PHANTOM HOLOMORPHIC PROJECTIONS ARISING FROM STURM'S FORMULA

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ABSTRACT. We show the analytic continuation of certain Siegel Poincaré series to their critical point for weight three in genus two. We proof that this continuation posesses a nonhomomorphic part and describe it. We show that Sturm's operator also produces a nonhomorphic share for weight three, we call it a phantom term. Weight three is the distinguished weight for genus two where this phenomenon arises.

CONTENTS

1. INTRODUCTION

For hermitian symmetric domains $\mathcal{H} = G/K$ the complex structure on $\mathcal{H}$ corresponds to a decomposition $Lie(G) \otimes_{\mathbb{R}} \mathbb{C}$ into subspaces $Lie(K) \oplus \mathfrak{p}_+ \oplus \mathfrak{p}_-$. The maximal compact subgroup K of G acts on the subspaces $\mathfrak{p}_\pm$, corresponding to the complexified holomorphic resp. antiholomorphic tangent space of $\mathcal{H}$ at the origin. For arithmetic subgroups Γ of the group G the Hilbert space $L^2(\Gamma\backslash G, dg)$ decomposes into the discrete spectrum $L^2_{\text{dis}}(\Gamma\backslash G, dg)$ and a continous spectrum. Part of this discrete spectrum is obtained as follows. By assumption the group G is of hermitian type, hence there exist discrete series representations G. Among these are the representations π of the holomorphic discrete series. The isotypical subspace $L^2(\Gamma\backslash G, dg)_\pi$ of $L^2_{\text{dis}}(\Gamma\backslash G, dg)$ on which G acts by one of these holomorphic discrete series representations π is isomorphic to a finite direct sum of π with multiplicity say $m(\pi)$. It is well known that these subspaces $L^2(\Gamma\backslash G, dg)_\pi$ occur in the cuspidal part of the spectrum. The projection operator

$$P_\pi : L^2(\Gamma\backslash G, dg) \longrightarrow L^2(\Gamma\backslash G, dg)_\pi$$

can be studied by various techniques. Since a holomorphic discrete series representation contains a unique lowest K-type τ, it often suffices to study the projection operator P_π on the τ-isotypic subspace $L^2(\Gamma\backslash G, dg)_\tau$ of the action of K on $L^2(\Gamma\backslash G, dg)$. Classically, the analysis of the projection operator P_π is then often achieved by passing from functions f in $L^2(\Gamma\backslash G, dg)_\tau$ to functions h on $\mathcal{H}$, defined by $h(gK) = J_\tau(g)f(g)$ using an explicit cocycle factor J_ρ whose definition involves the lowest K-type of π. The functions h on $\mathcal{H}$ so defined transform under the action of $\gamma \in \Gamma$ with respect to the modular transformation property $h(\gamma z) = J_\tau(\gamma, z)h(z)$. The functions f in the subspace $L^2(\Gamma\backslash G, dg)_\pi$ correspond to the holomorphic functions on $\mathcal{H}$ with this transformation property. Putting L^2-conditions aside (for simplicity), the projectors P_π thus become holomorphic projectors. Classically, they were studied in terms of Fourier expansions using the theory of Poincare series in [13], [4] for the classical case $G = \mathrm{SL}_2$ and in [10] for the case of the symplectic groups $G = \mathrm{Sp}_g$.

For the case $G = \mathrm{SL}_2$ this becomes more concrete as follows. The holomorphic discrete series $\pi = \pi_k$ of SL_2 is parametrized by the integers $k \geq 2$, the weight of their lowest K-type, and it is well known that $m(\pi_k)$ can be identified with the dimension of the space of cuspidal modular forms $[\Gamma, k]_0$ of weight k with respect to the arithmetic group Γ on the complex upper half plane $\mathcal{H}$. Suppose Γ contains translations, so that the modular transformation property $h(\gamma z) = j_k(\gamma, z)h(z)$ implies $h(z) = h(z+n)$ for some integer n that allows to expand $h(z)$ into a Fourier expansion

$$h(z) \;=\; \sum_t \; a(t,y)\exp(2\pi i t z) \;=\; \sum_t \; b(t,y)\exp(2\pi i t x)\;.$$

For $z = x + iy$ the Fourier coefficients $b(t, y)$ are functions of the imaginary part. In this special context it has been shown by Sturm [13] for $k > 2$ and Gross-Zagier [4] for $k = 2$ that, up to some explicit normalizing constant $c(k)$ depending only on k, the projector P_π for $\pi = \pi_k$ from above corresponds to the following holomorphic projector defined on the level of Fourier coefficients by

$$b(t,y) \quad \mapsto \quad b_{\mathrm{hol}}(t) \;=\; c(k)\cdot \int_0^\infty b(t,y)\exp(-2\pi t y)(ty)^k \frac{dy}{y}\;.$$

So the holomorphic projection of $h(z)$ is given by the holomorphic modular form $\sum_t b_{\mathrm{hol}}(t)\exp(2\pi i t z)$.

Concerning the higher dimensional cases studied in [10] and [8], it turned out that for the holomorphic discrete series of scalar weight high enough again the holomorphic projectors P_π are given by analogous holomorphic projections on the level of Fourier coefficients generalizing the Sturm projections from above. One could therefore believe, that this holds quite generally for all holomorphic discrete series representations of the symplectic group Sp_g. However, in the higher rank case new phenomena seem to occur. Although the Sturm projection operators are defined, in general they do not coincide with the corresponding projector operators in all cases. This may be the general situation for those holomorphic discrete series whose lowest K-type is small. For the special case $g = 2$ we analyze this in detail. The holomorphic discrete series $\pi = \pi_k$ of the group Sp_2 are indexed by their lowest K-types $k = (k_1, k_2)$ that are given by integers $k_1 \geq k_2 \geq 3$. The 'smallest' case is $k_1 = k_2 = 3$, the case of scalar weight 3. We show that in this case, as opposed to the cases of weight greater 3, the Sturm projection does not describe the projection operator P_π for the holomorphic discrete series of scalar weight 3. Indeed, it will realize this projection P_π only up to some additional phantom projection Q_π to a a space of phantom components

$$L^2(\Gamma\backslash G, dg)_{ph(\pi)} \subset L^2(\Gamma\backslash G, dg)$$

defined by a certain representations $ph(\pi)$ associated to π. In our particular case, this representation $ph(\pi)$ is an irreducible unitary representation of the group G not belonging to the discrete series. Expressed in terms of modular forms, the correction terms that are needed for Sturm's projection formula to hold in the case of scalar weight 3 correspond to derivatives of holomorphic modular forms of scalar weight one.

2. Notation and resolvents

We follow the notation of [8]. Let $G = \mathrm{Sp}_m(\mathbb{R})$ be the symplectic group of genus m. Apart from section 5.1 we restrict to the case $m = 2$. Realize G as the group of those $g \in M_{m,m}(\mathbb{R})$ satisfying $g'Wg = W$ for

$$W = \begin{pmatrix} 0 & -E_m \\ E_m & 0 \end{pmatrix}.$$

We have the usual action of G on the Siegel halfspace $\mathcal{H}$, for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$,

$$g\cdot z = (az+b)(cz+d)^{-1}\,.$$

The stabilizer K of $i = iE_m \in \mathcal{H}$ is a maximal compact subgroup of G. It is isomorphic to the unitary group $U(m)$. We denote by

$$g \;\mapsto\; g\cdot i =: z = x + iy$$

the obvious isomorphism of G/K to $\mathcal{H}$. Let $\mathfrak{g}$ be the Lie algebra of G realized as $\mathfrak{g}_{\mathbb{C}} \subset M_{2m,2m}(\mathbb{C})$ consisting of those g satisfying $g'W + Wg = 0$. Then $\mathfrak{g}_{\mathbb{C}} = \mathfrak{p}_+ \oplus \mathfrak{p}_- \oplus \mathfrak{k}_{\mathbb{C}}$, where $\mathfrak{k}_{\mathbb{C}}$ is the Lie algebra of K given by the matrices satisfying

$$\begin{pmatrix} A & S \\ -S & A \end{pmatrix}, \quad A' = -A\,, \quad S' = S\,,$$

and

$$\mathfrak{p}_{\pm} = \left\{ \begin{pmatrix} X & \pm iX \\ \pm iX & -X \end{pmatrix}, \quad X' = X \right\}.$$

