On Perrin-Riou's exponential map for (φ, Γ)-modules

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November 1, 2014

Abstract

Let $K/\mathbb{Q}_p$ be a finite Galois extension and D a (φ, Γ)-module over the Robba-ring $\mathbf{B}_{\mathrm{rig},K}^{\dagger}$. We give a generalization of the Bloch-Kato exponential map for D using continuous Galois-cohomology groups $H^i(G_K, \mathbf{W}(D))$ for the B-pair $\mathbf{W}(D)$ associated to D. We construct a big exponential map $\Omega_{D,h}$ $(h \in \mathbb{N})$ for cyclotomic extensions of K for D in the style of Perrin-Riou using the theory of Berger's B-pairs, which interpolates the generalized Bloch-Kato exponential maps on the finite levels.

Contents

1 Introduction

We fix some notation. Let K be a finite extension of $\mathbb{Q}_p$ and denote by F the biggest subextension of K that is unramified over $\mathbb{Q}_p$. Let μ_{p^n} denote the roots of unity in a fixed

algebraic closure $\overline{K}$ of K and set $K_n = K(\mu_{p^n})$ and $K_\infty = \bigcup_n K_n$. As usual G_K denotes the absolute Galois group of K, and we set $H_K = \mathrm{Gal}(\overline{K}/K_\infty)$ and $\Gamma_K = G_K/H_K$. Perrin-Riou considers a distribution algebra $\mathcal{H}(\Gamma_K)$ that contains the usual Iwasawa algebra $\Lambda(\Gamma_K)$.

Recall that by the theory of Fontaine one may then associate to any p-adic representation of V of G_K finite dimensional F-vector spaces

$$\mathbf{D}_{\mathrm{cris}}(V) \subset \mathbf{D}_{\mathrm{st}}(V) \subset \mathbf{D}_{\mathrm{dR}}(V),$$

via the $\mathbb{Q}_p$-algebras $\mathbf{B}_{\mathrm{cris}}, \mathbf{B}_{\mathrm{st}}, \mathbf{B}_{\mathrm{dR}}$, where the first two come equipped with an action of a Frobenius φ and a nilpotent monodromy operator N, and the third one is equipped with a filtration.

Bloch and Kato constructed the exponential map $\exp : \mathbf{D}_{\mathrm{dR}}(V) \longrightarrow H^1(K, V)$, which is nothing but a transition morphism arising from a long exact sequence of continuous Galois cohomology. They showed that there exists a deep connection between this map and the special values of the complex L-function attached to V.

Perrin-Riou set out to adapt this construction to the theory of p-adic L-functions. Explicitly, for $K/\mathbb{Q}_p$ unramified and V crystalline (i.e. $\dim_F \mathbf{D}_{\mathrm{cris}}(V) = \dim_{\mathbb{Q}_p} V$) she constructed a map $\Omega_{V(j),h}$ that fits into the following diagram

$$\begin{array}{ccc} \mathcal{H}(\Gamma_K) \otimes_{\mathbb{Q}_p} \mathbf{D}_{\mathrm{cris}}(V(j)) & \xrightarrow{\Omega_{V(j),h}} & \mathcal{H}(\Gamma_K) \otimes_\Lambda H^1_{\mathrm{Iw}}(K, V(j))/V(j)^{G_{\mathbb{Q}_{p,n}}} \\ \Big\downarrow{\Xi_{n,j}} & & \Big\downarrow{\mathrm{pr}_n} \\ K_n \otimes \mathbf{D}_{\mathrm{st}}(V(j)) & \xrightarrow{(h-1)!\exp_{K_n n, V(j)}} & H^1(K_n, V(j)) \end{array} \qquad (1)$$

for $h \gg 0$, $j \gg 0$ and all n, where $\Xi_{n,j}$ and pr_n are certain canonical projections and H^1_{Iw} denotes Iwasawa cohomology with respect to the tower $(K_n)_n$. The point here is that $\Omega_{V,h}$ interpolates infinitely many Bloch-Kato exponential maps on the finite levels.

In [28], Perrin-Riou extended her construction to semi-stable representations over unramified extensions. She gave a definition of a free $\mathcal{H}(\Gamma_K)$-module $\mathbf{D}_{\infty,g}(V)$ and a map

$$\Omega_{V,h} : \mathbf{D}_{\infty,g}(V) \longrightarrow \mathcal{H}(\Gamma_{\mathbb{Q}_p}) \otimes_\Lambda H^1_{\mathrm{Iw}}(K, V)/V^{G_{K_\infty}}$$

that has a similar interpolation property as (1) for $j \gg 0$ and $n \gg 0$.

It was Berger who gave an explicit description of a "big exponential map" for crystalline representations using these modules not only on the finite level, but on the whole of $\mathcal{H}(\Gamma_K) \otimes \mathbf{D}_{\mathrm{cris}}(V)$ and $H^1_{\mathrm{Iw}}(K, V)$. His fundamental insight is a comparison isomorphism depending on the construction of another ring $\mathbf{B}^\dagger_{\mathrm{rig},K}$ resp. $\mathbf{B}^\dagger_{\mathrm{log},K}$.

Berger considered in the crystalline case the element $\nabla_{h-1} \circ \ldots \circ \nabla_0$, where $\nabla_i \in \mathcal{H}(\Gamma_K)$ is Perrin-Riou's differential operator, and showed that one obtains a map

$$\nabla_{h-1} \circ \ldots \circ \nabla_0 : (\varphi-1)(\mathbf{B}^\dagger_{\mathrm{rig},K} \otimes \mathbf{D}_{\mathrm{cris}}(V(j)))^{\psi=1} \longrightarrow (\varphi-1)\mathbf{D}^\dagger_{\mathrm{rig}}(V(j))^{\psi=1}$$
$$= \mathcal{H}(\Gamma_{\mathbb{Q}_p}) \otimes_\Lambda H^1_{\mathrm{Iw}}(K, V(j))/V(j)^{G_{K_\infty}}$$

that actually coincides with Perrin-Riou's $\Omega_{V(j),h}$ (see [4], Theorem II.13).

Since one has an embedding of the category of p-adic representations into the category of all (φ, Γ)-modules over $\mathbf{B}_{\mathrm{rig},K}^{\dagger}$ via the functor $\mathbf{D}_{\mathrm{rig}}^{\dagger}(\ )$, one might be inclined to generalize the framework of exponential maps to this setting. Similarly as in the étale case, one may define finite-dimensional vector spaces $\mathbf{D}_{\mathrm{cris}}(D)$, $\mathbf{D}_{\mathrm{st}}(D)$ and $\mathbf{D}_{\mathrm{dR}}(D)$, generalized Bloch-Kato exponential maps

$$\exp : \mathbf{D}_{\mathrm{dR}}(D) \to H^1(K, D),$$

and develop the notion of a (φ, Γ)-module being crystalline, semi-stable or de Rham. We define a $\mathcal{H}(\Gamma_K)$-module $\mathbf{D}_{\infty,g}(D)$ and show that there exists a map for $h \gg 0$

$$\Omega_{D,h} := \nabla_{h-1} \circ \ldots \circ \nabla_0 : \mathbf{D}_{\infty,g}(D) \longrightarrow (\varphi - 1)D^{\psi=1}.$$

The main result of the third section is then the following interpolation property (see Theorem 3.41 for the precise statement):

Theorem. Let D be a de Rham (φ, Γ_K)-module over $\mathbf{B}_{\mathrm{rig},K}^{\dagger}$, $g \in \mathbf{D}_{\infty,g}(D)$ and G a "complete solution" (cf. Definition 3.32) for g in L and let $h \gg 0$. Then for $k \geq 1 - h$ and $n \gg 1$ one has

$$
\begin{aligned}
h^1_{K_n, D(k)} & (\nabla_{h-1} \circ \ldots \circ \nabla_0(g) \otimes e_k) \\
& = p^{-n(K_n)}(-1)^{h+k-1}(h+1-k)! \frac{1}{[L_n : K_n]} \mathrm{Cor}_{L_n/K_n} \exp_{K_n, D(k)}(\Xi_{n,k}(G)).
\end{aligned}
$$

If one is interested in the construction of p-adic L-functions, one needs to construct a certain "inverse" of the map Ω_h. This construction depends on the so-called reciprocity law for (φ, Γ)-modules, which we will return to in a future paper, using the results of this article.

We remark that during this work learned of the results of K. Nakamura [23], who gave a description of a "big exponential map" for (φ, Γ)-modules. We briefly outline how our constructions differ from [23]. Firstly, we show the existence of a fundamental exact sequence

$$0 \longrightarrow X^0(\widetilde{D}) \longrightarrow \widetilde{D}_{\mathrm{log}}[1/t] \longrightarrow X \longrightarrow X^1(\widetilde{D}) \longrightarrow 0$$

of continuous G_K-modules associated to any (φ, Γ)-module D, generalizing the Bloch-Kato fundamental exact sequence (cf. p. 19 for the definition of X). Taking continuous Galois-cohomology one obtains, in a completely analoguous fashion to the étale case, a generalized Bloch-Kato exponential map as the transition map for cohomology, which is automatically functorial by construction.

Secondly, we introduce certain finitely generated $\mathcal{H}(\Gamma)$-modules $\mathbf{D}_{\infty,*}(D)$ of the free $\mathcal{B}(\Gamma)$-module $\mathbf{N}_{\mathrm{dR}}(D)^{\psi=0}$ such that X arises in a natural way after projecting to some finite level K_n and looking at the Bloch-Kato exponential map on this level. Using these two different ingredients we are able to show the main theorem above.

Some important facts about these modules are:

- the $\mathbf{D}_{\infty,*}(D)$ are invariant under Tate-twists (as opposed to $(1 - \varphi)\mathbf{N}_{\mathrm{dR}}(D)^{\psi=1}$), and

- the $\mathbf{D}_{\infty,*}(D)$ remove the ambiguity in the statements [23], Theorem 3.10 (1) and [4], Theorem II.16 about the existence of an element y such that $(1 - \varphi)(y) = x$.

These points and further examples suggest that, in order to study reciprocity laws and the connection of exponential maps with p-adic L-functions, one should look at these modules instead of $(1 - \varphi)\mathbf{N}_{\mathrm{dR}}(D)^{\psi=1}$. We also refer to the introduction of [27] in the étale unramified case for some further motivation.

2 Rings and Modules

2.1 General notations

The general strategy of Fontaine is to study p-adic representations by certain *admissibility* conditions. Recall that if V is a finite dimensional $\mathbb{Q}_p$-vectorspace endowed with a continuous action of a topological group G and if B is a topological $\mathbb{Q}_p$-algebra which also carries an action of G, then Fontaine considers the B^G-modules $D_B(V) = (B \otimes_{\mathbb{Q}_p} V)^G$. It inherits actions from B and V. One says that V is B-**admissible** if $B \otimes_{\mathbb{Q}_p} V \cong B^d$ as G-modules.

Let k be a perfect field of characteristic p. We denote by $W(k)$ the ring of Witt-vectors for k and set $F = \mathrm{Quot}(W(k))$. Let K/F be a totally ramified extension of F. Fix an algebraic closure $\overline{F}$ of F and denote by $\mathbb{C}_p = \widehat{\overline{F}}$ the p-adic completion of this closure. Let $G_K = \mathrm{Gal}(\overline{K}/K)$ be the group of automorphisms of $\overline{K}$ which fix K. By continuity these are also the K-linear automorphisms of $\mathbb{C}_p$. Let $\mathcal{O}_{\mathbb{C}_p}$ be the ring of integers of $\mathbb{C}_p$ and $\mathfrak{m}_{\mathbb{C}_p}$ its maximal ideal. We have $\mathcal{O}_{\mathbb{C}_p}/\mathfrak{m}_{\mathbb{C}_p} = \overline{k}$.

We denote by μ_{p^n} the group of roots of unity of p^n-order in $\mathbb{C}_p$ and set $K_n = K(\mu_{p^n})$. Further we pose $K_\infty = \bigcup_n K_n$. We fix once and for all a compatible set of primitive p-th roots of unity $\{\zeta_{p^n}\}_{n\geq 0}$ such that $\zeta_1 = 1$, $\zeta_p \neq 1$, $\zeta_{p^{n+1}}^p = \zeta_{p^n}$. One has the cyclotomic character $\chi : G_K \to \mathbb{Z}_p^\times$ which is defined by the formula $g(\zeta_{p^n}) = \zeta_{p^n}^{\chi(g)}$ for $n \geq 1$ and $g \in G_K$. We set $H_K = \ker(\chi)$ and $\Gamma_K = G_K/H_K$, which is the Galois group of K_∞/K. We know that this can also be identified via the cyclotomic character with an open subgroup of $\mathbb{Z}_p^\times$.

If $K/\mathbb{Q}_p$ is a finite extension denote by $F = K_0$ the maximal unramified extension of $\mathbb{Q}_p$ in K. Further denote by K_0' the biggest unramified subextension of K_0 in K_∞.

By a p-adic representation we mean a finite dimensional $\mathbb{Q}_p$-vectorspace endowed with a continuous and linear action of G_K. A $\mathbb{Z}_p$-representation is a free $\mathbb{Z}_p$-module of finite rank equipped with a linear and continuous action of G_K. It is known that if V is a p-adic representation then there exists a $\mathbb{Z}_p$-lattice T in V that is stable under the action of G_K.

If $C^\bullet(-)$ denotes complex of R-modules for some commutative ring (for example, $C^\bullet(G_K, M)$) R we denote as usual $R\Gamma(-)$ the complex which we regard as an object in the derived category of R-modules.

2.2 Rings in p-adic Hodge theory

We first recall certain rings constructed by Fontaine, see for instance [16]. Let

$$\widetilde{\mathbf{E}} = \varprojlim_{x \mapsto x^p} \mathbb{C}_p = \{(x^{(0)}, x^{(1)}, \ldots) \mid x^{(i)} \in \mathbb{C}_p, (x^{(i+1)})^p = x^{(i)} \; \forall i\}.$$

Similarly, let

$$\widetilde{\mathbf{E}}^+ = \varprojlim_{x \mapsto x^p} \mathcal{O}_{\mathbb{C}_p} = \{(x^{(0)}, x^{(1)}, \ldots) \mid x^{(i)} \in \mathcal{O}_{\mathbb{C}_p}, (x^{(i+1)})^p = x^{(i)} \; \forall i\}$$

$$\cong \{(x_n)_{n \in \mathbb{N}} \mid x_n \in \mathcal{O}_{\mathbb{C}_p}/p\mathcal{O}_{\mathbb{C}_p}, x_{n+1}^p = x_n \; \forall n\}.$$

This is the set of elements of $\widetilde{\mathbf{E}}$ such that $x^{(0)} \in \mathcal{O}_{\mathbb{C}_p}$. One can define multiplication and addition on these sets. Also, one knows that $\widetilde{\mathbf{E}}$ is the fraction field of $\widetilde{\mathbf{E}}^+$.

With the choices of the primitive p^n-th roots of unity one defines the elements $\varepsilon = (1, \zeta_p, \ldots) \in \widetilde{\mathbf{E}}^+$ and $\overline{\pi} = \varepsilon - 1 \in \widetilde{\mathbf{E}}^+$. One has the usual commuting actions of a Frobenius φ and the Galois group $G_{\mathbb{Q}_p}$ on $\widetilde{\mathbf{E}}$, which restrict to actions of $\widetilde{\mathbf{E}}^+$. For $K/\mathbb{Q}_p$ finite we set

$$\mathbf{E}_K^+ = \{(x_n) \in \widetilde{\mathbf{E}}^+ \mid x_n \in \mathcal{O}_{K_n}/p\mathcal{O}_{K_n} \; \forall n \geq n(K)\},$$

where $n(K)$ is some constant depending on K which arises in the fields of norm theory of Fontaine-Wintenberger (cf. [15]). We put $\mathbf{E}_K = \mathbf{E}_K^+[1/\overline{\pi}]$. One can show that that $\mathbf{E}_F = \kappa((\overline{\pi}))$ and one defines $\mathbf{E}$ as the seperable closure of $\mathbf{E}_F$ in $\widetilde{\mathbf{E}}$. Let $\mathbf{E}^+ = \mathbf{E} \cap \widetilde{\mathbf{E}}^+$ and $\mathfrak{m}_E = \mathbf{E} \cap \mathfrak{m}_{\widetilde{\mathbf{E}}}$. One can show that $\mathbf{E}_K = \mathbf{E}^{H_K}$ and one knows that $\mathrm{Gal}(\mathbf{E}/\mathbf{E}_K) \cong H_K$.

Let W be the Witt functor. We set

$$\widetilde{\mathbf{A}}^+ = W(\widetilde{\mathbf{E}}^+), \quad \widetilde{\mathbf{A}} = W(\widetilde{\mathbf{E}}) = W(\mathrm{Frac}(\widetilde{\mathbf{E}}^+)), \quad \widetilde{\mathbf{B}}^+ = \widetilde{\mathbf{A}}^+[1/p].$$

We write elements $x \in \widetilde{\mathbf{B}}^+$ as $x = \sum_{k \gg -\infty}^{\infty} p^k[x_k]$ where $x_k \in \widetilde{\mathbf{E}}^+$ and $[x_k]$ is its Teichmüller representative. The commuting actions of φ and $G_{\mathbb{Q}_p}$ on $\widetilde{\mathbf{E}}^+$ extend to an action of $\widetilde{\mathbf{B}}^+$ (and $\widetilde{\mathbf{A}}, \widetilde{\mathbf{B}}, \ldots$).

