Tot}(\mathcal{C}^\bullet(G,B) \to \mathcal{C}^\bullet(G,BC))$.

Proof. The first item is immediately checked on cochains. For the second item we observe that the upper rectangle commutes due to the compatibility of cup products with change of coefficients, while the lower one commutes as the cup product for hyper-cohomology is combined from the usual cup products up to sign, see [Ne, (3.4.5.2)]. Since we consider the complex $B \to C$ as concentrated in degrees 0 and 1 the middle cup product contributes with the sign $(-1)^{\bullet\cdot 0} = 1$ to the lower one. □

Proposition A.46. *Let M be a φ_L-module over $\mathcal{R} = \mathcal{R}_K$ (cf. 2.19) and $c \in K^\times$. Then $M/(\psi - c)(M)$ is finite-dimensional over K.*

Proof. (The proof follows closely the proof of [KPX, Prop. 3.3.2] in the cyclotomic situation) We set $\psi_c := c^{-1}\psi$ and show that $M/(\psi_c - 1)(M)$ is finite-dimensional over K.

Choose a model $M^{[r_0,1)}$ of M with $1 > r_0 > p^{\frac{-1}{(q-1)e}}$ and put $r = r_0^{\frac{1}{q^2}}$. Recall that for all $1 > s \geqslant r$ we have maps $M^{[s,1)} \xrightarrow{\psi_c - 1} M^{[s,1)}$ (where strictly speaking we mean

ψ_c followed by the corresponding restriction). We first show that it suffices to prove that $\operatorname{coker}\left(M^{[r,1)} \xrightarrow{\psi_c - 1} M^{[r,1)}\right)$ has finite dimension over K. Indeed, given any $m \in M$ we have $m \in M^{[s,1)}$ for some $1 > s \geqslant r$. Then there exists $k \geqslant 0$ such that $r \geqslant s^{q^k} \geqslant r_0$, whence $\psi_c^k(m)$ belongs to $M^{[r,1)}$ and represents the same class in $M/(\psi_c - 1)(M)$ as m.

Choose a basis $\mathbf{e}'_1, \ldots, \mathbf{e}'_n$ of $M^{[r_0,1)}$ and take $\mathbf{e}_i := \varphi(\mathbf{e}'_i) \in M^{[r^q,1)}$; by the φ-module property the latter elements also form a basis of $M^{[r^q,1)}$. Note that by base change these two bases also give rise to bases in $M^{[s,1)}$ for $1 > s \geqslant r^q$. Thus we find a matrix F' with entries in $\mathcal{R}^{[r^q,1)}$ such that $\mathbf{e}_j = \sum_i F'_{ij}\mathbf{e}'_i$ and we put $F = \varphi(F')$ with entries in $\mathcal{R}^{[r,1)}$, i.e., $\varphi(\mathbf{e}_j) = \sum F_{ij}\mathbf{e}_i$. Similarly let G be a matrix with values in $\mathcal{R}^{[r^q,1)} \subseteq \mathcal{R}^{[r,1)}$ such that $\mathbf{e}'_j = \sum_i G_{ij}\mathbf{e}_i$ and hence $\mathbf{e}_j = \varphi\left(\sum_i G_{ij}\mathbf{e}_i\right)$.

We identify $M^{[r,1)}$ with $\left(\mathcal{R}^{[r,1)}\right)^n$ by sending $(\lambda_i)_i$ to $\sum_i \lambda_i \mathbf{e}_i$ and endow it for each $r \leqslant s < 1$ with the norm given by $\max_i |\lambda_i|_s$. Note that then the "semi-linear" map ψ_c (followed by the corresponding restriction) on $\left(\mathcal{R}^{[r,1)}\right)^n$ is given by the matrix G as follows from the projection formula (43):

$$\psi_c(\sum_j \lambda_j \mathbf{e}_j) = \sum_j \psi_c(\lambda_j \varphi(\sum_i G_{ij}\mathbf{e}_i)) = \sum_{i,j} \psi_c(\lambda_j) G_{ij}\mathbf{e}_i.$$

Moreover, the restriction of $\varphi : M^{[r,1)} \to M^{[r^{\frac{1}{q}},1)}$ to $\sum_I \mathcal{R}^+ e_i$ becomes the semi-linear map $(\mathcal{R}^+)^n \to \left(\mathcal{R}^{[r,1)}\right)^n$ given by the matrix F.

Consider, for I any subset of the reals $\mathbf{R}$, the K-linear map $P_I : \mathcal{R}^{[r,1)} \to \mathcal{R}^{[r,1)}, \sum a_i Z^i \mapsto \sum_{i \in \mathbf{Z} \cap I} a_i Z^i$. We then introduce K-linear operators P_I and Q_k, $k \geqslant 0$, on $M^{[r,1)}$ by

$$\begin{aligned} P_I((\lambda)_i) &:= (P_I(\lambda_i))_i, \\ Q_k &:= P_{(-\infty,-k)} - \frac{c\pi_L}{q}\varphi \circ P_{(k,\infty)}, i.e., \\ Q_k((\lambda)_i) &= (P_{(-\infty,-k)}(\lambda)_i)_i - \frac{c\pi_L}{q} F \cdot (\varphi(P_{(k,\infty)}(\lambda_i)))_i, \end{aligned}$$

because $P_{(k,\infty)}$ factorises through $\mathcal{R}^+$. Then the K-linear operator $\Psi_k := \mathrm{id} - P_{[-k,k]} + (\psi_c - 1)Q_k$ of $M^{[r,1)}$ satisfies

$$\begin{aligned} \Psi_k &= \psi_c \circ P_{(-\infty,-k)} + \frac{c\pi_L}{q}\varphi \circ P_{(k,\infty)}, i.e., \\ \Psi_k((\lambda_i)_i) &= G \cdot (\psi_c(P_{(-\infty,-k)}(\lambda_i)))_i + F \cdot (\frac{c\pi_L}{q}\varphi(P_{(k,\infty)}(\lambda_i)))_i, \end{aligned}$$

whence its operator norm satisfies

$$\|\Psi_k\|_s \leqslant \max\{\|G\|_s \|\psi_c \circ P_{(-\infty,-k)}\|_s, \frac{c\pi_L}{q}\|F\|_s \|\varphi \circ P_{(k,\infty)}\|_s\}.$$

It is easy to check that, for $1 > s > q^{\frac{-1}{q-1}}$, we have $\|\varphi \circ P_{(k,\infty)}\|_s \leqslant |Z|_s^{(q-1)k} = s^{(q-1)k}$ (using the norm relation after (31)) and $\|\psi_c \circ P_{(-\infty,-k)}\|_s \leqslant C_s s^{k(1-q^{-1})}$ for some constant $C_s > 0$.