Let $e_{kl} \in M_{m,m}(\mathbb{C})$ be the elementary matrix having entries $(e_{kl})_{ij} = \delta_{ik}\delta_{jl}$ and let $X^{(kl)} = \frac{1}{2}(e_{kl} + e_{lk})$. The elements $(E_{\pm})_{kl} = (E_{\pm})_{lk}$ of $\mathfrak{p}_{\pm}$ are defined to be those corresponding to $X = X^{(kl)}$, $1 \leq k,l \leq m$. Then $(E_{\pm})_{kl}$, $1 \leq k \leq l \leq m$ form a basis of $\mathfrak{p}^{\pm}$. A basis of $\mathfrak{k}_{\mathbb{C}}$ is given by B_{kl}, for $1 \leq k,l \leq m$, where B_{kl} corresponds to $A_{kl} = \frac{1}{2}(e_{kl} - e_{lk})$ and $S_{kl} = \frac{i}{2}(e_{kl} + e_{lk})$. For abbreviation, let $E_{\pm}$ be the matrix having entries $(E_{\pm})_{kl}$. Similarly, let $B = (B_{kl})_{kl}$ be the matrix with entries B_{kl} and let B^* be its transpose having entries $B^*_{kl} = B_{lk}$. Thus, E_+, E_-, B and B^* are matrix valued matrices. Taking formal traces of them and their formal products, e.g. $\mathrm{tr}(E_+E_-)$, such traces are not invariant under cyclic permutations of their arguments. The center $\mathfrak{z}_{\mathbb{C}}$ of the universal enveloping Lie algebra $\mathfrak{U}(\mathfrak{g}_{\mathbb{C}})$ is generated by m elements. The Casimir elements C_1, C_2 belong to $\mathfrak{z}_{\mathbb{C}}$,

$$C_1 = \frac{1}{2}(\mathrm{tr}(E_+E_-) + \mathrm{tr}(E_-E_+)) + \mathrm{tr}(BB)\,,$$

$$\begin{aligned} C_2 &= \frac{1}{2}\big(\mathrm{tr}(E_+E_-E_+E_-) + \mathrm{tr}(E_-E_+E_-E_+) + \mathrm{tr}(B^4) + \mathrm{tr}((B^*)^4)\big) \\ &\quad +2\big(\mathrm{tr}(E_+E_-BB) + \mathrm{tr}(E_-E_+B^*B^*)\big) \\ &\quad -\sum_{i,j,k,l}\{(E_+)_{kl},(E_-)_{ij}\}B_{jk}B_{il} \\ &\quad +\frac{(m+1)^2}{2}(\mathrm{tr}(E_+E_-) + \mathrm{tr}(E_-E_+))\,. \end{aligned}$$

For a smooth representation π of G the actions of the Casimir elements restricted to scalar K-types $(\kappa,\dots,\kappa)$ are given by

$$\pi(C_1) \;=\; \pi(\mathrm{tr}(E_+E_-)) - \kappa m(m+1-\kappa)$$

and

$$\begin{aligned} \pi(C_2) &= \pi(\mathrm{tr}(E_+E_-E_+E_-)) + m\kappa^4 \\ &\quad +((m+1)^2 - 2\kappa(m+1) + 2\kappa^2)\big(\pi(\mathrm{tr}(E_+E_-)) - \kappa m(m+1)\big)\,. \end{aligned}$$

For genus $m = 2$ let

$$\mathfrak{h}_{\mathbb{C}} \;=\; \mathbb{C}B_{11} + \mathbb{C}B_{22} \;\subset\; \mathfrak{k}_{\mathbb{C}}$$

be a common Cartan subalgebra for $\mathfrak{k}_{\mathbb{C}}$ and $\mathfrak{g}_{\mathbb{C}}$. Let Δ^+ be the set of positive roots for $\mathfrak{h}_{\mathbb{C}}$ such that their root spaces belong to $\mathbb{C}B_{12} + \mathfrak{p}^-$. Writing $\Lambda = (\Lambda_1,\Lambda_2)$ for $\Lambda \in \mathfrak{h}_{\mathbb{C}}^*$, where $\Lambda_j = \Lambda(B_{jj})$, these root spaces are

$$\begin{aligned} \mathfrak{g}_{(1,-1)} = \mathbb{C}B_{12}, \quad \mathfrak{g}_{(2,0)} = \mathbb{C}(E_-)_{11}\,, \\ \mathfrak{g}_{(1,1)} = \mathbb{C}(E_-)_{12}, \quad \mathfrak{g}_{(0,2)} = \mathbb{C}(E_-)_{22}\,. \end{aligned}$$

Half the sum of positive root is

$$\delta \;=\; \delta_G \;=\; \frac{1}{2}\sum_{\Lambda\in\Delta^+}\Lambda = (2,1)\,,$$

while $\delta_K = \frac{1}{2}(1,-1)$ is half the sum of positive compact roots. Applying the linear form Λ to the images of C_1 and C_2 under the Harish-Chandra homomorphism we get

$$\begin{aligned} \Lambda(C_1) \;=\; \Lambda(\gamma(C_1)) &= \Lambda_1^2 + \Lambda_2^2 - 5\,, \\ \Lambda(C_2) \;=\; \Lambda(\gamma(C_2)) &= \Lambda_1^4 + \Lambda_2^4 - 17 + 3\Lambda(C_1)\,. \end{aligned}$$

The diagonal subalgebra $\mathfrak{a}_{\mathbb{C}}$ is another Cartan subalgebra, and by choosing Euclidean coordinates $\Lambda = (\Lambda_1,\Lambda_2)$ there we get an isometric isomorphism to $\mathfrak{h}_{\mathbb{C}}$. So the above formulas retain valid. Choosing the system of positive roots correspondingly,

$$\Delta^+ := \{\alpha_1 = (0,2), \alpha_2 = (1,-1), \alpha_1+\alpha_2, \alpha_1+2\alpha_2\} \subset \mathfrak{a}_{\mathbb{C}}^*\,,$$

$\mathfrak{a}$ is the split component of the Borel subgroup

$$B \;=\; \left\{\begin{pmatrix} T & X \\ 0 & T'^{-1}\end{pmatrix} \mid T \text{ upper triangular}\right\} \subset G\,.$$

The Weyl group W of G acts on $\mathfrak{a}_{\mathbb{C}}^*$. It is generated by the simple reflections s_{α_1} and s_{α_2}. We also define $\mathfrak{a}_1 = \ker(\alpha_1)$ to be the split component of the

Klingen parabolic $P_1 \supset B$, and $\mathfrak{a}_2 = \ker(\alpha_2)$ to be the split component of the Siegel parabolic $P_2 \supset B$.
Let u and v be complex variables. In [8, sec. 3] there are chosen elements

$$\begin{aligned} D_+(u,\Lambda) &= \prod_{\alpha\in\Delta \text{ long}} \big(\check{\alpha}(\Lambda) - u\big)\,, \\ D_-(v,\Lambda) &= \prod_{\alpha\in\Delta \text{ short}} \big(\check{\alpha}(\Lambda) - v\big)\,, \end{aligned}$$

or equivalently,

$$\begin{aligned} D_+(u,\Lambda) &= (\Lambda_1^2 - u^2)(\Lambda_2^2 - u^2)\,, \\ D_-(v,\Lambda) &= \big((\Lambda_1+\Lambda_2)^2 - v^2\big)\big((\Lambda_1-\Lambda_2)^2 - v^2\big)\,. \end{aligned}$$

So they are the images of the Casimir elements

$$\begin{aligned} D_+(u) &:= \frac{1}{2}\big(C_1^2 - C_2 + 11C_1 - 2(u^2-1)C_1 + 2(u^2-1)(u^2-4)\big)\,, \\ D_-(v) &:= 2C_2 - C_1^2 - 34C_1 - 2(v^2-9)C_1 + (v^2-9)(v^2-1) \end{aligned}$$

under the Harish-Chandra homomorphism. Let Γ be any subgroup of finite index in the full modular group $\mathrm{Sp}_2(\mathbb{Z})$ containing the group

$$\Gamma_\infty = \{\begin{pmatrix} \pm E_2 & * \\ 0 & \pm E_2 \end{pmatrix} \in \mathrm{Sp}_2(\mathbb{Z})\}$$

of translations. The space $L^2(\Gamma\backslash G)$ is a representation space for $G = \mathrm{Sp}_2(\mathbb{R})$ by right translations. This G-action comes along with an action of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g}_\mathbb{C})$ on $\mathcal{C}^\infty$-vectors and action of the elements $D_+(u)$ and $D_-(v)$ allows extension to $L^2(\Gamma\backslash G)$.
The behavior and existence of the resolvents $R_\pm$ of the Casimir operators $D_\pm$ on $L^2(\Gamma\backslash G)$ is regulated by their spectrum. We are interested in the scalar K-type $\kappa = (3,3)$.

Proposition 2.1. *The resolvent $R_-(v)$ exists as a meromorphic function on $\mathrm{Re}\, v > 1$. The resolvent $R_+(u)$ exists as a meromorphic function on $\mathrm{Re}\, u > \frac{1}{2}$. Its spectral pole locus within $L^2(\Gamma\backslash G)_{(3,3)}$ at $u = 1$ is given by the two discrete parameters $\Lambda = (2,1)$ and $\Lambda = (0,1)$.*

Proof of Proposition 2.1. The meromorphicity of the resolvents on the given domains is shown in [8, sec. 3]. The poles of $R_+(u)$ in $u = 1$ are given by the 1-dimensional continuous spectral component $K_{\alpha_1}(1) = (i\mathbb{R}, 1)$ and discrete components indexed by $\Lambda = (s,1)$. By Proposition 3.1 the first does not occur in $L^2(\Gamma\backslash G)_{(3,3)}$. By Theorem 3.2, the remaining discrete parameters are the claimed. □

3. On the spectrum of $L^2(\Gamma\backslash G)$

We give results on the occurrence of the K-type $(3,3)$ in $L^2(\Gamma\backslash G)$.