We have a ring homomorphism

$$\theta : \widetilde{\mathbf{B}}^+ \longrightarrow \mathbb{C}_p, \quad \sum_{k \gg -\infty}^{\infty} p^k[x_k] \longmapsto \sum_{k \gg -\infty}^{\infty} p^k x_k^{(0)}.$$

We set $\pi = [\overline{\pi}] = [\varepsilon] - 1$, $\pi_n = [\varepsilon^{p^{-n}}] - 1$, $\omega = \pi/\pi_1$ and $q = \varphi(\omega) = \varphi(\pi)/\pi$. Then $\ker(\theta)$ is a principal ideal generated by ω.

The ring $\mathbf{B}_{\mathrm{dR}}^+$ is defined by completing $\widetilde{\mathbf{B}}^+$ with the $\ker(\theta)$-adic topology, i.e., $\mathbf{B}_{\mathrm{dR}}^+ = \varprojlim_{n \geq 0} \widetilde{\mathbf{B}}^+/(\ker(\theta)^n)$. This gives a complete discrete valuation ring with maximal ideal $\ker(\theta)$. One can show that $\log([\varepsilon])$ converges in $\mathbf{B}_{\mathrm{dR}}^+$, and we denote this element by t. It is a generator of the maximal ideal, hence we can form the field $\mathbf{B}_{\mathrm{dR}} = \mathbf{B}_{\mathrm{dR}}^+[1/t]$. This field is equipped with an action of $G_{\mathbb{Q}_p}$ and a canonical filtration defined by $\mathrm{Fil}^i(\mathbf{B}_{\mathrm{dR}}) = t^i\mathbf{B}_{\mathrm{dR}}^+$, $i \in \mathbb{Z}$.

We say that a p-adic representation V of G_K is **de Rham** if it is $\mathbf{B}_{\mathrm{dR}}$-admissible. We put
$$\mathbf{D}_{\mathrm{dR}}(V) = (\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{G_K}, \quad \mathrm{Fil}^i \mathbf{D}_{\mathrm{dR}}(V) = (\mathrm{Fil}^i \mathbf{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{G_K}.$$

From Fontaine's theory it is known that $\mathbf{D}_{\mathrm{dR}}(V)$ is finite dimensional K-vectorspace which we endow with the above (exhaustive, seperated and decreasing) filtration.

We say that a p-adic representation V is **Hodge-Tate** with Hodge-Tate weights $h_1, \dots, h_d$ if one has a decomposition $\mathbb{C}_p \otimes_{\mathbb{Q}_p} V \cong \bigoplus_{i=1}^d \mathbb{C}_p(h_i)$. We say that V is **positive** if its Hodge-Tate weights are negative. It is known that every de Rham representation is Hodge-Tate and that the Hodge-Tate weights are those integers h such that there is a jump in the filtration at $-h$, i.e. $\mathrm{Fil}^{-h}\mathbf{D}_{\mathrm{dR}}(V) \neq \mathrm{Fil}^{-h+1}\mathbf{D}_{\mathrm{dR}}(V)$. With this convention the representation $\mathbb{Q}_p(1)$ is of weight 1.

Let
$$\mathbf{A}_{\mathbb{Q}_p} = \mathbb{Z}[\widehat{[\overline{\pi}]}][1/\pi] = \left\{ \sum_{k \in \mathbb{Z}} a_k \pi^k \,\middle|\, a_k \in \mathbb{Z}_p, \lim_{k \to -\infty} v_p(a_k) = +\infty \right\} \hookrightarrow \widetilde{\mathbf{A}},$$

and set $\mathbf{B}_{\mathbb{Q}_p} = \mathbf{A}_{\mathbb{Q}_p}[1/p]$. Then $\mathbf{B}_{\mathbb{Q}_p}$ is a field, complete for the p-adic valuation with ring of integers $\mathbf{A}_{\mathbb{Q}_p}$ and residue field $\mathbf{E}_{\mathbb{Q}_p}$. Let $\mathbf{B}$ be the p-adic completion of the maximal unramified extension of $\mathbf{B}_{\mathbb{Q}_p}$ in $\widetilde{\mathbf{B}}$. We define $\mathbf{A} = \mathbf{B} \cap \widetilde{\mathbf{A}}$, $\mathbf{A}^+ = \mathbf{A} \cap \widetilde{\mathbf{A}}^+$. These rings still have the commuting action of φ and $G_{\mathbb{Q}_p}$. We put $\mathbf{A}_K = \mathbf{A}^{H_K}$ and $\mathbf{B}_K = \mathbf{A}_K[1/p]$. By Hensel's Lemma there exists a unique lift $\pi_K \in \mathbf{A}_K$ such that the reduction mod p is equal to $\overline{\pi}_K$, viewed as an element in $\widetilde{\mathbf{A}}$.

Colmez has defined the ring
$$\mathbf{B}_{\max}^+ = \{\sum_{n \geq 0} a_n \frac{\omega^n}{p^n} \mid a_n \in \widetilde{\mathbf{B}}^+, \ a_n \to 0 \text{ for } n \to \infty\}$$

which is "very close" to $\mathbf{B}_{\mathrm{cris}}^+$. We set $\mathbf{B}_{\max} = \mathbf{B}_{\max}^+[1/t]$. There is a canonical injection of $\mathbf{B}_{\max}$ into $\mathbf{B}_{\mathrm{dR}}$ and it is therefore equipped with a canonical filtration. There are actions of φ and $G_{\mathbb{Q}_p}$ on $\mathbf{B}_{\max}$, which extend the actions on $\widetilde{\mathbf{A}}^+ \to \widetilde{\mathbf{A}}^+$. Let
$$\widetilde{\mathbf{B}}_{\mathrm{rig}}^+ = \bigcap_{n=0}^\infty \varphi^n(\mathbf{B}_{\max}^+)$$

and $\widetilde{\mathbf{B}}_{\mathrm{rig}} = \widetilde{\mathbf{B}}_{\mathrm{rig}}^+[1/t]$. We remark that one has $\widetilde{\mathbf{B}}_{\mathrm{rig}} = \bigcap_{n=0}^\infty \varphi^n(\mathbf{B}_{\mathrm{cris}})$ and hence in particular $\widetilde{\mathbf{B}}_{\mathrm{rig}}^{\varphi=1} = \mathbf{B}_{\max}^{\varphi=1} = \mathbf{B}_{\mathrm{cris}}^{\varphi=1}$. We say that a representation is **crystalline** if it is $\mathbf{B}_{\max}$-admissible, which is the same as asking that it be $\widetilde{\mathbf{B}}_{\mathrm{rig}}^+[1/t]$-admissible. We put
$$\mathbf{D}_{\mathrm{cris}}(V) = (\mathbf{B}_{\max} \otimes_{\mathbb{Q}_p} V)^{G_K} = (\widetilde{\mathbf{B}}_{\mathrm{rig}}^+[1/t] \otimes_{\mathbb{Q}_p} V)^{G_K}.$$

This is a K_0-vectorspace of dimension d, equipped with a filtration induced by $\mathbf{B}_{\mathrm{dR}}$ and an action of Frobenius induced by $\mathbf{B}_{\max}$. If V is crystalline we have $\mathbf{D}_{\mathrm{dR}}(V) = K \otimes_{K_0} \mathbf{D}_{\mathrm{cris}}(V)$ which shows that a crystalline representation is also de Rham.

Following Berger the series $\log(\overline{\pi}^{(0)}) + \log(\overline{\pi}/\overline{\pi}^{(0)})$, after a choice of $\log p$, converges in $\mathbf{B}_{\mathrm{dR}}^+$, and we denote the limit by $\log[\overline{\pi}]$. This element is transcendent over $\mathrm{Frac}(\mathbf{B}_{\mathrm{max}}^+)$, and we set $\mathbf{B}_{\mathrm{st}} = \mathbf{B}_{\mathrm{max}}[\log[\overline{\pi}]]$ and $\widetilde{\mathbf{B}}_{\mathrm{log}}^+ = \widetilde{\mathbf{B}}_{\mathrm{rig}}^+[\log[\overline{\pi}]]$. We say that a representation is **semistable** if it is $\mathbf{B}_{\mathrm{st}}$-admissible, which is the same as asking it being $\widetilde{\mathbf{B}}_{\mathrm{log}}^+[1/t]$-admissible. Similarly, as in the crystalline case we put

$$\mathbf{D}_{\mathrm{st}}(V) = (\mathbf{B}_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V)^{G_K} = (\widetilde{\mathbf{B}}_{\mathrm{log}}^+[1/t] \otimes_{\mathbb{Q}_p} V)^{G_K}.$$

Again this is a K_0-vectorspace of dimension d, equipped with a filtration and an action of Frobenius induced by $\mathbf{B}_{\mathrm{st}}$. As before we have in this case $\mathbf{D}_{\mathrm{dR}}(V) = K \otimes_{K_0} \mathbf{D}_{\mathrm{st}}(V)$. Additionally one can define the monodromy operator $N = -d/d\log[\overline{\pi}]$ on $\mathbf{B}_{\mathrm{st}}$ which induces a nilpotent endomorphism on $\mathbf{D}_{\mathrm{st}}(V)$ and satisfies the relation $N\varphi = p\varphi N$. We also make use of the finite dimensional K_0-vectorrspace $\mathbf{D}_{\mathrm{st}}^+(V) = (\widetilde{\mathbf{B}}_{\mathrm{log}}^+ \otimes_{\mathbb{Q}_p} V)^{G_K}$.

Recall that elements $x \in \widetilde{\mathbf{B}}$ may be written in the form $x = \sum_{k \gg -\infty} p^k[x_k]$ with $x_k \in \widetilde{\mathbf{E}}$. For $r > 0$ we set

$$\widetilde{\mathbf{B}}^{\dagger,r} = \left\{ x \in \widetilde{\mathbf{B}} \;\middle|\; \lim_{k \to +\infty} v_{\mathbf{E}}(x_k) + \frac{pr}{p-1}k = +\infty \right\}.$$

We note that x as above converges in $\mathbf{B}_{\mathrm{dR}}$ if and only if $\sum_{k \gg -\infty} p^k x_k^{(0)}$ converges in $\mathbb{C}_p$.

For $n \geq 0$ we set once and for all $r_n = (p-1)p^{n-1}$. Colmez and Cherbonnier showed that for n big enough such that $r_n \geq r$ there is an injection

$$\iota_n = \varphi^{-n} : \widetilde{\mathbf{B}}^{\dagger,r} \longrightarrow \mathbf{B}_{\mathrm{dR}}^+, \qquad \sum_{k \gg -\infty} p^k[x_k] \longmapsto \sum_{k \gg -\infty} p^k[x_k^{p^{-n}}].$$

We put $\widetilde{\mathbf{B}}^{\dagger,n} = \widetilde{\mathbf{B}}^{\dagger,r_n}$. Let $\mathbf{B}^{\dagger,r} = \mathbf{B} \cap \widetilde{\mathbf{B}}^{\dagger,r}$, $\widetilde{\mathbf{B}}^\dagger = \bigcup_{r \geq 0} \widetilde{\mathbf{B}}^{\dagger,r}$, $\mathbf{B}^\dagger = \bigcup_{r \geq 0} \mathbf{B}^{\dagger,r}$. Let $\widetilde{\mathbf{A}}^{\dagger,r}$ be the elements of $\widetilde{\mathbf{B}}^{\dagger,r} \cap \widetilde{\mathbf{A}}$ such that $v_E(x) + \frac{pr}{p-1}k \geq 0$ for all $k \geq 0$. Let $\mathbf{A}^{\dagger,r} = \widetilde{\mathbf{A}}^{\dagger,r} \cap \mathbf{A}$, $\mathbf{A}^\dagger = \widetilde{\mathbf{A}}^\dagger \cap \mathbf{A}$, $\widetilde{\mathbf{A}}^\dagger = \widetilde{\mathbf{B}}^\dagger \cap \widetilde{\mathbf{A}}$. Let $\mathbf{B}_K^{\dagger,r} = (\mathbf{B}^{\dagger,r})^{H_K}$, $\mathbf{A}_K^{\dagger,r} = (\mathbf{A}^{\dagger,r})^{H_K}$, $\widetilde{\mathbf{B}}_K^{\dagger,r} = (\widetilde{\mathbf{B}}^{\dagger,r})^{H_K}$, $\widetilde{\mathbf{A}}_K^{\dagger,r} = (\widetilde{\mathbf{A}}^{\dagger,r})^{H_K}$.

Proposition 2.1. If L/K be a finite extension then $\mathbf{B}_L^\dagger$ is a finite field extension of $\mathbf{B}_K^\dagger$ of degree $[L_\infty : K_\infty] = [H_K : H_L]$, and if L/K is Galois, then the same holds for $\mathbf{B}_L^\dagger/\mathbf{B}_K^\dagger$, which then has galois group $\mathrm{Gal}(L_\infty/K_\infty)$.

Proof. See [10], Proposition II.4.1. $\qquad\qquad\square$

If A is a ring which is complete for the p-adic topology and X, Y are indeterminantes we let

$$A\{X,Y\} = \varprojlim_n A[X,Y]/p^n A[X,Y],$$

that is, $A\{X,Y\}$ is the p-adic completion of $A[X,Y]$. Every element of $A\{X,Y\}$ can be written as $\sum_{i,j \geq 0} a_{ij} X^i Y^j$ where a_{ij} is a sequence in A tending to 0 in the p-adic

topology. We let $r, s \in \mathbb{N}[1/p] \cup \{+\infty\}$ such that $r \leq s$. By definition one has (in $\mathrm{Fr}(\widetilde{\mathbf{B}})$) $p/[\overline{\pi}]^{+\infty} = 1/[\overline{\pi}]$ and $[\overline{\pi}]^{+\infty}/p = 0$. Let

$$\begin{aligned}
\widetilde{\mathbf{A}}_{[r;s]} &= \widetilde{\mathbf{A}}^+ \{ \frac{p}{[\overline{\pi}]^r}, \frac{[\overline{\pi}]^s}{p} \} \\
&= \widetilde{\mathbf{A}}^+ \{X, Y\} / ([\overline{\pi}]^r X - p, pY - [\overline{\pi}]^s, XY - [\overline{\pi}]^{s-r}), \\
\widetilde{\mathbf{B}}_{[r;s]} &= \widetilde{\mathbf{A}}_{[r;s]}[1/p].
\end{aligned}$$

If I is any interval of $\mathbb{R} \cup \{+\infty\}$ we let $\widetilde{\mathbf{B}}_I = \bigcap_{[r;s] \subset I} \widetilde{\mathbf{B}}_{[r;s]}$. It is clear that if $I \subset J$ are two closed intervals then $\widetilde{\mathbf{B}}_J \subset \widetilde{\mathbf{B}}_I$. One has a p-adic valuation V_I on $\widetilde{\mathbf{B}}_I$ defined by the condition $V_I(x) = 0$ if and only if $x \in \widetilde{\mathbf{A}}_I \setminus p\widetilde{\mathbf{A}}_I$ and such that the image of V_I is $\mathbb{Z}$. With this valuation $\widetilde{\mathbf{B}}_I$ becomes a p-adic Banach space.

The action of G_F on $\widetilde{\mathbf{A}}^+$ extends to $\widetilde{\mathbf{A}}^+[p/[\overline{\pi}]^r, [\overline{\pi}]^s/p]$ and by continuity further extends to $\widetilde{\mathbf{A}}_I$ and $\widetilde{\mathbf{B}}_I$. The Frobenius φ extends to a morphism

$$\varphi : \widetilde{\mathbf{A}}^+ [\frac{p}{[\overline{\pi}]^r}, \frac{[\overline{\pi}]^s}{p}] \longrightarrow \widetilde{\mathbf{A}}^+ [\frac{p}{[\overline{\pi}]^{pr}}, \frac{[\overline{\pi}]^{ps}}{p}]$$

and finally to a map $\varphi : \widetilde{\mathbf{A}}_I \to \widetilde{\mathbf{A}}_{pI}$ for every I.

Berger defines $\widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r} = \widetilde{\mathbf{B}}_{[r,+\infty[}$, $\widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger} = \bigcup_{r \geq 0} \widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r}$. $\widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r}$ is endowed with the Fréchet topology defined by the family of valuations V_I for closed subsets $I \subset [r, +\infty[$, and subsequently $\widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger}$ is an LF-space. One can define $\widetilde{\mathbf{A}}_{\mathrm{rig}}^{\dagger,r}$ as the ring of integers of $\widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r}$ with respect to the valuation $V_{[r;r]}$. We put $\widetilde{\mathbf{A}}_{\mathrm{rig}}^{\dagger} = \bigcup_{r \geq 0} \widetilde{\mathbf{A}}_{\mathrm{rig}}^{\dagger,r}$. One defines $\mathbf{B}_{\mathrm{rig},K}^{\dagger}$ to be the LF-space arising from the completion of the $\mathbf{B}_K^{\dagger,r}$ with respect to the Fréchet topology induced by the V_I. Further, let $\mathbf{B}_{\mathrm{rig}}^{\dagger} = \mathbf{B}_{\mathrm{rig},F}^{\dagger} \otimes_{\mathbf{B}_F^{\dagger}} \mathbf{B}^{\dagger}$.