E.g. for the latter we have for $\lambda = \sum_i a_i Z^i \in \mathcal{R}^{[r,1)}$

$$\begin{aligned}
|\sum_{i<-k} \psi_c(a_i Z^i)|_s &\leqslant \sup_{i<-k} |a_i||\psi_c(Z^i)|_s \\
&\leqslant \sup_{i<-k} |a_i| C_s s^{\frac{i}{q}} \\
&\leqslant C_s \sup_{i<-k} |a_i||Z|_s^i s^{i(q^{-1}-1)} \\
&\leqslant C_s |\lambda|_s s^{-k(q^{-1}-1)},
\end{aligned}$$

where we use that by continuity of ψ_c there exists C_s such that

$$|\psi_c(Z^i)|_s \leqslant C_s |Z^i|_{s^{\frac{1}{q}}} = C_s s^{\frac{i}{q}}.$$

Thus we may and do choose k sufficiently big such that $\|\Psi_k\|_r \leqslant \frac{1}{2}$. Given $m_0 \in M^{[r,1)}$ we define inductively $m_{i+1} := \Psi_k(m_i)$. This sequence obviously converges to zero with respect to the r-Gauss-norm. We shall show below that also for all $s \in (r^{\frac{1}{q}}, 1)$ the series $(m_i)_i$ tends to zero with respect to the Gauss norm $|\ |_s$, i.e., by cofinality the sum $m := \sum_{i\geqslant 0} m_i$ converges in $M^{[r,1)}$ for the Frechét-topology and satisfies

$$m - m_0 = m - P_{[-k,k]}(m) + (\psi_c - 1)Q_k(m),$$

i.e. $P_{[-k,k]}(m)$ represents the same class as m_0 in $M^{[r,1)}/(\psi_c - 1)(M)$. Since the image of $P_{[-k,k]}$ has finite dimension, the proposition follows, once we have shown the following
Claim: For all $s \in (r^{\frac{1}{q}}, 1)$ we have

$$|\Psi_k(m)|_s \leqslant \max\{\frac{1}{2}|m|_s, C_s\|G\|_s \left(\frac{s^{\frac{1}{q}}}{r}\right)^{-k} |m|_r, |\frac{c\pi_L}{q}|\|F\|_s \left(\frac{s^q}{r}\right)^{k'} |m|_r\}.$$

Indeed, we fix such s and may choose $k' \geqslant k$ such that $\|\Psi_{k'}\|_s \leqslant \frac{1}{2}$. Then $\Psi_k = \Psi_{k'} - \psi_c \circ P_{[-k',-k)} - \frac{c\pi_L}{q}\varphi \circ P_{(k,k']}$, whence the claim as for $\lambda \in \mathcal{R}^{[r,1)}$

$$|\psi_c \circ P_{[-k',-k)}(\lambda)|_s \leqslant C_s \left(\frac{s^{\frac{1}{q}}}{r}\right)^{-k} |\lambda|_r$$

$$|\varphi \circ P_{(k,k']}(\lambda)|_s \leqslant \left(\frac{s^q}{r}\right)^{k'} |\lambda|_r$$

by similar estimations as above. □

Remark A.47. *This result answers the expectation from [BF, Remark 2.3.7.] positively.*

Corollary A.48. *Let $V^*(1)$ be L-analytic and $M := D^\dagger_{rig}(V^*(1))$.*

(i) *The cohomology group $h^2(K^\bullet_{\psi,U'}((M)^\Delta)[d-1])$ is finite dimensional over L.*

(ii) *We have isomorphisms*

$$\begin{aligned} h^1(K^\bullet_{\psi,U'}(D^\dagger_{rig}(V(\tau^{-1}))^\Delta)[d-1])^* &\cong h^1(K^\bullet_{\varphi,U'}(M^\Delta)) \\ &\cong H^1_\dagger(L', V^*(1)), \end{aligned}$$

and

$$\begin{aligned} h^2(K^\bullet_{\psi,U'}(D^\dagger_{rig}(V(\tau^{-1}))^\Delta)[d-1])^* &\cong h^0(K^\bullet_{\varphi,U'}(M^\Delta)) \\ &= (V^*(1))^{G_{L'}}. \end{aligned}$$

Proof. (i) Since $h^2(K^\bullet_{\psi,U'}((M)^\Delta)[d-1])$ is a quotient of $(M/(\psi-1)(M))^\Delta$ by (128) this follows from the Proposition. (ii) We are in the situation of Remark 3.7 (i) with regard to $\mathcal{C} = K^\bullet_{\psi,U'}(D^\dagger_{rig}(V(\tau^{-1}))^\Delta)[d-1]$ and $i = 2, 3$ in the notation of the remark. Indeed, for $h^3(\mathcal{C}) = \mathcal{C}^4 = 0$ by construction and $\mathcal{C}^3 = 0$ as well as $h^2(\mathcal{C})$ is finite by (i). Hence the first isomorphism follows in both cases from (125) using the reflexivity of M. The second isomorphisms arise by Lemma A.44 together with (117) and (116), respectively. $\square$

In accordance with diagram at the end of subsection A.1.4 we may visualize the relations between the various Herr-complexes by the following diagram:

$$\begin{tikzcd}
K^\bullet_{\varphi,U'}(\tilde{D}^\dagger_{rig}(V)^\Delta) & K^\bullet_{\varphi,U'}(\tilde{D}^\dagger(V)^\Delta) \arrow[l, Leftarrow] \arrow[r, Rightarrow] & K^\bullet_{\varphi,U'}(\tilde{D}(V)^\Delta) \\
K^\bullet_{\varphi,U'}(D^\dagger_{rig}(V)^\Delta) \arrow[u, "b"] & K^\bullet_{\varphi,U'}(D^\dagger(V)^\Delta) \arrow[l, Leftarrow] \arrow[r] \arrow[u] & K^\bullet_{\varphi,U'}(D(V)^\Delta) \arrow[u, Rightarrow] \\
\mathcal{T}^{an}_{\varphi_L,\Gamma_L}(D^\dagger_{rig}(V)) \arrow[u, dashed] & & \mathcal{C}^\bullet(G_{L'}, V) \arrow[u, dashed, leftrightarrow]
\end{tikzcd}$$

Here all arrows represent injective maps of complexes, among which the arrows => represent quasi-isomorphisms, while the arrows −> need not induce isomorphisms on cohomology, in general. The interrupted arrow − −> means a map in the derived category while < − −> means a quasi-isomorphism in the derived category. By Lemma A.44 we have a analogous diagram for $\mathcal{T}_{\varphi,U}(?(V))$ with $? \in \{D, \tilde{D}, D^\dagger, \tilde{D}^\dagger, D^\dagger_{rig}, \tilde{D}^\dagger_{rig}\}$.