Proposition 3.1. *In the* 1*-dimensional continuous spectral component included in the parameter* $K_{\alpha_1}(1) = (i\mathbb{R}, 1)$ *the* K*-type* $(3,3)$ *is trivial.*

Proof of Proposition 3.1. By [6, Sec. 7.1] this spectral component for the general symplectic group $\tilde{G} = GSp_2(\mathbb{A})$ is globally given by

$$\int_{i\mathbb{R}} ind_{P_1}^G(\omega_1|\cdot|^{it} \otimes \omega(\det))|\det|^{-it/2})\ dt\ ,$$

for unitary characters ω_1, ω of $\mathbb{G}_m$, where for an element

$$m(\lambda, g)\ =\ \begin{pmatrix} \lambda & & & \\ & a & & b \\ & & \nu/\lambda & \\ & c & & d \end{pmatrix}$$

of $\tilde{M}_{Kl}$, where $\lambda \in \mathbb{G}_m$ and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2$ with $\nu = \det g$, the character $\omega_1|\cdot|^{it} \otimes \omega(\det))|\det|^{-it/2}$ is defined as

$$\omega_1|\cdot|^{it} \otimes \omega(\det))|\det|^{-it/2}(m(\lambda, g))\ =\ \omega_1(\lambda)|\lambda|^{it}\omega(\nu)|\nu|^{-it/2}\ .$$

So at the real place the element

$$m_0\ =\ m(1, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix})\ \in\ K \cap M_1$$

always produces the value

$$\omega_1|\cdot|^{it} \otimes \omega(\det))|\det|^{-it/2}(m_0)\ =\ 1\ ,$$

while for the K-types $\pm(3,3)$ which equal $\det^{\pm 3}$ on the unitary group $K \cong U(2)$ we must have (see 5.1)

$$\det{}^{\pm 3}(\psi(m_0))\ =\ \pm i\ .$$

So this continuous spectral component does not contain this K-type. □

Theorem 3.2. *Let* π *be an irreducible unitary representation of* $G = \mathrm{Sp}_2(\mathbb{R})$ *with infinitesimal character in the Weyl group orbit of* $\Lambda = (1, s)$ *containing the* K*-type* $(3,3)$ *nontrivially. Then either* $\Lambda = (2,1)$ *and* π *is the holomorphic discrete series representation* $\pi^-_{(3,3)}$ *of minimal weight* $(3,3)$*, or* $\Lambda = (0,1)$ *and* π *is the holomorphic representation* $\pi^{hol}_{(1,1)}$ *of weight one (non-discrete series).*

Theorem 3.2 is proven by the following series of lemmas.

Lemma 3.3. [16, Theorem 1.1] *If* π *and* π' *are irreducible representations of the same infinitesimal character containing the same scalar* K*-type nontrivially, then* π *and* π' *are equivalent.*

We include a simple proof of this lemma.

Proof of Lemma 3.3. It follows from Casselman's subrepresentation theorem that π and π' are constituents of the same induced representation. By Peter Weyl's theorem a scalar K-type occurs with at most multiplicity one in an induced representation. So π equals π' there. □

Lemma 3.4. *Any irreducible unitary representation π of G with infinitesimal character Λ Weyl conjugated to $(1, s)$ occurring in $L^2(\Gamma\backslash G)$ is either a discrete series representation or occurs in the following list of irreducible Langlands quotients $J'(P, \sigma, \nu)$ for parabolic subgroups $P \neq G$.*

(a) *For the Siegel parabolic subgroup $P = P_2$, $\sigma = \sigma_2^+$ is the discrete series representation of M_2 of minimal $(K \cap M_2)$-type 2 and $\nu = 0$. Then π has infinitesimal character $\Lambda = (1, -1)$.*

(b) *Let P=P_1 be the Klingen parabolic subgroup. Either $\sigma = (\sigma_1^\pm, \pm)$ is a discrete series representation of M_1 of minimal $(K \cap M_1)$-type 1 and $\nu = e_1$. Then π has infinitesimal character $\Lambda = (1, 0)$. Or $\sigma = (\sigma_2^\pm, -)$ is a discrete series representation of M_1 of minimal $(K \cap M_1)$-type 2 and $\nu = e_1$. Then π has infinitesimal character $\Lambda = (1, 2)$.*

(c) *If $P = B$ is the Borel subgroup, $\sigma = 1$ is the trivial representation of M_B and $\nu = (1, 0)$ or $\nu = (2, 1)$. Then π has infinitesimal character equal to ν.*

Proof of Lemma 3.4. We follow Nzoukoudi's [9] classification of the irreducible unitary representations via Langlands quotients. The parabolic subgroup $P = G$ produces the discrete series.

For the Siegel parabolic, the quotients $J'(P_2, \sigma, \nu)$ belong to limits discrete series representations $\sigma = \sigma_n^+$ of $M_2 \cong \mathrm{SL}_2(\mathbb{R})^\pm$ with infintesimal character $(n-1)(e_1-e_2)$ and characters $\nu = z(e_1-e_2)$ for complex z. For the infinitesimal character $\Lambda = (n+z, -n+z)$ of the quotient to belong to the Weyl orbit $(1, s)^W$ we must have $z \in \mathbb{Z}$. The unitary constraint then is $0 \leq z \leq 1$ and n odd. The only possible choice is $\Lambda = (1, -1)$, $\nu = 0$ and $\sigma = \sigma_2^+$.

For the Klingen parabolic, $\sigma = (\sigma_n^\pm, \pm)$ is a limit of discrete series of $M_2 \cong \mathrm{SL}_2(\mathbb{R}) \times Z_2$ with infinitesimal character $2(n-1)e_2$, and $\nu = 2ze_1$, z complex, is the character. So the quotient has infinitesimal character $\Lambda = (2z, 2n)$, which belongs to the Weyl orbit $(1, s)^W$ only if $2z = \pm 1$. The unitary constraint then forces $\sigma = (\sigma_n^\pm, -)$ for arbitrary $n \geq 0$ or $\sigma = (\sigma_1^\pm, +)$. As $J'(P_2, \sigma, \nu)$ isn't discrete series, we have the Eisenstein constraint $\|\Lambda\|^2 \leq 5$ for the Langlands quotient to belong to the residual spectrum of $L^2(\Gamma\backslash G)$. So $n = 0, 1$.

The subgroup M_B of the Borel group is isomorphic to $Z_2 \times Z_2$, and any representation $\sigma = \sigma^{\epsilon_1, \epsilon_2}$ of M_B is described by two signs ϵ_1, ϵ_2 on generators. The infinitesimal character Λ of the Langlands quotient equals $\nu = (z_1, z_2)$. The unitary contraint implies that ν is either purely imaginary, or of the form $(x+iy, x-iy)$ with $0 < x \leq \frac{1}{2}$ and $y \in \mathbb{R}$, or of the form (x, iy) with $0 < x \leq 1$ and $y \in \mathbb{R}^\times$, or $z_1 \geq z_2 \geq 0$ are real and $z_1 + z_2 \leq 1$ or $(z_1, z_2) = (2, 1)$ and $\sigma = 1$. So only in the last two cases it may belong to the Weyl orbit $(1, s)^W$. In case $\Lambda = \nu = (1, iy)$ the infinitesimal character of the quotient is not real. So it doesn't appear in the residual spectrum of $L^2(\Gamma\backslash G)$. The remaining possibilities are $\Lambda = (1,0)$ or $\Lambda = (1,2)$ with $\sigma = 1$. $\square$

Lemma 3.5. *Among the discrete series of G there is a unique one carrying the K-type $(3,3)$ nontrivially and having infinitesimal character in the Weyl orbit $(1,s)^W$. This is the holomorphic discrete series represenation π_3^- of minimal K-type $(3,3)$.*

Proof of Lemma 3.5. For a semisimple real Lie group G with $rank\, G = rank\, K$ the discrete series representations are parametrized by Harish-Candra parameters (infinitesimal characters) Λ which belong the weight lattice and satisfy $\langle\check{\alpha}, \Lambda\rangle \neq 0$ for all roots $\alpha \in \Delta$ and $\langle\check{\alpha}, \Lambda\rangle > 0$ for all positive compact roots $\alpha \in \Delta_c^+$. There is a unique choice of positive roots Δ_Λ^+ for which Λ is dominant. Then the Blattner weight for Λ is given by

$$\beta_\Lambda = \frac{1}{2}\sum_{\alpha\in\Delta_\Lambda^+}\alpha - \sum_{\alpha\in\Delta_c^+}\alpha\,.$$

