A short note on Dedekind sums

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1. Introduction

For integers m and n with $n \geq 1$ and $(m,n)=1$, we denote by $s(m,n)$ the classical Dedekind sum defined by

$$s(m,n) = \sum_{\ell=1}^{n}((\frac{\ell}{n}))((\frac{\ell m}{n})),$$

where as usual

$$((x)) := \begin{cases} x-[x]-\frac{1}{2} & \text{if } x \in \mathbf{R} \setminus \mathbf{Z} \\ 0 & \text{if } x \in \mathbf{Z}, \end{cases}$$

see for instance [8, p.1]. Note that $s(m_1,n)=s(m_2,n)$ if $m_1 \equiv m_2 \pmod{n}$.

It was proved by Hickerson [7] that the values $s(m,n)$ are dense on the real line. In fact, he showed that the set of points $(\frac{m}{n}, s(m,n))$ is dense in the plane. The main tool in the proof were the reciprocity law for Dedekind sums and the theory of continued fractions.

In this short note we want to give a different and very short proof of the first result mentioned above, using the so-called three-term relation of Rademacher and Dieter for Dedekind sums, coupled with Dirichlet's theorem on primes in arithmetic progressions. We wonder if our method can be suitably generalized to obtain the full result proved in [7] this way.

The importance of the three-term relation for example was nicely demonstrated in [6] in connection with detecting Dedekind sums of equal values. This note should be viewed as another example for the usefulness of this relation.

For more general and deeper distribution properties of values of Dedekind sums we refer the reader e.g. to [3] and the literature given there, and to [2,4].

Finally, let us state another related question: are the values $s(m,n)$ dense in the field $\mathbf{Q}_p$ of p-adic numbers, where p is a prime? It seems that this problem has not been discussed in the literature so far at all.

Remark. After finishing this paper, we found that a similar (though not completely identical) argument was also given in [5].

2. Statement of result and proof

We shall prove

Theorem. *The values* $s(m,n)$ $(m,n\in \mathbf{Z}, n>0, (m,n)=1)$ *are dense in* $\mathbf{R}$.

Proof. We find it more convenient to work with

$$S(m,n) := 12s(m,n)$$

instead of $s(m,n)$.

Since $\mathbf{Q}$ is dense in $\mathbf{R}$, it is sufficient to show that any rational number can be approximated by values $S(m,n)$.

Recall the three-term relation for Dedekind sums (cf. e.g. [4, Lemma 1]) which says that

$$S(m,n) = S(c,d) + S(r,q) + \frac{n}{dq} + \frac{d}{nq} + \frac{q}{nd} - 3. \tag{1}$$

Here $n,d\in \mathbf{N}, m,c\in \mathbf{Z}$ with $(m,n)=(c,d)=1$ and $q := md-nc$ is supposed to be positive. Further, if $j,k\in\mathbf{Z}$ and $-cj+dk=1$, then $r := -nk+mj$. (One checks that under the given conditions $(r,q)=1$.)

We remark that identity (1) is a straightforward consequence of the classical evaluation of the automorphy factor of the function $\log\eta(\tau)$ under the action of $SL_2(\mathbf{Z})$, where η is the Dedekind η-function, in terms of Dedekind sums and the fact that the latter action indeed is a group action. This fact has been noticed independently by several authors before.

We now choose $n=d$ in (1) and let m be an inverse of c modulo d. Then

$$S(m,n) = S(c,d)$$

(cf. e.g. [1, Thm. 3.6. (a)]). We will suppose without loss of generality that $m-c>0$.

The assumptions $-cj+dk=1$ and $mc\equiv 1 \pmod d$ imply that

$$r = -dk+mj = -1+j(m-c)$$

$$\equiv -1-m(m-c) \pmod{d(m-c)}.$$

Thus altogether from (1) we obtain that

$$S(-1-m(m-c), d(m-c)) = 3 - \frac{2}{d(m-c)} - \frac{m-c}{d}. \tag{2}$$

Now let x be a positive rational number and write $x=\frac{a}{b}$ with $a,b\in\mathbf{N}$, $(a,b)=1$. Let $\overline{a}$ be an inverse of a modulo b. Since $(4b^2+1,b)=1$ and $(\overline{a},b)=1$ we conclude from Dirichlet's theorem on primes in arithmetic progressions (coupled with the Chinese remainder theorem) that there are infinitely many primes p satisfying

$$p \equiv 1 \pmod{(4b^2+1)}, \quad p \equiv \overline{a} \pmod{b}. \tag{3}$$

Fix a prime p as above and let $d = p$ in (2). We put

$$e := \frac{ap-1}{b}$$

which is an integer by the second congruence in (3). We then find for the Jacobi symbols

$$(\frac{e^2+4}{p}) = (\frac{(ap-1)^2+4b^2}{p})$$

(since p does not divide b)

$$= (\frac{4b^2+1}{p})$$

$$= (\frac{p}{4b^2+1})$$

(by quadratic reciprocity)

$$= 1$$

(by the first congruence in (3)).

Thus e^2+4 is a square modulo p. Choose an integer c such that

$$e^2 + 4 \equiv (2c+e)^2 \pmod{p}.$$

(Note that p must be odd and $(c,p) = 1$.) It follows that

$$c^2 + ec - 1 \equiv 0 \pmod{p},$$

i.e.

$$c(c+e) \equiv 1 \pmod{p}. \tag{4}$$

We now return to (2) and (because of (4)) can choose $m := c+e$. Note that $m - c = e = \frac{ap-1}{b} > 0$ since $a, b > 0$. With the above choices we then find

$$S(-1-me, pe) = 3 - \frac{2}{pe} - \frac{e}{p}$$

and

$$\frac{e}{p} = \frac{ap-1}{bp} \to x \quad (p \to \infty).$$

Thus

$$S(-1-me, pe) \to 3-x \quad (p \to \infty). \tag{5}$$

Since

$$S(-h,k) = -S(h,k)$$

(cf. e.g. [1, Thm. 3.6. (a)]) we also derive from (5) that

$$S(1+me, pe) \to x-3 \quad (p \to \infty). \tag{6}$$

Altogether from (5) and (6) we see that any rational number is an accumulation point of values of Dedekind sums. This finishes the proof.

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