Remark A.49. *The image of*

$$h^i(\mathcal{T}_{\varphi,U}(D^\dagger_{rig}(V))) \cong h^i(K^\bullet_{\varphi,U'}(D^\dagger_{rig}(V)^\Delta)) \cong h^i(K^\bullet_{\varphi,U'}(D^\dagger(V)^\Delta)) \cong h^i(\mathcal{T}_{\varphi,U}(D^\dagger(V)))$$

in $H^i(L', V)$ is independent of the composite (= path) in above diagram.

We quickly discuss the analogues of some results of §I.6 in [ChCo2]. Consider the subring $A = \mathbf{A}_L^+[[\frac{\pi_L}{Z^{q-1}}]] = \{x = \sum_k a_k Z^k \in \mathbf{A}_L | v_{\pi_L}(a_k) \geqslant -\frac{k}{q-1}\} \subseteq \mathbf{A}_L$. For $x \in \mathbf{A}_L$ and each inter $n \geqslant 0$, we define $w_n(x)$ to be the smallest integer $k \geqslant 0$ such that $x \in Z^{-k}A + \pi_L^{n+1}\mathbf{A}_L$. It satisfies $w_n(x+y) \leqslant \max\{w_n(x), w_n(y)\}$ and $w_n(xy) \leqslant w_n(x) + w_n(y)$ (since A is a ring) and $w_n(\varphi(x)) \leqslant q w_n(x)$ (use that $\frac{\varphi(Z)}{Z^q} \in A^\times$, whence $\varphi(Z^{-k})A = Z^{-qk}A$).

Set for $n \geqslant 2, m \geqslant 0$ the integers $r(n) := (q-1)q^{n-1}$, $l(m,n) = m(q-1)(q^{n-1}-1) = m(r(n)-(q-1))$ and define $\mathbf{A}_L^{\dagger,n} = \{x = \sum_k a_k Z^k \in \mathbf{A}_L | v_{\pi_L}(a_k) + \frac{k}{r(n)} \to \infty \text{ for } k \mapsto -\infty\}$. By Remark A.24 we obtain that $\mathbf{A}_L^\dagger = \bigcup_{n \geqslant 2} \mathbf{A}_L^{\dagger,n}$.

Lemma A.50. *Let $x = \sum_k a_k Z^k \in \mathbf{A}_L$ and $l \geqslant 0, n \geqslant 2$. Then*

(i) *we have*

$$w_m(x) \leqslant l \Leftrightarrow v_{\pi_L}(a_k) \geqslant \min\{m+1, -\frac{k+l}{q-1}\} \text{ for } k < -l. \tag{173}$$

(ii) *$x \in \mathbf{A}_L^{\dagger,n}$ if and only if $w_m(x) - l(m,n)$ goes to $-\infty$ when m runs to ∞.*

Item (ii) of the Lemma is an analogue of [ChCo2, Prop. III 2.1 (ii)] for $\mathbf{A}_L^{\dagger,n}$ instead of $\mathbf{A}_{L,\leqslant 1}^{\dagger,n} = \{x = \sum_k a_k Z^k \in \mathbf{A}_L^{\dagger,n} | v_{\pi_L}(a_k) + \frac{k}{r(n)} \geqslant 0 \text{ for all } k \leqslant 0\}$.

Proof. (i) follows from the fact that $x \in Z^{-l}A$ if and only if $v_{\pi_L}(a_k) \geqslant -\frac{k+l}{q-1}$ for $k < -l$. (ii) Let $M, N = M(q-1) \gg 0$ be arbitrary huge integers and assume first that $x \in \mathbf{A}_L^{\dagger,n}$. Then

$$w_m(x) - l(m,n) \leqslant -N \tag{174}$$

is equivalent to

$$v_{\pi_L}(a_k) \geqslant \min\{m+1, -\frac{k + l(m,n) - N}{q-1}\} \text{ for } k < -l(m,n) + N. \tag{175}$$

by (i). To verify this relation for m sufficiently huge, we choose a $k_0 \in \mathbb{Z}$ such that $v_{\pi_L}(a_k) + \frac{k}{r(n)} \geqslant N \geqslant 0$ for all $k \leqslant k_0$. Now choose m_0 with $-l(m_0,n) < k_0$ and fix $m \geqslant m_0$. For $\frac{-k}{r(n)} > m$ we obtain $v_{\pi_L}(a_k) \geqslant m+1$, because $k < k_0$ holds. For k with

$$k \geqslant -r(n)m \Leftrightarrow \frac{k}{r(n)} - \frac{k + l(m,n)}{q-1} \leqslant 0 \tag{176}$$

we obtain $v_{\pi_L}(a_k) \geqslant -\frac{k+l(m,n)-N}{q-1}$. Thus the above relation holds true.
Vice versa choose m_0 such that (174) holds for all $m \geqslant m_0$, and fix $k \leqslant k_0 := -r(n)\max\{Mq^{n-1}, m_0\}$. Let m_1 be the unique integer satisfying $r(n)M - k \geqslant r(n)m_1 \geqslant r(n)M - k - r(n)$. In particular, we have $m_1 + 1 + \frac{k}{r(n)} \geqslant M$ and $k \geqslant -r(n)m_1$, which implies $-\frac{k+l(m_1,n)-N}{q-1} + \frac{k}{r(n)} \geqslant M$ by (176). Moreover, it holds $m_1 \geqslant m_0$ and $k < -l(m_1,n) + N$ (using $k \leqslant r(n)M - r(n)m_1 = -l(m_1,n) + q^{n-1}N - (q-1)m_1$ and $m_1 > (q^{n-1}-1)M$ by our assumption on k). Hence we can apply (175) to conclude $v_{\pi_L}(a_k) + \frac{k}{r(n)} \geqslant M$ as desired. □

