NONVANISHING OF KOECHER-MAASS SERIES ATTACHED TO SIEGEL CUSP FORMS

SOUMYA DAS, WINFRIED KOHNEN, AND JYOTI SENGUPTA

Abstract. We prove a nonvanishing result for Koecher-Maass series attached to Siegel cusp forms of weight k and degree n in certain strips on the complex plane. When $n = 2$, we prove such a result for forms orthogonal to the space of the Saito-Kurokawa lifts 'up to finitely many exceptions', in bounded regions.

1. Introduction

In the theory of modular forms, L-functions play a central role. For example, in the case of a holomorphic elliptic cusp form f of weight k on a congruence subgroup of $\mathrm{SL}_2(\mathbf{Z})$ with Fourier coefficients $a(f, n)$, the corresponding L-function

$$L_f(s) := \sum_{n \geq 1} a(f, n) n^{-s} \tag{1.1}$$

defined originally in the half-plane $\mathrm{Re}(s) > \frac{k+1}{2}$, has an analytic continuation to the whole complex plane $\mathbf{C}$ and satisfies a functional equation under $s \mapsto k - s$. Moreover, if f is a normalized Hecke eigenform which is a newform then $L_f(s)$ has an Euler product and determines the newform completely. However in the case of higher degrees, namely for Siegel modular forms for congruence subgroups of $\mathrm{Sp}_n(\mathbf{Z})$, the situation is rather different for $n \geq 2$. On the one hand, even though there is a rich theory of L-functions defined by means of the Satake parameters attached to a cuspidal eigenform, the basic Dirichlet series that one can associate with the Fourier coefficients $a(F, T)$ of a cusp form F on $\mathrm{Sp}_n(\mathbf{Z})$ of weight k ($T > 0$), namely the Koecher-Maass series $D_F(s)$ defined as

$$D_F(s) := \sum_{T \in \Lambda_n^+ \backslash \mathrm{GL}_n(\mathbf{Z})} \frac{1}{\epsilon(T)} a(F, T) (\det T)^{-s}, \quad (\mathrm{Re}(s) > \tfrac{k+n+1}{2}) \tag{1.2}$$

has an analytic continuation to $\mathbf{C}$ and a functional equation under $s \mapsto k - s$ (see [15]), but in general does not have an Euler product, even if F is a Hecke eigenform. Here, Λ_n^+ is the set of positive definite half-integral matrices of size n and the equivalence relation '$\sim$' indicates the right-action of $\mathrm{GL}_n(\mathbf{Z})$ on Λ_n^+ by

2010 *Mathematics Subject Classification.* Primary 11F30; Secondary 11F46.
Key words and phrases. Koecher-Maass series, Siegel Modular forms.

$T \mapsto T[U] := U'TU$. Further $\epsilon(T)$ denotes the number of units of the quadratic form associated with T, i.e.,

$$\epsilon(T) = \#\{U \in \mathrm{GL}_n(\mathbf{Z}) \mid T[U] = T\}.$$

On the other hand $D_F(s)$, by definition is built out of the average of the Fourier coefficients of F. If F is a Hecke eigenform of degree 2, then the squares of these averages are important, through its relation to Böcherer's conjecture [5] on special values of twisted spinor zeta functions at the central point. Moreover, a result of Choie-Kohnen [8] states that the special values of certain character twists of Koecher-Maass series in a region reminiscent of a "critical strip" (in the sense of Deligne), lie in a finite $\mathbf{Z}$-module. The Koecher-Maass series also has significant applications in proving sign change results for the Fourier coefficients of Siegel cusp forms, see [12].

In the theory of Dirichlet series, knowledge about the existence of zero-free regions has many applications and is a question of much interest. In (1.1), the existence of an Euler product ensures that $L_f(s)$ does not vanish in $\mathrm{Re}(s) > \frac{k+1}{2}$. Because of the functional equation

$$L_f^*(s) = \pm L_f^*(k-s), \quad \text{where } L_f^*(s) := (2\pi)^{-s}\Gamma(s)L_f(s)$$

is the completed L function, and the above-mentioned nonvanishing property, it is enough to focus on the 'critical strip'

$$\frac{k-1}{2} \leq \mathrm{Re}(s) \leq \frac{k+1}{2}.$$

In this connection, a result due to W. Kohnen [17] says that at each point s of the 'critical strip', one can find a Hecke eigenform f such $L(f,s)$ does not vanish. More precisely,

Theorem A. *Let $t_0 \in \mathbf{R}$ and $\varepsilon > 0$. Then there exist a constant $C(t_0, \varepsilon)$ such that for $k > C(t_0, \varepsilon)$ and any $s = \sigma + it$ with $t = t_0$, $(k-1)/2 < \sigma < k/2 - \varepsilon$, $k/2 + \varepsilon < \sigma < (k+1)/2$, there exists a cuspidal Hecke eigenform f of weight k on $\mathrm{SL}_2(\mathbf{Z})$ such that $L^*(f,s) \neq 0$.*

The aim of this paper is to generalize the above result to Siegel cusp forms of arbitrary degree $n \geq 2$ in the context of Koecher-Maass series. Our strategy will be to consider the kernel-function $F^n_{k,s}$, which essentially produces the values $D_F(s)$ when integrated against a cusp form F. More precisely, we follow the method as in [17], i.e., analyze the Fourier expansion of $F^n_{k,s}$ after expanding it in terms of an orthogonal basis of the space of cusp forms. However in the case of higher degrees, the computation of the Fourier expansion of $F^n_{k,s}$ is rather non-trivial and technically much more complicated, see section 3.

In the rest of the paper we will always assume that k is even and $k > 2(n+1)$. To state the result in a concise way, let us first define for $\varepsilon > 0$ small, the region

$\mathcal{S}_{n,k,\varepsilon}$, which is the union of two infinite strips, in which we would state our result, by

$$\mathcal{S}_{n,k,\varepsilon} := \{s \in \mathbf{C} \mid \frac{k}{2} - \frac{n+1}{4} \leq \sigma \leq \frac{k}{2} - \varepsilon\} \cup \{s \in \mathbf{C} \mid \frac{k}{2} + \varepsilon \leq \sigma \leq \frac{k}{2} + \frac{n+1}{4}\}. \tag{1.3}$$

Let S_k^n denote the space of Siegel cusp forms of weight k and degree n for the group $\mathrm{Sp}_n(\mathbf{Z})$. Denote the line $\{s \in \mathbf{C} \mid \mathrm{Im}(s) = t_0\}$ by $\mathcal{L}(t_0)$. Then the main results that we prove in this paper can be stated as follows.

Theorem 1.1. *Let $t_0 \in \mathbf{R}$, $\varepsilon > 0$ and $n \geq 2$. Then there exists a positive constant $C_n(t_0, \varepsilon)$ such that for $k > C_n(t_0, \varepsilon)$ and for each s on the line-segment $\mathcal{L}(t_0) \cap S_{n,k,\varepsilon}$, there exists a cuspidal Hecke eigenform $F \in S_k^n$ depending on s, such that $D_F(s) \neq 0$ and $a(F, 1_n) \neq 0$.*

We mention here that the above theorem will follow from the more general statement that

$$\sum_{F \in \mathcal{B}_k^n} \frac{D_F^*(s)}{\langle F^\rho, F^\rho \rangle} F^\rho \neq 0 \tag{1.4}$$

for s varying in suitable strips under consideration, where $\mathcal{B}_k^n$ is any orthogonal basis for S_k^n and $f^\rho := \overline{f(-\bar{Z})}$. Here D_F^* is the completed Koecher-Maass series defined by (see (2.3))

$$D_F^*(s) := \gamma_n(s) D_F(s), \text{ where } \gamma_n(s) := (2\pi)^{-ns} \prod_{\nu=0}^{n-1} \Gamma(s - \nu/2). \tag{1.5}$$

Thus the above theorem can be thought of as an analogue of the "Riemann Hypothesis for modular L-functions on average" (cf. Theorem A) in the context of Koecher-Maass series. We note here that even though when $n \geq 2$ there is apriori no "critical strip" in the sense of Deligne, but one can define a possible substitute for it, as in Remark 4.13.

For $R > 0$, let use denote the closed ball of radius R centered at the origin, by B_R. Also, when $n = 2$, we denote the space of cusp forms in S_k^2 which are orthogonal to the space spanned by Saito-Kurokawa lifts (also known as Maass lifts) by $S_k^\flat$.

Theorem 1.2. *There exists an absolute constant $k_0 > 20$ such that for all $k \geq k_0$, one can find an eigenform $F \in S_k^\flat$ (depending only on k) such that $D_F^*(s) \neq 0$ for all points s in B_R with at most $O_{k,\varepsilon}(R^{1+\varepsilon})$ exceptions.*

We now outline the structure of the paper along with a brief overview of some of the interesting intermediate results. Namely, after recalling some preliminaries, we start computing the Fourier expansion of the kernel function in section 3, which is divided into several subsections. The final expression can be seen in Proposition 3.8. Along the way we introduce an equivalence relation $\diamond$ on the set of n-rowed coprime symmetric pairs (G, H) and determine 'nice' representatives in each equivalence class under $\diamond$ when the rank of, say, G is fixed and less than n. See Proposition 3.2.

The next step is to show that the kernel function $F^n_{k,s}$ does not vanish in suitable strips of the complex plane, so that Theorem 1.1 would then follow from (2.3). This is done by showing that the 'first' Fourier coefficient $a(F^n_{k,s}, 1_n)$ is non-zero in such a region. See section 4. However, we find that proving $a(F^n_{k,s}, 1_n) \neq 0$ is a rather challenging task. To mention a few instances, we note that the Fourier expansion of $F^n_{k,s}$ involves a certain kind of hypergeometric function ${}_1\mathcal{F}_1^{(\cdot)}(\cdot, \cdot, M)$, where $M \in M_n(\mathbf{C})$, introduced in [13]; see section 4.1 for the definition. These functions are related to the confluent hypergeometric functions treated by Shimura [25], however only in the cases when M is real.

In our case M is purely imaginary, and we seek a non-trivial estimate for it so that the Fourier expansion can be used to prove that $a(F^n_{k,s}, 1_n) \neq 0$. It should be mentioned here that the 'trivial bound' $|{}_1\mathcal{F}_1^{(\cdot)}(\cdot, \cdot, M)| \leq 1$, does not suffice when $n > 1$ (unlike when $n = 1$, cf. [17]). We believe such a result is not yet available in the literature and section 4.1 is devoted to this. The relevant estimate that we obtain, and which is sufficient for us, is as follows.

Proposition. *Let $M \in M_n(\mathbf{R})$ be symmetric and let $\alpha, \beta \in \mathbf{C}$ be such that $\mathrm{Re}(\beta - \alpha) > n$. Then*

$$|{}_1\mathcal{F}_1^{(n)}(\alpha, \beta, iM)| \leq \min\left\{1, 2\Gamma_n(\beta)\left(1 + \mathrm{tr}(\frac{M^2}{2})\right)^{-\kappa} \eta_\kappa(i1_n)\right\},$$

where $2\kappa = \min\{\mathrm{Re}(\alpha), \mathrm{Re}(\beta - \alpha)\}$, $\eta_\kappa(i1_n)$ is a certain constant (cf. (4.2)*) and $\Gamma_n(\beta)$ is the generalized gamma function of degree n (cf.* (3.16)*).*

Another delicate point regarding the Fourier expansion involves infinite sums over a certain $\mathrm{GL}_p(\mathbf{Z})$-variable ($p < n$) and certain subsets of 'co-prime symmetric pairs'; see section 3.3. Then, along with other technical lemmas on estimates for some arithmetical counting functions (see section 4.2), we are able to prove that $a(F^n_{k,s}, 1_n) \neq 0$ for the relevant s in section 4.4.

In order to prove Theorem 1.2 (recall that $n = 2$ in this case), we need to have some preparatory results, see section 5. In this context, we observe that using the main theorem in [20], one obtains as a corollary (cf. Corollary 5.3) that for all sufficiently large k, there exists an eigenform in $S_k^\flat$ whose first Fourier-Jacobi coefficient is non-zero. Therefore this result gives evidence towards a folkore conjecture that (see [16]):

"The first Fourier Jacobi coefficient of an eigenform $F \in S_2^k$ does not vanish."

This in turn follows from the fact that such an eigenform has $a(F, 1_2) \neq 0$, see Corollary 5.2. We next show that the completed Koecher-Maass series $D_F^*(s)$ is an entire function of order at most one. Theorem 1.2 follows from these observations and some standard analytic arguments.

Apart from their own intrinsic interest and importance, one of the reasons for focusing on the space $S_k^\flat$ is that one has an explicit form of the Koecher-Maass series for Saito-Kurokawa lifts [14] (and more generally for Ikeda lifts). One may ask, if in Theorem 1.1, one could simply take F to be an appropriate Ikeda-lift and use the explicit formula just mentioned. Indeed such a result would involve studying the zero-free regions of sums of products of L-functions in appropriate 'critical strips', not all of which have Euler products. We do not see an immediate proof of Theorem 1.1 along these lines. In any case, Theorem 1.1 is derived from (1.4), which is stronger than Theorem 1.1. Also, we feel that the explicit Fourier expansion of $F_{k,s}^n$ might be of some use in other applications.

We also note that in Theorem 1.1, we get hold of eigenforms with $a(F, 1_n) \neq 0$. This follows almost like a by-product when we equate the 1_n-th Fourier coefficients on both sides of (1.4). Thus (1.4) gives a method, in general, to prove that given $T \in \Lambda_n^+$, the T-th Poincaré series $P_{k,T}^n$ does not vanish identically. Indeed, this is an old conjecture on Poincaré series. Such questions were treated in [6, 9], and it would be interesting to compare these results with what can be obtained from this method.

Acknowledgements. The authors thank the Department of Mathematics IISc., Bangalore and the School of Mathematics TIFR, Mumbai for providing a very congenial working atmosphere. The first author also acknowledges financial support from UGC-DST (India) and IISc., Bangalore.

2. Notation and setup

2.1. **General notation.** (1) For a commutative ring R with 1, we denote by $M_{m,n}(R)$ to be set of $m \times n$ matrices with coefficients in R. If $m = n$, we put $M_{m,n}(R) = M_n(R)$. We denote by $\mathrm{Sym}_n(\mathbf{R})$ (resp. $\mathrm{Sym}_n(R)^+$) the space of symmetric (resp. positive definite) matrices over the reals $\mathbf{R}$. Further, the $n \times n$ identity matrix over a subring of $\mathbf{C}$ is denoted by 1_n.

The rank of a matrix M is denoted by $\mathrm{rank}(M) = \mathrm{r}(M)$. The notation $M[N] := N'MN$ for matrices of appropriate size is used, where N' denotes the transpose of N.

For an invertible matrix U, we set $U^* := U'^{-1}$.

For a square matrix $M \in \mathrm{Sym}_n(R)$, we usually write $M = \begin{pmatrix} M_1^{(p)} & M_2 \\ M_2' & M_4 \end{pmatrix}$, where $M_1^{(p)}$ denotes a square matrix of size p etc..