Lemma 2.2.   a) $\mathbf{B}_{\mathrm{rig},K}^{\dagger} = \mathbf{B}_{\mathrm{rig},F}^{\dagger} \otimes_{\mathbf{B}_F^{\dagger}} \mathbf{B}_K^{\dagger}$.

  b) $\mathbf{B}_{\mathrm{rig}}^{\dagger} = \mathbf{B}_{\mathrm{rig},K}^{\dagger} \otimes_{\mathbf{B}_K^{\dagger}} \mathbf{B}^{\dagger}$.

  c) $(\mathbf{B}_{\mathrm{rig}}^{\dagger})^{H_K} = \mathbf{B}_{\mathrm{rig},K}^{\dagger}$.

Proof. See [3], section 3.4. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ □

Berger has shown the existence of unique map $\log : \widetilde{\mathbf{A}}^+ \to \widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger}[X]$ such that $\log([x]) = \log[x]$, $\log(p) = 0$ and $\log(xy) = \log(x) + \log(y)$. Hence one defines $\log \pi := \log(\pi)$ and sets $\widetilde{\mathbf{B}}_{\log}^{\dagger} = \widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger}[\log \pi]$, $\mathbf{B}_{\log}^{\dagger} = \mathbf{B}_{\mathrm{rig}}^{\dagger}[\log \pi]$ and $\mathbf{B}_{\log,K}^{\dagger} = \mathbf{B}_{\mathrm{rig},K}^{\dagger}[\log \pi]$. One defines a monodromy operator N on $\widetilde{\mathbf{B}}_{\log}^{\dagger}$ by extending $N \log \pi := -p/(p-1)$ in the usual way.

2.3 (φ, Γ_K)-modules over $\mathbf{B}_{\mathrm{rig},K}^{\dagger}$

We describe how to extend certain results of [3] to (in general non-étale) (φ, Γ_K)-modules, cf. also [6].

We make use of the following notation: Suppose R is a commutative ring equipped with an endomorphism $f : R \to R$, and M is a R-module. We may then consider the R-module $R \otimes_{f,R} M$, where R is considered as an R-module via $r \cdot s := f(r)s$ $(r, s \in R)$.

a) A (φ, Γ_K)-**module** D over $\mathbf{B}^\dagger_{\mathrm{rig},K}$ is a free, finitely generated $\mathbf{B}^\dagger_{\mathrm{rig},K}$-module with a semi-linear continuous map φ_D (i.e. $\varphi_D(\lambda x) = \varphi(\lambda)\varphi_D(x)$ for $\lambda \in \mathbf{B}^\dagger_{\mathrm{rig},K}, x \in D$) and a continuous action of Γ_K which commutes with φ_D, such that the map

$$\varphi^* : \mathbf{B}^\dagger_{\mathrm{rig},K} \otimes_{\varphi,\mathbf{B}^\dagger_{\mathrm{rig},K}} D \longrightarrow D, \quad a \otimes x \longmapsto a\varphi(x)$$

is an isomorphism of $\mathbf{B}^\dagger_{\mathrm{rig},K}$-modules.

b) (φ, Γ_K)-module D over $\mathbf{B}^\dagger_{\mathrm{rig},K}$ is **étale** (or **of slope 0**) if there exists p-adic representation V such that $D = \mathbf{D}^\dagger_{\mathrm{rig},K}(V)$.

For example, for a p-adic representation V we set $\mathbf{D}^\dagger_{\mathrm{rig},K}(V) := (\mathbf{B}^\dagger_{\mathrm{rig},K} \otimes_{\mathbb{Q}_p} V)^{H_K}$. Furthermore, let us define $\mathbf{D}^\dagger_{\log,K}(V) := (\mathbf{B}^\dagger_{\log} \otimes_{\mathbb{Q}_p} V)^{H_K}$ and $\mathbf{D}^+_{\mathrm{rig},K} := (\mathbf{B}^+_{\mathrm{rig}} \otimes_{\mathbb{Q}_p} V)^{H_K}$. Then $\mathbf{D}^\dagger_{\mathrm{rig},K}(V)$ is a (φ, Γ)-module over $\mathbf{B}^\dagger_{\mathrm{rig},K}$.

Let D be a (φ, Γ_K)-module over $\mathbf{B}^\dagger_{\mathrm{rig},K}$. φ_D will henceforth simply be denoted by φ. For the ring $\mathbf{B}^\dagger_{\mathrm{rig},K}$ we have a decomposition $\mathbf{B}^\dagger_{\mathrm{rig},K} = \bigoplus_{i=0}^{p-1} (1 + \pi)^i \varphi(\mathbf{B}^\dagger_{\mathrm{rig},K})$ so that one may define an operator ψ on $\mathbf{B}^\dagger_{\mathrm{rig},K}$ by sending $\sum_{i=0}^{p-1} (1 + \pi)^i \varphi(x_i)$ to x_0, that extends a similarly defined operator ψ on $\mathbf{B}^\dagger_K$. More generally, if D is a (φ, Γ_K)-module over $\mathbf{B}^\dagger_{\mathrm{rig},K}$ we have thanks to condition a) in the definition of (φ, Γ)-modules that there exists a unique operator ψ on D that is defined by the same formula and that and commutes with the action of Γ_K.

Proposition 2.3. If $0 \to D' \to D \to D'' \to 0$ is an exact sequence of (φ, Γ_K)-modules over $\mathbf{B}^\dagger_{\mathrm{rig},K}$ then $0 \to D'^{\psi=0} \to D^{\psi=0} \to D''^{\psi=0} \to 0$ is an exact sequence of Γ_K-modules.

Proof. For the proof of the right-exactness one just uses the fact that if $x \in D^{\psi=0}$ then (uniquely) $x = \sum_{i=1}^{p-1} (1 + \pi)^i \varphi(x_i)$ with $x_i \in D$. The compatibility with the action of Γ_K is clear since it commutes with ψ. $\square$

If L/K is a finite extension, we denote the **restriction** $D|_L$ by

$$D|_L := \mathbf{B}^\dagger_{\mathrm{rig},L} \otimes_{\mathbf{B}^\dagger_{\mathrm{rig},K}} D,$$

with actions of φ and Γ_L defined diagonally. Hence, $D|_L$ is a (φ, Γ_L)-module over $\mathbf{B}^\dagger_{\mathrm{rig},L}$.

The **dual** D^* of a (φ, Γ_K)-module D over $\mathbf{B}^\dagger_{\mathrm{rig},K}$ is defined by

$$D^* := \mathrm{Hom}_{\mathbf{B}^\dagger_{\mathrm{rig},K}}(D, \mathbf{B}^\dagger_{\mathrm{rig},K}),$$

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where for $f \in D^*$ the actions of Γ_K and φ are defined via

$$\gamma(f)(x) := \gamma(f(\gamma^{-1}x)), \ \gamma \in \Gamma_K, x \in D, \quad \varphi(f)(x) := \sum a_i \varphi(f(x_i)), \ x = \sum a_i \varphi(x_i) \in D.$$

If D_1, D_2 are two (φ, Γ_K)-modules over $\mathbf{B}_{\mathrm{rig},K}^\dagger$ then the **tensor product** of D_1 and D_2 is defined by

$$D_1 \otimes D_2 := D_1 \otimes_{\mathbf{B}_{\mathrm{rig},K}^\dagger} D_2,$$

where φ and Γ_K act diagonally. Note that this does *not* imply that ψ acts diagonally.

Let D be a (φ, Γ_K)-module over $\mathbf{B}_{\mathrm{rig},K}^\dagger$ of rank d. By [6], Theorem I.3.3 there exists an $n(D)$ and a unique finite free $\mathbf{B}_{\mathrm{rig},K}^{\dagger,r_{n(D)}}$-module $D^{(n(D))} \subset D$ of rank d with

a) $\mathbf{B}_{\mathrm{rig},K}^\dagger \otimes_{\mathbf{B}_{\mathrm{rig},K}^{\dagger,r_{n(D)}}} D^{(n(D))} = D$,

b) Let $D^{(n)} = \mathbf{B}_{\mathrm{rig},K}^{\dagger,r_n} \otimes_{\mathbf{B}_{\mathrm{rig},K}^{\dagger,r_{n(D)}}} D^{(n(D))}$ for each $n \geq n(D)$. Then $\varphi(D^{(n)}) \subset D^{(n+1)}$ and the map

$$\mathbf{B}_{\mathrm{rig},K}^{\dagger,r_{n+1}} \otimes_{\varphi, \mathbf{B}_{\mathrm{rig},K}^{\dagger,r_n}} D^{(n)} \to D^{(n+1)}, \quad a \otimes x \mapsto a\varphi(x),$$

   is an isomorphism.

2.4 $\widetilde{\mathbf{B}}_{\mathrm{rig}}^\dagger$-modules and B-pairs

Let us collect some facts about φ-modules over $\widetilde{\mathbf{B}}_{\mathrm{rig}}^\dagger$.

Definition 2.4. Let $h \geq 1$ and $a \in \mathbb{Z}$. The **elementary φ-module** $M_{a,h}$ is the φ-module over $\widetilde{\mathbf{B}}_{\mathrm{rig}}^\dagger$ with basis $e_0, \ldots, e_{h-1}$ and $\varphi(e_0) = e_1, \ldots, \varphi(e_{h-2}) = e_{h-1}, \varphi(e_{h-1}) = p^a e_0$.

Proposition 2.5. If M is a φ-module over $\widetilde{\mathbf{B}}_{\mathrm{rig}}^\dagger$ then there exist integers a_i, h_i such that $M \cong \bigoplus_i M_{a_i,h_i}$.

Proof. See [19], Theorem 4.5.7. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Definition 2.6. Let M be a φ-module over $\widetilde{\mathbf{B}}_{\mathrm{rig}}^\dagger$. If $M = M_{a,h}$ is elementary one defines the **slope** of M as $\mu(M) = a/h$ and one says that M is **pure** of this slope. In general if $M \equiv \bigoplus M_{a_i,h_i}$ one define $\mu(M) = \sum \mu(M_{a_i,h_i})$, so that μ is compatible with short exact sequences.

Let D now be a (φ, Γ)-module over $\mathbf{B}_{\mathrm{rig},K}^\dagger$. One sets $\mathbf{B}_e := (\widetilde{\mathbf{B}}_{\mathrm{rig}}^\dagger[1/t])^{\varphi=1}$. From [5], Proposition 2.2.6, we know that

a) $\mathbf{W}_e(D) := (\widetilde{\mathbf{B}}_{\mathrm{rig}}^\dagger[1/t] \otimes_{\mathbf{B}_{\mathrm{rig},K}^\dagger} D)^{\varphi=1}$ is a free $\mathbf{B}_e$-module of rank d which inherits an action of G_K,

b) $\mathbf{W}_{\mathrm{dR}}^+(D) := \mathbf{B}_{\mathrm{dR}}^+ \otimes_{\iota_n, \mathbf{B}_{\mathrm{rig},K}^{\dagger,r_n}} D^{(n)}$ does not depend on $n \gg 0$ and is a free $\mathbf{B}_{\mathrm{dR}}^+$-module of rank d which inherits an action of G_K.

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With this in mind, Berger defined:

Definition 2.7. A tuple $W = (W_e, W_{\mathrm{dR}}^+)$, where W_e is a free $\mathbf{B}_e$-module of finite rank equipped with an semi-linear action of G_K and W_{dR}^+ is a $\mathbf{B}_{\mathrm{dR}}^+$-lattice in $\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{B}_e} W_e$ that is stable under the action of G_K, is called a **B-pair**.

From [5], Proposition 2.2.6 it follows that the tuple $\mathbf{W}(D) = (\mathbf{W}_e(D), \mathbf{W}_{\mathrm{dR}}^+(D))$ actually is a B-pair. Furthermore, Berger proved:

Theorem 2.8. The functor $D \mapsto \mathbf{W}(D)$ gives rise to an equivalence of categories between the category of (φ, Γ_K)-modules over $\mathbf{B}_{\mathrm{rig},K}^\dagger$ and the category of B-pairs.

One knows (cf. [6], section 2.2.) how to construct a functor $\widetilde{\mathbf{D}}$ from the category of B-pairs to the category of (φ, G_K)-modules over $\widetilde{\mathbf{B}}_{\mathrm{rig}}^\dagger$ such that there exists a unique (φ, Γ_K)-module $\mathbf{D}(W)$ over $\mathbf{B}_{\mathrm{rig},K}^\dagger$ with $\widetilde{\mathbf{B}}_{\mathrm{rig}}^\dagger \otimes_{\mathbf{B}_{\mathrm{rig},K}^\dagger} \mathbf{D}(W) = \widetilde{\mathbf{D}}(W)$. Hence, one has, similarly as in the preceding theorem:

Theorem 2.9. The functor $D \mapsto \widetilde{D} := \widetilde{\mathbf{B}}_{\mathrm{rig}}^\dagger \otimes_{\mathbf{B}_{\mathrm{rig},K}^\dagger} D$ gives rise to an equivalence of categories between the category of (φ, Γ_K)-modules over $\mathbf{B}_{\mathrm{rig},K}^\dagger$ and the category of (φ, G_K)-modules over $\mathbf{B}_{\mathrm{rig}}^\dagger$.

We shall also abbreviate $\widetilde{D}_{\log} = \widetilde{\mathbf{B}}_{\log}^\dagger \otimes_{\mathbf{B}_{\mathrm{rig},K}^\dagger} D$ and $\mathbf{W}_{\mathrm{dR}}(D) := \mathbf{B}_{\mathrm{dR}} \otimes_{\iota_n \mathbf{B}_{\mathrm{rig},K}^{\dagger,r_n}} D^{(n)}$, which is independent of the choice of n for $n \gg 0$.

It is known that the canonical map

$$\widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r_n} \otimes_{\mathbf{B}_e} \mathbf{W}_e(D) \to \widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r_n} \otimes_{\mathbf{B}_{\mathrm{rig},K}^{\dagger,r_n}} D^{(n)},$$

induced by $a \otimes x \mapsto ax$, is an isomorphism of G_K-modules for every $n \geq n(D)$. One defines the following map of G_K-modules:

$$\beta : \mathbf{W}_e(D) \hookrightarrow \mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{B}_e} \mathbf{W}_e(D) \cong \mathbf{B}_{\mathrm{dR}} \otimes_{\iota_n, \mathbf{B}_{\mathrm{rig},K}^{\dagger,r_n}} (\widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r_n} \otimes_{\mathbf{B}_e} \mathbf{W}_e(D)) \cong \mathbf{W}_{\mathrm{dR}}(D). \quad (2)$$

We use the same symbol for the map $\beta : \mathbf{W}_e(D) \to \mathbf{B}_{\mathrm{dR}}/\mathbf{B}_{\mathrm{dR}}^+ \otimes_{\mathbf{B}_e} \mathbf{W}_e(D)$. Set $\mathbf{W}_e^+(D) = (\widetilde{\mathbf{B}}_{\mathrm{rig}}^\dagger \otimes_{\mathbf{B}_{\mathrm{rig},K}^\dagger} D)^{\varphi=1}$. $\mathbf{W}_e(D)$.

Let now W be a B-pair and set $X^0(W) = W_e \cap W_{\mathrm{dR}}^+ \subset W_{\mathrm{dR}}$ and $X^1(W) = W_{\mathrm{dR}}/(W_e + W_{\mathrm{dR}}^+)$, which are nothing but the kernel and cokernel respectively of the natural map $W_e \to W_{\mathrm{dR}}/W_{\mathrm{dR}}^+$. Hence, one has ([7], Theorem 3.1):

Theorem 2.10. If W is a B-pair and $\widetilde{D} = \widetilde{\mathbf{D}}(W)$, there are natural identifications

a) $X^0(W) \cong \mathbf{W}_e^+(D)$ and $X^1(W) \cong \widetilde{D}/(1 - \varphi)$,

b) $X^0(W) = 0$ if and only if all slopes of $\widetilde{D}$ are > 0; $X^1(W) = 0$ if and only if all slopes of $\widetilde{D}$ are ≤ 0.

We recall the following definition, introduced by Fontaine (see [14]):

Definition 2.11. An **almost $\mathbb{C}_p$-representation** is a p-adic Banach space X equipped with a linear and continuous action of G_K such that there exists a $d \geq 0$ and two (finite-dimensional) p-adic representations $V_1 \subset X$, $V_2 \subset \mathbb{C}_p^d$ such that $X/V_1 \cong \mathbb{C}_p^d/V_2$.

Berger has shown that $X^0(W)$ and $X^1(W)$ are almost $\mathbb{C}_p$-representations, cf. [7].

2.5 Cohomology of (φ, Γ_K)-modules

Liu (cf. [22]) has worked out reasonable definitions for cohomology of (in general non-étale) (φ, Γ_K)-modules over $\mathbf{B}_K, \mathbf{B}_K^\dagger$ and $\mathbf{B}_{\mathrm{rig},K}^\dagger$.