The analogue of Lemma 6.2 in (loc. cit.) holds by the discussion in [SV15] after Remark 2.1. This can be used to show the analogue of Corollary 6.3, viz $w_n(\psi(x)) \leqslant 1 + \frac{w_n(x)}{q}$. Now fix a basis $(e_1, \ldots, e_d)$ of $D(T)$ over $\mathbf{A}_L$ and denote by $\Phi = (a_{ij})$ the matrix defined by $e_j = \sum_{i=1}^d a_{ij}\varphi(e_i)$. The proof of Lemma 6.4 then carries over to show that for $x = \psi(y) - y$ with $x, y \in D(T)$ we have

$$w_n(y) \leqslant \max\{w_n(x), \frac{q}{q-1}(w_n(\Phi)+1)\}, \tag{177}$$

where $w_n(\Phi) = \max_{ij} w_n(a_{ij})$ and $w_n(a) = \max_i w_n(a_i)$ for $a = \sum_{i=1}^d a_i\varphi(e_i)$ with $a_i \in \mathbf{A}_L$.

Lemma A.51. *Let $T \in \mathrm{Rep}_{o_L}(G_L)$ such that $V = T \otimes_{o_L} L$ is overconvergent. Then the canonical map $D^\dagger(T) \to D(T)$ induces an isomorphism $D^\dagger(T)/(\psi-1)(D^\dagger(T)) \cong D(T)(\psi - 1)(D(T))$.*

Proof. We follow closely the proof of [Li, Lem. 3.6], but note that he claims the statement for $D^\dagger_{\leqslant 1}(T)$ Choose a basis $e_1, \ldots, e_d$ of $D^\dagger(T)$, which is free by Lemma **??**. Since V is overconvergent it is also a basis of $D(T)$. Due to étaleness and since $(\mathbf{A}_L^\dagger) \cap \mathbf{A}_L^\times = (\mathbf{A}_L^\dagger)^\times$ also $\varphi(e_1), \ldots, \varphi(e_d)$ is a basis of all these modules. Given $x = \psi(y) - y$ with $x \in D^\dagger(T)$ and $y \in D(T)$ there is an $m > 0$ such that all x_i, a_{ij} lie in $\mathbf{A}_L^{\dagger,m}$ for some m. Since $q \geqslant 2$ it follows from the criterion in Lemma A.50 (ii) combined with (177) that all y_i belong to $\mathbf{A}_L^{\dagger,m+1}$, whence $y \in D^\dagger(T)$. This shows injectivity. In order to show surjectivity we apply Nakayamas Lemma with regard to the ring o_L upon recalling that $D(T)/(\psi-1)$ is of finite type over it. Indeed, by left exactness of $D^\dagger$ we obtain $D^\dagger(T)/\pi_L D^\dagger(T) \subseteq D^\dagger(T/\pi_L T) = D(T/\pi_L T)$. Since these are vector spaces over $\mathbf{E}_L$ of the same dimension, they are equal, whence

$$(D^\dagger(T)/(\psi-1))/(\pi_L) = (D^\dagger(T)/(\pi_L))/(\psi-1) = (D(T)/(\pi_L))/(\psi-1) = (D(T)/(\psi-1))/(\pi_L).$$

□

Corollary A.52. *Under the assumption of Lemma 1.30 for $V(\tau^{-1})$, the inclusion of complexes*

$$K^\bullet_{\psi,U'}(D^\dagger(V(\tau^{-1}))^\Delta) \subseteq K^\bullet_{\psi,U'}(D(V(\tau^{-1}))^\Delta)$$

is a quasi-isomorphism.

Proof. Use the automorphism $\psi - 1$ of $D(V(\tau^{-1}))/D^\dagger(V(\tau^{-1}))$ and proceed as in the proof of Proposition A.43. □

Remark A.53. *Instead of using Lemma 1.30 (for crystalline, analytic representations) one can probably show by the same techniques as in [ChCo2, Prop. III.3.2(ii)] that for any overconvergent representation V we have $D^\dagger(V)^{\psi=1} = D(V)^{\psi=1}$.*

The interest in the following diagram, the commutativity of which is shown before Lemma A.58, stems from the discrepancy that the reciprocity law has been formulated and proved in the setting of $K^\bullet_{\psi,U'}(D^\dagger_{rig}(V(\tau^{-1}))^\Delta)[d-1]$ while the regulator map originally lives in the setting of $K^\bullet_{\psi,U'}(D(V(\tau^{-1}))^\Delta)[d-1]$:

(178)
$$\begin{tikzcd}
\mathcal{C}^\bullet(G_{L'},V^*(1)) \arrow[d, dashed, "\simeq"] & \times & \mathcal{C}^\bullet(G_{L'},V) \arrow[r, "\cup_{G_{L'}}"] \arrow[d, dashed, "e"', "\simeq"] & \mathcal{C}^\bullet(L',L(1)) \arrow[r, dashed, "\mathrm{tr}_{\mathcal{C}}"] & L[-2] \arrow[d, equal] \\
K^\bullet_{\varphi,U'}(M^\Delta) & \times & K^\bullet_{\psi,U'}(D(V(\tau^{-1}))^\Delta)[d-1] \arrow[rr, "\cup_{K,\psi}"] & & L[-2] \arrow[d, equal] \\
K^\bullet_{\varphi,U'}((M^\dagger)^\Delta) \arrow[u, hook] \arrow[d, hook, "\simeq"] & \times & K^\bullet_{\psi,U'}(D^\dagger(V(\tau^{-1}))^\Delta)[d-1] \arrow[u, hook] \arrow[d, hook] \arrow[rr, "\cup_{K,\psi}"] & & L[-2] \arrow[d, equal] \\
K^\bullet_{\varphi,U'}((M^\dagger_{rig})^\Delta) & \times & K^\bullet_{\psi,U'}(D^\dagger_{rig}(V(\tau^{-1}))^\Delta)[d-1] \arrow[rr, "\cup_{K,\psi}"] & & L[-2]
\end{tikzcd}$$