We define the set of half-integral, symmetric, non-negative matrices, denoted by Λ_n, by

$$\Lambda_n := \{S = (s_{i,j}) \in M(n, \tfrac{1}{2}\mathbf{Z}) \cap \mathrm{Sym}_n(\mathbf{R}) \mid s_{i,i} \in \mathbf{Z}, \text{ and } S \text{ is positive semi-definite}\};$$

and denote the subset of positive definite matrices in Λ_n by Λ_n^+.

Throughout the paper, ε denotes a small positive number which may vary at different places. Moreover the symbols $A \ll_c B$ and $O_S(T)$ have their standard meaning; implying that the constants involved depends on c or the set S.

(2) For T real and $Z \in M_n(\mathbf{C})$ we define $e(TZ) := \exp(2\pi i \mathrm{tr}(TZ))$, where $\mathrm{tr}(M)$ is the trace of the matrix M. We denote by

$$\mathbf{H}_n := \{Z \in M_n(\mathbf{C}) \mid Z = Z', \mathrm{Im}(Z) > 0\},$$

the Siegel upper half-space of degree n. For $Z \in \mathbf{H}_n$ we usually write $Z = X + iY$, with $X = \mathrm{Re}(Z), Y = \mathrm{Im}(Z)$. For $Z \in \mathbf{H}_n$ and $s = \sigma + it \in \mathbf{C}$, we denote by $\det(Z)^s$ the complex number $\exp\{s \log(\det(Z))\}$. Here $\log(\det(Z))$ denotes the unique holomorphic function $h(Z)$ such that $\exp\{h(Z)\} = \det(Z)$ and $h(iY) = \frac{\pi i n}{2} + \log(\det(Y))$. Moreover the imaginary part of $h(Z)$ is bounded, see [18].

(3) We put $\Gamma_n := \mathrm{Sp}_n(\mathbf{Z})$ and denote by S_k^n the space of Siegel cusp forms of weight k and degree n with respect to Γ_n. We refer the reader to [11] for basic facts on the theory of Siegel modular forms.

2.2. **The kernel-function.** Let $\langle\ ,\ \rangle$ be the standard Petersson inner product on S_k^n. In [18], the authors define a 'kernel-function', which represents the linear functional $F \mapsto D_F^*(s)$ for a fixed $s \in \mathbf{C}$ with respect to the Petersson inner product. We denote the kernel function by $F_{k,s}^n$ and recall some of its important analytic properties for the convenience of the reader.

Let us put

$$G_{0,n} := \{\left(\begin{smallmatrix} U & 0 \\ 0 & U^* \end{smallmatrix}\right) \mid U \in \mathrm{GL}_n(\mathbf{Z})\} \subset \Gamma_n.$$

For an even integer $k > 2(n+1)$, $Z \in \mathbf{H}_n$ and $s = \sigma + it \in \mathbf{C}$ with

$$n + 1 < \sigma < k - (n+1),$$

we define

$$F_{k,s}^n(Z) := e^{\pi i n s/2} \gamma_n(s) \gamma_n(k-s) \cdot$$

$$\cdot \sum_{\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in G_{0,n} \backslash \Gamma_n} \det(CZ + D)^{-k} \det((AZ+B)(CZ+D)^{-1})^{-s}, \tag{2.1}$$

where $\gamma_n(s)$ was defined in (1.5). We introduce the function $A_{n,k}(s)$, which is defined by

$$A_{n,k}(s) := e^{\pi ins/2}\gamma_n(s)\gamma_n(k-s).$$

The most relevant properties of the function $F^n_{k,s}$ are summarized below. Let the Fourier expansion of $f \in S^n_k$ be written as

$$F(Z) = \sum_{T\in\Lambda_n^+} a(F,T)e(TZ).$$

Proposition 2.1. *Suppose that $k > 3(n+1)$ be even. Then the following hold:*

(i) In the region $n+1 < \sigma < k-(n+1)$, $F^n_{k,s}$ is a cusp form of weight k in the variable Z.

(ii) One has $F^n_{k,k-s}(Z) = (-1)^{\frac{nk}{2}} F^n_{k,s}$.

(iii) In the region $(k+n+1)/2 < \sigma < k-(n+1)$, one has the identity

$$\langle F_{k,s}(\cdot), F^\rho\rangle = C_{n,k} D^*_F(s), \tag{2.2}$$

where $F^\rho(Z) := \sum_{T>0} \overline{a(F,T)} e(TZ)$ and where $C_{n,k}$ is given by

$$C_{n,k} = (-1)^{\frac{nk}{2}} 2^{1+n(1-k)} (2\pi)^{\frac{n(n+1)}{2}} \prod_{\nu=1}^{n} \Gamma(k - \frac{n+\nu}{2}).$$

From Proposition 2.1 (i) and (iii), we can write

$$F^n_{k,s} = C_{n,k} \sum_{F\in\mathcal{B}^n_k} \frac{D^*_F(s)}{\langle F^\rho, F^\rho\rangle} F^\rho, \tag{2.3}$$

when $\frac{k+n+1}{2} < \sigma < k-(n+1)$. By analytic continuation, this holds in the region $n+1 < \sigma < k-(n+1)$. Here $\mathcal{B}^n_k$ is any orthogonal basis for S^n_k.

2.3. **Co-prime symmetric pairs.** We say that a pair of $n\times n$ integral matrices (G,H) is a co-prime symmetric pair if $GH' = HG'$ and if for $Q \in M_n(\mathbf{Q})$ one has

$$QG, QH \in M_n(\mathbf{Z}),$$

then $Q \in M_n(\mathbf{Z})$. They can also be characterized as the 'last row-block' of an integral symplectic matrix, i.e., $\left(\begin{smallmatrix} * & * \\ G & H\end{smallmatrix}\right) \in \Gamma_n$. The set of such pairs is denoted by $\mathrm{CS}(n)$.

Next we recall that the unimodular group is naturally embedded inside $\mathrm{Sp}_n(\mathbf{Z})$:

$$\mathrm{GL}_n(\mathbf{Z}) \hookrightarrow \mathrm{Sp}_n(\mathbf{Z}), \quad U \to \begin{pmatrix} U & 0 \\ 0 & U^* \end{pmatrix}.$$

Of course $\mathrm{GL}_n(\mathbf{Z})$ acts on $\mathrm{CS}(n)$ on the left by

$$U\cdot(G,H) = (UG, UH).$$

We say that $(G,H) \sim (G_1,H_1)$ if there exists $U \in \mathrm{GL}_n(\mathbf{Z})$ such that

$$(G,H) = (UG_1, UH_1). \tag{2.4}$$

We denote the set of representatives in $\mathrm{CS}(n)$ modulo this relation by $\sim \backslash \mathrm{CS}(n)$. It is also useful to note, from the first definition of $\mathrm{CS}(n)$, that $(G,H) \in \mathrm{CS}(n)$ if and only if $(H,G) \in \mathrm{CS}(n)$.

Moreover, $\mathrm{GL}_n(\mathbf{Z})$ also acts on 'columns' $\left(\begin{smallmatrix} A \\ C \end{smallmatrix}\right)$ of matrices in Γ_n from the left by

$$U \cdot \left(\begin{smallmatrix} A \\ C \end{smallmatrix}\right) = \left(\begin{smallmatrix} UA \\ U^*C \end{smallmatrix}\right). \tag{2.5}$$

We now have the following lemma.

Lemma 2.2. *$\left(\begin{smallmatrix} A \\ C \end{smallmatrix}\right)$ is the first column block of an element of Γ_n if and only if the row block $(A', C') \in \mathrm{CS}(n)$.*

Proof. The lemma follows from the computation

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix} = \begin{pmatrix} * & * \\ A' & C' \end{pmatrix} \in \Gamma_n. \qquad \square$$

In view of the above lemma, it is then natural to define the following equivalence relation '$\diamond$' on $\mathrm{CS}(n)$ by taking transpose in (2.5): we say that $(G,H) \diamond (G_1,H_1)$ if there exists $U \in \mathrm{GL}_n(\mathbf{Z})$ such that

$$(G,H) = (G_1U', H_1U^{-1}). \tag{2.6}$$

We denote the set of representatives in $\mathrm{CS}(n)$ modulo this relation by $\mathrm{CS}(n)/\diamond$. We will need another lemma on the set of representatives for Γ_n modulo $G_{0,n}$.

Lemma 2.3. *A set of left coset representatives in $G_{0,n}\backslash\Gamma_n$ is given by each of the following the set of matrices:*

$$(i) \quad \left\{ \begin{pmatrix} * & * \\ C & D \end{pmatrix} \in \Gamma_n \mid (C,D) \in \sim \backslash \mathrm{CS}(n) \right\}.$$

$$(ii) \quad \left\{ \begin{pmatrix} A & * \\ C & * \end{pmatrix} \in \Gamma_n \mid (A',C') \in \mathrm{CS}(n)/\diamond \right\}.$$

Proof. (i) is clear by what was said above. For (ii), we need only observe that as $\left(\begin{smallmatrix} A \\ C \end{smallmatrix}\right)$ varies over the first column of a symplectic matrix modulo the left action of $\mathrm{GL}_n(\mathbf{Z})$ as in (2.5), the associated row blocks (A',C') vary over $\mathrm{CS}(n)/\diamond$. $\square$

3. The Fourier expansion of $F^n_{k,s}$

3.1. **Splitting according to the ranks.** Let us denote the Fourier coefficients of $F^n_{k,s}(Z)$ by $b_s(T)$. Then for any fixed real $K = K' > 0$, we have

$$b_s(T) = \int\limits_{X \bmod 1_n, Y=K} F_{k,s}(Z) e(-TZ) dZ. \tag{3.1}$$

We would like to rewrite the series defining $F_{k,s}$ as a sum on the first columns $\left(\begin{smallmatrix} A \\ C \end{smallmatrix}\right)$ or rather their transposes (A', C'), by appealing to Lemma 2.3.

First, we state a useful lemma on the representatives for the orbits under the left action of $\mathrm{GL}_n(\mathbf{Z})$ on the set

$$\mathrm{CS}(n,p) := \{(G,H) \in \mathrm{CS}(n) \mid \mathrm{rank}(G) = p\},$$

see [26, § 12] and also [4, Lem. 1]. For $0 \le p \le n$ we define

$$\mathrm{GL}_n(\mathbf{Z})_p := \{U \in \mathrm{GL}_n(\mathbf{Z}) \mid U^{(n-p,p)} = 0\}, \text{ where } U = \begin{pmatrix} *^{(p)} & * \\ U^{(n-p,p)} & * \end{pmatrix}.$$

Lemma 3.1. *For $0 \le p \le n$, a system of $\mathrm{GL}_n(\mathbf{Z})$-representatives for the equivalence classes $\widetilde{(G,H)}$ inside $\sim \backslash \mathrm{CS}(n,p)$ is given by the set*

$$\left\{ \left(\begin{pmatrix} G_1^{(p)} & 0 \\ 0 & 0_{n-p} \end{pmatrix} W', \begin{pmatrix} H_1^{(p)} & 0 \\ 0 & 1_{n-p} \end{pmatrix} W^{-1} \right) \middle| \begin{matrix} (G_1, H_1) \in \sim \backslash \mathrm{CS}(p,p) \\ W \in \mathrm{GL}_n(\mathbf{Z})/\mathrm{GL}_n(\mathbf{Z})_p \end{matrix} \right\}.$$

Next, note that Lemma 3.1 shows that for the pair $(G,H) \in \mathrm{CS}(n)$ and $0 \le q \le n$,

$$\mathrm{rank}(G) = q \implies \mathrm{rank}(H) \ge n - q.$$

We now split up the series defining $F^n_{k,s}$ in (2.1) according to the ranks of C and A by appealing to Lemma 2.3 (ii) and the above observation with $(G,H) = (C', A')$. Thus we write

$$F^n_{k,s}(Z) = \sum_{q=0}^{n} \sum_{p=n-q}^{n} F^n_{p,q;s}(Z), \tag{3.2}$$

where $F^n_{p,q;s}(Z)$ defined as in (2.1), but with rank of C equal to q and that of A equal to p:

(3.3)
$$F^n_{p,q;s}(Z) := A_{n,k}(s) \sum_{\substack{\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in G_{0,n} \backslash \Gamma_n \\ \mathrm{r}(A)=p, \mathrm{r}(C)=q}} \det(CZ+D)^{-k} \det((AZ+B)(CZ+D)^{-1})^{-s}.$$

Accordingly, we denote the contribution of the piece $F^n_{p,q;s}(Z)$ in the Fourier coefficient $b_s(T)$ in (3.4) as $b_{p,q;s}(T)$ so that:

$$b_s(T) = \sum_{q=0}^{n} \sum_{p=n-q}^{n} b_{p,q;s}(T). \tag{3.4}$$

We shall end the section by proving another lemma similar to (and actually a consequence of) Lemma 3.1 which gives the representatives in $\mathrm{CS}(n)$ modulo the equivalence relation $\diamond$ as defined in (2.6).

Lemma 3.2. *For $0 \leq p \leq n$, a system of representatives of the equivalence classes $\overline{(G, H)}$ in $\mathrm{CS}(n,p)/\diamond$ is given by the set*

$$\left\{ \left(U \begin{pmatrix} G_1 & 0 \\ 0 & 0_{n-p} \end{pmatrix}, U \begin{pmatrix} H_1 & 0 \\ 0 & 1_{n-p} \end{pmatrix} \right) \middle| \begin{matrix} (G_1, H_1) \in \mathrm{CS}(p,p)/\diamond \\ U \in \mathrm{GL}_n(\mathbf{Z})/\mathrm{GL}_n(\mathbf{Z})_p \end{matrix} \right\}.$$

Remark 3.3. This lemma will be later applied to the pair $(G, H) = (A', C')$, where $\left(\begin{smallmatrix} A \\ C \end{smallmatrix}\right)$ is the first column block of an element in Γ_n.

Proof. We start with the equivalence class $\widetilde{(G, H)}$ of $(G,H) \in \mathrm{CS}(n,p)$ under the equivalence relation under $\sim$. By Lemma 3.1, we know that $\widetilde{(G, H)} = \widetilde{(PW', QW^{-1})}$, for a certain $W \in \mathrm{GL}_n(\mathbf{Z})$; here we have denoted the block-matrices in Lemma 3.1 by P, Q respectively. Thus we get hold of $U \in \mathrm{GL}_n(\mathbf{Z})$ such that

$$(UG, UH) = (PW', QW^{-1}).$$

Therefore $(G, H) \diamond (U^{-1}P, U^{-1}Q)$.

We now need to show that the elements of our set are inequivalent modulo the relation $\diamond$. Suppose that we have for some $W \in \mathrm{GL}_n(\mathbf{Z})$

$$(U_1 P_1, U_1 Q_1) = (U_2 P_2 W', U_2 Q_2 W^{-1}), \tag{3.5}$$

where as before, P_i, Q_i $(i = 1, 2)$ denote the block matrices as in the statement of the lemma, $U_i \in \mathrm{GL}_n(\mathbf{Z})/\mathrm{GL}_n(\mathbf{Z})_p$.