The minimal K-type of π_Λ is given by the Blattner paramter

$$k \;=\; \Lambda + \beta_\Lambda\,,$$

and all other K-types of π_Λ are of the form

$$k + \sum_{\alpha\in\Delta_\Lambda^+} n_\alpha \alpha\,,$$

for nonnegative integers n_α ([5, IX.7]). For $G = \mathrm{Sp}_2(\mathbb{R})$ we choose the Cartan subalgebra of both G and K

$$\mathfrak{h}_\mathbb{C} \;=\; \{H(t_1,t_2) = \begin{pmatrix} & & it_1 & 0 \\ & & 0 & it_2 \\ -it_1 & 0 & & \\ 0 & -it_2 & & \end{pmatrix}\}\,,$$

and let $e_j(H(t_1,t_2)) = t_j$. Chooinge simple roots $\alpha_1 = 2e_2$ and $\alpha_2 = e_1 - e_2$ as before, the short root α_2 is compact (i.e. its root space belongs $Lie(K)_\mathbb{C} = \mathfrak{k}_\mathbb{C}$) and we choose $\Delta_c^+ = \{\alpha_2\}$. The root system of $\mathfrak{g}_\mathbb{C}$ with respect to $\mathfrak{h}_\mathbb{C}$ is

$$\begin{aligned}\Delta &= \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1+\alpha_2), \pm(\alpha_1+2\alpha_2)\}\\ &= \{(0,\pm 2), (\pm 1, \pm 1), (\pm 2, 0)\}\,.\end{aligned}$$

There are four sectors of weight vectors satisfying the above conditions for Δ_c^+ corresponding to the dominant weights of the following four choices of positive roots. For the general symplectic group $\mathrm{GSp}_2(\mathbb{R})$ these reduce by equivalence to the choices

$$\Delta_1^+ \;=\; \{(0,2), (1,-1), (1,1), (2,0)\}\,,$$

here the dominant weights $\Lambda = (\Lambda_1, \Lambda_2)$ satisfy $\Lambda_1 > \Lambda_2 > 0$, and

$$\Delta_2^+ \;=\; \{(0,-2), (1,-1), (1,1), (2,0)\}\,,$$

for which dominant weights $\Lambda = (\Lambda_1, \Lambda_2)$ satisfy $\Lambda_1 > -\Lambda_2 > 0$.

The holomorphic discrete series π_k^- of $\mathrm{GSp}_2(\mathbb{R})$ belong to Δ_1^+. Here $\beta_1 = (1,2)$ and the minimal K-type k is given by $k = (\Lambda_1+1, 3)$ for dominant infinitesimal character $(\Lambda_1, 1)$. For the K-type l to occur we must have that l or $-l$ is contained in $k + \mathbb{Z}_{\geq 0}(0,2) + \mathbb{Z}_{\geq 0}(1,-1)$. So the K-type $l = (3,3)$ only occurs in π_k^- if $k = (3,3)$ with infinitesimal character $(2,1)$.
The nonholomorphic discrete series π_k^+ belong to Δ_2^+. Then $\beta_2 = (1,0)$, the dominant infinitesimal characters are $\Lambda = (\Lambda_1, \Lambda_2)$, where $\Lambda_1 > -\Lambda_2 > 0$. The Blattner parameter for $\Lambda = (\Lambda_1, -1)$ is $k = (\Lambda_1 + 1, -1)$. The arising K-types l are of the form $\pm l \in k + \mathbb{Z}_{\geq 0}(1,1) + \mathbb{Z}_{\geq 0}(0,-2)$. So $l = (3,3)$ can occur as a K-type in π_k^+ at most in case $k = (3,-1)$, which has infinitesimal character $(2,-1)$ Weyl conjugated to that of $\pi_{(3,3)}^-$. But this is impossible by Lemma 3.3. □

Lemma 3.6. *Concerning case (b) of Lemma 3.4, the Langlands quotient belonging to* $\sigma = (\sigma_1^-, +)$ *with* $\nu = e_1$ *has nontrivial K-type* $(3,3)$*. It is the holomorphic representation* $\pi_{(1,1)}^{hol}$ *of weight one.In all other cases of (a)-(c) the K-type* $(3,3)$ *is zero.*

Proof of Lemma 3.6. The $(K \cap M_S)$-types of discrete series representation σ_2^+ in case (a) are included in $2 + 2\mathbb{Z}$. By Frobenius reciprocity, the induced representation $ind_{P_S}^G(\sigma_2^+)$ does not contain the odd scalar K-type $(3,3)$, nor does its Langlands quotient. Concerning the limits of discrete series in case (b), the holomorphic limit of discrete series σ_1^- of $\mathrm{SL}_2(\mathbb{R})$ contains the K-type 3 nontrivially, as well does the limit of discrete series $(\sigma_1^-, +)$ of $M_{Kl} \cong \mathrm{SL}_2(\mathbb{R}) \times Z_2$. Again by Frobenius reciprocity, the corresponding Langlands quotient contains the K-type $(3,3)$ nontrivially.
The irreducible Langlands quotients are pairwise inequivalent. The quotients left in cases (b)-(c) have infinitesimal characters conjugated to either $(2,1)$ or $(1,0)$. If one of them contained a nontrivial K-type $(3,3)$, it was equivalent to either $\pi_{(3,3)}^-$ or $\pi_{(1,1)}^{hol}$ in contradiction to Lemma 3.3. □

4. Poincaré series for weight three

We define Poincaré series of weight $\kappa = 3$. For complex variables u and v and positive definite $(2,2)$-matrices T with half-integral entries let

$$P_T(g,u,v) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} H_T(\gamma g, s_1, s_2) \ ,$$

where

$$H_T(g,s_1,s_2) \ = \ \frac{\exp(2\pi i \operatorname{tr}(Tz))}{j(g,i)^\kappa} \operatorname{tr}(Ty)^{s_1} \det(y)^{s_2} \ ,$$

and

$$s_1 \ = \ \frac{v-2u-1}{2} \quad \text{and} \quad s_2 \ = \ \frac{u-(\kappa-m)}{2} \ = \ \frac{u-1}{2} \ .$$

In [8, Cor. 4.4] we showed that these Poincaré series belong to $L^2(\Gamma\backslash G)$ within their area of convergence, which is

$$A \ = \ \{(u,v) \in \mathbb{C}^2 \mid \operatorname{Re} u > 2 \text{ and } \operatorname{Re} v > 5\} \ .$$

The function $H_T(g,s_1,s_2)$ is nonholomorphic (in the variable g) apart from $(s_1,s_2) = (0,0)$. One expects the Poincaré series to have analytic continuation to the critical point $(s_1,s_2) = (0,0)$, equivalently $(u,v) = (1,3)$, which is holomorphic with respect to g. By the same method as for case $\kappa = 4$ ([8, Sec. 6]) we show that indeed this analytic continuation exists along the line $s_1 = 0$, but that there is a nonholomorphic share.

Theorem 4.1. *The Poincaré series $P_T(\cdot,u,v)$ admit meromorphic continuation in $L^2(\Gamma\backslash G)_\kappa$ to the cone*

$$\{(u,v) \in \mathbb{C}^2 \mid \operatorname{Re} u > \frac{1}{2}, \operatorname{Re} v > 1\} \ .$$

The poles are contained in a finite number of lines $u = const.$ and $v = const.$. The limit

$$P_T(\cdot,1,3) \ := \ \lim_{u\to 1} P_T(\cdot,u,2u+1)$$

exists as a function of $L^2(\Gamma\backslash G)_\kappa$. It has a $\mathcal{C}^\infty$-representative. Its nonzero spectral components belong to the isotypical components of irreducible representations with infinitesimal characters $\Lambda = (2,1)$ and $\Lambda = (1,0)$.

Proof of Theorem 4.1. By abuse of notation, we omit the dependence on T in our notations, i.e. $P(g,u,v) := P_T(g,u,v)$. As in [8] we use Casimir operators and their resolvents for the continuation. Their actions depend on the weight $\kappa = 3$. For the two Casimir operators C_1 and C_2 we calulate

$$\begin{aligned} C_1(P(g,u,v)) \ = \ & 4(s_1^2 + 2s_1s_2 + 2s_2^2 + 2s_1 + 3s_2)P(g,u,v) \\ & -16\pi(s_1+s_2)P(g,u,v+2) \\ & -8\det(T)s_1(s_1-1)P(g,u+2,v) \\ & +32\pi\det(T)s_1P(g,u+2,v+2) \end{aligned}$$

and

$$
\begin{aligned}
C_2(P(g,u,v)) \;=\; & \\
& \big(17u^4 + 2v^4 - 12uv^3 + 30u^2v^2 - 36u^3v + 15u^2 \\
& \qquad\qquad +6v^2 - 18uv - 32\big)P(g,u,v) \\
& +256\pi^2(s_1+s_2)(s_1+s_2+1)P(g,u,v+4) \\
& -128\pi(s_1+s_2)\big((s_1+s_2)^2 + 3(s_1+s_2) + \frac{23}{8}\big)P(g,u,v+2) \\
& +32\det(T)^2 s_1(s_1-1)(s_1-2)(s_1-3)P(g,u+4,v) \\
& -256\pi\det(T)^2 s_1(s_1-1)(s_1-2)P(g,u+4,v+2) \\
& -16\det(T)s_1(s_1-1)(7u^2+3v^2-9uv-u+\frac{7}{2})P(g,u+2,v) \\
& +512\pi^2\det(T)^2 s_1(s_1-1)P(g,u+4,v+4) \\
& -64\pi\det(T)s_1(4u^2-3uv-10u+9v-8)P(g,u+2,v+2) \\
& -256\pi^2\det(T)(s_1+s_2)(4s_1+2s_2+1)P(g,u+2,v+4)\,.
\end{aligned}
$$