which in turn induces the commutativity of the lower rectangle in the following diagram (the upper rectangles commute obviously)
(179)
$$\begin{tikzcd}
D^\dagger_{rig}(V(\tau^{-1}))^{\psi=1} \arrow[d, "pr_U"'] & D^\dagger(V(\tau^{-1}))^{\psi=1} \arrow[l, hook'] \arrow[r, hook, "a"] \arrow[d, "pr_U"'] & D(V(\tau^{-1}))^{\psi=1} \arrow[d, "pr_U"] \\
h^1\left(K^\bullet_{\psi,U'}(D^\dagger_{rig}(V(\tau^{-1}))^\Delta)[d-1]\right) \arrow[d, "\cong"', "a"] & h^1\left(K^\bullet_{\psi,U'}(D^\dagger(V(\tau^{-1}))^\Delta)[d-1]\right) \arrow[l] \arrow[r, "b"] & h^1\left(K^\bullet_{\psi,U'}(D(V(\tau^{-1}))^\Delta)[d-1]\right) \arrow[d, "c"', "\cong"] \\
H^1_{/\dagger}(L',V) & & H^1(L',V) \arrow[ll, two heads, "pr"']
\end{tikzcd}$$

Here the vertical maps pr_U are defined as in (129), a and pr are taken from Proposition 3.20 while the isomorphism c stems from (185). The map a is bijective under the assumption of Lemma 1.30, which extends to the map b by Corollary A.52.

A.3 Iwasawa cohomology and descent

In this subsection we recall a crucial observation from [Ku, KV], which is based on [Ne] and generalizes [SV15, Thm. 5.13]. As before let U be an open subgroup of Γ_L. We set $\mathbb{T} := \Lambda(U) \otimes_{o_L} T$ with actions by $\Lambda(U) := o_L[[U]]$ via left multiplication on the left factor and by $g \in G_{L'}$ given as $\lambda \otimes t \mapsto \lambda \bar{g}^{-1} \otimes g(t)$, where $\bar{g}$ denotes the image of g in U. We write $R\Gamma_{Iw}(L_\infty/L', T)$ for the continuous cochain complex $C^\bullet(U, \mathbb{T})$ and recall that its cohomology identifies with $H^\bullet_{Iw}(L_\infty/L', T)$ by [SV15, Lem. 5.8]. For any continuous endomorphism f of M, we set $\mathcal{T}_f(M) := [M \xrightarrow{f-1} M]$, a complex concentrated in degree 0 and 1.

The map $p : \mathbb{T} \to o_L \otimes_{\Lambda(U)} \mathbb{T} \cong T,\ t \mapsto 1 \otimes t$, and its dual $i : T^\vee(1) \to \mathbb{T}^\vee(1)$ induce on cohomology the corestriction and restriction map, respectively, and they are linked by the following commutative diagram

$$
\begin{tikzcd}
\mathcal{C}^\bullet(G_{L'},\mathbb{T}) \arrow[d, "p_*"] & \times & \mathcal{C}^\bullet(G_{L'},\mathbb{T}^\vee(1)) \arrow[r, "\cup_{G_{L'}}"] & \mathcal{C}^\bullet(L',L/o_L(1)) \arrow[r, dashed, "\mathrm{tr}_{\mathcal{C}}"] & L/o_L[-2] \arrow[d, equal] \\
\mathcal{C}^\bullet(G_{L'},T) & \times & \mathcal{C}^\bullet(G_{L'},\mathbb{T}^\vee(1)) \arrow[u, "i_*"] \arrow[r, "\cup_{G_{L'}}"] & \mathcal{C}^\bullet(L',L/o_L(1)) \arrow[r, dashed, "\mathrm{tr}_{\mathcal{C}}"] & L/o_L[-2]
\end{tikzcd}
\tag{180}
$$

By [FK, Prop. 1.6.5 (3)] (see also [Ne, (8.4.8.1)]) we have a canonical isomorphism

$$o_L \otimes^{\mathbb{L}}_{\Lambda(U)} R\Gamma(L',\mathbb{T}) \cong R\Gamma(L', o_L \otimes_{\Lambda(U)} \mathbb{T}) \cong R\Gamma(L',T) \tag{181}$$

where we denote by $R\Gamma(L',-)$ the complex $\mathcal{C}^\bullet(G_{L'},-)$ regarded as an object of the derived category. Dually, by a version of Hochschild-Serre, there is a canonical isomorphism

$$R\operatorname{Hom}_\Lambda(o_L, R\Gamma(L',\mathbb{T}^\vee(1))) \cong R\Gamma(L',T^\vee(1)). \tag{182}$$

It follows that the isomorphism

$$R\Gamma_{Iw}(L_\infty/L',T) \cong R\operatorname{Hom}_{o_L}(R\Gamma(L',\mathbb{T}^\vee(1)), L/o_L)[-2]$$

induced by the upper line of (180) induces an isomorphism

$$o_L \otimes^{\mathbb{L}}_{\Lambda(U)} R\Gamma_{Iw}(L_\infty/L',T) \cong R\operatorname{Hom}_{o_L}(R\operatorname{Hom}_\Lambda(o_L, R\Gamma(L',\mathbb{T}^\vee(1))), L/o_L)[-2], \tag{183}$$

which is compatible with the lower cup product pairing in (180) via the canonical identifications (181) and (182).

Lemma A.54. *There is a canonical isomorphism $R\Gamma(L',\mathbb{T}^\vee(1)) \cong \mathcal{T}_\varphi(D(T^\vee(1)))$ in the derived category.*

Proof. See [KV, Thm. 5.1.11]. □

For the rest of this section we assume that $U \subseteq \Gamma_L$ is an open *torsionfree* subgroup.

Lemma A.55. *Let T be in $\operatorname{Rep}_{o_L}(G_L)$ of finite length. Set $\Lambda := \Lambda(U)$ and let $\gamma_1,\dots,\gamma_d$ be topological generators of U. Then we have a up to signs canonical isomorphism of complexes*

$$\operatorname{Hom}^\bullet_\Lambda(K_\bullet(\gamma), \mathcal{T}_\varphi(D(T^\vee(1))))^\vee[-2] \cong \operatorname{tot}\left(\mathcal{T}_\psi(D(T(\tau^{-1})))[-1] \otimes_\Lambda K_\bullet(\gamma^{-1})(\Lambda)^\bullet\right)$$

where $-^\vee$ denotes forming the Pontrjagin dual.