Let D be a (φ, Γ_K)-module over one of these rings and let Δ_K be a torsion subgroup of Γ_K. Γ_K is an open subgroup of $\mathbb{Z}_p^\times$ and Δ_K is a finite group of order dividing $p-1$ (or 2 if $p = 2$). Define the idempotent operator p_{Δ_K} by $p_{\Delta_K} = (1/|\Delta_K|) \sum_{\delta \in \Delta_K} \delta$, so that p_{Δ_K} is the projection from D to $D' := D^{\Delta_K}$. If $\Gamma_K' := \Gamma_K/\Delta_K$ is procyclic with generator γ_K define the exact sequence

$$C_{\varphi,\gamma_K}^\bullet(D): \ 0 \longrightarrow D' \xrightarrow{\ d_1\ } D' \oplus D' \xrightarrow{\ d_2\ } D' \longrightarrow 0 \tag{3}$$

with

$$d_1(x) = ((\varphi - 1)x, (\gamma_K - 1)x), \quad d_2(x, y) = (\gamma_K - 1)x - (\varphi - 1)y.$$

Define for $i \in \mathbb{Z}$

$$H^i(K, D) := H^i(C_{\varphi,\gamma_K}^\bullet(D)),$$

which is, up to canonical isomorphism, independent of the choice of γ_K (cf. [22], section 2), so that we shall now fix a choice of Δ_K and γ_K.

For applications in Iwasawa-theory one also considers the following complex:

$$C_{\psi,\gamma_K}^\bullet(D): \ 0 \longrightarrow D' \xrightarrow{\ d_1\ } D' \oplus D' \xrightarrow{\ d_2\ } D' \longrightarrow 0 \tag{4}$$

with

$$d_1(x) = ((\psi - 1)x, (\gamma_K - 1)x), \quad d_2(x, y) = (\gamma_K - 1)x - (\psi - 1)y.$$

If D_1 and D_2 are two (φ, Γ_K)-modules over $\mathbf{B}_{\mathrm{rig},K}^\dagger$ one may, following Herr ([17]), define the following cup products (we always mean classes where appropriate):

$$
\begin{aligned}
H^0(K, D_1) \times H^0(K, D_2) &\longrightarrow H^0(K, D_1 \otimes D_2), & (x, y) &\mapsto (x \otimes y), \\
H^0(K, D_1) \times H^1(K, D_2) &\longrightarrow H^1(K, D_1 \otimes D_2), & (x, (y, z)) &\mapsto (x \otimes y, x \otimes z), \\
H^0(K, D_1) \times H^2(K, D_2) &\longrightarrow H^2(K, D_1 \otimes D_2), & (x, y) &\mapsto (x \otimes y), \\
H^1(K, D_1) \times H^1(K, D_2) &\longrightarrow H^2(K, D_1 \otimes D_2), & ((x, y), (w, v)) &\mapsto y \otimes \gamma_K(w) - x \otimes \varphi(v).
\end{aligned}
\tag{5}
$$

We note that some authors swap the maps of the sequence $C_{\varphi,\gamma_K}^\bullet(D)$ so that of course one has to adjust the definition of the cup-product. We adhere to the conventions made in [17].

Liu's result is then ([22], Theorem 0.1 and Theorem 0.2):

Theorem 2.12. Let D be a (φ, Γ_K)-module over $\mathbf{B}_{\mathrm{rig},K}^\dagger$.

a) If $D = \mathbf{D}_{\mathrm{rig}}^\dagger(V)$ is étale one has canonical functorial isomorphisms $H^i(K, \mathbf{D}_{\mathrm{rig}}^\dagger(V)) \cong H^i(G_K, V)$ for all $i \in \mathbb{Z}$ that are compatible with cup-products.

b) $H^i(K, D)$ is a finite dimensional $\mathbb{Q}_p$-vectorspace and vanishes for $i \neq 0, 1, 2$.

c) For $i = 0, 1, 2$ the pairing

$$H^i(K, D) \times H^{2-i}(K, D^*(1)) \longrightarrow H^2(K, D \otimes D^*(1)) = H^2(K, \mathbf{B}_{\mathrm{rig},K}^\dagger(1))$$
$$= H^2(K, \mathbb{Q}_p(1)) \cong \mathbb{Q}_p$$

where $D \otimes D^*(1) \to \mathbf{B}_{\mathrm{rig},K}^\dagger(1)$ is the map $x \otimes f \mapsto f(x)$, is perfect.

Recall that $D|_L = \mathbf{B}_{\mathrm{rig},L}^\dagger \otimes_{\mathbf{B}_{\mathrm{rig},K}^\dagger} D$. Let $m = [\Delta_K : \Delta_L]$ and n be such that $\gamma_K^{p^n} = \gamma_L$. Define $\tau_{L/K} = \sum_{i=0}^{p^n-1} \gamma_K^i$ and $\sigma_{L/K} = \sum_{g \in \Gamma_K / \Gamma_L} g$. We define the restriction maps Res : $H^i(K, D) \to H^i(L, D|_L)$ via the map induced by the following map on complexes (where $*'$ means the invariants with respect to the "right" Δ):

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & D' & \xrightarrow{d_1} & D' \oplus D' & \xrightarrow{d_2} & D' & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id} \oplus (m \cdot \tau_{L/K})} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & D|_L' & \xrightarrow{d_1} & D|_L' \oplus D|_L' & \xrightarrow{d_2} & D|_L' & \longrightarrow & 0
\end{array}
$$

Similarly, we define the corestriction map Cor : $H^i(K, D) \to H^i(L, D|_L)$ via the map induced by the following map on complexes:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & D|_L' & \xrightarrow{d_1} & D|_L' \oplus D|_L' & \xrightarrow{d_2} & D|_L' & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle \sigma_{L/K}} & & \downarrow{\scriptstyle \sigma_{L/K} \oplus \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & D' & \xrightarrow{d_1} & D' \oplus D' & \xrightarrow{d_2} & D' & \longrightarrow & 0
\end{array}
$$

Proposition 2.13. The map Cor $\circ$ Res on $H^i(K, D)$ is nothing but multiplication by $[L : K]$.

Proof. It is clear that on $H^0(K, D) = D^{\varphi=1, \gamma_K=1}$ (thus γ_K acts trivially) the map Cor$\circ$Res is just the trace map and equal to multiplication by $[L : K]$. Since the $H^i(K, D)$ are cohomological δ-functors (see [21], Theorem 8.1) we get the claim. $\square$

2.6 $(\varphi, N, \mathbf{Gal}(L/K))$-modules associated to (φ, Γ_K)-modules

We begin with a series of definitions (see [3], section 5, and [6]).

Definition 2.14. Let D be (φ, Γ_K)-module and $n \geq n(D)$. Set

$$\mathbf{D}_{\mathrm{dif},n}^+(D) := K_n[[t]] \otimes_{\iota_n, \mathbf{B}_{\mathrm{rig},K}^{\dagger,r_n}} D^{(n)}, \quad \mathbf{D}_{\mathrm{dif},n}(D) := K_n((t)) \otimes_{\iota_n, \mathbf{B}_{\mathrm{rig},K}^{\dagger,r_n}} D^{(n)}$$

and, via the transition maps $\mathbf{D}_{\mathrm{dif},n}^+(D) \hookrightarrow \mathbf{D}_{\mathrm{dif},n+1}^+$, $f(t) \otimes x \mapsto f(t) \otimes \varphi(x)$ (and similarly for $\mathbf{D}_{\mathrm{dif},n}(D) \hookrightarrow \mathbf{D}_{\mathrm{dif},n+1}$)

$$\mathbf{D}_{\mathrm{dif}}^+(D) := \varinjlim_n \mathbf{D}_{\mathrm{dif},n}^+(D), \quad \mathbf{D}_{\mathrm{dif}}(D) := \varinjlim_n \mathbf{D}_{\mathrm{dif},n}(D).$$

Note that $\mathbf{D}_{\mathrm{dif}}^+(D)$ (resp. $\mathbf{D}_{\mathrm{dif}}(D)$) is a free $K_\infty[[t]] := \bigcup_{n=1}^\infty K_n[[t]]$- (resp. $K_\infty((t)) = K_\infty[[t]][1/t]$-)module of rank d with a semi-linear action of Γ_K. One defines a Γ_K-equivariant injection

$$\iota_n : D^{(n)} \hookrightarrow \mathbf{D}_{\mathrm{dif},n}^+(D), \quad x \mapsto 1 \otimes x.$$

Definition 2.15. Let D be a (φ, Γ_K)-module. Set

$$\mathbf{D}_{\mathrm{cris}}^K(D) := (\mathbf{B}_{\mathrm{rig},K}^\dagger[1/t] \otimes_{\mathbf{B}_{\mathrm{rig},K}^\dagger} D)^{\Gamma_K},$$

$$\mathbf{D}_{\mathrm{st}}^K(D) := (\mathbf{B}_{\mathrm{log},K}^\dagger[1/t] \otimes_{\mathbf{B}_{\mathrm{rig},K}^\dagger} D)^{\Gamma_K},$$

$$\mathbf{D}_{\mathrm{dR}}^K(D) := (\mathbf{D}_{\mathrm{dif}}(D))^{\Gamma_K},$$

and

$$\mathrm{Fil}^i \mathbf{D}_{\mathrm{dR}}^K(D) := \mathbf{D}_{\mathrm{dR}}^K(D) \cap t^i \mathbf{D}_{\mathrm{dif}}^+(D) \subset \mathbf{D}_{\mathrm{dif}}(D), \ i \in \mathbb{Z}.$$

The filtration $\mathrm{Fil}^i \mathbf{D}_{\mathrm{dR}}^K(D)$ is decreasing, separated and exhaustive. We also set $\mathbf{D}_{\mathrm{dR}}^{K,+}(D) := \mathrm{Fil}^0(\mathbf{D}_{\mathrm{dR}}^K(D)) = \mathbf{D}_{\mathrm{dif}}^+(D)^{\Gamma_K}$.

One has canonical maps which we will denote by α_* for $* \in \{\mathrm{cris}, \mathrm{st}, \mathrm{dR}\}$, induced by $a \otimes d \mapsto ad$:

$$\mathbf{B}_{\mathrm{rig},K}^\dagger[1/t] \otimes \mathbf{D}_{\mathrm{cris}}^K(D) \to D[1/t]$$
$$\mathbf{B}_{\mathrm{log},K}^\dagger[1/t] \otimes \mathbf{D}_{\mathrm{st}}^K(D) \to \mathbf{B}_{\mathrm{log},K}^\dagger[1/t] \otimes D,$$
$$K_\infty((t)) \otimes \mathbf{D}_{\mathrm{dR}}^K(D) \to \mathbf{D}_{\mathrm{dif}}(D).$$

Proposition 2.16. All maps α_* above are injective. Hence, one always has inequalities

$$\dim_{K_0} \mathbf{D}_{\mathrm{cris}}^K(D) \leq \dim_{K_0} \mathbf{D}_{\mathrm{st}}^K(D) \leq \dim_K \mathbf{D}_{\mathrm{dR}}^K(D) \leq \mathrm{rank}_{\mathbf{B}_{\mathrm{rig},K}^\dagger} D,$$

and equalities $\dim \mathbf{D}_*^K(D) = \mathrm{rank}_{\mathbf{B}_{\mathrm{rig},K}^\dagger} D$ for $* \in \{\mathrm{cris}, \mathrm{st}, \mathrm{dR}\}$ if and only if the corresponding α is an isomorphism.

Proof. Standard proof. $\qquad\square$

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Definition 2.17. The **Hodge-Tate weights** of a (φ, Γ_K)-module are those integers h such that $\text{Fil}^{-h} \mathbf{D}_{\text{dR}}^K(D) \neq \text{Fil}^{-h+1} \mathbf{D}_{\text{dR}}^K(D)$. We say that D is **positive** if $h \leq 0$ for all weights h, and that D is **negative** if $h \geq 0$ for all weights h.

Proposition 2.18. Let D be a de Rham (φ, Γ_K)-module over $\mathbf{B}_{\text{rig},K}^\dagger$. If D is positive then $\mathbf{D}_{\text{dR}}^{K,+}(D) = \mathbf{D}_{\text{dR}}^K(D)$. More generally, let $h \geq 0$ be such that $\text{Fil}^{-h} \mathbf{D}_{\text{dR}}^K(D) = \mathbf{D}_{\text{dR}}^K(D)$. Then $t^h \mathbf{D}_{\text{dR}}^K(D) = \mathbf{D}_{\text{dR}}^{K,+}(D(-h))$ (in $\mathbf{D}_{\text{dif}}(D)$).

Proof. The first part is obvious from the definitions and can be shown the same way as in the étale case. The second follows similarly from Lemma 2.19. □

One can define the **Tate-twist** for a (φ, Γ_K)-module D: if $k \in \mathbb{Z}$, then $D(k)$ is the (φ, Γ_K)-module with D as $\mathbf{B}_{\text{rig},K}^\dagger$-module, but with

$$\varphi|_{D(k)} = \varphi|_D, \quad \gamma x = \chi^k(\gamma) \gamma x, \ x \in D.$$

Analoguously one define a **Tate-twist** for a filtered (φ, N)-module D over K_0. If $k \in \mathbb{Z}$, then $D[k]$ is the filtered (φ, N)-module with D as K_0-vectorspace and filtration $\text{Fil}^r(D[k])_K = \text{Fil}^{r-k} D_K$ and

$$N|_{D[k]} = N|_D, \quad \varphi|_{D[k]} = p^k \varphi|_D.$$

Lemma 2.19. One has $\mathbf{D}_{\text{st}}^K(D(k)) = \mathbf{D}_{\text{st}}^K(D)[-k]$.

Proof. One has $D(k) = D \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(k)$, and if e_k is a generator of $\mathbb{Z}_p(k)$, the isomorphism

$$(\mathbf{B}_{\text{log},K}^\dagger[1/t] \otimes_{\mathbf{B}_{\text{rig},K}^\dagger} D)^{\Gamma_K}[-k] \to (\mathbf{B}_{\text{log},K}^\dagger[1/t] \otimes_{\mathbf{B}_{\text{rig},K}^\dagger} D(k))^{\Gamma_K}$$

is given by

$$d = \sum a_n \otimes d_n \mapsto \sum a_n e_{-k} \otimes (d_n \otimes e_k) = (e_{-k} \otimes e_k)d.$$
□

Definition 2.20. A (φ, Γ_K)-module D is defined to be **crystalline** (resp. **semi-stable**, resp. **de Rham**) if $\dim_{K_0} \mathbf{D}_{\text{cris}}^K(D) = \text{rank}_{\mathbf{B}_{\text{rig},K}^\dagger} D$ (resp. $\dim_{K_0} \mathbf{D}_{\text{st}}^K(D) = \text{rank}_{\mathbf{B}_{\text{rig},K}^\dagger} D$, resp. $\dim_K \mathbf{D}_{\text{dR}}^K(D) = \text{rank}_{\mathbf{B}_{\text{rig},K}^\dagger} D$).

Similarly, we define D to be **potentially crystalline** (resp. **potentially semi-stable**) if there exists a finite extension L/K such that $D|_L$ is cristalline (resp. semistable).

Definition 2.21. Let D be a de Rham (φ, Γ_K)-module of rank d. If $n \geq n(D)$, set

$$\mathbf{N}_{\text{dR}}^{(n)}(D) := \{x \in D^{(n)}[1/t] | \ \iota_m(x) \in K_m[[t]] \otimes_K \mathbf{D}_{\text{dR}}^K(D) \text{ for any } m \geq n\}$$

and $\mathbf{N}_{\text{dR}}(D) = \varinjlim_n \mathbf{N}_{\text{dR}}^{(n)}(D)$.

Definition 2.22.   a) For a torsion free element γ_K of Γ_K Perrin-Riou's differential operator ∇ is defined as

$$\nabla = -\frac{\log(\gamma)}{\log_p(\chi(\gamma_K))} = -\frac{1}{\log_p(\chi(\gamma_K))} \sum_{n \geq 1} \frac{(1-\gamma_K)^n}{n} \in \mathcal{H}(\Gamma_K).$$

   b) The operator ∂ (on $\mathbf{B}^\dagger_{\mathrm{rig},K}[1/t]$) is defined as $\partial := 1/t \cdot \nabla$.

We remark that ∇ is independent of the choice of γ, which may be checked with the series properties of log. The module $\mathbf{N}_{\mathrm{dR}}(D)$ is denoted by $\mathbf{D}$ in [8], Theorem III.2.3. This theorem also implies:

Theorem 2.23. Let D be a de Rham (φ, Γ_K)-module of rank d. Then $\mathbf{N}_{\mathrm{dR}}(D)$ is a (φ, Γ_K)-module of rank d with the following properties:

   a) $\mathbf{N}_{\mathrm{dR}}(D)[1/t] = D[1/t]$,

   b) $\nabla_0(\mathbf{N}_{\mathrm{dR}}(D)) \subset t\mathbf{N}_{\mathrm{dR}}(D)$.

The following proposition is analoguous to [3], Theorem 3.6.