We used the computer algebra system Magma to verify these results. But for continuation we need to apply operators which produce Poincaré series of better convergence in either u or v. These operators are the known $D_+(u)$ and $D_-(v)$, respectively:

$$
D_+(u) \;:=\; \frac{1}{2}\big(C_1^2 - C_2 + 11C_1 - 2(u^2-1)C_1 + 2(u^2-1)(u^2-4)\big)
$$

and

$$
D_-(v) \;:=\; 2C_2 - C_1^2 - 34C_1 - 2(v^2-9)C_1 + (v^2-9)(v^2-1)\,,
$$

respectively. We get

$$
\begin{aligned}
(1)\qquad D_+(u)P(g,u,v) \;=\; & \\
& +16\det(T)^2 s_1(s_1-1)(s_1-2)(s_1-3)P(g,u+4,v) \\
& -128\pi\det(T)^2 s_1(s_1-1)(s_1-2)P(g,u+4,v+2) \\
& +256\pi^2\det(T)^2 s_1(s_1-1)P(g,u+4,v+4) \\
& +8\det(T)s_1(s_1-1)(u+1)(v-2)P(g,u+2,v) \\
& -32\pi\det(T)s_1(6s_1s_2+3s_1+8s_2^2-8)P(g,u+2,v+2) \\
& +64\pi^2\det(T)(v-u-2)(u-2)P(g,u+2,v+4)\,,
\end{aligned}
$$

respectively,

$$
\begin{aligned}
(2)\quad D_-(v)P(g,u,v) \;=\; & +64\pi^2(v-u)(v-u-2)P(g,u,v+4) \\
& +32\pi(u-1)(v+1)(v-u-2)P(g,u,v+2) \\
& +128\pi\det(T)s_1(s_1-2)(v+1)P(g,u+2,v+2) \\
& -256\pi^2\det(T)(v-u-2)^2P(g,u+2,v+4)\,.
\end{aligned}
$$

Step 1. Meromorphic continuation. In [8, sec. 3] the spectral poles of the resolvents $R_+(u)$ and $R_-(v)$ of $D_+(u)$ and $D_-(v)$, respectively, were studied: Accordingly, they exist as meromorphic functions with the following properties:

Proposition 4.2. [8, Prop. 3.1, 3.2, 3.3] *The resolvent $R_+(u)$ of $D_+(u)$ is meromorphic on $\operatorname{Re} u > \frac{1}{2}$. On the 2-dimensional continuous spectrum, which is included in the parameters $\operatorname{Re}\Lambda = 0$, it is holomorphic. On the 1-dimensional spectrum it is meromorphic with a finite number of simple poles $u = c$ including $u = 1$, which arise from components $K_{\alpha_1}(x)$ or $K_{\alpha_1+2\alpha_2}(c)$. On the discrete spectrum $R_+(u)$ is meromorphic with a finite number of poles corresponding to the roots of $(\Lambda_1^2 - u^2)(\Lambda_2^2 - u^2)$ for infinitesimal characters $\Lambda = (\Lambda_1, \Lambda_2)$.*
The resolvent $R_-(v)$ of $D_-(v)$ is meromorphic on $\operatorname{Re} v > 1$. On the 2-dimensional as well as on the 1-dimensional continuous spectral components it is holomorphic. On the discrete spectrum $R_+(u)$ is meromorphic with a finite number of poles corresponding to the roots of $((\lambda_1+\Lambda_2)^2 - v^2)((\lambda_1-\Lambda_2)^2 - v^2)$ for infinitesimal characters $\Lambda = (\Lambda_1, \Lambda_2)$.

Iterated application (see [8, sec. 4]) of the resolvents $R_+(u)$ and $R_-(v)$ to the Poincaré series yields their meromorphic continuation as L^2-functions to the largest area on which the resolvents exist, that is to the cone

$$\{(u,v) \in \mathbb{C}^2 \mid \operatorname{Re} u > \frac{1}{2},\ \operatorname{Re} v > 1\} .$$

Step 2. The L^2-limit $P(\cdot, 1, 3) := \lim_{u\to 1} P(\cdot, u, 2u+1)$ exists. Let

$$L^2(\Gamma\backslash G) \ = \ L^2_{\operatorname{Re}\Lambda=0}(\Gamma\backslash G) \bigoplus_{\gamma,c} L^2_{\gamma,c}(\Gamma\backslash G) \bigoplus_{\Lambda} L^2_{\Lambda}(\Gamma\backslash G)$$

be the spectral decomposition of $L^2(\Gamma\backslash G)$ and denote by

$$P_\bullet(\cdot, u, v)$$

the according spectral components of the Poincaré series. Notice that $(u, v) = (1, 3)$ is an inner point of the area of meromorphicity. We analyze the operator $D_+(u)$. We choose $v = 2u+1$, so the limit series has equation $s_1 = 0$. Equation (1) simplifies on the intersection of $s_1 = 0$ with the cone of convergence to

$$D_+(u)P(\cdot, u, 2u+1) \ = \ 64\pi^2 \det(T)(u-1)(u-2)P(g, u+2, 2u+5) ,$$

where $P(\cdot, u+2, 2u+5)$ actually is convergent in $(u, v) = (1, 3)$. As the meromorphic continuation is unique, this holds everywhere on $s_1 = 0$. Equivalently, as $D_+(1) = D_+(u) - (u^2-1)(C_1 - (u^2-4))$,

$$\begin{aligned} D_+(1)P(\cdot, u, 2u+1) \ &= \ (u^2-1)\big(C_1 - (u^2-4)\big)P(\cdot, u, 2u+1) \\ &\quad +64\pi^2 \det(T)(u-1)(u-2)P(\cdot, u+2, 2u+5) . \end{aligned}$$

Thus,

$$\begin{aligned} D_+(1)^n P(\cdot, u, 2u+1) \ &= \ (u^2-1)^n\big(C_1 - (u^2-4)\big)^n P(\cdot, u, 2u+1) \\ &\quad +(u-1)\mathcal{P}(\cdot, u) , \end{aligned}$$

where $\mathcal{P}(\cdot,u)$ is a symbol for a $\mathbb{C}[u]$-linear combination of Poincaré series which actually converge in $(u,v)=(1,3)$. Choosing n greater than the pole order of $P(\cdot,u,v)$ in $(u,v)=(1,3)$, we have

$$\lim_{u\to 1}\|(u^2-1)^n\big(C_1-(u^2-4)\big)^n P(\cdot,u)\| \;=\; 0$$

as well as

$$\lim_{u\to 1}\|(u-1)\mathcal{P}(\cdot,u)\| \;=\; 0\,.$$

Applying Schwarz' inequality we deduce

$$\lim_{u\to 1}\|D_+(1)^n P(\cdot,u,2u+1)\|^2 \;=\; 0\,.$$

Written according to the spectral decomposition,

$$\begin{aligned} 0 \;&=\; \sum_\Lambda |D_+(1,\Lambda)|^{2n}\lim_{u\to 1}\|P_\Lambda(\cdot,u,2u+1)\|^2 \\ &\quad+\sum_{\gamma,c}\lim_{u\to 1}\|D_+(1)^n P_{\gamma,c}(\cdot,u,2u+1)\|^2 \\ &\quad+\lim_{u\to 1}\|D_+(1)^n P_{\mathrm{Re}\,\Lambda=0}(\cdot,u,2u+1)\|^2\,. \end{aligned}$$

So any single summand is zero: The limit $\lim_{u\to 1}\|P_\Lambda(\cdot,u,2u+1)\|$ exits for any discrete parameter Λ. It is nonzero only if $D_+(1,\Lambda)=(\Lambda_1^2-1)(\Lambda_2^2-1)$ is zero. So the remaining nonzero discrete components $P_\Lambda(\cdot,u,v)$ have parameters $\Lambda=(1,\Lambda_2)$ and are indeed analytically continued in $(u,v)=(1,3)$. On any continuous component apart from $K_\alpha(1)$, the resolvent $R_+(1)$ exists, thus there we have

$$\begin{aligned} \lim_{u\to 1}\|P_\bullet(\cdot,u,2u+1)\|^2 \;&=\; \lim_{u\to 1}\|R_+(1)^n D_+(1)^n P_\bullet(\cdot,u,2u+1)\|^2 \\ &\le\; \|R_+(1)\|_\bullet^{2n}\cdot\lim_{u\to 1}\|D_+(1)^n P_\bullet(\cdot,u,2u+1)\|^2 = 0\,. \end{aligned}$$

On the component $K_\alpha(1)$ we have $D_+(u)=(u^2-1)M_+(u)$, where $M_+(u)$ can be parametrized by (u^2+t^2) and is bounded from below by u^2. So from

$$0 \;=\; \lim_{u\to 1}\|D_+(1)^n P_{\alpha,1}(\cdot,u,2u+1)\| \;=\; \lim_{u\to 1}(u^2-1)^n\|M_+(1)^n P_{\alpha,1}(\cdot,u,2u+1)\|$$

we deduce as before that $\|M_+(1)P_{\alpha,1}(\cdot,u,2u+1)\|$ exists. As the resolvent $M_+^{-1}(1)$ is an operator bounded by $u^{-1}=1$, the limit

$$\begin{aligned} \lim_{u\to 1}\|P_{\alpha,1}(\cdot,u,2u+1)\|^2 \;&=\; \lim_{u\to 1}\|M_+(1)^{-n}M_+(1)^n P_{\alpha,1}(\cdot,u,2u+1)\|^2 \\ &\le\; \|M_+^{-1}(1)\|_{\alpha,1}^{2n}\cdot\lim_{u\to 1}\|M_+(1)^n P_{\alpha,1}(\cdot,u,2u+1)\|^2 \end{aligned}$$

exists. So the limit $P(\cdot,1,3):=\lim_{u\to 1}P(\cdot,u,2u+1)$ exists as an L^2-function.

