Some indivisibility properties of generalized left-factorials

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Abstract: We shall prove certain p-indivisibility properties of so-called generalized left-factorials, where p is a prime

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1. Introduction

Let p be a prime. Recall that following Kurepa [3] one defines the so-called left-factorial $!p$ by

$$!p := 0! + 1! + \ldots + (p-1)!. \tag{1}$$

In [3] it was conjectured that that $!p$ is never divisible by p, for any $p > 2$. So far this is still an open question, cf. e.g. [1,2] and the literature given there for background information and equivalent formulations of the conjecture. In [1] some probability arguments were given that counterexamples to the conjecture should exist, however none was found so far. In fact, in [1] the conjecture was verified in the range $2 < p < 2^{34}$.

In [1] also a generalization of (1) was introduced, namely the k-th generalized left-factorial

$$!^k p := 0!^k + 1!^k + \ldots + (p-1)!^k, \tag{2}$$

for any $k \in \mathbf{N}$. It was shown that for all $1 < k < 100$ there exists an odd prime p such that $!^k p$ is divisible by p. To compute $!^k p$ modulo p, the authors use sophisticated algorithms and machine computations, apart from some trivial cases where the assertion is obvious, like e.g. $!^k 3 \equiv 0 \pmod 3$ for k even.

It seems that so far in general no theoretical results are known about the value $!^k p$ modulo p.

In this short note we want to consider the case opposite to the above, namely indivisibility of (2) by p. Again there are some trivial results, e.g. $!^k 3 \not\equiv 0 \pmod 3$ for k odd. We also observe that by Fermat's little theorem, if $!^k p \not\equiv 0 \pmod p$, then also $!^{kp^t} \equiv !^k p \not\equiv 0 \pmod p$ for any $t \in \mathbf{N}$ and so there exists arbitrarily large N with $!^N p \not\equiv 0$ $(\mathrm{mod}\ p)$.

Here we would like to show how in a simple way one can obtain odd natural numbers k such that there exist "several" odd primes p with $!^k p \not\equiv 0 \pmod{p}$. The proof which will be given in the next section is short and elementary. In the same way, using Sophie Germain primes and some standard conjectures about their distribution, one can prove (conditionally) a similar result also for *squarefree* odd natural numbers. We will comment on this in sect. 3 at the end of the paper.

2. Statement of main result and proof

We will prove

Theorem. *There exists a sequence* $(k_n)_{n\in\mathbf{N}}$ *of odd natural numbers such that*

$$k_n \le (n/2)^{cn} \quad (n \to \infty) \tag{3}$$

and for each n *there exist* n *odd primes* p *such that*

$$!^{k_n} p \not\equiv 0 \pmod{p}.$$

In (3), $c > 0$ *is an absolute constant.*

Proof. Let $p_1 < p_2 < \ldots < p_n < \ldots$ be the primes congruent to 3 modulo 4 in ascending order. We put

$$k_n := \prod_{\nu=1}^{n} \frac{p_\nu - 1}{2}.$$

Then k_n is odd. Also

$$k_n \le \frac{p_1 \cdots p_n}{2^n},$$

hence

$$\log k_n \le n \log p_n - n \log 2$$

$$= \pi^*(p_n) \log p_n - n \log 2 \tag{4}$$

where $\pi^*(x)$ denotes the number of primes congruent to 3 modulo 4 up to x.

By Dirichlet's prime number theorem (see e.g. [4, p. 73, Thm. 2])

$$\pi^*(x) \approx \frac{1}{\phi(4)} \frac{x}{\log x} = \frac{1}{2} \cdot \frac{x}{\log x} \quad (x \to \infty). \tag{5}$$

Therefore with $x = p_n$ we obtain from (4) that

$$\log k_n \ll p_n - n \log 2. \tag{6}$$

Also from (5) we deduce in the usual manner that

$$p_n \approx 2n \log n \quad (n \to \infty).$$

Inserting into (6) we therefore find

$$\log k_n \ll n \log n - n \log 2$$
$$= n \log \frac{n}{2}$$

as claimed.

Now let $k = k_n$ and for fixed $\nu \in \{1, \ldots, n\}$ write $p = p_\nu$. With $q := \frac{p-1}{2}$ we then find

$$!^k p = \sum_{j=0}^{p-1} j!^k = \sum_{j=0}^{p-1} \Big((j!)^q\Big)^{k/q}$$

$$\equiv \sum_{j=0}^{p-1} \Big(\frac{j!}{p}\Big)^{k/q} \pmod p \equiv \sum_{j=0}^{p-1} \Big(\frac{j!}{p}\Big) \pmod p$$

where in the last two lines we have used Euler's criterion for quadratic residues and the fact that k/q is odd.

Note that

$$\Big(\frac{0!}{p}\Big) = 1$$

and also that

$$\Big(\frac{(p-1)!}{p}\Big) = \Big(\frac{-1}{p}\Big) = -1$$

by Wilson's theorem and since $p \equiv 3 \pmod 4$.

We therefore find that $!^k p$ modulo p is a sum of $p-2$ numbers all of which are equal to 1 or -1. Since $p-2$ is odd and $p-2 < p$, it follows that

$$!^k p \not\equiv 0 \pmod p$$

as claimed.

3. A simple generalization

Recall that a Sophie Germain prime is a prime q such that $p = 2q+1$ is also a prime, cf. e.g. [4]. If in the proof of the Theorem we replace $\frac{p_\nu - 1}{2}$ by the ν-th odd Sophie Germain prime q_ν and assume the (widely believed) conjecture that the number of Sophie Germain primes up to x for $x \to \infty$ is asymptotically equal to

$$2C \frac{x}{\log^2 x}$$

where $C := .660161...$ is Hardy-Littlewood's twin prime constant [4, p. 123], then in a similar way as before one can obtain a sequence $(t_n)_{n\in\mathbf{N}}$ of *squarefree* natural numbers such that

$$t_n \leq \exp(c_1 \frac{n\log^2 n}{\log\log n})$$

(where $c_1 > 0$ is an absolute constant) and

$$!^{t_n} p \not\equiv 0 \pmod p$$

for n odd primes p. We leave the details of the proof to the reader.

References

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