From these relations it follows that

$$U_2^{-1} U_1 \left(\begin{smallmatrix} \mathcal{P}_1 & 0 \\ 0 & 0_{n-p} \end{smallmatrix}\right) = \left(\begin{smallmatrix} \mathcal{P}_2 & 0 \\ 0 & 0_{n-p} \end{smallmatrix}\right) W', \quad \left(\begin{smallmatrix} \mathcal{Q}_1 & 0 \\ 0 & 0_{n-p} \end{smallmatrix}\right) = \left(\begin{smallmatrix} \mathcal{Q}_2 & 0 \\ 0 & 0_{n-p} \end{smallmatrix}\right) W^{-1}, \tag{3.6}$$

where we have put $P_i = \left(\begin{smallmatrix} \mathcal{P}_i & 0 \\ 0 & 1_{n-p} \end{smallmatrix}\right)$ and similarly for Q_i, $i = 1, 2$.

Writing

$$U_2^{-1} U_1 = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right), \quad W = \begin{pmatrix} \widetilde{\alpha} & \widetilde{\beta} \\ \widetilde{\gamma} & \widetilde{\delta} \end{pmatrix},$$

we get from (3.6) that

$$\begin{pmatrix} \alpha \mathcal{P}_1 & 0 \\ \gamma \mathcal{P}_1 & 0_{n-p} \end{pmatrix} = \begin{pmatrix} \mathcal{P}_2 \widetilde{\alpha}' & \mathcal{P}_2 \widetilde{\gamma}' \\ 0 & 0_{n-p} \end{pmatrix}. \tag{3.7}$$

Now notice that $\mathcal{P}_1$ is non-singular, since $\mathrm{r}(P) = p$. So, (3.7) implies that $\gamma = 0$. Thus $U_2^{-1}U_1 \in \mathrm{GL}_n(\mathbf{Z})_p$, which forces $U_1 = U_2$. Moreover $\widetilde{\gamma}' = 0$ as well, since $\mathcal{P}_2$ is non-singular.

Going back to (3.7), we have (since $W \in \mathrm{GL}_n(\mathbf{Z})$),

$$\mathcal{P}_1 = \mathcal{P}_2\widetilde{\alpha}', \quad \widetilde{\alpha} \in \mathrm{GL}_p(\mathbf{Z}).$$

Then the second equality in (3.6) also implies that $\mathcal{Q}_1 = \mathcal{Q}_2\widetilde{\alpha}^{-1}$.

Thus we have

$$(\mathcal{P}_1, \mathcal{Q}_1) \diamond (\mathcal{P}_2, \mathcal{Q}_2),$$

where we have used the same symbol $\diamond$ to denote the appropriate equivalence relation as defined in (2.6) for different degrees. Thus $\mathcal{P}_1 = \mathcal{P}_2$ and $\mathcal{Q}_1 = \mathcal{Q}_2$. This completes the proof. □

3.2. **The case of maximal rank.** Here we consider the case where the rank of C is n and so we have pieces $F_{p,n}(Z, s)$, where $0 \le p \le n$. First we state a lemma, which characterizes symplectic matrices having the same 'first column-block'.

Lemma 3.4. *If* $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ *and* $\left(\begin{smallmatrix} A & B_1 \\ C & D_1 \end{smallmatrix}\right) \in \Gamma_n$, *then*

$$\begin{pmatrix} A & B_1 \\ C & D_1 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 1_n & S \\ 0 & 1_n \end{pmatrix}, \ \textit{where } S \in \mathrm{Sym}_n(\mathbf{Z}).$$

Proof. The statement easily follows from the symplectic relations for the block entries of a symplectic matrix. □

Then by Lemma 2.3 and taking Lemma 3.4 into account, we can write

$$F_{p,n}(Z, s) = A_{n,k}(s) \sum_{\substack{(A',C') \in \mathrm{CS}(n)/\diamond \\ \mathrm{r}(A)=p, \mathrm{r}(C)=n}} \sum_{S \in \mathrm{Sym}_n(\mathbf{Z})} \det(C(Z+S)+D_0)^{-k}$$
$$\cdot \det((A(Z+S)+B_0)(C(Z+S)+D_0)^{-1})^{-s},$$

where for each (A', C'), (B_0, D_0) is a fixed choice such that $\left(\begin{smallmatrix} A & B_0 \\ C & D_0 \end{smallmatrix}\right) \in \Gamma_n$.

We would now like to proceed to compute the pieces of the Fourier coefficients by plugging in (3.8) in the formula

$$b_{p,n;s}(T) = \int_{X \bmod 1_n, Y=1_n} F_{p,n}(Z, s) e(-TZ) dZ, \tag{3.8}$$

and interchange the order of summation and integration. By the dominated convergence theorem, it is sufficient to show that the sum of the integrals over the terms of (3.8) taken in absolute values is convergent. Note that the series defined by taken absolute values in (3.8) is convergent on compact sets (cf. [18]) and hence defines a smooth and periodic function on $\mathbf{H}_n$ with Fourier coefficients given by the usual integral formula. In particular this is true for the constant term. In the latter, however we can interchange the order of integration and summation by the monotone convergence theorem. Thus our claim follows.

3.3. **Calculation of the Fourier coefficients.** Using the above expression in the integral (see (3.4)) which evaluates $b_{p,n;s}(n;T)$, interchanging the summation and integration, and making the substitution $X \mapsto X - S$ we arrive at (with $K = 1_n$)

$$b_{p,n;s}(n;T) = A_{n,k}(s) \sum_{\substack{(A',C')\in \mathrm{CS}(n)/\diamond \\ \mathrm{r}(A)=p,\mathrm{r}(C)=n}} \int_{Y=1_n} \det(CZ+D_0)^{-k} \cdot$$
$$\cdot \det((AZ+B_0)(CZ+D_0)^{-1})^{-s} e(-TZ)dZ.$$

In the above integral, we make the substitution $Z \mapsto W := Z - C^{-1}D_0$. We compute using the symplectic relations:

$$CW + D_0 = CZ, \quad (AW+B_0)(CW+D_0)^{-1} = -Z^{-1}[C^{-1}] + AC^{-1}. \tag{3.9}$$

This shows that

$$\begin{aligned} b_{p,n;s}(n;T) = A_{n,k}(s) &\sum_{(A',C')\in \mathrm{CS}(n)/\diamond} e(TC^{-1}D_0)\det(C)^{-k+2s} \cdot \\ &\cdot \int_{Y=1_n} \det(Z)^{-k}\det(-Z^{-1}+A'C)^{-s} e(-TZ)dZ. \end{aligned} \tag{3.10}$$

The above formula in fact does not depend on D_0, as the sum is only over (A', C'). The following lemma makes this more transparent.

Lemma 3.5. *In the term $e(TC^{-1}D_0)$ we can replace D_0 by any integral matrix $\widetilde{D}_0$ such that $\widetilde{D}_0 A' \equiv 1_n \bmod C$ and $\widetilde{D}_0 C' = C\widetilde{D}_0'$.*

Proof. From the symplectic relation $D_0A' - CB_0' = 1_n$, it follows that $D_0A' \equiv 1_n \bmod C$. Conversely, suppose that the conditions on $\widetilde{D}_0$ are satisfied. Then we have $(\widetilde{D}_0 - D_0)A' \equiv 0 \bmod C$, hence $(\widetilde{D}_0 - D_0)A'D_0 = CSD_0$ with $S \in M_n(\mathbf{Z})$. Therefore we calculate using the symplectic relations:

$$\begin{aligned} CSD_0 &= (\widetilde{D}_0 - D_0)(1_n + C'B_0) \\ &= \widetilde{D}_0 - D_0 + C\widetilde{D}_0'B_0 - CD_0'B_0, \end{aligned}$$

using the hypothesis. Thus, $e(TC^{-1}D_0) = e(TC^{-1}\widetilde{D}_0)$. □

Also at this point, to consider the cases when $\mathrm{r}(A) < n$, it is convenient to choose representatives in the class containing (A', C') according to Lemma 3.2. Recall that $\mathrm{r}(C) = n$ in this subsection. We apply Lemma 3.2 with $(G, H) =$

(A', C') and write $(G_1, H_1) = (a_1', c_1')$ (note that both a_1 and c_1 are invertible). Therefore, we can rewrite the equation (3.10) as

$$b_{p,n;s}(n;T) = A_{n,k}(s) \sum_{U \in \mathrm{GL}_n(\mathbf{Z})/\mathrm{GL}_n(\mathbf{Z})_p} \sum_{(a_1',c_1') \in \mathrm{CS}(n,p)/\diamond} e(T[U^*]_1 c_1^{-1} d_1) \det(c_1)^{-k+2s} \cdot$$
$$\cdot \int_{Y=1_n} \det(Z)^{-k} \det(-Z^{-1} + U \left(\begin{smallmatrix} a_1' c_1 & 0 \\ 0 & 0 \end{smallmatrix}\right) U')^{-s} e(-TZ) dZ,$$

where d_1 is any integral matrix of size p satisfying the requirements of Lemma 3.5 (denoted $\tilde{D}_0$ in that context). Also recall that $T[U^*]_1$ denotes the left upper block of $T[U^*]$ of size p.

Henceforth, when there is no danger of confusion, we will sometimes drop the sets on which the variables run, for ease of notation. Making a change of variable $Z \mapsto Z[U^{-1}]$ we get

$$b_{p,n;s}(n;T) = A_{n,k}(s) \sum_{U,(a_1',c_1')} e(T[U^*]_1 c_1^{-1} d_1) \det(c_1)^{-k+2s} \cdot$$
$$\cdot \int_{Y=U'U} \det(Z)^{-k} \det(-Z^{-1} + \left(\begin{smallmatrix} a_1' c_1 & 0 \\ 0 & 0 \end{smallmatrix}\right))^{-s} e(-T[U^*]Z) dZ. \tag{3.11}$$

Let us call the integral inside the summation in the above equation $\Phi_{p,n}(U, a_1, c_1; T)$. We next modify it to express the resulting formula in terms of hypergeometric functions in many variables. For further use and to save on notation, let us define

$$S_1 := a_1' c_1, \quad \mathcal{T} := T[U^*] = \begin{pmatrix} \mathcal{T}_1^{(p)} & \mathcal{T}_2/2 \\ \mathcal{T}_2'/2 & \mathcal{T}_4 \end{pmatrix}, \quad Z = \begin{pmatrix} Z_1^{(p)} & Z_2 \\ Z_2' & Z_4 \end{pmatrix}.$$

Then from (3.11) we have

$$\begin{aligned} \Phi_{p,n}(U, a_1, c_1; T) &= \int_{Y=U'U} \det(Z)^{-k} \det(-Z^{-1} + \left(\begin{smallmatrix} a_1' c_1 & 0 \\ 0 & 0 \end{smallmatrix}\right))^{-s} e(-\mathcal{T} Z) dZ \\ &= \int_{Y=U'U} \det(Z)^{-(k-s)} \det(-I_p + a_1' c_1 Z_1)^{-s} e(-\mathcal{T} Z) dZ \end{aligned}$$

Now, the Jacobi-decomposition for Z (w.r.t. Z_1) says that

$$Z = \begin{pmatrix} Z_1 & 0 \\ 0 & Z_4 - Z_1^{-1}[Z_2] \end{pmatrix} \left[\begin{pmatrix} 1_p & Z_4^{-1} Z_2 \\ 0 & 1_{n-p} \end{pmatrix} \right]. \tag{3.12}$$

Therefore,

$$\det(Z) = \det(Z_1) \det(Z_4 - Z_1^{-1}[Z_2]).$$

Using this, we get the following expression for $\Phi_{p,n}(U, a_1, c_1; T)$:

$$\Phi_{p,n}(U, a_1, c_1; T) = \int\limits_{Y=U'U} \det(Z_1)^{-(k-s)} \cdot \det(Z_4 - Z_1^{-1}[Z_2])^{-(k-s)}$$

$$\cdot \det(-1_p + S_1 Z_1)^{-s} e(-\mathcal{T}_1 Z_1 - \mathcal{T}_2 Z_2' - \mathcal{T}_4 Z_4)) dZ, \tag{3.13}$$

3.4. **Calculation of I_4.** In the integral on the r.h.s. of (3.13), we would first like to consider the integral over Z_4, hence we look at

$$I_4 := \int\limits_{X_4} \det(Z_4 - Z_1^{-1}[Z_2])^{-(k-s)} e(-\mathcal{T}_4 Z_4) dZ_4,$$

here the integral is over $X_4 = \mathrm{Re}(Z_4)$ and $\mathrm{Im}(Z_4) = K_4 > 0$ is fixed. In I_4, we first make a change of variable $Z_4 \to Z_4 + Z_1^{-1}[Z_2]$ followed by $Z_4 \mapsto iZ_4$. This gives

$$I_4 = i^{(n-p)(s-k)} e(-\mathcal{T}_4 \cdot Z_1^{-1}[Z_2]) \int\limits_{\mathcal{Y}_4} \det(\mathcal{Z}_4)^{-(k-s)} \exp(2\pi \mathrm{tr}\,(T_4 \mathcal{Z}_4)) d\mathcal{Z}_4, \tag{3.14}$$

where $\mathcal{Z}_4 = \mathcal{X}_4 + i\mathcal{Y}_4$ and $\mathcal{X}_4 > 0$ is fixed in this integration. It is known that (3.14) is upto a constant, the 'inverse Laplace transform' of the function $T_4 \mapsto \det(2\pi\mathcal{T}_4)$, where $T_4 > 0$ is of size $n-p$ (see, e.g. [13]) and is independent of $\mathcal{X}_4$. More precisely we have,

$$I_4 = i^{(n-p)(s-k)} \Gamma_{n-p}(k-s)^{-1} \det(2\pi\mathcal{T}_4)^{k-s-\frac{n-p+1}{2}} e(-\mathcal{T}_4 Z_1^{-1}[Z_2]), \tag{3.15}$$

where the generalized gamma function $\Gamma_m(s)$ is defined by (cf. [13, p. 480])

$$\Gamma_m(s) := \int_{R=R'>0} \det(R)^{s-\frac{m+1}{2}} \exp(-\mathrm{tr}\ R) dR, \quad (\mathrm{Re}(s) > \frac{m-1}{2})$$

and equals the product (see [21, Lem. 1.6, p. 142])

$$\Gamma_m(s) = \pi^{\frac{m(m-1)}{4}} \prod_{t=0}^{m-1} \Gamma(s - \frac{t}{2}). \tag{3.16}$$