Step 3. The spectral locus of $P(\cdot,u,v)$ *in* $(1,3)$. We examine the possible spectral components left from Step 2. Within the continuous spectrum, the only remaining component is indexed by $\Lambda = (1, i\mathbb{R}) = K_{\alpha_1}(1)$. But by Proposition 3.1, in this component the K-type $\kappa = (3,3)$ does not occur. As the K-type is passed on the continuation, the $P(\cdot, 1, 3)$ has weight κ. So its continuous spectral component is identically zero. The remaining discrete spectral components are indexed by $\Lambda = (\Lambda_1, 1)$. By Proposition 3.2, only two components occur within $L^2(\Gamma\backslash G)_\kappa$: The holomorphic discrete series representation π_κ of minimal K-type $\kappa = (3,3)$ and of infinitesimal character $\Lambda = (2,1)$, and a non-discrete series representation of infinitesimal character $\Lambda = (0,1)$.

Step 4. There is a C^∞-representative. The limit $P(\cdot, 1, 3)$ is the solution of an elliptic differential equation with C^∞-coefficients. So itself is C^∞ by regularity theory. □

5. Phantom holomorphic projection

5.1. **Differential operators.** Let ρ be an irreducible unitary representation of $U(2)$ on a finite dimensional vector space V_ρ. By the homomorphism $J : G \to \mathrm{GL}_m(\mathbb{C})$, $J(g) = (ci+d)$, we get an isomorphism $\psi = J\mid_K : K \tilde{\to} U_m(\mathbb{C})$. So $\rho(g) := \rho \circ \psi(g)$ is an irreducible unitary representation of K. As ρ is the restriction of an irreducible representation of $\mathrm{GL}_m(\mathbb{C})$, the element $J_\rho(g) = \rho(ci+d) \in Aut(V_\rho)$ is well-defined for all $g \in G$.
In the following we make use of formulas developed in [15, §3]. The notation there is according to the choice of the isomorphism $K \cong U(m)$ given by the complex conjugate $\bar{\psi}$ of ψ. This implies to work instead of ρ with the representation $\rho(\bar{\psi}(g)) = \rho(\bar{g})$ on V_ρ, which is isomorphic to the contragredient representation $\rho^* \circ \psi$ of K. So we must carfully replace ρ^* by ρ in some of the formulas in [15, §3].
For the Siegel upper halfspace $\mathcal{H} = G/K$ we have the isomorphism [15, p. 30]

$$\begin{array}{rcl} C^\infty(G, V_{\rho\circ\bar{\psi}})^K & \tilde{\to} & C^\infty(\mathcal{H}, V_\rho)\,, \\ f(g) & \mapsto & J_\rho(g) f(g)\,. \end{array}$$

Here $C^\infty(G, V_\rho)^K$ is the subspace of K-invariant functions in $C^\infty(G, V_\rho) = C^\infty(G) \otimes V_\rho$, on which K acts by right trasnlations $R_g f(x) = f(xg)$. By Schur's lemma $C^\infty(G, V_\rho)^K$ is the K-isotypical component for the representation $(\rho \circ \bar{\psi})^*$. But this is isomorphic to $\rho \circ \psi$ and we can identify $C^\infty(G, V_\rho)^K$ with $C^\infty(G)_\rho$, the ρ-isotypical component of $C^\infty(G)$ on which K acts by right translations. So we get isomorphisms

$$\phi_\rho \,:\, C^\infty(G)_\rho \tilde{\to} C^\infty(\mathcal{H}_2, V_\rho)\,.$$

The universal envelopping algebra $\mathfrak{U}(\mathfrak{g}_\mathbb{C})$ of the complex Lie algebra $\mathfrak{g}_\mathbb{C}$ of G acts from the right on $C^\infty(G)$ and this action commutes with the left action of G. The abelian Lie algebra $\mathfrak{p}_+$ (respectively $\mathfrak{p}_-$) can be identified with the holomorphic (respectively antiholomorphic) tangent space of $\mathcal{H}$ in $Z = iE$. So $\mathfrak{U}(\mathfrak{p}_-)$ acts on $C^\infty(G)$ by leftinvariant differential operators.

Lemma 5.1. *For the adjoint representation of K on $\mathfrak{U}(\mathfrak{p}_+)$ it holds*

$$\mathfrak{U}(\mathfrak{p}_+) \;\cong\; \mathrm{Symm}^\bullet\,\mathrm{Symm}^2(\mathbb{C}^m) \;=\; \bigoplus_{\rho_k} V_{\rho_k}\,,$$

where ρ_k runs through the irreducible repesentations of K of highest weight $k = (k_1, \dots, k_m)$ for $k_i \in 2\mathbb{Z}$ and $k_1 \geq k_2 \geq \cdots \geq k_m \geq 0$.

Proof of Lemma 5.1. This is [15, Lemma 3] with respect to the change from $\bar{\psi}$ to ψ. □

The representation of K on $\mathfrak{U}(\mathfrak{p}_-)$ is dual to that on $\mathfrak{U}(\mathfrak{p}_+)$. For any irreducible representation ρ in $\mathfrak{U}(\mathfrak{p}_+)$ there are operators E_+^ρ on the ρ-isotypical component of $\mathfrak{U}(\mathfrak{p}_+)$, respectively E_-^ρ in $\mathfrak{U}(\mathfrak{p}_-)$ on the ρ^*-isotypical component of $\mathfrak{U}(\mathfrak{p}_-)$

[15, p.43f.]. They map a K-isotypical subspace $C^\infty(G)_\tau$ of $C^\infty(G)$ to the direct sum of K-isotypical subspaces $C^\infty(G)_{\tilde\tau}$, where

$$\rho \otimes \tau \;=\; \bigoplus \tilde\tau \,,$$

respectively $\rho^* \otimes \tau$ in case E^ρ_-. We get the Maass operators E_+ respectively E_- by choosing $\rho = \rho_k$ of highest weight $k = (2, 0, \ldots, 0)$. More generally, we get Maass operators $E^{[\mu]}_+$ respectively $E^{[\mu]}_-$ by choosing

$$k = (2, \ldots, 2, 0, \ldots, 0)$$

where μ is the number of 2s occurring. By the above identifications ϕ_τ and $\phi_{\rho\otimes\tau} = \bigoplus_{\tilde\tau} \phi_{\tilde\tau}$ we have the commutative diagramm

$$\begin{array}{ccc} C^\infty(G)_\tau & \xrightarrow{\phi_\tau} & C^\infty(\mathcal{H}, V_\tau) \\ \downarrow E^\rho_+ & & \downarrow \Delta^\rho_+ \\ C^\infty(G)_{\rho\otimes\tau} & \xrightarrow{\phi_{\rho\otimes\tau}} & C^\infty(\mathcal{H}, V_{\rho\otimes\tau}) \end{array} .$$

An explicit description of the operators $\Delta^{[\mu]}_+$ respectively $\Delta^{[\mu]}_-$ is found in [15, pp. 33, 44]. $\Delta^{[\mu]}_+$ acts on $C^\infty(\mathcal{H}, V_\tau) = C^\infty(\mathcal{H}) \otimes V_\tau$ with values in $C^\infty(\mathcal{H}) \otimes V_{(2,\ldots,2,0,\ldots,0)} \otimes V_\tau$ via

$$\Delta^{[\mu]}_+ \,(h(Z) \otimes v) \;=\; \Delta^{[\mu]}_+ \,(h(Z)) \otimes v$$

for $h \in C^\infty(\mathcal{H})$ and $v \in V_\tau$ and is defined up to a factor 2^μ by

$$\Delta^{[\mu]}_+ \,(h(Z)) \;=\; (2i)^\mu (\tau \otimes det^{\frac{1-\mu}{2}})(Y^{-1}) \cdot \partial^{[\mu]}_Z \Big((\tau \otimes det^{\frac{1-\mu}{2}})(Y) \cdot h(Z) \Big) \;.$$

Here $\partial_Z = \frac{1}{2}(\partial_X - i\partial_Y)$ is a matrix valued operator with components $\frac{1}{2}(1 + \delta_{ij})\frac{\partial}{\partial_{Z_{ij}}}$ for the symmetric matrix $Z = X + iY \in \mathcal{H}$ with components Z_{ij}, and for a matrix M let $M^{[\mu]} = \bigwedge^\mu(M)$ be the matrix of the μ-th exterior power of M, i.e. the $\binom{m}{\mu} \times \binom{m}{\mu}$-matrix of the minors of M of size μ (see [2, p. 208ff]).