Proposition 2.24. Let D be a semistable (φ, Γ_K)-module. Then one has

$$(\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}} \otimes_{\mathbf{B}^\dagger_{\mathrm{rig},K}} D)^{G_K} = D^{\Gamma_K},$$

$$(\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}[1/t] \otimes_{\mathbf{B}^\dagger_{\mathrm{rig},K}} D)^{G_K} = (\mathbf{B}^\dagger_{\mathrm{rig},K}[1/t] \otimes_{\mathbf{B}^\dagger_{\mathrm{rig},K}} D)^{\Gamma_K},$$

$$(\widetilde{\mathbf{B}}^\dagger_{\mathrm{log}}[1/t] \otimes_{\mathbf{B}^\dagger_{\mathrm{rig},K}} D)^{G_K} = (\mathbf{B}^\dagger_{\mathrm{log},K}[1/t] \otimes_{\mathbf{B}^\dagger_{\mathrm{rig},K}} D)^{\Gamma_K}.$$

Proof. We only treat the first case, as the proof of the others is similar.

One has $(\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}} \otimes_{\mathbf{B}^\dagger_{\mathrm{rig},K}} D)^{G_K} \subset (\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}} \otimes_{\mathbf{B}^\dagger_{\mathrm{rig},K}} D)^{H_K} = \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig},K} \otimes_{\mathbf{B}^\dagger_{\mathrm{rig},K}} D$ since H_K acts trivially on D (it is a free $\mathbf{B}^\dagger_{\mathrm{rig},K}$-module). Let $\{e_i\}_{1 \leq i \leq d}$ be a $\mathbf{B}^\dagger_{\mathrm{rig},K}$-basis of D and $\{d_i\}_{1 \leq i \leq r}$ be a K_0-basis for $(\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}} \otimes_{\mathbf{B}^\dagger_{\mathrm{rig},K}} D)^{G_K}$ and $M \in M_{r \times d}(\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig},K})$ defined by the relation $(d_i) = M(e_i)$. M has rang r (that is, the image of a basis of D under M form a free $\mathbf{B}^\dagger_{\mathrm{rig},K}$-module of rank r) and satisfies $\gamma_K(M)G = M$ (since the elements d_i are fixed under γ_K), where $G \in \mathrm{GL}_d(\mathbf{B}^\dagger_{\mathrm{rig},K})$ is the matrix of γ_K with respect to the basis $\{e_i\}$. The operator R_m of Colmez/Berger (cf. loc.cit., §2.6) give $\gamma_K(R_m(M))G - R_m(M) = 0$ for every $m \in \mathbb{N}$. Further $R_m(M) \overset{m \to \infty}{\longrightarrow} M$ and $N = \varphi^m(R_m(M)) \in \mathbf{B}^\dagger_{\mathrm{rig},K}$ since R_m is a section of $\varphi^{-m}(\mathbf{B}^{\dagger,p^k r}_{\mathrm{rig},K}) \subset \widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig},K}$. Hence, $\gamma(N)\varphi^m(G) = N$ and since the actions of φ and Γ_K commute on D one has $\varphi(G)P = \gamma_K(P)G$, where $P \in M_d(\mathbf{B}^\dagger_{\mathrm{rig},K})$ is the matrix of φ with respect to the basis $\{e_i\}$. If one sets $Q = \varphi^{m-1}(P)\ldots\varphi(P)P$ then $\varphi^m(G)Q = \gamma_K(Q)G$ and hence $\gamma_K(NQ)G = NQ$, so that NQ determines r elements in D that are fixed under Γ_K. But since for m big enough the matrix M has rank r and P has full rank, since it is an injection and $\mathbf{B}^\dagger_{\mathrm{rig},K} \cdot \varphi(D) = D$, one sees that these elements give a rank r-submodule of D. Hence, the K_0-vectorspace generated by these elements is also of dimension r, whence the claim. $\square$

Before stating the next result we recall the notion of a p-adic differential equation. If D is any (φ, Γ_K)-module over $\mathbf{B}^\dagger_{\mathrm{rig}, K}$ it is known that the same definition as for ∇ gives rise to differential operator $\nabla_D : D \to D$ that commutes with the action of φ and Γ_K such that $\nabla_D(\lambda x) = \nabla(\lambda)x + \lambda\nabla_D(x)$ (see [6], Proposition III.1.1). With this one may also consider the operator $\partial_D = 1/t \cdot \nabla_D$ on $D[1/t]$. A p-**adic differential equation** is a (φ, Γ_K)-module D over $\mathbf{B}^\dagger_{\mathrm{rig}, K}$ that is stable under the operator ∂_D.

If there is no confusion we will drop the index D of the operators ∇_D and ∂_D.

Theorem 2.25. Let M be a p-adic differential equation equipped with a Frobenius. Then there exists a finite extension L/K such that the natural map

$$\mathbf{B}^\dagger_{\log, L} \otimes_{L'_0} (\mathbf{B}^\dagger_{\log, L} \otimes_{\mathbf{B}^\dagger_{\mathrm{rig}, K}} D)^{\partial=0} \to \mathbf{B}^\dagger_{\log, L} \otimes_{\mathbf{B}^\dagger_{\mathrm{rig}, K}} D.$$

is an isomorphism.

Proof. [1]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Recall that a ∇-**crystal** over $\mathbf{B}^\dagger_{\mathrm{rig}, K}$ is a free $\mathbf{B}^\dagger_{\mathrm{rig}, K}$-module equipped with an action of a Frobenius and a connection (also denoted by ∇), compatible with ∇ on $\mathbf{B}^\dagger_{\mathrm{rig}, K}$, that commutes with the Frobenius. A ∇-crystal over $\mathbf{B}^\dagger_{\mathrm{rig}, K}$ is called **unipotent** if it admits a filtration of sub-crystals such that each successive quotient has a basis consisting of elements in the kernel of ∇. More generally, a ∇-crystal M is called **quasi-unipotent** if there exists a finite extension L/K such that $\mathbf{B}^\dagger_{\mathrm{rig}, L} \otimes_{\mathbf{B}^\dagger_{\mathrm{rig}, K}} M$ (which is a ∇-crystal over $\mathbf{B}^\dagger_{\mathrm{rig}, L}$ in a natural way) is unipotent.

We note the following result, which is known by the experts and may be proved as in the étale case ([3], Proposition 5.6):

Proposition 2.26. Every de Rham (φ, Γ_K)-module is potentially semi-stable.

Proof. One defines the (faithful, exact, ...) functor $D \mapsto \mathbf{N}_{\mathrm{dR}}(D)$ from the category of de Rham (φ, Γ_K)-modules into the category of p-adic differential equations equipped with a Frobenius. Since by André's theorem 2.25 one knows that any such equation is quasi-unipotent, it suffices to show that D is potentially semistable if and only if $\mathbf{N}_{\mathrm{dR}}(D)$ is quasi-unipotent.

Now D is potentially semistable if and only if there exists a finite extension L/K such that

$$\dim_{L_0}(\mathbf{B}^\dagger_{\log, L}[1/t] \otimes_{\mathbf{B}^\dagger_{\mathrm{rig}, K}} D)^{\Gamma_L} = \mathrm{rank}_{\mathbf{B}^\dagger_{\mathrm{rig}, K}} D =: d.$$

This gives via [3], Proposition 5.5 a unipotent ∇-subcrystal of $D|_L[1/t]$, which is nothing else but $\mathbf{N}_{\mathrm{dR}}(D|_L) \cong \mathbf{B}^\dagger_{\mathrm{rig}, L} \otimes_{\mathbf{B}^\dagger_{\mathrm{rig}, K}} \mathbf{N}_{\mathrm{dR}}(D)$.

Conversely if $D|_{L'}[1/t]$ contains a unipotent ∇-subcrystal of rank d for some finite extension L'/K then the again by loc.cit. there exist elements $e_0, \ldots, e_{d-1}$ which generate an L'_0-vectorspace of dimension d on which $\log(\gamma)$ acts trivially. Hence, there exists a finite extension L/L' such that Γ_L acts trivially on this basis, so that we obtain a basis of $(\mathbf{B}^\dagger_{\log, L}[1/t] \otimes_{\mathbf{B}^\dagger_{\mathrm{rig}, K}} D)^{\Gamma_L}$ of the right dimension, i.e. D is potentially semistable. $\qquad\square$

We briefly review the slope theory of φ-modules over $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$ or $\mathbf{B}^{\dagger}_{K}$.

Definition 2.27. Let M is a φ-module over one of these rings. If M is of rank 1 and v a generator, then $\varphi(v) = \lambda v$ for some $\lambda \in (\mathbf{B}^{\dagger}_{\mathrm{rig},K})^{\times} = (\mathbf{B}^{\dagger}_{K})^{\times}$ (cf. [19]; see also [20], Hypothesis 1.4.1. resp. Example 1.4.2). We define the **degree** $\deg(M)$ of M to be $w(\lambda)$, where w is the p-adic valuation of $\mathbf{B}_K$. If M is of rank n then $\bigwedge^n M$ has rank 1. We define the **slope** $\mu(M)$ of M as $\mu(M) = \deg(M)/\mathrm{rk}M$.

We remark that the definition of the degree (hence the slope) is independent of the choice of the generator. Under the equivalence of Theorem 2.9 we have the following correspondence of the slope theory: If D is a (φ, Γ_K)-module over $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$, one may consider the φ-module $\widetilde{D}$ over $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}$. Then the two definitions of the slope for D coincide. Hence, we have the notion of a (φ, Γ_K)-module that is **pure** of some slope. The fundamental theorem is the following result by Kedlaya:

Theorem 2.28. (Slope filtration theorem) Let M be a φ-module over $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$. Then there exists a unique filtration $0 = M_0 \subset M_1 \subset \ldots \subset M_l = M$ by saturated φ-submodules whose successive quotients are pure with $\mu(M_1/M_0) < \ldots < \mu(M_{l-1}/M_l)$. If M is a (φ, Γ_K)-module all M_i are (φ, Γ_K)-submodules.

Proof. See [20]. $\qquad\qquad\square$

3 Exponential maps

3.1 Bloch-Kato exponential maps for (φ, Γ_K)-modules

In this section we define short exact sequences associated to (φ, Γ_K)-modules, generalizing the "classical" Bloch-Kato sequence (see [9]) which one may use to study cohomological questions relating to p-adic representations (i.e. the slope zero case). One interesting phenomenon that occurs in this more general setting is that, in order to get the general versions of the exponential maps, it is necessary to distinguish between the slope ≤ 0-case and the slope > 0-case.

We are interested in the long exact sequences for continuous Galois-cohomology induced by these sequences. Let us briefly recall the machinery. Let M be continuous G_K-module and define the continuous imhomogeneous cochains in the usual way ($q \geq 0$):

$$C^q_{\mathrm{cont}}(G_K, M) := C^q_{\mathrm{cont}}(K, M) := \{x : G^n \longrightarrow M \mid x \text{ continuous}\}$$

with differential $\delta^q : C^q_{\mathrm{cont}}(K, M) \to C^{q+1}_{\mathrm{cont}}(K, M)$ defined by

$$\delta^q(x)(g_1, \ldots, g_{q+1}) = g_1 x(g_2, \ldots, g_{q+1}) + (-1)^{q+1} x(g_1, \ldots, g_q)$$
$$+ \sum_{i=1}^{q} (-1)^i x(g_1, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_{q+1}).$$

By convention $C^{-i}(G_K, M) = 0$ for $i > 1$. The continuous cochain complex is then defined via

$$C^\bullet_{\mathrm{cont}}(K, M) := \left[C^0_{\mathrm{cont}}(K, M) \xrightarrow{\delta^0} C^1_{\mathrm{cont}}(K, M) \xrightarrow{\delta^1} \dots \right],$$

and one defines continuous cohomology via

$$H^q_{\mathrm{cont}}(K, M) := H^q(C^\bullet_{\mathrm{cont}}(K, M)).$$

Lemma 3.1. If $0 \to M' \longrightarrow M \xrightarrow{f} M'' \longrightarrow 0$ is an exact sequence of G_K-modules such that f admits a continuous (but not necessarily G_K-equivariant) splitting, then continuous cohomology induces a long exact sequence

$$\dots \to H^i_{\mathrm{cont}}(K, M') \to H^i_{\mathrm{cont}}(K, M) \to H^i_{\mathrm{cont}}(K, M'') \to H^{i+1}_{\mathrm{cont}}(K, M') \to \dots$$

Proof. This is standard, see for example [30], §2. $\qquad\square$

If there is no possibility of confusion we will drop the subscript "cont". The splitting property in our setting will be granted by the following

Proposition 3.2. If $f : B_1 \longrightarrow B_2$ be a linear continuous surjective map of p-adic Banach spaces, there exists a continuous splitting $s : B_2 \longrightarrow B_1$ of f, i.e. $f \circ s = \mathrm{id}_{B_2}$.

Proof. See [12], Proposition I.1.5, (iii). $\qquad\square$

We define the following set X, which will be used in the next few statements:

$$X := \{(x, y, z) \in \widetilde{D}_{\log}[1/t] \oplus \widetilde{D}_{\log}[1/t] \oplus \mathbf{W}_e(D)/\mathbf{W}^+_{\mathrm{dR}}(D)|\ N(y) = (p\varphi - 1)(x)\}.$$

Lemma 3.3. Let D be a (φ, Γ)-module over $\mathbf{B}^\dagger_{\mathrm{rig}, K}$. We assume D is pure of slope $\mu(D) \le 0$. Then one has the following exact sequences of G_K-modules (cf. (2) for the definition of β):

$$0 \longrightarrow \mathbf{W}^+_e(D) \xrightarrow{f} \mathbf{W}_e(D) \xrightarrow{g} \mathbf{W}_{\mathrm{dR}}(D)/\mathbf{W}^+_{\mathrm{dR}}(D) \longrightarrow 0$$
$$x \longmapsto \beta(x)$$

$$0 \longrightarrow \mathbf{W}^+_e(D) \xrightarrow{f} \widetilde{D}[1/t] \xrightarrow{g} \widetilde{D}[1/t] \oplus \mathbf{W}_{\mathrm{dR}}(D)/\mathbf{W}^+_{\mathrm{dR}}(D) \longrightarrow 0$$
$$x \longmapsto ((\varphi - 1)(x), \beta(x))$$

$$0 \longrightarrow \mathbf{W}^+_e(D) \xrightarrow{f} \widetilde{D}_{\log}[1/t] \xrightarrow{g} X \longrightarrow 0$$
$$x \longmapsto (N(x), (\varphi - 1)(x), \beta(x))$$

Additionally, each g above admits a continuous (not necessarily G_K-equivariant) splitting.

Proof. The exactness of the first sequence is tautological, see Theorem 2.10. For the second recall that for a φ-module M over $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$ the map $\varphi - 1 : M[1/t] \to M[1/t]$ is surjective. This implies the exactness of the second sequence. For the exactness of the last sequence first observe that the map g is well-defined. Recall that $N : \widetilde{D}_{\log} \to \widetilde{D}_{\log}$ is extended linearly from the operator N on $\widetilde{\mathbf{B}}^\dagger_{\log}$, so that

$$N(\sum_{i\geq 0} d_i \log^i \pi) = -\sum_{i\geq 1} i \cdot d_i \log^{i-1} \pi \tag{6}$$

for $\sum_{i\geq 0} d_i \log^i \pi \in \widetilde{D}_{\log}$. The exactness at $\widetilde{D}_{\log}[1/t]$ is clear since from (6) one has $(\widetilde{D}_{\log}[1/t])^{N=0} = \widetilde{D}[1/t]$, so we only have to check the exactness at X. The surjectivity of $N : \widetilde{\mathbf{B}}^\dagger_{\log}[1/t] \to \widetilde{\mathbf{B}}^\dagger_{\log}[1/t]$, which again follows from (6), implies that it is enough to check that if $(0, y, z) \in X$ then there exists $x' \in \widetilde{D}[1/t]$ such that $g(x') = (0, y, z)$, which is nothing but exactness of the second sequence.

The splitting property follows from Proposition 3.2 for the first sequence. For the remaining ones one has to observe that continuous surjections $1 - \varphi$ and N on $\widetilde{D}[1/t]$ have continuous sections, which follows for example from the proof of Proposition 2.1.5 of [19] for the first map, and is obvious for the monodromy operator. $\square$

Lemma 3.4. Let D be a (φ, Γ)-module over $\mathbf{B}^\dagger_{\mathrm{rig},K}$. We assume D is pure of slope $\mu(D) > 0$. Then one has the following exact sequences of G_K-modules (cf. (2) for the definition of β):

$$0 \longrightarrow \mathbf{W}_e(D) \xrightarrow{f} \mathbf{W}_{\mathrm{dR}}(D)/\mathbf{W}^+_{\mathrm{dR}}(D) \xrightarrow{g} \mathbf{W}_{\mathrm{dR}}(D)/(\mathbf{W}_e(D) + \mathbf{W}^+_{\mathrm{dR}}(D)) \longrightarrow 0$$
$$x \longmapsto \overline{x}$$

$$0 \longrightarrow \widetilde{D}[1/t] \xrightarrow{f} \widetilde{D}[1/t] \oplus \mathbf{W}_{\mathrm{dR}}(D)/\mathbf{W}^+_{\mathrm{dR}}(D) \xrightarrow{g} \mathbf{W}_{\mathrm{dR}}(D)/(\mathbf{W}_e(D) + \mathbf{W}^+_{\mathrm{dR}}(D)) \longrightarrow 0$$
$$f : x \longmapsto ((1 - \varphi)(x), \overline{x})$$
$$g : (x, y) \longmapsto \overline{y}$$

$$0 \longrightarrow \widetilde{D}_{\log}[1/t] \xrightarrow{f} X \xrightarrow{g} \mathbf{W}_{\mathrm{dR}}(D)/(\mathbf{W}_e(D) + \mathbf{W}^+_{\mathrm{dR}}(D)) \longrightarrow 0$$
$$f : x \longmapsto (N(x), (\varphi - 1)(x), \overline{x})$$
$$g : (x, y, z) \longmapsto \overline{z}$$

Additionally, each g above admits a continuous (not necessarily G_K-equivariant) splitting.

