A short remark on consecutive coincidences of a certain multiplicative function

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1. Introduction

If $f : \mathbf{N} \to \mathbf{N}$ is a multiplicative number-theoretic function taking positive integral values, such as Euler's totient function, the divisor function or functions closely related, a quite intriguing problem is to study those n for which $f(n) = f(n+1)$ or more generally $f(n) = f(n+k)$ where $k \in \mathbf{N}$ is fixed. If e.g. $f(n) = \phi(n)$ is Euler's totient function, there has been quite a number of interesting results on this subject, cf. [4] and the literature given there. Though in general it seems quite difficult to find explicit solutions of $\phi(n) = \phi(n+k)$, one conjectures that there are always infinitely many [5]. So far this conjecture has not been proved in a single case. Similar assertions concern the divisor function $\sigma(n)$ and in this connection analogous questions about perfect numbers or amicable numbers, cf. e.g. [3].

On the other hand, one may ask for simple and explicit examples of mutiplicative functions $f : \mathbf{N} \to \mathbf{N}$, closely related to the classical ones, such that there are only finitely many n with $f(n) = f(n+k)$. In this short note we will construct such a function in a simple way. In fact, using Jordan's totient functions (which are simple generalizations of Euler's phi-function , the definition will be recalled in sect. 2), we will define a multiplicative function $A : \mathbf{N} \to \mathbf{N}$ in a similar way as the classical ones are defined, however with the main differences that the Euler factor at the prime 3 has been slightly modified and also the product of the Euler p-factors for all primes p converges to a non-zero value. We shall prove that for each given odd $k \in \mathbf{N}$, there are only finitely many n with $A(n) = A(n+k)$. Once the proper definition of A has been found, the proof is quite simple and only relies on the unequal parity of n and $n + k$ and some elementary estimates.

2. Statement of result

We define $A : \mathbf{N} \to \mathbf{N}$ by

$$A(n) := n^2 \prod_{p|n, p\neq 3} (1 + \frac{1}{p^2}). \tag{1}$$

Then A clearly is multiplicative.

Recall the definition of the ℓ-th Jordan totient function

$$J_\ell(n) := n^\ell \prod_{p|n}(1-\frac{1}{p^\ell}) \quad (n \in \mathbf{N})$$

[2, p. 46]. As is easy to see $J_\ell(n)$ counts the number of ℓ-tuples of positive integers all less or equal to n that form a coprime $(\ell+1)$-tuple together with n. Clearly $J_1(n) = \phi(n)$.

Regarding (1) we note that

$$n^2 \prod_{p|n}(1+\frac{1}{p^2}) = \frac{J_4(n)}{J_2(n)}.$$

We remind the reader that $J_\ell(n)$ is a useful and interesting number-theoretic function which for example (among other things) is demonstrated by the classical identiy

$$\#Sp_m(\mathbf{Z}/n\mathbf{Z}) = n^{m^2} \prod_{\ell=1}^{m} J_{2\ell}(n)$$

(cf. [1]). Here as usual $Sp_m \subset GL_{2m}$ denote the symplectic group of degree $2m$.

Theorem. *Let k be a fixed odd positive integer. Then the equation $A(n) = A(n+k)$ has only finitely many solutions $n \in \mathbf{N}$.*

Remark. It would be interesting to investigate solutions of the equations $J_\ell(n) = J_\ell(n+k)$, for fixed ℓ and k, in a similar way as was done in the case $\ell = 1$ for the Euler phi-function.

3. Proof of Theorem

We rewrite the equality $A(n) = A(n+k)$ as

$$\frac{A(n)}{n^2} = (1+\frac{k}{n})^2 \frac{A(n+k)}{(n+k)^2}. \tag{2}$$

Let us first suppose that n is even. Then $n+k$ is odd and from (1) and (2) we obtain

$$(1+\frac{1}{2^2}) \prod_{p|n, p\neq 3, p\ odd} (1+\frac{1}{p^2}) = (1+\frac{k}{n})^2 \prod_{p|n+k, p\neq 3, p\ odd} (1+\frac{1}{p^2}). \tag{3}$$

The left-hand side in (3) is bounded from below by $(1+\frac{1}{2^2})$. Hence we find that

$$1+\frac{1}{2^2} < (1+\frac{k}{n})^2 \prod_{p|n+k, p\neq 3, p\ odd} (1+\frac{1}{p^2})$$

$$< (1+\frac{k}{n})^2 \prod_{p\ odd, p\neq 3} (1+\frac{1}{p^2})$$

and hence

$$(1+\frac{1}{2^2})^2 < (1+\frac{k}{n})^2 \prod_{p\neq 3}(1+\frac{1}{p^2})$$

$$= (1+\frac{k}{n})^2 \cdot \frac{1}{1+\frac{1}{3^2}} \prod_p (1+\frac{1}{p^2}).$$

We have

$$\prod_p (1+\frac{1}{p^2}) = \frac{\prod_p (1-\frac{1}{p^4})}{\prod_p (1-\frac{1}{p^2})}$$

$$= \frac{\zeta(2)}{\zeta(4)}.$$

Inserting the values

$$\zeta(2) = \frac{\pi^2}{6}, \zeta(4) = \frac{\pi^4}{90}$$

we therefore finally get

$$c < (1+\frac{k}{n})^2 \tag{4}$$

with

$$c := (1+\frac{1}{2^2})^2 (1+\frac{1}{3^2}) \cdot \frac{\pi^2}{15}$$

$$= \frac{\pi^2}{8.64}.$$

Since $c > 1$ we get a contradiction from (4) when n is large.

If n is odd, then $n+k$ is even, and we can proceed as before with the roles of n and $n+k$ interchanged. This proves our assertion.

References

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