Proof. Upon noting that $\mathcal{T}_\varphi(D(T^\vee(1)))^\vee[-2] \cong \mathcal{T}_\psi(D(T(\tau^{-1})))[-1]$ (canonically up to a sign!) this is easily reduced to the following statement

$$\operatorname{Hom}^\bullet_\Lambda(K_\bullet(\gamma), M)^\vee \cong M^\vee \otimes_\Lambda K_\bullet(\gamma^{-1})(\Lambda)^\bullet,$$

which can be proved in the same formal way as (107), and a consideration of signs. □

Remark A.56. *For every $M \in \mathfrak{M}(\mathbf{A}_L)$ we have a canonical isomorphism*

$$\operatorname{Hom}^\bullet_\Lambda(K^U_\bullet, \mathcal{T}_\varphi(M)) \cong K_{\varphi,U}(M)$$

up to the sign $(-1)^n$ in degree n and a non-canonical isomorphism

$$\operatorname{tot}\left(\mathcal{T}_\psi(M)[-1] \otimes_\Lambda K^U_\bullet(\Lambda)^\bullet\right) \cong K_{\psi,U}(M)[d-1]$$

(involving the self-duality of the Koszul complex). Here, the right hand sides are formed with respect to the same sequence of topological generators as the left hand sides.

Proof. By our conventions in section 3.2.1 $K_{\varphi,U}(M)$ is the total complex of the double complex $\mathrm{Hom}^\bullet(K_\bullet(\Lambda)^\bullet, M) \xrightarrow{1-\varphi_*} \mathrm{Hom}^\bullet(K_\bullet(\Lambda)^\bullet, M)$. A comparison with the total Hom-complex (with the same sign rules as in secton 3.2.1) shows the first claim. For the second statement we have

$$\begin{aligned}
\mathrm{tot}\left(\mathcal{T}_\psi(M)[-1] \otimes_\Lambda K_\bullet(\Lambda)^\bullet\right) &\cong \mathrm{tot}\left(\mathcal{T}_\psi(M) \otimes_\Lambda K_\bullet(\Lambda)^\bullet\right)[-1] \\
&= \mathrm{tot}\left(\mathcal{T}_\psi\left(M \otimes_\Lambda K_\bullet(\Lambda)^\bullet\right)\right)[-1] \\
&\cong \mathrm{tot}\left(\mathcal{T}_\psi\left(M \otimes_\Lambda K^\bullet(\Lambda)[d]\right)\right)[-1] \\
&= \mathrm{tot}\left(\mathcal{T}_\psi\left(K^\bullet(M)[d]\right)\right)[-1] \\
&= \mathrm{cone}\left(K_U^\bullet(M)[d] \xrightarrow{1-\psi} K_U^\bullet(M)[d]\right)[-2] \\
&\cong K_{\psi,U}(M)[d-1].
\end{aligned}$$

The first isomorphism involves a sign on $\mathcal{T}_\psi^1(M)$. The third isomorphisms stems from (106) while the last isomorphism again involves signs. □

Theorem A.57. *There are canonical isomorphisms*

$$R\Gamma_{Iw}(L_\infty/L, T) \cong \mathcal{T}_\psi\left(D(T(\tau^{-1}))\right)[-1] \tag{184}$$

$$K_{\psi,U}(D(T(\tau^{-1})))[d-1] \xrightarrow{\simeq} R\Gamma(L', T). \tag{185}$$

in the derived category $D_{perf}(\Lambda_{o_L}(\Gamma_L))$ of perfect complexes and in the derived category $D^+(o_L-\mathrm{Mod})$ of bounded below cochain complexes of o_L-modules, respectively.

Proof. The first isomorphism is [KV, Thm. 5.2.54] while the second one follows from this and (181) as

$$\begin{aligned}
R\Gamma_{Iw}(L_\infty/L, T) \otimes^{\mathbb{L}}_{\Lambda_{o_L}(U)} o_L &\cong \mathcal{T}_\psi\left(D(T(\tau^{-1}))\right)[-1] \otimes^{\mathbb{L}}_\Lambda K_\bullet(\Lambda)^\bullet \\
&= \mathrm{tot}\left(\mathcal{T}_\psi\left(D(T(\tau^{-1}))\right)[-1] \otimes_\Lambda K_\bullet(\Lambda)^\bullet\right) \\
&= K_{\psi,U}(D(T(\tau^{-1})))[d-1].
\end{aligned}$$

by Remark A.56. □

By Lemma A.55 and Remark A.56 we see that, for T be in $\mathrm{Rep}_{o_L}(G_L)$ of finite length,

$$K_{\varphi,U}(D(T^\vee(1))) = R\,\mathrm{Hom}_\Lambda(o_L, \mathcal{T}_\varphi(D(T^\vee(1)))))[2] \tag{186}$$

is dual to

$$K_{\psi,U}(D(T(\tau^{-1}))) = o_L \otimes^{\mathbb{L}}_{\Lambda(U)} \mathcal{T}_\psi(D(T(\tau^{-1})))[-1], \tag{187}$$

such that the upper rectangle in the diagram (178) commutes by (183), taking inverse limits and inverting π_L.

Lemma A.58. *Let T be in $\mathrm{Rep}_{o_L}(G_L)$. Then the left rectangle in* (143) *is commutative.*