3.5. **Calculation of I_2.** Now, an inspection of (3.13) and taking into account (3.15) shows that next we need to consider the integral

$$I_2 := \int\limits_{X_2} e(\mathcal{T}_2' Z_2 - \mathcal{T}_4 \cdot Z_1^{-1}[Z_2]) dZ_2, \tag{3.17}$$

where $Z_2 = X_2 + iY_2$ and Y_2 is fixed. Let $\mathbf{z} := -Z_1^{-1}$. We make a change of variable $Z_2 \mapsto \omega := Z_2 \mathcal{T}_4^{\frac{1}{2}}$, and observe that the differentials transform by

$$d\omega = \det(\mathcal{T}_4)^{\frac{p}{2}} dZ_2.$$

Let $\omega = (\omega_1, \dots, \omega_{n-p})$, i.e., ω_l are the columns of ω. Then (3.17) reads

$$I_2 = \det(\mathcal{T}_4)^{-\frac{p}{2}} \int\limits_{\mathrm{Im}(\omega)=\mathrm{const.}} e(-\mathcal{T}_2'\omega\mathcal{T}_4^{-\frac{1}{2}} - \mathbf{z}[\omega])d\omega$$

$$= \det(\mathcal{T}_4)^{-\frac{p}{2}} \prod_{l=1}^{n-p} \int\limits_{\mathrm{Im}(\omega_l)=\mathrm{const.}} \exp\left(i\{-2\pi\mathcal{T}_4^{-\frac{1}{2}}\mathcal{T}_2'\omega_l\} - \frac{1}{2}\{4\pi\cdot(-i\mathbf{z})[\omega_l]\}\right)d\omega_l.$$

We can now use the following formula for the inner integrals, see [10, p. 32] for positive definite matrices; the complex case follows from this by analytic continuation. Let $\mathcal{C} = \mathcal{C}' \in M_p(\mathbf{C})$ be such that $\mathrm{Re}(\mathcal{C}) > 0$. Then for $b \in \mathbf{C}^{p,1}$

$$\int\limits_{\mathrm{Im}(\omega_l)=\mathrm{const.}} \exp\left(-\frac{1}{2}\mathcal{C}[\omega_l] + ib'\omega_l\right)d\omega_l = \frac{(2\pi)^{n/2}}{\det(\mathcal{C})^{1/2}}\exp\left(-\frac{1}{2}\mathcal{C}^{-1}[b]\right). \tag{3.18}$$

Therefore, with $\mathcal{C} = 4\pi\cdot(-i\mathbf{z})$ and $b = 2\pi\mathcal{T}_2\mathcal{T}_4^{-\frac{1}{2}}e_\ell$ where e_ℓ is the ℓ-th unit column vector of size $(n-p,1)$, the final expression for I_2 is given by

$$I_2 = \det(\mathcal{T}_4)^{-\frac{p}{2}}\frac{(2\pi)^{\frac{p(n-p)}{2}}}{\det(4\pi i Z_1^{-1})^{\frac{n-p}{2}}}\exp\left(\frac{i}{2}\mathrm{tr}\ (\frac{Z_1}{4\pi})[2\pi\mathcal{T}_2\mathcal{T}_4^{-\frac{1}{2}}]\right)$$

$$= (2i)^{\frac{-p(n-p)}{2}}\det(\mathcal{T}_4)^{-\frac{p}{2}}\det(Z_1)^{\frac{n-p}{2}}e\left(\mathcal{T}_4^{-1}[\mathcal{T}_2'/2]Z_1\right). \tag{3.19}$$

3.6. **Calculation of I_1.** To compute $\Phi_{p,n}(U,a_1,c_1;T)$ it is thus left to compute the 'integral over Z_1' and to put everything together, which we will do now. From (3.13) and the previous subsection, we see that the following integral has to be determined:

(3.20)

$$I_1 := \int\limits_{\mathrm{Im}(Z_1)=\mathrm{const.}} \det(Z_1)^{-(k-s)+\frac{n-p}{2}}\det(-1_p + S_1Z_1)^{-s}e\left(-(\mathcal{T}_1 - \mathcal{T}_4^{-1}[\mathcal{T}_2'/2])Z_1\right)dZ_1$$

To proceed further, let us define the hypergeometric function of degree n (cf. [13, p. 487]) by:

(3.21)

$${}_1\mathcal{F}_1^{(n)}(\alpha,\beta,\mathbf{M}) := \frac{\Gamma_n(\beta)}{(2\pi i)^{n(n+1)/2}}\int\limits_{\mathrm{Re}(\mathfrak{Z})=\mathfrak{X}_0}\exp(\mathrm{tr}(\mathfrak{Z}))\det(I_n - \mathbf{M}\mathfrak{Z}^{-1})^{-\alpha}\det(\mathfrak{Z})^{-\beta}d\mathfrak{Z}.$$

Here M is a symmetric complex matrix of size n, $\mathfrak{X}_0 > 0$ is a fixed matrix such that $\mathfrak{X}_0 > \mathrm{Re}(M)$ and α,β are complex numbers such that $\mathrm{Re}(\beta) > n$.

Let us put $\mathbf{T} := \mathcal{T}_1 - \mathcal{T}_4^{-1}[\mathcal{T}_2'/2]$ and note that $\mathbf{T}$ is positive definite from the Jacobi decomposition of the matrix $\mathcal{T}[\left(\begin{smallmatrix}0 & 1_n\\ 1_n & 0\end{smallmatrix}\right)] = \left(\begin{smallmatrix}\mathcal{T}_4 & \mathcal{T}_2'/2\\ \mathcal{T}_2/2 & \mathcal{T}_1\end{smallmatrix}\right)$ with respect to the variable in the upper left corner. Then (3.20) can be written as

$$\begin{aligned} I_1 &= \det(S_1)^{-s} \int\limits_{\mathrm{Im}(Z_1)=\mathrm{const.}} \det(Z_1)^{-(k-s)+\frac{n-p}{2}} \det(-S_1^{-1}+Z_1)^{-s} e\left(-\mathbf{T}Z_1\right) dZ_1 \\ &= \det(S_1)^{-s} \int\limits_{\mathrm{Im}(Z_1)=\mathrm{const.}} \det(Z_1)^{-k+\frac{n-p}{2}} \det(1_p - S_1^{-1}Z_1^{-1})^{-s} e\left(-\mathbf{T}Z_1\right) dZ_1 \\ &= \frac{(-1)^{\frac{p(p+1)}{2}} i^u}{\det(S_1)^s} \int\limits_{\mathrm{Re}(\mathfrak{Z}_1)=\mathrm{const.}} \det(\mathfrak{Z}_1)^{-k+\frac{n-p}{2}} \det(1_p + iS_1^{-1}\mathfrak{Z}_1^{-1})^{-s} e^{(2\pi \mathrm{tr}\ \mathbf{T}\mathfrak{Z}_1)} d\mathfrak{Z}_1, \end{aligned}$$

where in the middle line we have substituted $Z_1 = i\mathfrak{Z}_1$ and the minus sign comes from the change in the orientation of the integrals arising from this change of variables. Further, $u := p(-k+\frac{n+1}{2})$. Substituting $\mathfrak{Z}_1 \mapsto (2\pi\mathbf{T})^{-\frac{1}{2}}\mathfrak{Z}_1(2\pi\mathbf{T})^{-\frac{1}{2}}$ in the last equation above and keeping in mind (3.21), gives

$$I_1 = \frac{i^v (2\pi)^{\frac{p(p+1)}{2}} \det(2\pi\mathbf{T})^{k-\frac{n+1}{2}}}{\det(S_1)^s \Gamma_p(k-\frac{n-p}{2})}\ {}_1\mathcal{F}_1^{(p)}\left(s, k-\frac{n-p}{2}, -iS_1^{-1}[(2\pi\mathbf{T})^{\frac{1}{2}}]\right), \tag{3.22}$$

where we have put $v := p(-k+\frac{n-p}{2})$. Also note that in (3.22), the hypergeometric function is well defined since $k-\frac{n-p}{2} > p$, from our assumption that $k > 2(n+1)$ (cf. Introduction).

3.7. The final expression for $\Phi_{p,n}(U, a_1, c_1; T)$.

Putting together the results of the previous sections 3.4, 3.5 and 3.6 in equation (3.13), we now arrive at the final expression for $\Phi_{p,n}(U, a_1, c_1; T)$. Let us first introduce the following notation, which will come in handy to deal with the Fourier expansion later in the paper. Namely, for a positive definite matrix $M = \left(\begin{smallmatrix} A & \alpha \\ \alpha' & B \end{smallmatrix}\right)$, we define

$$\mathrm{Ja}(M) := A - B^{-1}[\alpha']. \tag{3.23}$$

Thus, $\mathrm{Ja}(M)$ is nothing but the upper left corner of the diagonal block appearing in the Jacobi decomposition of $M[\left(\begin{smallmatrix} 0 & 1_n \\ 1_n & 0 \end{smallmatrix}\right)]$ with respect to B. Then the final expression for $\Phi_{p,n}(U, a_1, c_1; T)$ can be stated as (recall that $\mathcal{T} = T[U^*], \mathbf{T} = \mathrm{Ja}(\mathcal{T})$ from the previous subsection):

$$\begin{aligned} \Phi_{p,n}(U, a_1, c_1; T) = &\frac{i^{\beta_p} \det(T)^{k-\frac{n+1}{2}}}{2^{p(n-p)/2} \det(\mathcal{T}_4)^s \det(S_1)^s} \cdot \frac{(2\pi)^{h_{n,p}(k,s)}}{\Gamma_p(k-\frac{n-p}{2})\Gamma_{n-p}(k-s)} \\ &\cdot {}_1\mathcal{F}_1^{(p)}\left(s, k-\frac{n-p}{2}, -i(2\pi\mathrm{Ja}(\mathcal{T})^{\frac{1}{2}} S_1^{-1} (2\pi \mathrm{Ja}(\mathcal{T})^{\frac{1}{2}}\right), \end{aligned} \tag{3.24}$$

where $\beta_p := -pk + \frac{p(n-p)}{2} + (n-p)(s-k-p/2)$, and we have used the fact that $\det(\mathbf{T})\det(\mathcal{T}_4) = \det(T)$. Further, from the expressions of I_4 and I_1 in (3.15) and

(3.22), we can compute the exponent $h_{n,p}(k,s)$ above to be

$$\begin{aligned} h_{n,p}(k,s) &:= (k-s)(n-p) - (n-p)(\tfrac{n-p+1}{2}) + \tfrac{p(p+1)}{2} + p(k - \tfrac{n+1}{2}) \\ &= nk - s(n-p) - \tfrac{(n+1)(n-p)}{2}. \end{aligned} \tag{3.25}$$

3.8. **The cases, $0 < \mathrm{rank}(C) = q < n$.** We will now deal with the cases when the rank q of C in the summation (2.1) does not equal 0 or n. We first prove a lemma that expresses the piece $F^n_{p,q;s}(Z)$ (see (3.3)) as an infinite linear combination of kernel functions (as in section 2.2) corresponding to the lower dimension q. This enables us to conclude later in Proposition 3.7 that the pieces $b_{p,q;s}(T)$ with $T > 0$ do not contribute to the Fourier coefficient $b_s(T)$ for $1 \le p, q \le n-1$.

Lemma 3.6.

$$F^n_{p,q;s}(Z) = \sum_{U \in \mathrm{GL}_n(\mathbf{Z})/\mathrm{GL}_n(\mathbf{Z})_q} \det(U)^{s-k} F^q_{p+q-n,q;s}(Z[U]_1).$$

Note that on the r.h.s. of the above equation, the function $F^q_{p+q-n,q;s}(\cdot)$ is defined on $\mathbf{H}_q$, is invariant under translations, and so possesses a Fourier expansion, see e.g., (3.8).

Proof. We appeal to Lemma 3.2, and apply it to the pair $(C', A') \in \mathrm{CS}(n,q)$ with $\mathrm{rank}(C) = q < n$, in the definiton (3.3) of the series defining $F^n_{p,q;s}(Z)$. *Notice the flipping of the usual pair (A', C').* We use the substitutions

$$C' = U \begin{pmatrix} C_1' & 0 \\ 0 & 0_{n-q} \end{pmatrix}, \quad A' = U \begin{pmatrix} A_1' & 0 \\ 0 & 1_{n-q} \end{pmatrix},$$

and make the obvious choices of the matrices

$$B_0 = \begin{pmatrix} B_1' & 0 \\ 0 & 0_{n-q} \end{pmatrix} U^{-1}, \quad D_0 = \begin{pmatrix} D_1' & 0 \\ 0 & 0_{n-q} \end{pmatrix} U^{-1}$$

to complete (C', A') into a symplectic matrix $\begin{pmatrix} D_0 & B_0 \\ C' & A' \end{pmatrix}$. We also note that since the rank of A is p,

$$\mathrm{r}(A_1) = \mathrm{r}(A) + q - n = p + q - n.$$

We then calculate

$$\begin{aligned} \det(CZ + D) &= \det(U)^{-1} \det\left\{ \begin{pmatrix} C_1 & 0 \\ 0 & 0_{n-q} \end{pmatrix} U'ZU + \begin{pmatrix} D_1 & 0 \\ 0 & 1_{n-q} \end{pmatrix} \right\} \\ &= \det(U)^{-1} \det(CZ[U]_1 + D_1), \end{aligned}$$

and similarly that

$$(AZ + B_0)(CZ + D_0)^{-1} = (A_1 Z[U]_1 + B_1)(C_1 Z[U]_1 + D_1)^{-1}. \tag{3.26}$$

Since U varies in $\mathrm{GL}_n(\mathbf{Z})/\mathrm{GL}_n(\mathbf{Z})_q$, the proposition follows. ☐

Proposition 3.7. *$b_{p,q;s}(T) = 0$ for all $1 \le p, q \le n-1$ and $T \in \Lambda_n^+$, when $n + 1 < \sigma < k - (n+1)$.*

Proof. The proof follows from Lemma 3.6. Namely, we will show that the Fourier expansion of $F^n_{p,q;s}(Z)$ for each p,q as in the hypothesis, is supported on matrices in Λ_n which are *not* positive definite. This will clearly imply the proposition, since the kernel function $F^n_{k,s}(Z)$ is known to be a cusp form in the given range of σ, see Proposition 2.1.