Proof. The exactness of the first sequence is again tautological by Theorem 2.10. The rest of the proof follows analoguously to the previous proposition. $\square$

Putting everything together, we also see:

Corollary 3.5. Let D be a (φ, Γ_K)-module over $\mathbf{B}^\dagger_{\mathrm{rig},K}$. Then one has the following exact sequence of G_K-modules:

$$0 \longrightarrow X^0(\widetilde{D}) \xrightarrow{\ i\ } \widetilde{D}_{\log}[1/t] \xrightarrow{\ f\ } X \xrightarrow{\ p\ } X^1(\widetilde{D}) \longrightarrow 0$$

$$i : x \longmapsto x$$
$$f : x \longmapsto (N(x), (\varphi - 1)(x), \overline{x})$$
$$p : (x, y, z) \longmapsto \overline{z}$$

Following Nakamura, we now define for a B-pair $W = (W_e, W^+_{\mathrm{dR}})$ the following complex:

$$C^\bullet(G_K, W) := \mathrm{cone}(C^\bullet(G_K, W_e) \longrightarrow C^\bullet(G_K, W_{\mathrm{dR}}/W^+_{\mathrm{dR}})),$$

which is induced by the canonical inclusion $W_e \xrightarrow{\ i\ } W_{\mathrm{dR}}$. That is, we have

$$C^i(G_K, W) = C^i(G_K, W_e) \oplus C^{i-1}(G_K, W_{\mathrm{dR}}/W^+_{\mathrm{dR}})$$

with differentials

$$\delta^i_C : C^i(G_K, W) \ni (a, b) \mapsto (\delta^i_{C^i(G_K, W_e)}(a), i(a) - \delta^{i-1}_{C^i(G_K, W_e)}(b))$$

More generally, one may define the following complexes:

$$C^\bullet(G_K, W') := \mathrm{cone}(C^\bullet(G_K, \widetilde{D}[1/t]) \xrightarrow{(1-\varphi, i)} C^\bullet(G_K, \widetilde{D}[1/t] \oplus W_{\mathrm{dR}}/W^+_{\mathrm{dR}})),$$

$$C^\bullet(G_K, W'') := \mathrm{cone}(C^\bullet(G_K, \widetilde{D}_{\log}[1/t]) \xrightarrow{(N, 1-\varphi, i)} C^\bullet(G_K, X)),$$

We recall:

Lemma 3.6. Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a short exact sequence of continuous G_K-modules such that g admits a continuous, but not necessarily G_K-equivariant, splitting. We write (by abuse of notation)

$$\mathrm{cone}(g) := \mathrm{cone}(C^\bullet(G_K, B) \xrightarrow{g_*} C^\bullet(G_K, C))$$

$$\mathrm{cone}(f) := \mathrm{cone}(C^\bullet(G_K, A) \xrightarrow{f_*} C^\bullet(G_K, B)).$$

a) The natural map of complexes

$$
\begin{array}{ccccc}
C^\bullet(G_K, A): & C^0(G_K, A) & \longrightarrow & C^1(G_K, A) & \longrightarrow \cdots \\
& \downarrow & & \downarrow{\scriptstyle f} & \quad \downarrow{\scriptstyle (f,0)} \\
\mathrm{cone}(g): & C^0(G_K, B) & \longrightarrow & C^1(G_K, B) \oplus C^0(G_K, C) & \longrightarrow \cdots
\end{array}
$$

is a quasi-isomorphism that is compatible with the long exact sequence, i.e. the following diagram is commutative:

$$
\begin{array}{ccccccccc}
\cdots \longrightarrow & H^i(G_K, A) & \longrightarrow & H^i(G_K, B) & \longrightarrow & H^i(G_K, C) & \xrightarrow{\delta} & H^{i+1}(G_K, A) & \longrightarrow \cdots \\
& \downarrow & & \| & & \| & & \downarrow & \\
\cdots \longrightarrow & H^i(\mathrm{cone}(g)) & \longrightarrow & H^i(G_K, B) & \longrightarrow & H^i(G_K, C) & \xrightarrow{\delta} & H^{i+1}(\mathrm{cone}(g)) & \longrightarrow \cdots
\end{array}
$$

b) The natural map of complexes

$$
\begin{array}{ccccc}
C^\bullet(G_K,C)[-1]: & 0 = C^{-1}(G_K,C) & \longrightarrow & C^0(G_K,C) & \longrightarrow \dots \\
& \Big\uparrow & & \Big\uparrow & \\
\mathrm{cone}(f): & \Big\uparrow 0 & & \Big\uparrow (0,g) & \\
& C^0(G_K,A) & \longrightarrow & C^1(G_K,A)\oplus C^0(G_K,B) & \longrightarrow \dots
\end{array}
$$

is a quasi-isomorphism that is compatible with the long exact sequence, i.e. the following diagram is commutative:

$$
\begin{array}{ccccccccc}
\dots \longrightarrow & H^i(G_K,A) & \longrightarrow & H^i(G_K,B) & \longrightarrow & H^i(G_K,C) & \overset{\delta}{\longrightarrow} & H^{i+1}(G_K,A) & \longrightarrow \dots \\
& \| & & \| & & \uparrow & & \| & \\
\dots \longrightarrow & H^i(G_K,A) & \longrightarrow & H^i(G_K,B) & \overset{\delta}{\longrightarrow} & H^{i+1}(\mathrm{cone}(f)) & \overset{\delta}{\longrightarrow} & H^{i+1}(G_K,A) & \longrightarrow \dots
\end{array}
$$

Proof. This is left as an exercise, see for example [31], 1.5.8. □

Lemma 3.7. We have canonical quasi-isomorphisms

$$
C^\bullet(G_K,W) \cong C^\bullet(G_K,W') \cong C^\bullet(G_K,W'').
$$

Proof. Let $W = \mathbf{W}(D)$. Observe that the inclusions $\mathbf{W}_e(D) \subset \widetilde{D}[1/t] \subset \widetilde{D}_{\log}[1/t]$ and $\mathbf{W}_{\mathrm{dR}}(D)$ induce canonical maps on these complexes. If $W = \mathbf{W}(D)$ with D pure of some slope the statement then follows from Lemmas 3.3, 3.4 and 3.6.

For general D we are by Kedlaya's slope filtration theorem reduced to the case of an exact sequence $0 \to D_1 \to D \to D_2 \to 0$ such that the statement is true for D_1, D_2, hence the claim follows by considering the long exact sequences associated to this. □

With this statement and the properties of the cone we obtain a long exact sequence of cohomology groups:

$$
\dots \to H^i(G_K,W) \to H^i(G_K,\widetilde{D}_{\log}[1/t]) \to H^i(G_K,X) \overset{\delta}{\to} H^{i+1}(G_K,W) \to \dots
$$

With these exact sequences in mind we suggest the following

Definition 3.8. Let D be a (φ,Γ_K)-module over $\mathbf{B}^\dagger_{\mathrm{rig},K}$. The transition map

$$
\exp_{K,D}: H^0(K,X) \to H^1(K,\mathbf{W}(D))
$$

from the exact sequence above is called **generalized Bloch-Kato exponential map** for D.

Remark 3.9. Let D be an étale (φ,Γ_K)-module, so that $D = \mathbf{D}^\dagger_{\mathrm{rig},K}(V)$ for some p-adic representation V of Γ_K. Then since the slope of D is equal to zero, the first exact sequence in Lemma 3.3 computes to

$$
0 \longrightarrow V \to \mathbf{B}_e \otimes_{\mathbb{Q}_p} V \longrightarrow \mathbf{B}_{\mathrm{dR}}/\mathbf{B}^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V \longrightarrow 0
$$

This is nothing but the usual Bloch-Kato short exact sequence associated to the p-adic representation V.

Recall that if D is any (φ, Γ_K)-module over $\mathbf{B}^\dagger_{\mathrm{rig},K}$ the map $\varphi - 1 : \widetilde{D}[1/t] \to \widetilde{D}[1/t]$ is surjective. If $x \in \widetilde{D}$ we write $(\varphi - 1)^{-1}(x)$ for a choice of an element $y \in \widetilde{D}[1/t]$ such that $(\varphi - 1)(y) = x$. We want to consider the following maps:

$$\alpha : \widetilde{D} \longrightarrow \mathbf{W}_e(D), \quad x \mapsto \begin{cases} x, & \varphi(x) = x, \\ 0, & \text{otherwise.} \end{cases}$$

$$\beta : \widetilde{D} \longrightarrow \mathbf{W}_{\mathrm{dR}}(D)/\mathbf{W}^+_{\mathrm{dR}}(D), \quad x \mapsto \iota_n((\varphi - 1)^{-1}(x)),$$

where the second map is well-defined due to the discussion in [7], Remark 3.4. α and β are continuous and fit into the following commutative diagram of G_K-modules:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \widetilde{D}^{\varphi=1} & \longrightarrow & \widetilde{D} & \xrightarrow{\varphi-1} & \widetilde{D} & \longrightarrow & \widetilde{D}/(\varphi-1)\widetilde{D} & \longrightarrow & 0 \\
& & \| & & \downarrow{\alpha} & & \downarrow{\beta} & & \| & & \\
0 & \longrightarrow & X^0(\widetilde{D}) & \longrightarrow & \mathbf{W}_e(D) & \longrightarrow & \mathbf{W}_{\mathrm{dR}}(D)/\mathbf{W}^+_{\mathrm{dR}}(D) & \longrightarrow & X^1(\widetilde{D}) & \longrightarrow & 0,
\end{array}$$

where we use the identifications for X^0 and X^1 from Theorem 2.10.

Proposition 3.10. One has a quasi-isomorphism

$$\mathrm{cone}(C^\bullet(G_K, \widetilde{D}) \xrightarrow{\varphi-1} C^\bullet(G_K, \widetilde{D})) \cong C^\bullet(G_K, \mathbf{W}(D))$$

that is functorial in D.

Proof. We denote by $A^\bullet$ the complex on the left hand side of the statement. One checks that the commutativity of the preceeding diagram and the cohomological version of [31], Exercise 1.5.9 show that one has a commutative diagram

$$\begin{array}{ccccccccc}
\ldots \to & H^n(G_K, \widetilde{D}^{\varphi=1}) & \longrightarrow & H^n(A^\bullet) & \longrightarrow & H^{n-1}(G_K, \frac{\widetilde{D}}{(\varphi-1)\widetilde{D}}) & \to & H^{n+1}(G_K, \widetilde{D}^{\varphi=1}) & \to \ldots \\
& \| & & \downarrow & & \| & & \| & \\
\ldots \to & H^n(G_K, X^0(\widetilde{D})) & \to & H^n(G_K, \mathbf{W}(D)) & \to & H^{n-1}(G_K, X^1(\widetilde{D})) & \to & H^{n+1}(G_K, X^0(\widetilde{D})) & \to \ldots
\end{array}$$

which gives the proof. $\qquad\square$

Recall the following property of continuous cohomology: If $f : M^\bullet \to N^\bullet$ is map of complexes of continuous G-modules for some profinite group G one has an identification of complexes

$$C^\bullet_{\mathrm{cont}}(G, \mathrm{cone}(M^\bullet \xrightarrow{f} N^\bullet)) = \mathrm{cone}\left(C^\bullet_{\mathrm{cont}}(G, M^\bullet) \xrightarrow{f_*} C^\bullet_{\mathrm{cont}}(G, N^\bullet)\right) \qquad (7)$$

(cf. the discussion in [24], 3.4.1.3, 3.4.1.4; it holds in this general setting).

We recall that in the derived category of $\mathbf{B}^\dagger_{\mathrm{rig},K}$-modules, the complex $C^\bullet_{\varphi,\gamma}$ is also represented by

$$R\Gamma(K, D) = R\Gamma_{\mathrm{cont}}\left(\Gamma_K, \mathrm{cone}\left[D \xrightarrow{\varphi-1} D\right]\right) \cong \mathrm{cone}\left[R\Gamma_{\mathrm{cont}}(\Gamma_K, D) \xrightarrow{\varphi-1} R\Gamma_{\mathrm{cont}}(\Gamma_K, D)\right],$$

cf. [29], section 3.3, where the last identification is due to (7).

The following is then a generalization of Proposition 2.24:

Proposition 3.11. One has an isomorphism

$$R\Gamma(K, D) \cong R\Gamma(K, \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig},K} \otimes_{\mathbf{B}^\dagger_{\mathrm{rig},K}} D)$$

that is functorial in D.

Proof. The proof is similar to [29], Proposition 3.8. It suffices to show that that the natural map

$$R\Gamma_{\mathrm{cont}}(\Gamma_K, D) \longrightarrow R\Gamma_{\mathrm{cont}}(\Gamma_K, \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig},K} \otimes_{\mathbf{B}^\dagger_{\mathrm{rig},K}} D)$$

is an isomorphism, since applying cone $\left[\bullet \xrightarrow{\varphi-1} \bullet\right]$ induces the morphism in the statement again due to (7). We apply the techniques of [2], Appendix I and use the notation there, as follows: Let $\widetilde{\Lambda} := \widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}}$, $\mathcal{G} = G_K$, $\mathcal{H} = H_K$ so that $d = 0$. Further, $\mathcal{H}' = \mathcal{H}$, $\Lambda^{(i)}_{m,\mathcal{H}'} = \varphi^{-m}(\mathbf{B}^{\dagger,p^m r}_{\mathrm{rig},K})$ (since $i = 0$ is the only possible choice) and the maps $\tau^{(i)}_{m,\mathcal{H}'}$ correspond to the maps $R_m : \widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig},K} \to \varphi^{-m}(\mathbf{B}^{\dagger,p^m r}_{\mathrm{rig},K})$ (cf. [3], Proposition 2.32). As in [2], section 7.6, the maps R_m induce maps (by the usual process of taking the direct limit over all sufficiently big r) $R_m : \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig},K} \otimes_{\mathbf{B}^\dagger_{\mathrm{rig},K}} D \to \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig},K} \otimes_{\mathbf{B}^\dagger_{\mathrm{rig},K}} D$ for $m \geq 0$, and as in loc.cit. one obtains a decomposition of Γ_K-modules

$$\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig},K} \otimes_{\mathbf{B}^\dagger_{\mathrm{rig},K}} D \cong (1 - R_m)(\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig},K} \otimes_{\mathbf{B}^\dagger_{\mathrm{rig},K}} D) \oplus (\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig},K} \otimes_{\mathbf{B}^\dagger_{\mathrm{rig},K}} D)^{R_m=1}.$$

By construction of the map R_m it is clear that $(\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig},K} \otimes_{\mathbf{B}^\dagger_{\mathrm{rig},K}} D)^{R_0=1} = D$. Furthermore, as in the proof of loc.cit., Proposition 7.7, one may infer that $\gamma_K - 1$ acts invertibly on $(1 - R_0)(\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig},K} \otimes_{\mathbf{B}^\dagger_{\mathrm{rig},K}} D)$, so that $R\Gamma_{\mathrm{cont}}(\Gamma_K, (1 - R_0)(\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig},K} \otimes_{\mathbf{B}^\dagger_{\mathrm{rig},K}} D)) = 0$, which gives the claim. $\qquad\square$

Putting everything together, we see:

Corollary 3.12. One has an isomorphism

$$R\Gamma(K, D) \cong R\Gamma(G_K, \mathbf{W}(D)).$$

that is functorial in D.

Proof. We observe that the natural map

$$\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig},K} \cong R\Gamma_{\mathrm{cont}}(H_K, \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}) \tag{8}$$

is a quasi-isomorphism. This, together with the preceeding isomorphisms implies

$$\begin{aligned}
R\Gamma(K, D) &\cong R\Gamma_{\mathrm{cont}}(\Gamma_K, \mathrm{cone}(D \xrightarrow{\varphi-1} D)) \\
&= R\Gamma_{\mathrm{cont}}(\Gamma_K, \mathrm{cone}(\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig},K} \otimes D \xrightarrow{\varphi-1} \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig},K} \otimes D)) \\
&= R\Gamma_{\mathrm{cont}}(\Gamma_K, R\Gamma_{\mathrm{cont}}(H_K, \mathrm{cone}(\widetilde{D} \xrightarrow{\varphi-1} \widetilde{D}))) \\
&\overset{(*)}{=} R\Gamma_{\mathrm{cont}}(G_K, \mathrm{cone}(\widetilde{D} \xrightarrow{\varphi-1} \widetilde{D})) \\
&= R\Gamma(G_K, \mathbf{W}(D)).
\end{aligned}$$

where $(*)$ holds since the natural map $H^i(G_K/H_K, \widetilde{D}^{H_K}) \to H^i(G_K, \widetilde{D})$ is an isomorphism, since again $H^n(H_K, \widetilde{D}) = 0$ for $n > 0$ due to (8): for $i = 1$ this follows from the five term exact sequence in low degree, which extends in this case for continuous cohomology similarly as in e.g. [25], §6, to higher degrees by induction. $\qquad\square$

Corollary 3.13. $H^i(G_K, \mathbf{W}(D)) = 0$ for $i \neq 0, 1, 2$ and $H^i(G_K, \mathbf{W}(D))$ is a finite-dimensional $\mathbb{Q}_p$-vectorspace.