5.2. **Sturm's operator.** Let $\mathcal{Y} = \{y = y' \in M_m(\mathbb{R}) \mid y > 0\}$ be the space of positive definite symmetric matrices. Let the genus m equal two. Let $F \in \tilde{\mathcal{M}}_\kappa(\Gamma)$ be a (nonholomorphic modular) form of bounded growth (see [10, 2.4])) of weight κ and let

$$F(Z) \;=\; \sum_{T=T'} A(T, Y) e^{2\pi i \,\mathrm{tr}(TX)}$$

be its Fourier expansion. Sturm's operator for weight κ for the Fourier coefficients $A(T, Y)$ belonging to positive definite T is

$$A(T, Y) \mapsto a(T) := c(\kappa)^{-1} \det(T)^{\kappa - \frac{3}{2}} \int_{\mathcal{Y}} A(T, Y) \exp(-2\pi \,\mathrm{tr}(TY)) \det(Y)^{\kappa-3} \, dY,$$

where $c(\kappa) = \sqrt{\pi}(4\pi)^{3-\kappa}\Gamma(\kappa - \frac{3}{2})\Gamma(\kappa - 2)$ ([10, p. 84]). In case $\kappa \geq 4$ Sturm's operator realizes the orthogonal projection to the holomorphic part of F. The Fourier expansion

$$\tilde{F}(Z) \;=\; \sum_{T>0} a(T) e^{2\pi i\,\mathrm{tr}(TZ)}$$

gives rise to a holomorphic cuspform $\tilde{F} \in [\Gamma, \kappa]_0$ of weight κ, and for all $f \in [\Gamma, \kappa]_0$ it holds

$$\langle F, f\rangle \;=\; \langle \tilde{F}, f\rangle\,,$$

where $\langle \cdot, \cdot\rangle$ is the scalar product of the Hilbert space $L^2_\kappa(\Gamma\backslash\mathcal{H})$. This is shown in [8] (see also [10] in case $\kappa \geq 5$) by using Poincaré series of weight κ defined analogously to ours which are holomorphic cuspforms for $\kappa \geq 4$.

We are interested in the action of Sturm's operator on images of $\Delta_+^{[\mu]}$ in the special case $\mu = 2$. So let $k = (2, 2)$ and $\rho_k = det^2$ then

$$\Delta_+^{[2]} \;:\; C^\infty(\mathcal{H}_2, V_\tau) \;\to\; C^\infty(\mathcal{H}_2, V_{\tau\otimes det^2})\,,$$

and $\Delta_+^{[2]} \circ \phi_\tau = \phi_{\tau\otimes det^2} \circ \det(E_+^{[2]})$, is explicitly given on $\mathcal{H}$ by

$$\Delta_+^{[2]}(h)(Z) \;=\; (2i)^2(\tau\otimes det^{-\frac{1}{2}})(Y^{-1})\, det(\partial_Z)\left((\tau\otimes det^{-\frac{1}{2}})(Y)h(Z)\right)\,.$$

Proposition 5.2. *Let $h \in [\Gamma, k]_0$ be a holomorphic cuspform on $\mathcal{H}$ of weight (k, k), where $\kappa = k + 2$. Sturm's operator applied to the Fourier coefficients of the nonholomorphic image $\Delta_+^{[2]}(h)$ is zero in case $\kappa \geq 4$. But for $\kappa = 3$ it does not vanish.*

Proof of Proposition 5.2. We apply $\Delta_+^{[2]}$ to a holomorphic cuspform $h \in [\Gamma, k]_0$ on $\mathcal{H}_2$ of weight (k, k). So $V_\tau = \mathbb{C}$ and

$$\tilde{h}(Z) \;=\; \Delta_+^{[2]}(h)(Z) \;=\; -4\det(Y^{-1})^{k-\frac{1}{2}}\, det(\partial_Z)\left(\det(Y)^{k-\frac{1}{2}}h(Z)\right)$$

is a function on $\mathcal{H}_2$. (This formula for Maass' operator is also due to Shimura [12] and can be found in [10, 3.1].) It has K-type $(k+2, k+2)$ and belongs to the automorphic representation generated by the holomorphic cuspform h. Let

$$h(Z) \;=\; \sum_{T=T'} a(T)\exp(2\pi i\,\mathrm{tr}(TZ))$$

be its Fourier expansion. The Fourier coefficients $b(T, Y)$ of the function $\tilde{h}$ are given by

$$-4a(T)\det(Y)^{\frac{1}{2}-k}\, det(\partial_Z)\left(\det(Y)^{k-\frac{1}{2}}\exp(2\pi i\,\mathrm{tr}(TZ))\right)\cdot\exp(-2\pi i\,\mathrm{tr}(TZ))$$

Let

$$\tilde{b}(T, Y) \;=\; b(T, Y)\exp(-2\pi\,\mathrm{tr}(TY))$$

be the coefficient in the Fourier expansion in X only. For Sturm's formula we study whether the limit

$$\lim_{s\to 0}\int_{\mathcal{Y}} \exp(-2\pi\,\mathrm{tr}(TY))\tilde{b}(T, Y)\det(Y)^{k-1+s}\; dY \tag{3}$$

is zero. We make use of the Lemmas 5.4, 5.5, and 5.6 below. Applying Lemma 5.4 to the functions $f(Z) = \det(Y)^{k-\frac{1}{2}}$ and $g(Z) = \exp(2\pi i \operatorname{tr}(TZ))$ we get

$$\begin{aligned} det(\partial_Z)\left(\det(Y)^{k-\frac{1}{2}}\exp(2\pi i \operatorname{tr}(TZ))\right) &= -\frac{1}{4}C_2(k-\frac{1}{2})\det(Y)^{k-\frac{3}{2}}g \\ &\quad -\frac{i}{2}C_1(k-\frac{1}{2})\det(Y)^{k-\frac{3}{2}}\operatorname{tr}(Y2\pi iT)g \\ &\quad +\det(Y)^{k-\frac{1}{2}}(2\pi i)^2\det(T)g\,, \end{aligned}$$

where $C_2(k-\frac{1}{2}) = (k-\frac{1}{2})k$ and $C_1(k-\frac{1}{2}) = k-\frac{1}{2}$ by Lemma 5.5. So the Fourier coefficient $b(T,Y)$ is

(4)
$$(4\pi)^2 a(T)\det(T)\left(\frac{k(k-\frac{1}{2})}{(4\pi)^2}\det(TY)^{-1} - \frac{(k-\frac{1}{2})}{4\pi}\det(TY)^{-1}\operatorname{tr}(YT)+1\right).$$

By a change of variables $Y \mapsto 4\pi TY$, for the limit (3) we have to compute the integral

$$\int_{\mathcal{Y}}\left(k(k-\frac{1}{2})\det(Y)^{-1} - (k-\frac{1}{2})\det(Y)^{-1}\operatorname{tr}(Y)+1\right)\exp(-\operatorname{tr}(Y))\det(Y)^{k-1+s}\,dY$$

up to the factor $(4\pi)^2 a(T)\det(T)^{2-k-s}$, which by Lemma 5.6 is given (up to a factor $\sqrt{\pi}$) by

$$\begin{aligned} &k(k-\frac{1}{2})\Gamma(s+k-\frac{1}{2})\Gamma(s+k-1) \\ &\quad -2(k-\frac{1}{2})(s+k-\frac{1}{2})\Gamma(s+k-\frac{1}{2})\Gamma(s+k-1)+\Gamma(s+k+\frac{1}{2})\Gamma(s+k)\,. \end{aligned}$$

But this equals

$$s(s-\frac{1}{2})\Gamma(s+k-\frac{1}{2})\Gamma(s+k-1)\,.$$

So for all $k>1$ respectively $\kappa = k+2 > 3$ the limit (3) is zero,

$$\lim_{s\to 0}\int_{\mathcal{Y}}\exp(-2\pi\operatorname{tr}(TY))\tilde{b}(T,Y)\det(Y)^{k-1+s}\,dY \;=\; 0\,.$$

While in case $k=1$ (i.e. $\kappa=3$) it is a multiple of

$$\lim_{s\to 0}s(s-\frac{1}{2})\Gamma(s+\frac{1}{2})\Gamma(s) \;=\; -\frac{1}{2}\Gamma(\frac{1}{2}) \;\neq\; 0\,.$$

Collecting constants, the Sturm operator maps

$$\tilde{b}(T,Y) \;\mapsto\; (4\pi i)^2 a(T)\det(T)^{\frac{5}{2}}$$

in case $k=1$, while it is zero for $k>1$. □

Remark 5.3. *In case of weights (k,k) for $k = 0, \frac{1}{2}$ the Fourier coefficients (4) vanish. For $k=\frac{1}{2}$ this follows from the theory of singular modular forms* [3], [11]. *So for these weights Sturm's operator is supposed to establish the holomorphic projection, too.*