Let $F^q_{p+q-n,q;s}(\mathbf{z}^{(q)})$ be as in Lemma 3.6 and let its Fourier coefficients be $b_{p+q-n,q;s}(q;\mathbf{t})$, where $\mathbf{t}\in\Lambda_q$. Then we see that

$$\begin{aligned} F^n_{p,q;s}(Z) &= \sum_{U\in \mathrm{GL}_n(\mathbf{Z})/\mathrm{GL}_n(\mathbf{Z})_q} \det(U)^{s-k} F^q_{p+q-n,q;s}(Z[U]_1,s)\\ &= \sum_{\mathbf{t}\in\Lambda_q} b_{p+q-n,q;s}(q;\mathbf{t})\sum_U \det(U)^{s-k}\ e(\mathbf{t}Z[U]_1)\\ &= \sum_{\mathcal{S}\in\Lambda_n,\mathrm{r}(\mathcal{S})\le q} b_{p,q;s}(\mathcal{S})e(\mathcal{S}Z), \end{aligned} \tag{3.27}$$

where the Fourier coefficients $b_{p,q;s}(\mathcal{S})$ (cf. (3.4)) are given by

$$b_{p,q;s}(\mathcal{S}) = \sum_{\mathbf{t},U:\ \left(\begin{smallmatrix}\mathbf{t}&0\\0&0\end{smallmatrix}\right)[U']=\mathcal{S}} b_{p+q-n,q;s}(q;\mathbf{t})\det(U)^{s-k}.$$

In (3.27), we have used the following computation:

$$e(\mathbf{t}Z[U]_1) = e\left(\left(\begin{smallmatrix}\mathbf{t}&0\\0&0\end{smallmatrix}\right)U'ZU\right) = e\left(U\left(\begin{smallmatrix}\mathbf{t}&0\\0&0\end{smallmatrix}\right)U'\cdot Z\right).$$

This shows that $b_{p,q;s}(T)=0$ for all $1\le p,q\le n-1$, and completes the proof of the proposition. □

3.9. **The extreme cases.** In view of the results obtained in the above subsections, it remains to calculate the Fourier coefficients $b_{0,n}(T)$ and $b_{n,0}(T)$. Since

$$\Gamma_{n,\infty} := \{\left(\begin{smallmatrix}A&B\\C&D\end{smallmatrix}\right)\in\Gamma_n \mid C=0\} = G_{0,n}\cdot\{\left(\begin{smallmatrix}1_n&S\\0&0\end{smallmatrix}\right)\mid S=S'\},$$

when $\mathrm{r}(C)=0$, it is easy to write down the Fourier expansion of $F^n_{n,0}(Z,s)$:

$$F^n_{n,0}(Z,s) = A_{n,k}(s)\sum_{S=S'}\det(Z+S)^{-s} \tag{3.28}$$

$$= C_n\Gamma_n(k-s)\sum_{T>0}\det(T)^{s-\frac{n+1}{2}}e(TZ), \tag{3.29}$$

where $C_n=(2\sqrt{\pi})^{-\frac{n(n-1)}{2}}$. The series occuring on the l.h.s. of (3.28) is usually refered to as the generalized Lipshhitz series and its Fourier expansion as in (3.29), the generalized Lipschitz formula.

Now we indicate how to compute $b_{n,q;s}(T)$ in terms of their symmetric counterparts $b_{q,n;s}(T)$. First we note that $J=J_n := \left(\begin{smallmatrix}0&-1_n\\1_n&0\end{smallmatrix}\right)$ normalizes the group $G_{0,n}$. Therefore, if $\{\gamma_i\}$ is a system of left coset representatives for Γ_n modulo $G_{0,n}$, then so is the set $\{J^{-1}\gamma_i\}$. Moreover, if we set $\gamma = \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix}$, then $\mathrm{r}(a_\gamma) = n, \mathrm{r}(c_\gamma) = q$ if and only if $\mathrm{r}(a_{J^{-1}\gamma}) = q, \mathrm{r}(c_{J^{-1}\gamma}) = n$. Therefore,

$$\begin{aligned}
F^n_{n,q;s}(Z) &= A_{n,k}(s) \sum_{\substack{\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in G_{0,n}\backslash\Gamma_n \\ \mathrm{r}(A)=n, \mathrm{r}(C)=q}} \det(Z)^{-s} |_k \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \\
&= A_{n,k}(s) \sum_{\substack{\left(\begin{smallmatrix} C & D \\ -A & -B \end{smallmatrix}\right) \in G_{0,n}\backslash\Gamma_n \\ \mathrm{r}(A)=n, \mathrm{r}(C)=q}} \det(Z)^{-s} |_k \, J \cdot \left(\begin{smallmatrix} C & D \\ -A & -B \end{smallmatrix}\right) \\
&= A_{n,k}(s) \sum_{\substack{\left(\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right) \in G_{0,n}\backslash\Gamma_n \\ \mathrm{r}(\mathcal{A})=q, \mathrm{r}(\mathcal{C})=n}} \det(Z)^{-(k-s)} |_k \cdot \left(\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right) \\
&= e^{\frac{\pi i n(2s-k)}{2}} F^n_{q,n;k-s}(Z). \qquad (3.30)
\end{aligned}$$

Therefore we have a formula for the Fourier coefficients $b_{p,q;s}(T)$ for all p, q. We summarize the final result in the following proposition. Recall the function $\mathrm{Ja}(M)$ from (3.23).

Proposition 3.8. *Let $n+1 < \sigma < k-(n+1)$. Then, for $1 \le p \le n$,*

$$\begin{aligned}
b_{p,n;s}(T) =& \frac{A_{n,k}(s)(-1)^{\frac{p(p+1)}{2}} i^{\beta_p} (2\pi)^{h_{n,p}(k,s)}}{2^{\frac{p(n-p)}{2}} \Gamma_p(k - \frac{n-p}{2}) \Gamma_{n-p}(k-s)} \det(T)^{k-\frac{n+1}{2}} \sum_{U \in \mathrm{GL}_n(\mathbf{Z})/\mathrm{GL}_n(\mathbf{Z})_p} \\
& \sum_{\substack{(a'_1, c'_1) \in \mathrm{CS}(p,p)/\diamond \\ \mathrm{r}(a_1)=p, \mathrm{r}(c_1)=p}} e(T[U^*]_1 c_1^{-1} d_1) \det(T[U^*]_4)^{-s} \det(c_1)^{-(k-s)} \det(a_p)^{-s} \\
& \cdot {}_1\mathcal{F}_1^{(p)}\left(s, k - \frac{n-p}{2}, -2\pi i (a'_1 c_1)^{-1} [\mathrm{Ja}(T[U^*])^{\frac{1}{2}}]\right),
\end{aligned}$$

where $\beta_p = -pk + \frac{p(n+p+2)}{2} + (n-p)(s-k-p/2), h_{n,p}(k,s) = nk - s(n-p) - \frac{(n+1)(n-p)}{2}$, and d_1 is as in Lemma 3.5. Moreover,

$$\begin{aligned}
b_{n,q;s}(T) &= e^{\frac{\pi i n(2s-k)}{2}} b_{q,n;k-s}(T) \text{ for } 0 \le q \le n; \\
b_{n,0;s}(T) &= C_n \Gamma_n(k-s) \det(T)^{s-\frac{n+1}{2}} \quad (C_n = (2\sqrt{\pi})^{-\frac{n(n-1)}{2}}).
\end{aligned}$$

4. Proof of Theorem 1.1

Before we embark on to the proof of Theorem 1.1, certain remarks are in order. We wish to prove the theorem by showing that the kernel function $F^n_{k;s}(Z)$ is non-zero for s in a certain region by comparing the 1_n-th Fourier coefficients on both sides of (2.3). Nonvanishing of $F^n_{k;s}(Z)$ will essentially follow from showing that the total contribution of the pieces $b_{p,n;s}(1_n)$ with $1 \leq p < n$ is less than that of $b_{0,n;s}(1_n)$ as $k \to \infty$ (and symmetrically for the $b_{n,p;s}(1_n)$). Thus we need to have good upper bounds for the $b_{p,n;s}(1_n)$. For this, we would proceed directly from the explicit formula in Proposition 3.8. However, we need to take care of the following points in order to proceed:

(i) Find a non-trivial estimate for ${}_1\mathcal{F}_1^{(p)}(\cdot,\cdot,i\mathcal{M})$, where $\mathcal{M} \in M_p(\mathbf{R})$,

(ii) Take care of the summation over the variable U.

We find both of these points rather difficult to deal with, indeed the next three subsections are devoted towards the items (i) and (ii) above.

4.1. **Non-trivial estimate for the hypergeometric function.** The hypergeometric function ${}_1\mathcal{F}_1^{(\cdot)}(\cdot,\cdot,\mathcal{M})$ is a quite familiar object for physicists and statisticians. Indeed some estimates for it are available in the literature (cf. [25]) when $\mathcal{M}$ is *real*. Nothing seems to be available when $\mathcal{M}$ is, say, purely imaginary; other than the trivial estimate:

$$|{}_1\mathcal{F}_1^{(n)}(\alpha,\beta,iM)| \leq 1 \quad (M \in \mathrm{Sym}_n(\mathbf{R})). \tag{4.1}$$

This follows from the following characterization of ${}_1\mathcal{F}_1{}^{(m)}$, see [13, p. 488]:

$$\begin{aligned} &{}_1\mathcal{F}_1^{(n)}(\alpha,\beta,iM) = \\ &= \frac{1}{\beta_n(\alpha,\beta-\alpha)} \int_0^{1_n} \exp(i\mathrm{tr}RM)) \det(R)^{\alpha-\frac{n+1}{2}} \det(1_n - R)^{\beta-\alpha-\frac{n+1}{2}} dR \end{aligned}$$

where $\beta_m(\alpha,\beta-\alpha)$ is the generalized beta function:

$$\beta_n(\alpha,\beta-\alpha) = \int_0^{1_n} \det(R)^{\alpha-\frac{n+1}{2}} \det(1_n - R)^{\beta-\alpha-\frac{n+1}{2}} dR = \frac{\Gamma_n(\alpha)\Gamma_n(\beta-\alpha)}{\Gamma_n(\beta)}.$$

The estimate (4.1) is not sufficient for our purposes. Indeed if we use it for $n > 1$, we can show, e.g., that the series obtained by putting absolute values in the terms in (3.10) (with $\mathrm{r}(A) = n$) diverges. This can be seen by noting that the pairs $(V, 1_n) \in \mathrm{CS}(n)$ with $V = V' \in \mathrm{SL}_n(\mathbf{Z})$ are inequivalent under $\diamond$, and the integral in (3.10) is up to some constants, the hypergeometric function ${}_1\mathcal{F}_1^{(n)}$.

However, (4.1) was sufficient to treat the case $n = 1$, see [17]. A non-trivial estimate for ${}_1\mathcal{F}_1^{(\cdot)}(\cdot,\cdot,\mathcal{M})$, when $\mathcal{M}$ is purely imaginary; will be seen to follow from the lemma given below.

4.1.1. *The function* $\eta_\alpha(W)$. Let us introduce the function $\eta_\alpha(W)$ for $W \in \mathbf{H}_n$ which, in the notation in [21], is defined by the integral

$$\eta_\alpha(W) = \int_{X=X'} |\det(W+X)|^{-\alpha} dX, \quad (\alpha > n). \tag{4.2}$$

Then it is proved in [21] that one has the formula:

Proposition 4.1.

$$\eta_\alpha(W) = \mathrm{Im}(W)^{-\alpha+\frac{n+1}{2}}\eta_\alpha(i1_n).$$

The constant $\eta_\alpha(i1_n)$ can be evaluated explicitly as a ratio of product of certain gamma-factors by an induction on n as in [7, p. 395]. We will not need the explicit form of it. The following lemma is one of the important technical inputs in our paper.

Lemma 4.2. *Let* $\alpha := \min\{\sigma, k-\sigma\}$*. Then,*

$$\int\limits_{\mathrm{Im}(Z)=1_n} |\det(Z)|^{-k+\sigma}|\det(Z-S)|^{-\sigma}dZ \le 2^{1+\alpha/2}\left(1+\mathrm{tr}(\frac{S^2}{2})\right)^{\alpha/2}\eta_\alpha(i1_n).$$

Proof. Then the integral I_n above can be estimated as

$$I_n \le \int\limits_{X=X'} \det(1_n+X^2)^{-\alpha/2}\det\left(1_n+(X-S)^2\right)^{-\alpha/2}dX.$$

We next observe that for any nonnegative symmetric matrix R one has

$$\det(1_n+R) \ge 1+\mathrm{tr}(R),$$

this can be seen by diagonalizing R.

Applying this to X^2 and $(X-S)^2$, we have that

$$\det(1_n+X^2)+\det(1_n+(X-S)^2) \ge 1+\mathrm{tr}(X^2+(X-S)^2) \tag{4.3}$$

Further one has the inequality for $X, S \in \mathrm{Sym}_n(\mathbf{R})$:

$$X^2+(X-S)^2 \ge \frac{S^2}{2}. \tag{4.4}$$

This follows from the computation

$$X^2+(X-S)^2 = 2(X-\frac{S}{2})^2+\frac{S^2}{2}.$$

For ease of notation, let us call

$$d := 1+\mathrm{tr}(\frac{S^2}{2}), A(X) := \det(1_n+X^2), B(X) := \det(1_n+(X-S)^2).$$

We then have from (4.3) that

$$A(X)+B(X) \ge d.$$

Let us now rewrite the estimate for I_n as follows:

$$\begin{aligned} I_n &\leq \int\limits_{A(X)>\frac{d}{2}} (A(X)B(X))^{-\alpha/2} + \int\limits_{A(X)\leq\frac{d}{2}} (A(X)B(X))^{-\alpha/2} \\ &\leq \left(\frac{d}{2}\right)^{-\alpha/2} (\int\limits_{X=X'} B(X)^{-\alpha/2} + A(X)^{-\alpha/2}) dX \\ &\leq 2\left(\frac{d}{2}\right)^{-\alpha/2} \eta_\alpha(i1_n). \end{aligned}$$

□

We can now use Lemma 4.2 to get the following non-trivial bound for the hypergeometric function with a purely imaginary argument:

Proposition 4.3. *Let* $M \in \mathrm{Sym}_n(\mathbf{R})$ *and* $\mathrm{Re}(\beta - \alpha) > n$. *Then*

$$|{}_1\mathcal{F}_1^{(n)}(\alpha, \beta, iM)| \leq \min\left\{1, 2^{1+\kappa}\Gamma_m(\beta)\left(1 + \mathrm{tr}(\frac{M^2}{2})\right)^{-\kappa} \eta_\kappa(i1_n)\right\},$$

where $2\kappa = \min\{\mathrm{Re}(\alpha), \mathrm{Re}(\beta - \alpha)\}$.

Proof. In the expression for the hypergeometric function defined in (3.21), one first makes a change of variable $\mathfrak{Z} \to i\mathfrak{Z}$, so that the integration is over $\mathfrak{Z} \in \mathbf{H}_n$ with $\mathrm{Im}(\mathfrak{Z}) = K$, where K is a fixed positive definite matrix. Taking $K = 1_n$ and noting that $|\det(X + i1_n)|^2 = \det(1 + X^2)$, the proposition follows from (4.1) and Lemma 4.2. □

4.2. **Estimates for some counting functions.** This subsection is devoted to provide estimates for certain arithmetical counting functions. These functions count the number of integral matrices or pairs of matrices subject to appropriate conditions. These will be required while estimating the pieces $b_{p,n;s}(1_n)$.

In this regard, our first lemma gives an estimate for the number of coprime symmetric pairs (G, H) modulo the relation $\diamond$, such that $GH' = S$, where $S \in \mathrm{Sym}_n(\mathbf{Z})$ is non-singular. To this end, let us make the following definitions.