Proof. (Sketch) By an obvious analogue of Remark 3.18 it suffices to show the statement for $U = \Gamma_n \cong \mathbb{Z}_p^d$. In this situation we have a homological spectral sequence

$$H_{i,cts}(U, H^{-j}_{Iw}(L_\infty/L, T)) \Longrightarrow H^{-i-j}_{cts}(L', T)$$

which is induced by (181), see [Ne, (8.4.8.1)] for the statement and missing notation. We may and do assume that T is of finite length. Then, on the one hand, the map $H^1_{Iw}(L_\infty/L,T) \xrightarrow{cor} H^1(L',T)$ is dual to $H^1(L',T^\vee(1)) \xrightarrow{res} H^1(L_\infty,T^\vee(1))$, which sits in the five term exact sequence of lower degrees associated with the Hochschild-Serre spectral sequence. As explained just before this lemma the above homological spectral sequence arises by dualizing from the latter. Hence *cor* shows up in the five term exact sequence of lower degrees associated with this homological spectral sequence. On the other hand via the isomorphisms (181) and (185) the latter spectral sequence is isomorphic to

$$H_{i,cts}(U, h^{-j}(\mathcal{T}_\psi\left(D(T(\tau^{-1}))\right)[-1]) \Longrightarrow h^{-i-j}(K_{\psi,U}(D(T(\tau^{-1})))[d-1])$$

and one checks by inspection that *cor* corresponds to pr_U. □

B Weakly decompleting towers

Kedlaya and Liu's developed in [KLII, §5] the concept of perfectoid towers and studied their properties in an axiomatic way. The aim of this section is to show that the Lubin-Tate extensions considered in this article form *weakly decompleting*, but not *decompleting* tower, properties which we will recall or refer to in the course of this section. Moreover, we have to show that the *axiomatic* period rings coincide with those of Appendix A.

In the sense of Def. 5.1.1 in (loc. cit.) the sequence $\Psi = (\Psi_n : (L_n, o_{L_n}) \to (L_{n+1}, o_{L_{n+1}}))_{n=0}^\infty$ forms a finite étale tower over (L, o_L) or $X := \mathrm{Spa}(L, o_L)$, which is perfectoid as $\hat{L}_\infty$ is by [GAL, Prop. 1.4.12].

Therefore we can use the perfectoid correspondence [KLII, Thm. 3.3.8] to associate with $(\hat{L}_\infty, o_{\hat{L}_\infty})$ the pair

$$(\tilde{R}_\Psi, \tilde{R}^+_\Psi) := (\hat{L}^\flat_\infty, o^\flat_{\hat{L}_\infty}).$$

Now we recall the variety of period rings, which Kedlaya and Liu attach to the tower, in our notation, starting with

Perfect period rings: $\tilde{\mathbf{A}}_\Psi := \tilde{\mathbf{A}}_L = W(\hat{L}^\flat_\infty)_L$, $\tilde{\mathbf{A}}^+_\Psi := W(o^\flat_{\hat{L}_\infty})_L \subseteq \tilde{\mathbf{A}}^{\dagger,r}_\Psi := \tilde{\mathbf{A}}^{\dagger,r}_L = \{x = \sum_{i\geqslant 0} \pi_L^i[x_i] \in W(\hat{L}^\flat_\infty)_L |$ for i to $\infty : |\pi^i_L||x_i|^r_\flat \to 0\}$, $\tilde{\mathbf{A}}^\dagger_\Psi := \bigcup_{r>0} \tilde{\mathbf{A}}^{\dagger,r}_\Psi = \tilde{\mathbf{A}}^\dagger_L$.

Imperfect period rings:

Recall the map $\Theta : W(o^\flat_{\mathbb{C}_p})_L \to o_{\mathbb{C}_p}$, $\sum_{i\geqslant 0} \pi_L^i[x_i] \mapsto \sum \pi_L^i x_i^\sharp$, which extends to a map $\Theta : \tilde{\mathbf{A}}^{\dagger,s}_\Psi \to \mathbb{C}_p$ for all $s \geqslant 1$; for arbitrary $r > 0$ and $n \geqslant -\log_q r$ the composite $\tilde{\mathbf{A}}^{\dagger,r}_\Psi \xrightarrow{\varphi_L^{-n}} \tilde{\mathbf{A}}^{\dagger,1}_\Psi \xrightarrow{\Theta} \mathbb{C}_p$ is well defined and continuous as it is easy to check. It is a homomorphism of o_L-algebras by [GAL, Lem. 1.4.18].

Following [KLII, §5] we set $\mathbf{A}^{\dagger,r}_\Psi := \{x \in \tilde{\mathbf{A}}^{\dagger,r}_\Psi | \Theta(\varphi_q^{-n}(x)) \in L_n$ for all $n \geqslant -\log_q r\}$, $\mathbf{A}^\dagger_\Psi := \bigcup_{r>0} \mathbf{A}^{\dagger,r}_\Psi$, its completion $\mathbf{A}_\Psi := (\mathbf{A}^\dagger_\Psi)^{\wedge \pi_L\text{-adic}}$, and residue field $R_\Psi := \mathbf{A}_\Psi/(\pi_L) = (\mathbf{A}^\dagger_\Psi)/(\pi_L) \subseteq \tilde{R}_\Psi$, $R^+_\Psi := R_\Psi \cap \tilde{R}^+_\Psi$.

Note that $\omega_{LT} = \{[\iota(t)]\} \in \tilde{\mathbf{A}}_\Psi^+ := W(o^\flat_{\hat{L}_\infty})_L \subseteq \tilde{\mathbf{A}}_\Psi^{\dagger,r}$ for all $r > 0$ (in the notation of [GAL]). [GAL, Lem. 2.1.12] shows

$$\Theta(\varphi_q^{-n}(\omega_{LT})) = \Theta(\{[\varphi_q^{-n}(\omega)]\}) = \lim_{i\to\infty}[\pi_L^i]_\varphi(z_{i+n}) = z_n \in L_n,$$

where $t = (z_n)_{n\geqslant 1}$ is a fixed generator of the Tate module T of the formal Lubin-Tate group and $\omega = \iota(t) \in W(o^\flat_{\mathbb{C}_p})_L$ is the reduction of ω_{LT} modulo π_L satisfying with $\mathbf{E}_L = k((\omega))$. Therefore ω_{LT} belongs to $\mathbf{A}_\Psi^+ := \mathbf{A}_\Psi \cap \tilde{\mathbf{A}}_\Psi^+$. Then it is clear that first $\mathbf{A}_L^+ := o_L[[\omega_{LT}]] \subseteq \tilde{\mathbf{A}}_\Psi^\dagger$ and by the continuity of $\Theta\circ\varphi_L^{-n}$ even $\mathbf{A}_L^+ \subseteq \mathbf{A}_\Psi^\dagger$ holds. Since $\omega_{LT}^{-1} \in \tilde{\mathbf{A}}_\Psi^{\dagger,\frac{q-1}{q}}$ by [St, Lem. 3.10] (in analogy with [ChCo1, Cor. II.1.5]) and $\Theta\circ\varphi_L^{-n}$ is a ring homomorphism, it follows that $\omega_{LT}^{-1} \in \mathbf{A}_\Psi^{\dagger,\frac{q-1}{q}}$ and $o_L[[\omega_{LT}]][\frac{1}{\omega_{LT}}] \subseteq \mathbf{A}_\Psi^\dagger$.