Proof. This follows from the preceeding Corollary and [21], Theorem 8.1. $\qquad\square$

We wish to give a more explicit description of the isomorphisms on cohomology which we will need in the characterizing property of the big exponential map, where actually only the map for the H^1's will be important for us. Hence, we may only sketch certain steps for the higher cohomology groups (that is, H^2).

We briefly describe how one may interpret, in the slope ≤ 0-case, the cohomology group $H^1(G_K, \mathbf{W}_e^+(D))$ as extensions of $\mathbb{Q}_p$ by $\mathbf{W}_e^+(D)$. So let $c \in H^1(G_K, \mathbf{W}_e^+(D))$ and consider the exact sequence of G_K-modules

$$0 \longrightarrow \mathbf{W}_e^+(D) \longrightarrow E_c \longrightarrow \mathbb{Q}_p \longrightarrow 0$$

where $E_c = \mathbb{Q}_p \oplus \mathbf{W}_e^+(D)$ as $\mathbb{Q}_p$-vectorspace and G_K acts on E_c via

$$\sigma(a, m) = (a, \sigma m + a c_\sigma).$$

Since c is a 1-cocycle one has

$$\sigma(\tau(a, x)) = \sigma(a, \tau x + a c_\tau) = (a, \sigma\tau x + a\sigma c_\tau + c_\sigma) = \sigma\tau(a, x),$$

so that one has a well-defined map $Z^1(K, D) \to \mathrm{Ext}(\mathbb{Q}_p, \mathbf{W}_e^+(D))$. E_c is trivial if and only if there exists an element $1 \in E_c$ such that $g1 = 1$ for all g, i.e.

$$1 = (1, x), \quad g1 - 1 = (0, gx - x + c_g) = 0,$$

so that $c_g = (1 - g)x$ is a coboundary, which implies that the above map factors through $B^1(K, D)$. The fact that this map is an isomorphism can be checked as in the p-adic representation case.

Proposition 3.14. Suppose we are in the situation of Lemma 3.3. Then the complex $C_{\varphi, \gamma_K}^\bullet(K, D)$ (functorially) computes the cohomology of $C_{\mathrm{cont}}^\bullet(G_K, X^0(D))$.

Proof. We may assume that Γ_K is pro-cyclic with generator γ_K. First we have

$$H^0(K, D) = D^{\Gamma_K, \varphi=1} = \widetilde{D}^{G_K, \varphi=1} = X^0(D)^{G_K} = H^0(G_K, X^0(D)).$$

thanks to Proposition 2.24.

For H^1 we apply the construction of Cherbonnier/Colmez ([11]). To wit, let $(x, y) \in H^1(K, D)$ and pick $b \in \widetilde{D}$ such that $(\varphi - 1)b = x$. Then

$$h_{K,D}^1((x, y)) = \log_p^0(\chi(\gamma)) \cdot \left(\sigma \longmapsto \frac{\sigma - 1}{\gamma_K - 1} y - (\sigma - 1)b \right)$$

defines a 1-cocycle with values in $\widetilde{D}$ but one easily checks that $(\varphi - 1)h^1_{K,D}((x,y)) = 0$ so that we actually have a cocycle in $H^1(G_K, X^0(D))$. Injectivity and surjectivity now follow in the same way as in loc.cit. if one uses the description of extensions of $\mathbb{Q}_p$ by $X^0(D)$ given above, so that we obtain the isomorphism in the H^1-case.

For H^2 one can show that since $X^0(D)$ is an almost $\mathbb{C}_p$-representation that one has a Hochschild-Serre spectral sequence $H^i(\Gamma_K, H^j(H_K, X^0(D))) \Rightarrow H^{i+j}(G_K, X^0(D))$ associated to the exact sequence $1 \to H_K \to G_K \to \Gamma_K \to 1$. Since the cohomology on the left vanishes for j or i greater or equal to 2 one has with the fact that $H^3(G_K, X^0(D)) = 0$

$$H^2(G_K, X^0(D)) \cong H^1(\Gamma_K, H^1(H_K, X^0(D))).$$

Now the exact sequence $0 \to X^0(D) \to \widetilde{D} \overset{\varphi-1}{\to} \widetilde{D} \to 0$ of G_K-modules gives rise to a sequence

$$\ldots \longrightarrow \widetilde{D}^{H_K} \overset{\varphi-1}{\longrightarrow} \widetilde{D}^{H_K} \longrightarrow H^1(H_K, X^0(D)) \longrightarrow 0,$$

since $H^1(H_K, \widetilde{D}) = H^1(H_K, \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}} \otimes D) \cong H^1(H_K, \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}})^d = 0$. Hence, by Iwasawa theory

$$H^2(G_K, X^0(D)) \cong \widetilde{D}^{H_K}/(\varphi - 1, \gamma_K - 1).$$

Looking at the quasi-isomorphisms in Corollary 3.12 one sees that using Lemma 3.6, since we are in the $X^1(D) = 0$-case, the map $H^2(K, D) \to H^2(G_K, X^0(D))$ is given by the canonical inclusion of finite-dimensional $\mathbb{Q}_p$-vectorspaces

$$H^2(K, D) = D/(\varphi - 1, \gamma_K - 1) \subset \widetilde{D}^{H_K}/(\varphi - 1, \gamma_K - 1) = H^2(G_K, X^0(D)),$$

that are of the same dimension. This gives the description of the map for H^2. $\qquad\square$

Lemma 3.15. Let D be a (φ, Γ_K)-module over $\mathbf{B}^\dagger_{\mathrm{rig},K}$ and assume that Γ_K is pro-cyclic with generator γ_K. Then one has an exact sequence

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \dfrac{D^{\varphi=1}}{(\gamma_K - 1)} & \overset{f}{\longrightarrow} & H^1(K, D) & \overset{g}{\longrightarrow} & \left(\dfrac{D}{\varphi - 1}\right)^{\Gamma_K} & \longrightarrow & 0 \\
& & y & \longmapsto & (0, y) & & & & \\
& & & & (x, y) & \longmapsto & x & &
\end{array}
$$

Proof. Recall that by definition

$$H^1(K, D) = \{(x,y) \in D \oplus D|\ (\gamma_K - 1)x = (\varphi - 1)y\}/\{((\varphi - 1)z, (\gamma_K - 1)z)|\ z \in D\},$$

so that the first map is well-defined an injective. One checks that the map g is well-defined and if $x \in D/(\varphi - 1)$ such that $(\gamma_K - 1)x \in (\varphi - 1)D$ then there exists an $y \in D$ such that $(x, y) \in H^1(K, D)$ and $g(x, y) = x$. Obviously $g \circ f = 0$. Let $g(x, y) = 0$ so that $x = (\varphi - 1)z$ for some $z \in D$, so that $(x, y) \sim (0, y - (\gamma_K - 1)z)$ in $H^1(K, D)$. Hence, (x, y) is in the image of f. $\qquad\square$

We remark that this sequence is nothing but the short exact sequence associated to the inflation-restriction sequence if D is étale, i.e.,

$$0 \longrightarrow H^1(\Gamma_K, V^{H_K}) \longrightarrow H^1(G_K, V) \longrightarrow H^1(H_K, V^{\Gamma_K}) \longrightarrow 0,$$

see for example [13], section 5.2.

Proposition 3.16. Suppose we are in the situation of Lemma 3.4. Then the complex $C_{\varphi,\gamma_K}^\bullet(K, D)$ computes the cohomology of $C_D^\bullet := C_{\text{cont}}^\bullet(G_K, X^1(D))[1]$

Proof. We may assume that Γ_K is procyclic with generator γ_K. Since the slope of D is > 0 one has $X^0(D) = 0$, so that $D^{\varphi=1} = 0$ since $D^{\varphi=1} \subset \widetilde{D}^{\varphi=1} = 0$, so that $H^0(K, D) = 0$. The same holds tautologically for $H^0(C_D^\bullet)$.

For the case of the H^1's observe that since $X^0(D) = 0$ Lemma 3.15 implies that the canonical map $H^1(K, D) \to (D/(\varphi - 1))^{\Gamma_K})$, $\overline{(x, y)} \mapsto \overline{x}$, is an isomorphism. From Theorem 2.10 we also know that $X^1(D) = \widetilde{D}/(\varphi - 1)$. Hence, from Corollary 3.12 and Lemma 3.6 we have that the map

$$H^0(G_K, X^1(D)) = \left(\frac{\widetilde{D}}{\varphi - 1}\right)^{G_K} = \left(\frac{D}{\varphi - 1}\right)^{\Gamma_K} \cong H^1(K, D).$$

gives the identification.

For H^2 one has similarly as in the slope ≤ 0-case a Hochschild-Serre spectral sequence $H^i(\Gamma_K, H^j(H_K, X^1(D))) \Rightarrow H^{i+j}(G_K, X^1(D))$. From the exact sequence in low degree terms one then has

$$0 \to H^1(\Gamma_K, H^0(H_K, \widetilde{D}/(\varphi - 1)) \to H^1(G_K, \widetilde{D}/(\varphi - 1)) \to H^0(\Gamma_K, H^1(H_K, \widetilde{D}/(\varphi - 1)).$$

From the sequence $0 \longrightarrow \widetilde{D} \xrightarrow{\varphi-1} \widetilde{D} \to X^1(D) \longrightarrow 0$ one infers the vanishing of $H^1(H_K, X^1(D))$ since $H^1(H_K, \widetilde{D}) = H^2(H_K, TD) = H^2(H_K, \widetilde{\mathbf{B}}_{\text{rig}}^\dagger)^d = 0$. Hence, we see

$$H^1(G_K, X^1(D)) = H^1(\Gamma_K, H^0(H_K, \widetilde{D}/(\varphi - 1)) = \widetilde{D}^{H_K}/(\varphi - 1, \gamma_K - 1).$$

so that again by Corollary 3.12 and Lemma 3.6 the canonical inclusion of finite-dimensional $\mathbb{Q}_p$-vectorspaces

$$H^2(K, D) = D/(\varphi - 1, \gamma_K - 1) \subset \widetilde{D}^{H_K}/(\varphi - 1, \gamma_K - 1) = H^1(G_K, X^1(D)),$$

gives the description of the map for H^2. $\qquad\square$

Finally we describe how one may piece together the isomorphisms $H^i(K, D) \xrightarrow{h^1} H^i(K, \mathbf{W}(D))$ in the general case (where we only make the case H^1 explicit, which is all we need for the application to Perrin-Riou's exponential map): If $(x, y) \in H^1(K, D)$ write $x = (\varphi - 1)(b') + s(b'')$, where $s : \widetilde{D}/(\varphi - 1)\widetilde{D} \to \widetilde{D}$ is a continuous splitting of the natural projection (which exists thanks to Proposition 3.2), $b' \in \widetilde{D}$ and $b'' \in \widetilde{D}/(\varphi - 1)\widetilde{D}$. Putting the two constructions together, we may consider the tuple

$$h^1(x, y)) := \left(\log_p^0(\chi(\gamma)) \cdot \left(\sigma \longmapsto \tfrac{\sigma-1}{\gamma_K-1}y - (\sigma - 1)b'\right), (0, 0, \varphi^{-n}((\varphi - 1)^{-1}(s(b''))))\right)$$
$$\in C^1(G_K, \widetilde{D}_{\log}) \oplus C^0(G_K, X),$$

(9)

and one sees that actually $h^i((x, y)) \in H^1(K, \mathbf{W}(D))$, which gives the description of the isomorphism in the general case by the properties of the mapping cone.

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We will briefly describe, similarly as in the slope ≤ 0-case before, how one may interpret the cohomology group $H^1(G_K, \mathbf{W}_e(D))$ as extensions of $\mathbf{B}_e$ by $\mathbf{W}_e(D)$ (note however that we do not make any assumptions about the slopes of D). So let $c \in H^1(G_K, \mathbf{W}_e(D))$ and consider the exact sequence of G_K-modules

$$0 \longrightarrow \mathbf{W}_e(D) \longrightarrow E_c \longrightarrow \mathbf{B}_e \longrightarrow 0,$$

where $E_c = \mathbf{B}_e \oplus \mathbf{W}_e(D)$ as a $\mathbf{B}_e$-module with G_K-action $\sigma(a, x) = (\sigma a, \sigma x + \sigma a \cdot c_\sigma)$. One has

$$\sigma(\tau(a, x)) = \sigma(\tau a, \tau x + \tau a \cdot c_\tau) = (\sigma \tau a, \sigma \tau x + \sigma \tau a \cdot \sigma c_\tau + \sigma \tau a \cdot c_\sigma) = \sigma \tau(a, x),$$

so that one has a well-defined map $Z^1(K, \mathbf{W}_e(D)) \to \mathrm{Ext}(\mathbb{Q}_p, \mathbf{W}_e^+(D))$. E_c is trivial if and only if there exists an element $1 \in E_c$ such that $g1 = 1$ for all g, i.e.

$$1 = (1, x), \quad g1 - 1 = (0, gx - x + c_g) = 0,$$

so that $c_g = (1 - g)x$ is a coboundary, which implies that the above map factors through $B^1(K, \mathbf{W}_e(D))$. The fact that this map is an isomorphism can be checked as before.

Proposition 3.17. Let D be a (φ, Γ_K)-module over $\mathbf{B}_{\mathrm{rig},K}^\dagger$. Then the complex $C_{\varphi,\gamma_K}^\bullet(K, D[1/t])$ computes the cohomology of $C_{\mathrm{cont}}^\bullet(G_K, \mathbf{W}_e(D))$.

Proof. The proof is similar to the ones before; in fact, one may reduce to the case of Corollary 3.5 by taking direct limits (see also [23], Theorem 4.5). We are interested in the explicit description of the maps. From Proposition 2.24 again we have:

$$H^0(K, D[1/t]) = D[1/t]^{\varphi=1,\Gamma_K} = \widetilde{D}[1/t]^{\varphi=1,G_K} = H^0(G_K, \mathbf{W}_e(D)).$$

For H^1 we apply the same construction as in Proposition 3.14. So let $(x, y) \in H^1(K, D[1/t])$ and pick $b \in \widetilde{D}[1/t]$ such that $(\varphi - 1)b = x$. Then

$$h_{K,D}^1((x, y)) = \log_p^0(\chi(\gamma)) \cdot \left(\sigma \longmapsto \frac{\sigma - 1}{\gamma_K - 1} y - (\sigma - 1)b\right)$$

defines a 1-cocycle with values in $\widetilde{D}[1/t]$ which lies actually in $\mathbf{W}_e(D)$. Injectivity and surjectivity now follow in the same way as in loc.cit. if one uses the description of extensions of $\mathbf{B}_e$ by $\mathbf{W}_e(D)$ given above, so that we obtain the isomorphism in the H^1-case.

The case of the H^2's follows in the same way as in Proposition 3.14. $\qquad \square$

Proposition 3.18. One has an identification $H^0(K, \mathbf{W}_{\mathrm{dR}}(D)) = \mathbf{D}_{\mathrm{dR}}^K(D)$

Proof. From [11], Proposition IV.1.1 (i) we know that $K_\infty[[t]]$ is dense in $(\mathbf{B}_{\mathrm{dR}}^+)^{H_K}$, and the inclusion is compatible the action of Γ_K. Also one has $(\mathbf{B}_{\mathrm{dR}}^+)^{G_K} = K_\infty[[t]]^{\Gamma_K} = K$. Since D is free as a $\mathbf{B}_{\mathrm{rig},K}^\dagger$-module with trivial H_K-action, we see that $(\mathbf{B}_{\mathrm{dR}}^+ \otimes D)^{G_K} = ((\mathbf{B}_{\mathrm{dR}}^+)^{H_K} \otimes D)^{\Gamma_K} = \mathbf{D}_{\mathrm{dif}}^+(D)^{\Gamma_K}$. Since $\mathbf{B}_{\mathrm{dR}} = \varinjlim_{n \geq 0} 1/t^n \cdot \mathbf{B}_{\mathrm{dR}}^+$ and $K_\infty((t)) = \varinjlim_{n \geq 0} 1/t^n \cdot K_\infty[[t]]$ the claim follows, since taking invariants is compatible with direct limits.

Alternatively, the claim also follows from [14], Theorem 2.14, B) i). $\qquad \square$

We shall make use of the following considerations. Let D be a semi-stable (φ, Γ_K)-module over $\mathbf{B}_{\mathrm{rig}, K}^\dagger$ and consider the following complex $\mathfrak{C}_{\mathrm{st}}(K, D)$ (concentrated in degrees 0, 1, 2):

$$
\begin{array}{ccccc}
\mathbf{D}_{\mathrm{st}}^K(D) & \to & \mathbf{D}_{\mathrm{st}}^K(D) \oplus \mathbf{D}_{\mathrm{st}}^K(D) \oplus \mathbf{D}_{\mathrm{dR}}^K(D)/\mathrm{Fil}^0 \mathbf{D}_{\mathrm{dR}}^K(D) & \to & \mathbf{D}_{\mathrm{st}}^K(D) \\
x & \mapsto & (N(x), (\varphi - 1)(x), \beta(x)) & & \\
& & (x, y, z) & \mapsto & N(x) - (p\varphi - 1)(y).
\end{array}
$$
(10)

Then an element in $H^1(\mathfrak{C}_{\mathrm{st}}(K, D))$ can be considered as an element in $H^0(K, X)$ and hence be mapped via the exponential map to $H^1(K, \mathbf{W}(D))$.