Definition 4.4. Let $S \in M_n(\mathbf{Z})$, with $\det(S) \neq 0$. We define,

$$\begin{aligned} \mathbf{A}_n(S) &:= \{\overline{(G, H)} \in (M_n(\mathbf{Z}) \times M_n(\mathbf{Z}))/\diamond \mid GH' = S\}, \\ \mathbf{A}_n^{\mathrm{cs}}(S) &:= \{\overline{(G, H)} \in \mathrm{CS}(n)/\diamond \mid GH' = S\}; \end{aligned}$$

where $\overline{(G, H)}$ denotes the equivalence class containing (G, H).

Clearly one has $\mathbf{A}_n^{\mathrm{cs}}(S) \subset \mathbf{A}_n(S)$, and

Lemma 4.5. *For* $S \in \mathrm{Sym}_n(\mathbf{Z})$ *non-singular, one has* $\#\mathbf{A}_n(S) \ll_\varepsilon |\det(S)|^{n-1+\varepsilon}$.

Proof. We first note that is it enough to prove the lemma for lower triangular matrices S. This follows from two facts. Firstly, since we can find an unimodular matrix U_0 such that SU_0 is lower triangular (see e.g., [21, Cor. (5.2)]) with positive diagonal entries. Secondly, since there is a bijection between $\mathbf{A}_n(S)$ and $\mathbf{A}_n(SU_0)$ given by

$$\mathbf{A}_n(S) \to \mathbf{A}_n(SU_0), \quad \overline{(G,H)} \mapsto \overline{(G, U_0'H)}.$$

Next, we observe that one can assume the matrix G in the equation $GH' = S$, to be lower triangular as well. This is because it is possible to replace the pair (G,H) by (GW', HW^{-1}) for any $W \in \mathrm{GL}_n(\mathbf{Z})$. We simply choose W such that GW' is lower triangular, and continue to call this new matrix to be $G = (g_{i,j})$. We also note that (cf. [21, Cor. (5.2)]) $g_{i,i} > 0$ with $0 \le g_{i,j} < g_{i,i}$.

With these reductions, it is clear that for any element $\overline{(G,H)}$ of $\mathbf{A}_n(S)$, one must have H to be lower triangular as well. Then we get the following system of equalities:

$$g_{i,i}h_{i,i} = s_{i,i}, \qquad \text{for all } 1 \le i \le n.$$

This implies in particular that $g_{i,i} \mid s_i$ for all i, where $s_i = s_{i,i}$. Hence fixing a set of divisors $\{d_i\}$ of $\{s_i\}$, we see that the number of choices for the matrix G is at most

$$\sum_{d_1|s_1}\sum_{d_2|s_2}\cdots\sum_{d_n|s_n} d_2d_3^2\ldots d_n^{n-1}.$$

This implies that for any $\varepsilon > 0$

$$\#\mathbf{A}_n(S) \ll_\varepsilon s_1^\varepsilon s_2^{1+\varepsilon}\ldots s_n^{n-1+\varepsilon} \ll_\varepsilon |\det(S)|^{n-1+\varepsilon},$$

since for each choice of G; H is fixed by $G'^{-1}S$. □

Lemma 4.6. *The map*

$$\mathrm{CS}(n)/\diamond \to \mathrm{Sym}_n(\mathbf{Z}), \quad \overline{(G,H)} \to GH',$$

is surjective.

Proof. First, let us note that the map is well-defined. From the elementary divisors theorem for S we get $U, V \in \mathrm{GL}_n(\mathbf{Z})$ be such that $USV = R$, where $R = \mathrm{diag}(r_1,\ldots,r_n)$. Then $G = U^{-1}$ and $H = V^*R$ does the job. □

The next lemma will be used in proving Theorem 1.1 in the next section, while summing over the $\mathrm{GL}_n(\mathbf{Z})$ variable U. For $U_1, W_1 \in M_p(\mathbf{Z})$, $U_3, W_3 \in M_{n-p,p}(\mathbf{Z})$), we say that the pairs (U_1,U_3) and (W_1,W_3) are equivalent, if there exists an $\alpha \in \mathrm{GL}_p(\mathbf{Z})$ such that $(U_1,U_3) = (W_1\alpha, W_3\alpha)$. We denote by $[U_1,U_3]_{p,n-p}$ the equivalence class containing (U_1,U_3).

Similarly for $V \in \mathrm{Sym}_n(\mathbf{Z})$, we denote by $[V]_p$ the equivalence class containg V, where V and V_1 are in the same class if there exists an $\alpha \in \mathrm{GL}_p(\mathbf{Z})$ such that $V_1 = \alpha' V\alpha$.

For a positive-definite matrix $V \in \mathrm{Sym}_n(\mathbf{Z})$, let the set $\mathcal{R}_{p,n-p}(V)$ be defined by

$$\mathcal{R}_{p,n-p}(V) := \{[U_1, U_3]_{p,n-p} \mid [U_1'U_1 + U_3'U_3]_p = [V]_p\}. \tag{4.5}$$

Lemma 4.7. *For V as above, one has $\#\mathcal{R}_{p,n-p}(V) \leq \det(V)^{\frac{n}{2}-1+\epsilon}$.*

Proof. To prove the lemma, clearly we can assume that V is Minkowski-reduced in the sense of [11]. Then replacing (U_1, U_3) by $(U_1\alpha, U_3\alpha)$ if necessary, we have to count the number of pairs (U_1, U_3) such that

$$U_1'U_1 + U_3'U_3 = V.$$

For this, we equate the diagonal elements in the above equation to find that

$$U_{1,j}'U_{1,j} + U_{3,j}'U_{3,j} = V_j \quad (1 \leq j \leq p), \tag{4.6}$$

where $U_{1,j}, U_{3,j}$ denotes the j-th column of the respective matrices, V_j is the j-th diagonal element of V. Now (4.6), says that the sum of n integral squares equals V_j and so the number of possibilities for the j-th columns of U_1 and U_3 is at most $r_n(V_j)$, where $r_n(m)$ is the number of ways of writing m as the sum of n sqaures. The theory of theta series (see [24, p. 109] for example) shows that

$$r_n(m) \ll_{n,\varepsilon} m^{\frac{n}{2}-1+\varepsilon}.$$

Taking this into account and the fact that

$$V_1V_2 \dots V_p \ll_n \det(V)$$

since V is reduced; immediately gives the lemma. □

4.3. **Lemmas relating to the estimate for the Fourier coefficients $b_{p,n}(1_n)$.** In this subsection we collect together several prepatory lemmas that would be used to estimate the Fourier coefficient $b_{p,n}(1_n)$ by means of the series that defines it, see (3.8). This involves variables ranging over $\mathrm{Sym}_p(\mathbf{Z})$ and over sets related to $\mathrm{GL}_n(\mathbf{Z})$. and The next lemma will be used while summing over the $\mathrm{GL}_n(\mathbf{Z})$ variable U, which occurs in the argument of the hypergeometric function in (3.8). Recall the function $\mathrm{Ja}(\cdot)$ from (3.23).

Lemma 4.8. *For $M = \left(\begin{smallmatrix} A & \alpha \\ \alpha' & B \end{smallmatrix}\right) > 0$, one has $\mathrm{Ja}(M^{-1}) = A^{-1}$.*

Proof. The proof is by direct computation. Unfortunately, we could not find a less computational argument. Set $C := \mathrm{Ja}(M)$. Taking inverses on both sides of the Jacobi decomposition for M with respect to B, we get

$$\begin{aligned} M^{-1} &= \begin{pmatrix} 1 & 0 \\ -B^{-1}\alpha' & 1 \end{pmatrix} \begin{pmatrix} C^{-1} & 0 \\ 0 & B^{-1} \end{pmatrix} \begin{pmatrix} 1 & -\alpha B^{-1} \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} C^{-1} & -C^{-1}\alpha B^{-1} \\ -B^{-1}\alpha' C^{-1} & B^{-1}\alpha' C^{-1}\alpha B^{-1} + B^{-1} \end{pmatrix}. \end{aligned}$$

Then the identity $MM^{-1}=1$ gives the following identity:

$$AC^{-1}\alpha = \alpha B^{-1}(\alpha C^{-1}\alpha + B). \tag{4.7}$$

Then we compute:

$$\begin{aligned}
\mathrm{Ja}(M^{-1}) &= C^{-1} - C^{-1}\alpha B^{-1}(B^{-1}\alpha' C^{-1}\alpha B^{-1} + B^{-1})B^{-1}\alpha' C^{-1} \\
&= C^{-1} - C^{-1}\alpha(\alpha' C^{-1}\alpha B^{-1} + 1)^{-1}\alpha' C^{-1} \\
&= C^{-1} - C^{-1}\alpha(\alpha' C^{-1}\alpha + B)\alpha' C^{-1} \\
&= C^{-1} - A^{-1}\alpha B^{-1}\alpha' C^{-1} \qquad \text{(using (4.7))} \\
&= A^{-1}(A - \alpha B^{-1}\alpha')C^{-1} \\
&= A^{-1}.
\end{aligned}$$

□

The following lemma will be useful while dealing with the summation over the $\mathrm{GL}_n(\mathbf{Z})$ variable U. It parametrizes the set of equivalence classes $\mathrm{GL}_n(\mathbf{Z})/\mathrm{GL}_n(\mathbf{Z})_p$ by the 'first column-blocks' $(U_1, U_3)'$ which are 'primitive' and non-equivalent from the right by $\mathrm{GL}_p(\mathbf{Z})$. We say that $(U_1, U_3)' \in M_{n,p}$ is a primitive pair, if it can be completed to an unimodular matrix. The set of such primitive pairs is denoted by $\mathrm{Prim}_{p,n-p}$. The aforementioned equivalence relation was introduced just before Lemma 4.7 and the equivalence class containing $(U_1, U_3)'$ was denoted by $[U_1, U_3]_{p,n-p}$.

Lemma 4.9. *The natural map*

$$\theta\colon \mathrm{GL}_n(\mathbf{Z})/\mathrm{GL}_n(\mathbf{Z})_p \to \mathrm{Prim}_{p,n-p}/\mathrm{GL}_p(\mathbf{Z}), \quad U = \left(\begin{smallmatrix} U_1 & U_2 \\ U_3 & U_4 \end{smallmatrix}\right) \mapsto [(U_1, U_3)']_{p,n-p},$$

is a bijection of sets.

Proof. Clearly the map is well defined. Since (U_1, U_3) is primitive, surjectivity follows. To see that θ is injective, assume that $\theta(U) = \theta(V)$. Then we have, for some $\alpha \in \mathrm{GL}_p(\mathbf{Z})$ that,

$$(U_1, U_3)' = (V_1\alpha^{-1}, V_3\alpha^{-1})'.$$

To finish the proof, we note that one can solve for β and γ such that

$$\begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} = \begin{pmatrix} U_1\alpha & V_2 \\ U_3\alpha & V_4 \end{pmatrix} \quad \text{i.e.} \quad \begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix}\begin{pmatrix} \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} V_2 \\ V_4 \end{pmatrix}.$$

□

Lemma 4.10. *Let $m, n > 0$. Then for $S \in \mathrm{Sym}_n(\mathbf{R})$ with $\det(S) \neq 0$ one has*

$$\left(1_n + \mathrm{tr}(S^{-2})\right)^{-(m+1)/4} \leq \det(S)^{\frac{m+1}{2n}} \det\left(1_n + \frac{S^2}{n}\right)^{-(m+1)/4n}.$$

Proof. Diagonalizing S, we can assume that S is diagonal. Let the eigenvalues of S be $\mu_1, \dots, \mu_n$. We now apply the arithmetic-geometric mean inequality to get

$$1 + \mathrm{tr}(S^{-2}) = 1 + \sum_j \mu_j^{-2} \geq n \prod_j (\frac{1}{n} + \mu_j^{-2})^{1/n}$$

from which the lemma follows easily. ☐

4.3.1. *Estimate for the termwise majorant of the generalized Lipschitz series.* A result of A. Krieg [21, Cor. 1.5] gives an explicit estimate for the termwise majorant of the generalized Lipschitz series (see (3.28)), which we denote by $\mathscr{L}_n(\alpha, Z)$, and is defined for $\alpha > n$ by

$$\mathscr{L}_n(\alpha, Z) = \sum_{S \in \mathrm{Sym}_n(\mathbf{Z})} |\det(Z + S)|^{-\alpha}.$$

However, Krieg's result does not immediately serve our purpose, we need an even more effective version of that result. This is done in the next lemma, stated in a form that would be sufficient for us.

Lemma 4.11. *Let $V \in \mathrm{Sym}_n(\mathbf{Z})$ be positive definite. Then*

$$\mathscr{L}_n(\alpha, iV^{-1}) \ll_n \det(V)^{\alpha(n+1) - \frac{n+1}{2}}.$$

Proof. First of all, since both sides of the above inequality are invariant under $\mathrm{GL}_n(\mathbf{Z})$, we can assume that V is Minkowski-reduced. We start from [21, Prop. 1.4] with the choice $\varepsilon = 1$. First of all, it is easy to see that

$$\{R \in \mathrm{Sym}_n(\mathbf{R}) \mid R^2 \leq 1_n\} \subset \mathcal{C}(n, \mathbf{R}) \cup -\mathcal{C}(n, \mathbf{R}),$$

where $\mathcal{C}(n, \mathbf{R})$ is the fundamental domain for the action of $\mathrm{Sym}_n(\mathbf{Z})$ on $\mathrm{Sym}_n(\mathbf{R})$. Of course, $\mathcal{C}(n, \mathbf{R})$ has been taken as usual, to be the set of $X \in \mathrm{Sym}_n(\mathbf{R})$ whose coordinates lie in the interval $[0, 1)$.

Then, using [21, Prop. 1.4] we conclude that for any $R \in \mathrm{Sym}_n(\mathbf{R})$ such that $R^2 \leq 1_n$ and any $Z \in \mathbf{H}_n$ such that $\mathrm{Im}(Z) = V^{-1}$,

$$|\det(Z + R)|^{-\alpha} \gg_n (1 + \mathrm{tr}(V))^{-n\alpha} |\det(Z)|^{-\alpha} \gg_n \det(V)^{-n\alpha} |\det(Z)|^{-\alpha}. \tag{4.8}$$

In the above we have used that V is reduced, so that one has

$$\mathrm{tr}(V) = \sum_j V_j \leq \prod_j V_j \ll_n \det(V),$$

where $V_j > 0$ are the diagonal entries of V.