Lemma 5.4. *For quadratic matrices A, B define the symbol $2\cdot(A\cap B)$ by*
$$(A+B)^{[2]} \;=\; A^{[2]} + 2\cdot(A\cap B) + B^{[2]}\,.$$
Then it holds
$$\partial_Z^{[2]}(fg) \;=\; \partial_Z^{[2]}(f)g + 2(\partial_Z(f)\cap\partial_Z(g)) + f\partial_Z^{[2]}(g)\,.$$
Especially for $m=2$, it holds
$$2\cdot\left(Y^{-1}\cap T\right) \;=\; \det(Y)^{-1}\operatorname{tr}(YT)\,.$$

Proof of Lemma 5.4. See [2, pp. 208, 211]. In case of genus $m=2$ we have $A^{[2]}=\det(A)$, so $2(A\cap B) = A_{11}B_{22} + A_{22}B_{11} - A_{12}B_{21} - A_{21}B_{12}$ and the identity $2(Y^{-1}\cap T) = \det(Y)^{-1}\operatorname{tr}(YT)$ follows. □

Lemma 5.5. [2, p. 213] *It holds*
$$\partial_Y^{[h]}\det(Y)^{\alpha} \;=\; C_h(\alpha)\det(Y)^{\alpha}(Y^{-1})^{[h]}\,,$$
where $C_h(\alpha) = \alpha(\alpha+\frac{1}{2})\cdots(\alpha+\frac{(h-1)}{2})$.

Lemma 5.6. *It holds*
$$\int_{\mathcal{Y}} e^{-\operatorname{tr}(\tau Y)}\det(Y)^{s-3/2}\,dY \;=\; \sqrt{\pi}\det(\tau)^{-s}\Gamma(s)\Gamma(s-\frac{1}{2}) \tag{5}$$
as well as
$$\int_{\mathcal{Y}} e^{-\operatorname{tr}(Y)}\operatorname{tr}(Y)\det(Y)^{s-1}\,dY \;=\; 2\sqrt{\pi}(s+\frac{1}{2})\Gamma(s)\Gamma(s+\frac{1}{2})\,. \tag{6}$$

Proof of Lemma 5.6. The identity (5) is well-known (e.g. [12, p. 467]). Differentiating by ∂_τ in $\tau = E_2$ we get
$$\int_{\mathcal{Y}} e^{-\operatorname{tr}(Y)}Y\det(Y)^{s-3/2}\,dY \;=\; sE_2\sqrt{\pi}\Gamma(s)\Gamma(s-\frac{1}{2})\,,$$
so especially
$$\int_{\mathcal{Y}} e^{-\operatorname{tr}(Y)}\operatorname{tr}(Y)\det(Y)^{s-1}\,dY \;=\; 2\sqrt{\pi}(s+\frac{1}{2})\Gamma(s+\frac{1}{2})\Gamma(s\frac{1}{2})\,.$$
□

5.3. **Poincaré series.** Let ρ be the irreducible representation of K of minimal weight (κ,κ) for $\kappa = 3$. The images
$$p_T(g\cdot i, s) \;=\; \phi_\rho\left(P_T(g,u,2u+1)\right) \;=\; j(g,i)^{\kappa}P_T(g,u,2u+1)\,,$$
of the Poincaré series $P(g,u,2u+1)$ under the isomorphism ϕ_ρ define Poincaré series on $\mathcal{H}$,
$$p_T(z,s) \;:=\; \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} e^{2\pi i\operatorname{tr}(T\gamma\cdot z)}\frac{\det(\operatorname{Im}\gamma\cdot z)^s}{j(\gamma,z)^{\kappa}}\,.$$
Here s and u are related by $s=\frac{1}{2}(u-1)$. The Poincaré series p_T inherit the analytic properties of their preimages P_T.

Corollary 5.7. *For* $\operatorname{Re} s + \frac{\kappa}{2} > 2$ *the series* $p_T(z,s)$ *converge absolutely and locally uniformly in* s *and uniformly on the Siegel fundamental domain* $\mathcal{F}$ *for* Γ. *They belong to* $L^2_\kappa(\Gamma\backslash\mathcal{H})$, *the Hilbert space of functions on* $\Gamma\backslash\mathcal{H}$ *of weight* κ *with scalar product given by*

$$\langle f, g\rangle \;=\; \int_{\mathcal{F}} f(Z)\overline{g(Z)}\det(Y)^\kappa\; dv_Z\,.$$

They have meromorphic continuations to $\operatorname{Re} s > -\frac{1}{2}$ *as functions in* $L^2_\kappa(\Gamma\backslash\mathcal{H})$, *which are analytical in the critical point* $s=0$. *That is, the limit*

$$p_T(\cdot) \;:=\; \lim_{s\to 0} p_T(\cdot, s) \;\in\; L^2_\kappa(\Gamma\backslash\mathcal{H})$$

exists in $L^2_\kappa(\Gamma\backslash\mathcal{H})$ *and is* C^∞.

Theorem 5.8. *The analytic continuations* p_T *of the Poincaré series of weight* $(3,3)$ *to the critical point* $s=0$ *decompose into*

$$p_T \;=\; f_T \;+\; \Delta_+^{[2]}(h_T)\,,$$

where $f_T \in [\Gamma, 3]_0$ *is a holomorphic cuspform of weight* $(3,3)$, *and* $h_T \in [\Gamma, 1]$ *is a holomorphic modular form of weight* $(1,1)$. *In general the two forms* f_T *and* h_T *are nonzero.*

Let $\mathcal{X} = \{x = x' \in M_m(\mathbb{R}) \mid |x_{jk}| \le \frac{1}{2}, 1 \le j,k \le m\}$ be a stripe of symmetric matrices, so $\mathcal{F} \subset \mathcal{X} + i\mathcal{Y}$.

Proof of Theorem 5.8. The analytic continuations p_T of the Poincaré series to $s=0$ have the claimed decomposition by Theorem 4.1. Let

$$F(Z) \;=\; \sum_{T=T'} A(T,Y)e^{2\pi i\operatorname{tr}(TX)}$$

be the Fourier expansion of a nonholomorphic modular form of bounded growth of weight 3. We use unfolding (e.g. [4, IV.1] or [10, 2.4]) to get for $\operatorname{Re} s + \frac{3}{2} > 2$

$$\begin{aligned}
\langle F, p_T(\cdot,\bar{s})\rangle \;&=\; \int_{\mathcal{F}} F(Z) \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} e^{-2\pi i\operatorname{tr}(T\gamma\cdot\bar{Z})}\frac{\det(\operatorname{Im}\gamma\cdot Z)^s}{\overline{j(\gamma,Z)}^3}\det(Y)^3\; dv_Z\\
&=\; \int_{\mathcal{F}} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} F(\gamma\cdot Z)e^{-2\pi i\operatorname{tr}(T\gamma\cdot\bar{Z})}\det(\operatorname{Im}\gamma\cdot z)^{s+3}\; dv_Z\\
&=\; \int_{\mathcal{X}}\int_{\mathcal{Y}} F(Z)e^{-2\pi i\operatorname{tr}(T\bar{Z})}\det(Y)^{s+3}\; dv_Z\,.
\end{aligned}$$

But the last integral

$$\begin{aligned}
&\int_{\mathcal{X}}\int_{\mathcal{Y}} F(Z)e^{-2\pi i\operatorname{tr}(T\bar{Z})}\det(Y)^{s+3}\; dv_Z\\
&\quad= \int_{\mathcal{X}}\int_{\mathcal{Y}} \sum_{\tilde{T}} A(\tilde{T},Y)e^{2\pi i\operatorname{tr}((\tilde{T}-T)X)}e^{-2\pi\operatorname{tr}(TY)}\det(Y)^s\; dX\; dY\\
&\quad= \int_{\mathcal{Y}} A(T,Y)e^{-2\pi\operatorname{tr}(TY)}\det(Y)^s\; dY
\end{aligned}$$

exists for $\mathrm{Re}\, s \geq 0$ (this indeed is the definition of bounded growth), and its value at $s = 0$ is up to a factor the image

$$a(T) \;=\; c(3)^{-1} \det(T)^{\frac{3}{2}} \int_{\mathcal{Y}} A(T,Y) e^{-4\pi\,\mathrm{tr}(TY)}\; dY$$

of Sturm's operator. As analytic continuation is unique, we have

$$\langle F, p_T \rangle \;=\; c(3) \det(T)^{-\frac{3}{2}} a(T)\,. \tag{7}$$

This especially applies to the nonholomorphic function $F = \Delta_+^{[2]}(h)$ for a holomorphic cuspform $h \in [\Gamma, 1]_0$, where Sturm's operator is seen to be nonzero by Proposition 5.2. So the nonholomorphic component $\Delta_+^{[2]}(h_T)$ cannot be zero in general. Similary choosing $F \in [\Gamma, 3]_0$, Sturm's operator is the identity on F. So the holomorphic component f_T does not vanish in general. $\square$

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