Lemma B.1. *We have $R_\Psi^+ = \mathbf{E}_L^+$ and $R_\Psi = \mathbf{E}_L$.*

Proof. From the above it follows that $\mathbf{E}_L \subseteq R_\Psi$, whence $\mathbf{E}_L^{perf} \subseteq R_\Psi^{perf} \subseteq \tilde{R}_\Psi = \hat{L}_\infty^\flat$ the latter being perfect. Since $\widehat{\mathbf{E}_L^{perf}} = \hat{L}_\infty^\flat$ by [GAL, Prop. 1.4.17] we conclude that

$$R_\Psi^{perf} \text{ is dense in } \tilde{R}_\Psi. \tag{188}$$

By [KLII, Lem. 5.2.2] have the inclusion

$$R_\Psi^+ \subseteq \{x \in \tilde{R}_\Psi | x = (\bar{x}_n) \text{ with } \bar{x}_n \in o_{L_n}/(z_1) \text{ for } n >> 1\} \overset{(*)}{=} \mathbf{E}_L^+ = k[[\omega]]$$

where the equality (*) follows from work of Wintenberger as recalled in [GAL, Prop. 1.4.29]. Since $\mathbf{E}_L^+ \subseteq \tilde{R}_\Psi^+$ by its construction in (loc. cit.), we conclude that $R_\Psi^+ = \mathbf{E}_L^+$.

Since each element of R_Ψ is of the form $\frac{a}{\omega^m}$ with $a \in R_\Psi^+$ and $m \geqslant 0$ by [GAL, Lem. 1.4.6], we conclude that $R_\Psi = \mathbf{E}_L$. □

Thus for each $r > 0$ such that $\omega_{LT}^{-1} \in \mathbf{A}_\Psi^{\dagger,r}$, reduction modulo π_L induces a surjection $\mathbf{A}_\Psi^{\dagger,r} \twoheadrightarrow R_\Psi$. Recall that Ψ is called weakly decompleting, if

(i) R_Ψ^{perf} is dense in $\tilde{R}_\Psi$.

(ii) for some $r > 0$ we have a strict surjection $\mathbf{A}_\Psi^{\dagger,r} \twoheadrightarrow R_\Psi$ induced by the reduction modulo π_L for the norms $|-|_r$ defined by $|x|_r := \sup_i\{|\pi_L^i||x_i|_\flat^r\}$ for $x = \sum_{i\geqslant 0}\pi_L^i[x_i]$, and $|-|_\flat^r$.

We recall from [FF, Prop. 1.4.3.] or [KLI, Prop. 5.1.2 (a)] that $|-|_r$ is multiplicative.

Proposition B.2. *The above tower Ψ is weakly decompleting.*

Proof. Since (188) gives (i), only *(ii)* is missing: Since ω_{LT} has $[\omega]$ in degree zero of its Teichmüller series, we may and do choose $r > 0$ such that $|\omega_{LT} - [\omega]|_r < |\omega|_\flat^r$. Then $|\omega_{LT}|_r = \max\{|\omega_{LT} - [\omega]|_r, |\omega|_\flat^r\} = |\omega|_\flat^r$. Consider the quotient norm $||b||^{(r)} = \inf_{a\in\mathbf{A}_\Psi^{\dagger,r}, a\equiv b \mod \pi_L} |a|_r$.

Now let $b = \sum_{n\geqslant n_0} a_n\omega^n \in R_\Psi = k((\omega))$ with $a_{n_0} \neq 0$. Lift each $a_n \neq 0$ to $\breve{a}_n \in o_L^\times$ and set $\breve{a}_n = 0$ otherwise. Then, for the lift $x := \sum_{n\geqslant n_0}\breve{a}_n\omega_{LT}^n$ of b we have by the multiplicativity of $|-|_r$ that

$$||b||^{(r)} \leqslant |x|_r = (|\omega_{LT}|_r)^{n_0} = (|\omega|_\flat^r)^{n_0} = |b|_\flat^r.$$

Since, the other inequality $|b|_\flat^r \leqslant ||b||^{(r)}$ giving by continuity is clear, the claim follows. □

Proposition B.3. $\mathbf{A}_L = \mathbf{A}_\Psi$.

Proof. Both rings have the same reduction modulo π_L. And using that the latter element is not a zero-divisor in any of these rings we conclude inductively, that $\mathbf{A}_L/\pi_L^n\mathbf{A}_L = \mathbf{A}_\Psi/\pi_L^n\mathbf{A}_\Psi$ for all n. Taking projective limits gives the result. $\square$

Proposition B.4. $\mathbf{A}_L^\dagger = \mathbf{A}_\Psi^\dagger$.

Proof. By [KLII, Lem. 5.2.10] we have that $\mathbf{A}_\Psi^\dagger = \tilde{\mathbf{A}}_L^\dagger \cap \mathcal{R}_L$. On the other hand $\mathbf{A}_L^\dagger = (\tilde{\mathbf{A}}^\dagger \cap \mathbf{A})^{H_L} = \tilde{\mathbf{A}}_L^\dagger \cap \mathbf{A}$ is contained in $\mathcal{R}_L$ by Remark A.24, whence $\mathbf{A}_L^\dagger \subseteq \mathbf{A}_\Psi^\dagger$ while the inclusion $\mathbf{A}_\Psi^\dagger \subseteq \tilde{\mathbf{A}}^\dagger \cap \mathbf{A}_L = \mathbf{A}_L^\dagger$ follows from Proposition B.3. $\square$

This interpretion allows to partly deduce results of Appendix A from the machinery of §5.2-5 in (loc. cit.). In Definition 5.6.1 they define the property *decompleting* for a tower Ψ, which we are not going to recall here as it is rather technical. The cyclotomic tower over $\mathbb{Q}_p$ is of this kind for instance. If our Ψ would be decompleting, the machinery of (loc. cit.), in particular Theorems 5.7.3/4, adapted to the Lubin-Tate setting would imply that all the categories at the end of subsection A.1.4 are equivalent, which contradicts Remark A.29.

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