Now let us take $Z = S + iV^{-1}$, $S \in \mathrm{Sym}_n(\mathbf{Z})$ and $R \in \mathcal{C}(n, \mathbf{R})$. Then we have from (4.8) that

$$\begin{aligned} |\det(iV^{-1}+S)|^{-\alpha} &\ll_n \det(V)^{n\alpha} \int\limits_{R^2 \le 1_n} |\det(iV^{-1}+S+R)|^{-\alpha} dR \\ &\ll_n \int\limits_{R \in \mathcal{C}(n,\mathbf{R})} |\det(iV^{-1}+S+R)|^{-\alpha} dR. \end{aligned}$$

Then summing over all $S \in \mathrm{Sym}_n(\mathbf{Z})$ and using Proposition 4.1, we get

$$\begin{aligned} \mathscr{L}_n(\alpha, iV^{-1}) &\ll_n \det(V)^{n\alpha} \int\limits_{X=X'} |\det(iV^{-1}+X)|^{-\alpha} dX \\ &\ll_n \det(V)^{n\alpha} \eta_\alpha(iV^{-1}) \\ &\ll_n \det(V)^{\alpha(n+1)-\frac{n+1}{2}}. \end{aligned}$$

□

4.4. **Setup and completion of the proof.** *Throughout this subsection, p is a natural number such that $1 \le p \le n-1$.* We would now proceed to prove Theorem 1.1. To start with, let us recall and set the following conditions to be followed throughout this section. Namely we remind the reader that the 'strip' we would be considering is *contained* in a strip of the form

$$m' + 1 \le \sigma \le k - (m+1) - \frac{n}{2} \tag{4.9}$$

where $m' > m$ are positive integers, both of which depend only on n and would be specified later on by optimizing our estimates suitably. Further k is assumed to be large from now on as compared to n; the explicit range of k will be specified in (4.17) in course of the proof of Theorem 1.1.

For convenience, we find it very useful at this point to abbreviate some notation. We explicitly define them below. For $U \in \mathrm{GL}_n(\mathbf{Z})$ and $(a_1', c_1') \in \mathrm{CS}(p,p)$,

$$V_U^{-1} := \mathrm{Ja}([U^*]), \quad S := a_1' c_1. \tag{4.10}$$

Then Lemma 4.8 tells us that for $U = \left(\begin{smallmatrix} U_1 & U_2 \\ U_3 & U_4 \end{smallmatrix}\right)$,

$$V_U = (\mathrm{Ja}((U'U)^{-1}))^{-1} = U_1'U_1 + U_3'U_3.$$

Also by definition we have

$$\det(V_U) = \det([U^*]_4), \quad \text{where } [U^*] := 1_n[U^*].$$

Our next proposition gives an effective estimate for the pieces $b_{p,n}(1_n)$. We would like to keep the gamma factors as it is in the expression for $b_{p,n}(1_n)$ in

Proposition 3.8, and so the relevant thing to estimate is denoted by $B := B_{p,n;s}$ given by

$$B_{p,n;s} = \sum_{U\in \mathrm{GL}_n(\mathbf{Z})/\mathrm{GL}_n(\mathbf{Z})_p} \sum_{\substack{(a_1',c_1')\in \mathrm{CS}(p,p)/\diamond \\ \mathrm{r}(a_1)=p,\mathrm{r}(c_1)=p}} e(\mathcal{U}_1 c_1^{-1} d_1) \det([U^*]_4)^{-s} \det(c_1)^{s-k} \det(a_1)^{-s}$$

$$\cdot\, {}_1\mathcal{F}_1^{(p)}\left(s, k - \frac{n-p}{2}, -i(V_U^{\frac{1}{2}} S V_U^{\frac{1}{2}})^{-1}\right), \tag{4.11}$$

where $\mathcal{U}_1 = [U^*]_1$.

Then keeping in mind (4.9), it is easy to see that one has

$$m + 1 \leq \min\{\sigma, k - \frac{n-p}{2} - \sigma\}.$$

Therefore, with $(\alpha, \beta) = (\sigma, k - \frac{n-p}{2})$ in Proposition 4.3 we see that (loc. cit.) $\kappa \geq m + 1$.

Therefore, for $M \in \mathrm{Sym}_p(\mathbf{R})$, one has

$$\left|{}_1\mathcal{F}_1^{(p)}\left(s, k - \frac{n-p}{2}, -iM\right)\right| \ll_n \Gamma_p(k - \frac{n-p}{2})^{\frac{1}{2}} \left(1 + \mathrm{tr}(\frac{M^2}{2})\right)^{-\frac{m+1}{4}}. \tag{4.12}$$

Notice that the exponents occuring in the r.h.s. of (4.12) are $\frac{1}{2}$ of those occuring in (4.3); this will be crucial for us to guarantee a decay in terms of k, as $k \to \infty$. In fact, we can replace $\frac{1}{2}$ by any number $0 \leq \xi \leq 1$ by similarly adjusting the term with trace. This follows from the fact that $x \leq \min\{1, y\}$ with $x, y > 0$, implies $x \leq y^{\xi}$.

Recall from Proposition 4.3 that ${}_1\mathcal{F}_1^{(p)}$ is majorized by a function of the form $\min\{1, y\}$ with $y > 0$. The reason for the above estimate is that each of the majorants 1 and y for ${}_1\mathcal{F}_1^{(p)}$ does not serve our purpose.

Proposition 4.12. *For $m' \geq m + \frac{n}{2}, m \geq 2n-4$ and $m'+1 \leq \sigma \leq k-(m+1)-\frac{n}{2}$, we have*

$$B_{p,n;s} \ll_n \Gamma_p(k - \frac{n-p}{2})^{\frac{1}{2}}.$$

Proof. We start by taking absolute values of each term in (4.11):

$$|B_{p,n;s}| \ll_n \sum_{(U_1,U_3)',(a_1',c_1')} \frac{\det(V_U)^{-(m'+1)} |\det(S)|^{-(m+1)} |\Gamma_p(k - \frac{n-p}{2})|^{\frac{1}{2}}}{\left(1 + \mathrm{tr}(V_U^{\frac{1}{2}} S V_U^{\frac{1}{2}})^{-2}\right)^{\frac{m+1}{4}}},$$

where by Lemma 4.9, $(U_1, U_3)'$ vary over $\mathrm{Prim}_{p,n-p}$, and as seen before, (a_1', c_1') vary over $\mathrm{CS}(p,p)/\diamond$. Moreover V_U is as defined in (4.10).

Clubbing together the pairs (a_1', c_1') such that $a_1' c_1 = S$ and using Lemma 4.6, we see that the above summation can be replaced by S running over $\mathrm{Sym}_p(\mathbf{Z})$ with each such S contributing the quantity $A_p^{\mathrm{cs}}(S)$.

In this way we get by using Lemma 4.5, Lemma 4.10 that

$$\begin{aligned}
\frac{|B_{p,n;s}|}{\Gamma_p(k-\frac{n-p}{2})^{\frac12}} &\ll_n \sum_{(U_1,U_3)'}\sum_S \frac{A_n^{\mathrm{cs}}(S)\det(V_U)^{-(m'+1)}|\det(S)|^{-(m+1)}}{\left(1_p+\mathrm{tr}(V_U^{\frac12}SV_U^{\frac12})^{-2}\right)^{\frac{m+1}{4}}}\\
&\ll_n \sum_{(U_1,U_3)'}\sum_S \frac{|\det(S)|^{-(m+1)+(p-1)+\frac{m+1}{2p}+\varepsilon}\det(V_U)^{-(m'+1)+\frac{m+1}{2p}}}{\det\left(1_p+(V_U^{\frac12}SV_U^{\frac12})^2\right)^{(m+1)/4p}}\\
&\ll_n \sum_{(U_1,U_3)'}\sum_S \frac{|\det(S)|^{-(m+1)/2+(p-1)+\varepsilon}\det(V_U)^{-(m'+1)+\frac{m+1}{2p}}}{|\det(i1_p+V_U^{\frac12}SV_U^{\frac12})|^{(m+1)/2p}} \qquad (4.13)\\
&\ll_n \sum_{(U_1,U_3)'}\det(V_U)^{-(m'+1)}\mathscr{L}_p(\tfrac{m+1}{2p}, iV_U^{-1}),\\
&\ll_n \sum_{(U_1,U_3)'}\det(V_U)^{-(m'+1)+\frac{(m+1)(p+1)}{2p}-\frac{p+1}{2}}
\end{aligned}$$

where in (4.13), it we have to choose m such that

$$-(m+1)/2+(p-1)+\varepsilon<0 \quad \text{for all } p, \quad \text{i.e.} \quad m+1>2(n-2).$$

Moreover, in the last inequality, we have used the estimate from Lemma 4.11.

Next, we invoke Lemma 4.7 and replace the sum over the $(U_1,U_3)'$ by a sum over V, where $V=V_U=U_1'U_1+U_3'U_3$. We have to count the number of $(U_1,U_3)'$ such that $V_U=V$ modulo the respective equivalence relations. Lemma 4.7 says that, as $(U_1,U_3)'$ vary over equivalence classes of primitive right non-associated integral matrices, V_U varies over a subset of $\mathrm{Sym}_p(\mathbf{Z})^+/\mathrm{GL}_p(\mathbf{Z})$, with multiplicity $\mathcal{R}_{p,n-p}(V)$ ($\ll_n \det(V)^{p/2-1+\varepsilon}$). Therefore,

$$\begin{aligned}
\frac{|B_{p,n;s}|}{|\Gamma_p(k-\frac{n-p}{2})|^{\frac12}} &\ll_{n,\varepsilon} \sum_V \det(V)^{-(m'+1)+\frac{(m+1)(p+1)}{2p}-\frac{p+1}{2}+\frac{p}{2}-1+\varepsilon}\\
&\ll_{n,\varepsilon} \sum_V \det(V)^{-m'+m-\frac32+\varepsilon} \qquad (4.14)\\
&\ll_{n,\varepsilon} \sum_{d=1}^{\infty}\#\{V\mid \det(V)=d\}\cdot d^{-m'+m-\frac32+\varepsilon}\\
&\ll_{n,\varepsilon} 1,
\end{aligned}$$

provided $m' \geq m + \frac{n}{2} - 1 + \varepsilon$. Moreover in the last inequality above, we have used the fact that

$$\#\{V \in \mathrm{Sym}_p(\mathbf{Z})^+/\mathrm{GL}_p(\mathbf{Z}) \mid \det(V) = d\} \leq_\varepsilon d^{\frac{p-1}{2}+\varepsilon}. \qquad \square$$

Proof of Theorem 1.1. Suppose to the contrary that for each $F \in \mathcal{B}_k^n$, the product $D_F^*(s)a(F, 1_n) = 0$ for some $s \in \mathcal{S}_{n,k,\varepsilon}$. Thus the 1_n-th Fourier coefficient of the r.h.s. of (2.3) is zero, hence so is that of the l.h.s.. From Proposition 3.8, this means that

$$0 = C_n\Gamma_n(k-s) + C_n\Gamma_n(s) + \Sigma_s + \Sigma_{k-s}, \tag{4.15}$$

where we have set

$$\Sigma_s = \sum_{1\leq p\leq n} \frac{A_{n,k}(s) i^{\beta_p}(2\pi)^{h_{n,p}(k,s)}}{\Gamma_p(k-\frac{n-p}{2})\Gamma_{n-p}(k-s)} B_{p,n;s},$$

the quantity $h_{n,p}(k,s)$ and β_p are as in Proposition 3.8. Dividing both sides of (4.15) by $C_n\Gamma_n(k-s)$ we get

$$0 = 1 + \frac{\Gamma_n(s)}{\Gamma_n(k-s)} + \frac{\Sigma_s}{C_n\Gamma_n(k-s)} + \frac{\Sigma_{k-s}}{C_n\Gamma_n(k-s)}. \tag{4.16}$$

We first want to analyze Σ_s in (4.16) as a function of k. Therefore the relevant functions of k occuring in Σ_s can be estimated as follows. Let us define

$$G_{p,s}(k) := \frac{(2\pi)^{h_{n,p}(k,s)-nk}\Gamma_n(s)}{\Gamma_p(k-\frac{n-p}{2})^{\frac{1}{2}}\Gamma_{n-p}(k-s)}.$$

Using the expression for $\Gamma_g(s)$ $(g \geq 1)$ from (3.16), we have

$$G_{p,s}(k) \ll_n \left| \frac{(2\pi)^{h_{n,p}(k,s)-nk}\prod_{t=0}^{p-1}\Gamma(s-\frac{t}{2})\prod_{u=0}^{n-p-1}\Gamma(s-\frac{u+p}{2})}{\prod_{t=0}^{p-1}\Gamma(k-\frac{n-p+t}{2})^{\frac{1}{2}}\prod_{u=0}^{n-p-1}\Gamma(k-s-\frac{u}{2})} \right|.$$

In view of the functional equation of $D_F^*(s)$, it is enough to prove the theorem for s in the left strip in $\mathcal{S}_{n,k,\varepsilon}$. So, let us now put $s = \frac{k}{2} - \delta - it_0$ in (4.16), where $\varepsilon \leq \delta \leq \frac{n+1}{2}$. For a discussion on the choice of the region $\mathcal{S}_{n,k,\varepsilon}$, see Remark 4.13. In (4.9), Proposition 4.12, we will finally choose now

$$m := 2n-4, \quad m' := \frac{5n}{2} - 4, \quad k > 5n. \tag{4.17}$$

These choices of the parameters ensure the following:

- one has
$$m' + 1 < \frac{k}{2} - \frac{(n+1)}{4} \leq \sigma \leq \frac{k}{2} - \varepsilon < k - (m+1) - \frac{n}{2};$$
so that $\mathcal{S}_{n,k,\varepsilon}$ is contained in the strip as in (4.9),
- the conditions of Proposition 4.12 are fulfilled.

We now use the duplication formula

$$\Gamma(z)\Gamma(z+\frac{1}{2}) = 2^{1-2z}\pi^{\frac{1}{2}}\Gamma(2z)$$

for the terms $\Gamma(k-\frac{n-p+t}{2})$ with $z=\frac{k}{2}-\frac{n-p+t}{4}$ along with the fact that $\Gamma(x)$ is increasing for $x\geq 3$ to get

$$\Gamma\left(k-\frac{n-p+t}{2}\right)^{\frac{1}{2}} \gg 2^{\frac{k}{2}-\frac{n-p+t}{4}-1}\Gamma\left(\frac{k}{2}-\frac{n-p+t}{4}\right).$$

Also note that $\mathrm{Re}(h_{n,p}(k,s)) < nk$ for s in $\mathcal{S}_{n,k,\varepsilon}$. Using these and the estimate $|\Gamma(s)|\leq\Gamma(\sigma)$ $(\mathrm{Re}(s)>1)$, we obtain

$$G_{p,s}(k) \ll \prod_{t=0}^{p-1}\left|\frac{\Gamma(\frac{k}{2}-(\frac{t}{2}+\delta+it_0))}{\Gamma(\frac{k}{2}-\frac{n-p+t}{4})}\right| \prod_{u=0}^{n-p-1}\left|\frac{\Gamma(\frac{k}{2}-(\frac{p+u}{2}+\delta+it_0))}{\Gamma(\frac{k}{2}+(\delta-\frac{u}{2}-it_0))}\right|.$$

In the above inequality, we have ignored all the terms which are either independent of k or are absolutely bounded; so the implied constant depends only on n and ε.

Recall that if $\min\{\mathrm{Re}(z),\mathrm{Re}(z+a),\mathrm{Re}(z+b)\}\geq\varepsilon$ for some $a,b\in\mathbf{C}$ and $\varepsilon>0$, then (see e.g., [1, 6.1.47])

$$\left|\frac{\Gamma(z+a)}{\Gamma(z+b)}\right| \ll_{\varepsilon,|a|,|b|} |z|^{\mathrm{Re}(a-b)}, \quad \text{as } |z|\to\infty.$$

Hence we finally have (keeping in mind that $1\leq p\leq n-1$)

$$\begin{aligned} G_{p,s}(k) &\ll_{n,t_0} \prod_{t=0}^{p-1}(\tfrac{k}{2})^{-\delta+\frac{n-p-t}{4}} \prod_{u=0}^{n-p-1}(\tfrac{k}{2})^{-2\delta-\frac{p}{2}} \\ &\ll_{n,t_0} (\tfrac{k}{2})^{-p\delta+\frac{p(n-p)}{4}-\frac{p(p-1)}{8}-2(n-p)\delta-\frac{p(n-p)}{2}} \\ &\ll_{n,t_0} (\tfrac{k}{2})^{-(2n-p)\delta-\frac{p(n-p)}{4}-\frac{p(p-1)}{8}} \\ &\ll_{n,t_0} (\tfrac{k}{2})^{-(2n-p)\delta-\frac{n-1}{4}}. \end{aligned}$$

Therefore if $\delta\geq-\frac{1}{12}+\varepsilon$, with $\varepsilon>0$, we will have a decay in terms of k as $k\to\infty$. This finishes the case of Σ_s.

For Σ_{k-s} one argues similarly. The ratio of gamma factors in this case becomes

$$H_{p,s}(k) := \frac{e^{\frac{\pi in(2s-k)}{2}}(2\pi)^{h_{n,p}(k,k-s)-nk}\Gamma_n(s)}{\Gamma_p(k-\frac{n-p}{2})^{\frac{1}{2}}\Gamma_{n-p}(s)},$$

and one finds after a computation that

$$H_{p,s}(k) \ll_{n,t_0} (\tfrac{k}{2})^{-p\delta-\frac{p(n-p)}{4}-\frac{p(p-1)}{8}}.$$

Therefore, we have a decay in terms of k if $\delta\geq-\frac{1}{4}+\varepsilon$.

In the remaining case, one has the estimate

$$\left|\frac{\gamma_n(s)}{\gamma_n(k-s)}\right| \ll_{t_0} \prod_{l=0}^{n-1} \left|\frac{\Gamma(\frac{k}{2}-\delta-it_0)}{\Gamma(\frac{k}{2}+\delta+it_0)}\right| \ll_{\delta,t_0} (\tfrac{k}{2})^{-2n\delta}, \tag{4.18}$$

which decays with k if $\delta \geq \varepsilon$.

Therefore comparing all the three ranges of δ together and letting $k \to \infty$, we get the assertion of Theorem 1.1. □

Remark 4.13. An inspection of the proof of Theorem 1.1 will show that we could have chosen δ to be anything in between ε and c_n (δ has to be independent of k, so that the estimates involving the ratio of gamma factors is valid), with $c_n < \frac{k}{2} - (m'+1)$ depending only on n. The strip in the statement of Theorem 1.1 would have been $\frac{k}{2} - c_n \leq \sigma \leq \frac{k}{2} - \varepsilon$.

However, in some analogy with the degree 1, we might say (on analytic grounds) that the strip defined by:

$$\frac{k}{2} - \frac{n+1}{4} \leq \sigma \leq \frac{k}{2} + \frac{n+1}{4}$$

around the point $s = \frac{k}{2}$, to be a possible substitute for the 'critical strip' (in the sense used before in Theorem A, Introduction) when the degree is 1, for the following reason. First of all this strip is invariant under $s \mapsto k - s$. Also note that under the *Resnikoff-Saldaña conjecture* on the sizes of Fourier coefficients of a cusp form in S^n_k, one finds that $D_F(s)$ converges absolutely for $\sigma > \frac{k}{2} + \frac{n+1}{4} + \varepsilon$.

Moreover, a recent result of W. Kohnen [19] says that under the *Resnikoff-Saldaña conjecture*, the abscissa of absolute convergence of $D_F(s)$ is exactly $\sigma_a = \frac{k}{2} + \frac{n+1}{4}$. This is consistent with the picture when $n = 1$ and is the reason for our consideration of the 'critical strip', and also the choice $c_n = \frac{n+1}{4}$.

5. Proof of Theorem 1.2

In this section, we first show in Proposition 5.4 that for suitably large k, there exists a cuspidal eigenform G in $S^\flat_k$ such that $D_G(s)$ does not vanish identically. Incidentally, we note that such a statement was not known before for $n = 2$ and certainly is not known in general for higher degrees. This will be proved using a result in [20] recalled below. Theorem 1.2 essentially follows from Proposition 5.4 and some standard analytic arguments.

We start by recalling the result in [20] and show how it implies the above claim. The result involves the (normalised) spinor zeta function $Z_F(s)$ attached to an eigenform $F \in S^2_k$. Here, we just recall that Z_F has the Euler product expansion

$\mathrm{Re}(s) > 1$ given by

$$Z_F(s) = \prod_p Z_{p,F}(s); \quad Z_{p,F}(s) = (1 - \alpha_p p^{-s})(1 - \beta_p p^{-s})(1 - \alpha_p^{-1} p^{-s})(1 - \beta_p^{-1} p^{-s}),$$

where $\alpha_p^{\pm 1}, \beta_p^{\pm 1}$ are the Satake p-parameters of F. They all are of absolute value 1, which follows from the proof of the Ramanujan conjecture for F by R. Weissauer [27]. This readily shows that $Z_F(s)$ is non-zero for $\mathrm{Re}(s) > 1$.

Let us define the weights ω_k^F, which will be of the main interest to us. Namely,

$$\omega_k^F = \sqrt{\pi}(4\pi)^{3-2k}\Gamma(k - \tfrac{3}{2})\Gamma(k-2)\frac{|a(F, 1_2)|^2}{4\langle F, F\rangle}. \tag{5.1}$$

The theorem that we wish to refer to is the following (cf. [20, Thm 1.1]).

Theorem 5.1. *Then, for any $s \in \mathbf{C}$ such that $\mathrm{Re}(s) > 1$ and k even, we have*

$$\lim_{k \to +\infty} \sum_{F \in S_k^\natural} \omega_k^F Z_F(s) = \zeta(s + \tfrac{1}{2}) L(\chi_4, s + \tfrac{1}{2}), \tag{5.2}$$

where $\zeta(s)$ denotes the Riemann zeta function and $L(\chi_4, s)$ is the L-function associated to the unique Dirichlet character of conductor 4.

Corollary 5.2. *There exists an absolute constant $k_0 > 20$ such that for all $k \geq k_0$, one has $a(F, 1_2) \neq 0$ for some eigenform $F \in S_k^\natural$.*

Proof. In view of the above theorem, it suffices to note that for s real and greater than 1, the factor $\zeta(s + \frac{1}{2})L(\chi_4, s + \frac{1}{2})$ is real and non-zero. Hence $\omega_k^F L(F, s) \neq 0$ for all k large enough. In particular, we demand $k \geq 20$, otherwise $S_k^\natural = \{0\}$, and this gives the proposition. □

The following corollary is immediate.

Corollary 5.3. *There exists an absolute constant $k_0 > 20$ such that for all $k \geq k_0$, the first Fourier-Jacobi coefficient ϕ_1 is non-zero for some eigenform $F \in S_k^\natural$.*

Corollary 5.2 implies the existence of eigenforms in $S_k^\natural$ whose Koecher-Maass series does not vanish identically.

Proposition 5.4. *There exists an absolute constant $k_0 \geq 1$ such that for all $k \geq 20k_0$, one has $D_F(s) \not\equiv 0$ for some eigenform $F \in S_k^\natural$.*

Proof. In the region $\mathrm{Re}(s) > \frac{k+3}{2}$, where $D_F(s)$ can be represented by an absolutely convergent Dirichlet series; let us look at the Dirichlet coefficient corresponding to $\det(T) = 1$. This amounts to considering equivalence classes of integral quadratic forms with discriminant -4 under the action of $\mathrm{GL}_2(\mathbf{Z})$.

There is only one class, represented by 1_2 and hence the Dirichlet coefficient is upto a non-zero constant, nothing but $a(F, 1_2)$. Therefore the proposition now follows from Corollary 5.2. □

Before stating the next proposition, we briefly recall the notion of the order of an entire function g. Namely if g has the property that

$$|g(s)| = O(e^{|s|^a}), \quad \text{as } s \to \infty, \tag{5.3}$$

for some $a \geq 0$, then the order of g is defined to be the infimum of all those a for which (5.3) holds.

The next result has nothing to do with the special case of $n = 2$, and so is stated for $n \geq 1$ arbitrary.

Proposition 5.5. *Let $F \in S_k^n$. Then the completed Koecher-Maass series $D_F^*(s)$ is an entire function of order at most* 1.

Proof. In view of the functional equation $D_F^*(s) = (-1)^{\frac{nk}{2}} D_F^*(k-s)$, we can assume without loss of generality that $\mathrm{Re}(s) \geq \frac{k}{2}$. We start from the integral representation for $D_F^*(s)$ due to T. Arakawa, as given in [15]. Namely

$$D_F^*(s) = \int\limits_{Y \text{reduced}, \det(Y) \geq 1} (\det(Y)^s + (-1)^{\frac{nk}{2}} \det(Y)^{k-s}) F(iY) \det(Y)^{-\frac{n(n+1)}{2}} dY.$$

Using this, we wish to find an estimate for $D_F^*(s)$. To do that, we look at the proof of [15, Prop. 1, p. 147]. We then have, for some constant $c_1 > 0$ depending only on n, that

$$\begin{aligned} D_F^*(s) &\ll_F \int_1^\infty (t^{n(\mathrm{Re}(s)-k-\frac{n(n+1)}{2})-1} + t^{n(-\mathrm{Re}(s)-\frac{n(n+1)}{2})-1}) e^{-c_1 t} dt \\ &\ll_F \int_1^\infty t^{n(|s|-k-\frac{n(n+1)}{2})-1} e^{-c_1 t} dt. \end{aligned}$$

Now let us assume that $|s|$ is large, and in particular that $|s| > k + \frac{n(n+1)}{2}$. Then

$$\begin{aligned} D_F^*(s) &\ll_F c_1^{n(|s|-k-\frac{n(n+1)}{2})-2} \Gamma(n(|s|-k-\tfrac{n(n+1)}{2})) \\ &\ll_F (n|s|)^{c_2 n|s|} \ll_{F,\varepsilon} e^{|s|^{1+\varepsilon}}. \end{aligned}$$

This completes the proof of the proposition. □

We would like to use Proposition 5.5 is a standard way to give an upper bound of the number of zeros of $D_F^*(s)$ in the closed ball of radius R. Let us define $N_R(F)$ to be the set of those zeros. Then (see e.g., [22, Exercise 6.1.5]),

Proposition 5.6. $\#N_R(F) \ll_{F,\varepsilon} R^{1+\varepsilon}$ *for any* $\varepsilon > 0$.

We now come back to the proof of Theorem 1.2, and set $F_{k,s} := F^2_{k,s}$.

Proof of Theorem 1.2. To prove the theorem, by Proposition 5.4 we first get hold of an eigenform $F_k \in S^\flat_k$ whose Koecher-Maass series does not vanish identically. Then the assertion of Theorem 1.2 with $F = F_k$ follows directly from Proposition 5.5 and Proposition 5.6. $\square$

Corollary 5.7. *Let $k \geq k_0$, with k_0 as in Proposition 5.4. Then with the exception of at most $O_k(R^{1+\varepsilon})$ points $s \in B_R$, one has $F_{k,s} \notin S^*_k$.*

Proof. First of all, we note that for any $n \geq 1$, $F^n_{k,s}$ can be analytically continued to the whole complex plane by defining it to be the r.h.s. in the equality in (2.3). This possible since (2.3) already holds in a non-empty open set and the $D^*_F(s)$ apriori have analytic continuation to all of $\mathbf{C}$ by the work of T. Arakawa as discussed in [15].

With this analytic continuation, clearly (2.2) holds for all $s \in \mathbf{C}$, since $F^n_{k,s}$ is by definition characterized to be the unique function satisfying (2.2). Therefore for any set $K \subset \mathbf{C}$, we have

$$\mathbf{S}_K := \{s \in K \mid F_{k,s} \in S^*_k\} = \bigcap_{F \in S^\flat_k} \{s \in K \mid D^*_F(s) = 0\}. \tag{5.4}$$

Therefore by (5.4) and Proposition 5.4, we see that $\mathbf{S}_{B_R}$ is contained in the zero set $N_R(D^*_F)$ for $F \in S^\flat_k$ such that $D^*_F \not\equiv 0$. Note that by (5.4), the implied constant in $O(R^{1+\varepsilon})$ depends only on k, ε. Hence the result. $\square$

Question. It is a pertinent question to describe the set of points s in $\mathbf{C}$, for which $F_{k,s} \in S^*_k$. Perhaps this set is always empty!

Let $F \in S_k$ be such that $D^*_F \not\equiv 0$. By the general theory of Dirichlet series (see eg. [3]), we know that there exists a real number $r \gg 1$ (depending on F) such that $D^*_F(s) \neq 0$ for all s such that $\mathrm{Re}(s) > r$. Therefore in view of the functional equation of D^*_F, the set $\mathbf{S}_{\mathbf{C}}$ is at most contained in a finite strip.

Remark 5.8. Perhaps it is useful to note that there could be a possibility of proving an analogue of Theorem 1.1 for elements in the space $S^\flat_k$ in a more direct way, by using the Fourier expansion of $F_{k,s}$ and by filtering out the Saito-Kurokawa lifts from the r.h.s of (2.3) by applying the Hecke operator (see [23])

$$\mathscr{T}(p) := T(p)^2 - (p^{k-1} + p^{k-2})T(p) + p^{2k-2} Id. - T(p^2)$$

on both sides of it. It is known that $\mathscr{T}(p)$ annihilates S^*_k. Here p is a prime and $T(p^r)$ is the usual similitude Hecke operator. Of course, we then have to prove that $\mathscr{T}(p)F_{k,s} \neq 0$ for suitable s.

This would allow us one the one hand to prove a point-wise nonvanishing result for $S^\flat_k$ at *any* point in $\mathcal{S}_{n,k,\varepsilon}$ as in Theorem 1.1. On the other hand, this would give an alternative proof of the existence of $F \in S_k^\flat$ such that $D_F(s) \not\equiv 0$. We hope that such a method will eventually be fruitful, even though nothing seems to follow immediately.

Incidentally, we note here that none of our main theorems, viz., Theorem 1.1 and Theorem 1.2 imply each other when $n = 2$.

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Department of Mathematics, Indian Institute of Science, Bangalore – 560012, India.

E-mail address: somu@math.iisc.ernet.in, soumya.u2k@gmail.com

Mathematisches Institut, Ruprecht-Karls-Universitat Heidelberg, D – 69120 Heidelberg, Germany.

E-mail address: winfried@mathi.uni-heidelberg.de

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai – 400005, India.

E-mail address: sengupta@math.tifr.res.in