We shall give two maps which will be important in the construction of the dual exponential map for de Rham (φ, Γ_K)-modules.

First we remark that the canonical inclusion $D \to \mathbf{W}_{\mathrm{dR}}(D)$ factors via $D \to D[1/t]$. This allows us to describe a map $H^1(K, D) \to H^1(G_K, \mathbf{W}_{\mathrm{dR}}(D))$ explicitly via the composition of the canonical map $H^1(K, D) \to H^1(K, D[1/t])$, the identification $H^1(K, D[1/t]) \xrightarrow{\sim} H^1(G_K, \mathbf{W}_e(D))$ (cf. Proposition 3.17) and the canonical map $H^1(K, \mathbf{W}_e(D)) \longrightarrow H^1(K, \mathbf{W}_{\mathrm{dR}}(D))$.

Secondly, we show that the map

$$
\mathbf{D}_{\mathrm{dR}}^K(D) \longrightarrow H^1(G_K, \mathbf{W}_{\mathrm{dR}}(D)), \quad x \longmapsto [g \mapsto \log(\chi(\bar{g}))x]
$$
(11)

which generalizes Kato's formula of [18], §II.1, is an isomorphism, which may be proved as follows. First observe that

$$
H^1(G_K, \mathbf{B}_{\mathrm{dR}} \otimes D) \cong H^1(G_K, \mathbf{B}_{\mathrm{dR}} \otimes_K \mathbf{D}_{\mathrm{dR}}^K(D)) = H^1(G_K, \mathbf{B}_{\mathrm{dR}}) \otimes_K \mathbf{D}_{\mathrm{dR}}^K(D).
$$

From [16], Proposition 5.25, one knows that $K = H^0(G_K, \mathbf{B}_{\mathrm{dR}}) \to H^1(G_K, \mathbf{B}_{\mathrm{dR}})$, $x \mapsto x \cdot \log \chi$ is an isomorphism. This gives the claim.

Definition 3.19. The **generalized Bloch-Kato dual exponential map** $\exp_{K, D^*(1)}^*$ is the composition of the above maps $H^1(K, D) \to H^1(G_K, \mathbf{W}_{\mathrm{dR}}(D))$ with the inverse of the isomorphism $\mathbf{D}_{\mathrm{dR}}^K(D) \xrightarrow{\sim} H^1(G_K, \mathbf{W}_{\mathrm{dR}}(D))$.

Of course, in the étale case this is nothing but the dual exponential map considered by Kato in [18]. But even in this more general case this map has the desired property with respect to adjunction via pairings. First recall that one may define the K-bilinear perfect pairing $[\ ,\ ]_{K, D}$ by the natural map

$$
[\ ,\ ]_{K, D} : \mathbf{D}_{\mathrm{dR}}^K(D) \times \mathbf{D}_{\mathrm{dR}}^K(D^*(1)) \xrightarrow{\mathrm{ev}} \mathbf{D}_{\mathrm{dR}}^K(\mathbf{B}_{\mathrm{rig}, K}^\dagger(1)) \longrightarrow K.
$$

For the next proposition we note that Nakamura uses a different definition of the dual exponential map (see [23], section 2.4), which we briefly recall (we refer to loc.cit for the proofs): one may define the cohomology groups $H^i(K, \mathbf{D}_{\mathrm{dif}}(D))$ by $H_{\mathrm{cont}}^i(\Gamma_K, \mathbf{D}_{\mathrm{dif}}(D))$, which is computed by the complex

$$
C_{\gamma, \Delta}^\bullet(\mathbf{D}_{\mathrm{dif}}(D)) : \mathbf{D}_{\mathrm{dif}}(D) \xrightarrow{\gamma - 1} \mathbf{D}_{\mathrm{dif}}(D).
$$

Since the natural map $K_\infty((t)) \otimes_K \mathbf{D}_{\mathrm{dR}}^K(D) \to \mathbf{D}_{\mathrm{dif}}(D)$ is an isomorphism one has an identification

$$g_D : \mathbf{D}_{\mathrm{dR}}^K(D) \xrightarrow{\sim} H^1(K, \mathbf{D}_{\mathrm{dif}}(D)), \quad x \mapsto \overline{(\log \chi(\gamma))1 \otimes x}.$$

The second definition of $\exp_{K,D}^*$ is then given by the composition of the map $H^1(K, D) \to H^1(K, \mathbf{D}_{\mathrm{dif}}(D))$, $[(x,y)] \mapsto \iota_n(y)$ (for n big enough) and the inverse of g_D. Since $H^i(H_K, \mathbf{B}_{\mathrm{dR}}) = 0$ for $i > 0$ the five term exact sequence gives $H^1(G_K, \mathbf{W}_{\mathrm{dR}}(D)) \cong H^1(\Gamma_K, \mathbf{B}_{\mathrm{dR}}^{H_K} \otimes D)$. Using the same argument as in Proposition 3.18 one sees that the natural map $H^1(K, \mathbf{D}_{\mathrm{dif}}(D)) \to H^1(G_K, \mathbf{W}_{\mathrm{dR}}(D))$ is an isomorphism. Further, the natural map $H^1(K, D) \to H^1(G_K, \mathbf{W}_{\mathrm{dR}}(D))$ defined before is also given by $[(x,y)] \mapsto \iota_n(y)$. Hence, using all these identifications one obtains a commutative diagram

$$
\begin{array}{ccccc}
H^1(K, D) & \longrightarrow & H^1(K, \mathbf{D}_{\mathrm{dif}}(D)) & \xleftarrow{\ \sim\ } & \mathbf{D}_{\mathrm{dR}}^K(D) \\
\| & & \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} \\
H^1(K, D) & \longrightarrow & H^1(G_K, \mathbf{W}_{\mathrm{dR}}(D)) & \xleftarrow{\ \sim\ } & H^0(G_K, \mathbf{W}_{\mathrm{dR}}(D)),
\end{array}
$$

which shows that the two definitions of $\exp^*$ coincide.

Proposition 3.20. Let D be a de Rham (φ, Γ_K)-module over $\mathbf{B}_{\mathrm{rig},K}^\dagger$ and let $x \in \mathbf{D}_{\mathrm{dR}}^K(D)$ and $y \in H^1(K, D^*(1))$. Then

$$\langle \exp_{K,D}(x), y \rangle_{K,D} = \mathrm{Tr}_{K/\mathbb{Q}_p}[x, \exp_{K,D}^*(y)]_{K,D}$$

Proof. See [23], Proposition 2.16. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proposition 3.21. Let D be a semi-stable (φ, Γ_K)-module over $\mathbf{B}_{\mathrm{rig},K}^\dagger$. Let $y \in D^{\psi=1}$ and consider y as $y \in (\mathbf{B}_{\mathrm{log},K}^\dagger[1/t] \otimes_F \mathbf{D}_{\mathrm{st}}^K(D))^{N=0,\psi=1}$ via the comparison isomorphism. Then for $n \gg 0$

$$\exp_{V^*(1)}^*(h_{D,K_n}^1(y)) = p^{-n}\varphi^{-n}(y)(0).$$

Proof. As before we have

$$h_{D,K_n}^1(y)(\sigma) = \frac{\sigma - 1}{\gamma_{K_n} - 1}y - (\sigma - 1)b,$$

with $(\gamma_{K_n} - 1)(\varphi - 1)b = (\varphi - 1)y$ for some $b \in \widetilde{D}[1/t]$. Further Let n be big enough so that we may embed this cocycle into $\mathbf{B}_{\mathrm{dR}} \otimes D$, hence $\varphi^{-n}(y) \in K_n((t)) \otimes \mathbf{D}_{\mathrm{st}}^K(D)$ and we may consider $\varphi^{-n}(b)$ as an element in $\mathbf{B}_{\mathrm{dR}} \otimes D$. Since $\gamma_{K_n} t = \chi(\gamma_{K_n})t$ the action of $\gamma_{K_n} - 1$ is invertible on $t^k K_n \otimes \mathbf{D}_{\mathrm{st}}^K(D)$ for every $k \neq 0$. Putting this together we see that h_{D,K_n}^1 is equivalent in $H^1(K_n, \mathbf{B}_{\mathrm{dR}} \otimes D)$ to

$$\sigma \longmapsto \frac{\sigma - 1}{\gamma_{K_n} - 1}(\varphi^{-n}(y))(0).$$

σ acts via its image $\overline{\sigma} \in \Gamma_K^n$ (trivially) on K_n. Furthermore, if $n_i \in \mathbb{Z}$ is a sequence such that $\overline{\sigma} = \lim_{i \to \infty} \gamma_{K_n}^{n_i}$ one checks by going to the limit that

$$\frac{\sigma - 1}{\gamma_{K_n} - 1} \frac{\log_p \chi(\gamma_{K_n})}{\log_p \chi(\overline{\sigma})}$$

acts trivially on K_n. Hence, the above cycle is equivalent to

$$\sigma \longmapsto p^{-n} \log(\chi(\overline{\sigma}))(\varphi^{-n}(y))(0)$$

The claim follows now from formula (11). $\qquad \square$

3.2 Perrin-Riou exponential maps for (φ, Γ_K)-modules

We make the following definitions:

Definition 3.22. Let M be a (φ, N)-module over F. Define $\mathbf{N}_{\mathrm{dR}}(M) = (\mathbf{B}_{\log,K}^{\dagger} \otimes_F M)^{N=0}$, where $N = 1 \otimes N + N \otimes 1$ on $\mathbf{B}_{\log,K}^{\dagger} \otimes_F M$.

If D is a semi-stable (φ, Γ_K)-module over $\mathbf{B}_{\mathrm{rig},K}^{\dagger}$ then $\mathbf{N}_{\mathrm{dR}}(\mathbf{D}_{\mathrm{st}}^K(D)) = \mathbf{N}_{\mathrm{dR}}(D)$ (see Definition 2.21).

Definition 3.23. Let D be a de Rham (φ, Γ_K)-module over $\mathbf{B}_{\mathrm{rig},K}^{\dagger}$.

a) Let $\mathbf{D}_{\infty,g}(D)$ be the submodule of elements $g \in \mathbf{N}_{\mathrm{dR}}(D)^{\psi=0}$ such that there exists an $r \in \mathbb{Z}$ such that the equation $(1 - p^r \varphi)G = \partial^r(g)$ has a solution in $G \in \mathbf{N}_{\mathrm{dR}}(D)^{\psi=p^r}$.

b) Let $\mathbf{D}_{\infty,f}(D)$ be the submodule of elements $g \in \mathbf{N}_{\mathrm{dR}}(D)^{\psi=0}$ such that there exists a family $(G_k)_{k \in \mathbb{Z}}$ of elements $G_k \in \mathbf{N}_{\mathrm{dR}}(D)$ with $\partial(G_k) = G_{k+1}$ and an $r \in \mathbb{Z}$ such that $(1 - p^r \varphi)G = \partial^r(g)$

c) Let $\mathbf{D}_{\infty,e}(D)$ be the submodule of elements $g \in \mathbf{N}_{\mathrm{dR}}(D)^{\psi=0}$ such that the equation $(1 - p^r \varphi)G = \partial^r(g)$ has a solution in $G \in \mathbf{N}_{\mathrm{dR}}(D)^{\psi=p^r}$ for every $r \in \mathbb{Z}$.

We first note that if $D \longrightarrow D'$ is a morphism of two de Rham (φ, Γ_K)-modules over $\mathbf{B}_{\mathrm{rig},K}^{\dagger}$ then this induces a map of Γ_K-modules $\mathbf{D}_{\infty,*}(D) \to \mathbf{D}_{\infty,*}(D')$. Also, one clearly has

$$\mathbf{D}_{\infty,e}(D) \subset \mathbf{D}_{\infty,f}(D) \subset \mathbf{D}_{\infty,g}(D) \subset \mathbf{N}_{\mathrm{dR}}(D)^{\psi=0}.$$

By the above definition one may also define the modules $\mathbf{D}_{\infty,*}(\ )$ by starting with a $(\varphi, \mathbf{N})$-module.

Definition 3.24. Let D be a de Rham (φ, Γ_K)-module over $\mathbf{B}_{\mathrm{rig},K}^{\dagger}$. We say that D is of **Perrin-Riou-type** (or of PR-type) if D is semistable and $K_0 = K_0'$.

Lemma 3.25. The map $\partial : \mathbf{B}_{\log,K}^{\dagger} \to \mathbf{B}_{\log,K}^{\dagger}$ is surjective.

Proof. This amounts to an integration of power-series, cf. [3], Proposition 4.4. $\qquad \square$

Lemma 3.26. Suppose $K_0 = K_0'$. Then the kernel of ∂ on $\mathbf{B}_{\log,K}^\dagger$ is equal to K_0.

Proof. Let $f \in \mathbf{B}_{\mathrm{rig},K}^\dagger$. Due to Proposition 2.1 and Lemma 2.2 there is a polynomial P in $\mathbf{B}_{\mathrm{rig},F}^\dagger$ such that $P(f) = 0$ and $P'(f) \neq 0$. Then $\partial(f) = -(\partial P)(f)/P'(f)$, so that $\partial(f) = 0$ if and only if $f \in K_0$.

Now suppose $f = \sum_{i=1}^r f_i \log^i \pi \in \mathbf{B}_{\log,K}^\dagger$ and $\partial(f) = 0$. Since $\log \pi$ is a transcendent element over any $\mathbf{B}_{\mathrm{rig},K}^\dagger$ this gives rise to relations $\partial(f_i) + (j+1)\frac{\pi+1}{\pi} f_{i+1} = 0$ with $f_{r+1} = 0$. For $i = r$ this implies $f_r = \lambda \in K_0$, hence $\partial(f_{r-1}) = -\lambda r \frac{\pi+1}{\pi}$. Suppose there exists an $f \in \mathbf{B}_{\mathrm{rig},K}^\dagger$ with $\partial(f) = \frac{1+\pi}{\pi}$. Then $\partial(\log \pi - f) = 0$, so that $\log \pi = f + a$ with $a \in K_0$, a contradiction to the transcendency property of $\log \pi$. Hence, $\frac{\pi+1}{\pi}$ is not an element in the image of ∂ on $\mathbf{B}_{\mathrm{rig},K}^\dagger$, and we obtain $\lambda = 0$. By recurrence this shows that the kernel of ∂ on $\mathbf{B}_{\log,K}^\dagger$ is contained in K_0. $\qquad\square$

Let again D be a de Rham (φ, Γ)-module over $\mathbf{B}_{\mathrm{rig},K}^\dagger$.

Lemma 3.27. Let D be of PR-type. Then the map $\partial : \mathbf{B}_{\log,K}^\dagger \otimes \mathbf{N}_{\mathrm{dR}}(D) \to \mathbf{B}_{\log,K}^\dagger \otimes \mathbf{N}_{\mathrm{dR}}(D)$ is surjective.

Proof. We have
$$\mathbf{B}_{\log,K}^\dagger \otimes_{\mathbf{B}_{\mathrm{rig},K}^\dagger} \mathbf{N}_{\mathrm{dR}}(D) = \mathbf{B}_{\log,K}^\dagger \otimes_{K_0} \mathbf{D}_{\mathrm{st}}^K(D),$$
whence the claim follows from the Lemma above. $\qquad\square$

Proposition 3.28. Let D be of PR-type. The map
$$\partial : \mathbf{N}_{\mathrm{dR}}(D)^{\psi=0} \longrightarrow \mathbf{N}_{\mathrm{dR}}(D[1])^{\psi=0}(1)$$
is an isomorphism of Γ_K-modules.

Proof. With our preparations, namely, Lemma 3.25 and Lemma 3.26, this proof works the same as in [27], Proposition 2.2.3. $\qquad\square$

Obviously the operator ∂ induces a map of Γ_K-modules
$$\partial : \mathbf{N}_{\mathrm{dR}}(D)^{\psi=1} \to \mathbf{N}_{\mathrm{dR}}(D[1])^{\psi=1}(1)$$
which however is in general neither injective nor surjective. This should be contrasted with the étale case where $D^{\psi=1} = \mathbf{D}_{\mathrm{rig}}^\dagger(V)^{\psi=1} = H^1(K, V \otimes_{\mathbb{Q}_p} \mathcal{H}(\Gamma_K))$ and the fact that ∂ in this setting corresponds to the Tate-twist isomorphism.

For a semistable (φ, Γ_K)-module consider the following complex:
$$\mathfrak{C}_K(D): \quad 0 \to \mathbf{D}_{\mathrm{st}}^K(D) \xrightarrow{\delta_0} \mathbf{D}_{\mathrm{st}}^K(D) \times \mathbf{D}_{\mathrm{st}}^K(D) \xrightarrow{\delta_1} \mathbf{D}_{\mathrm{st}}^K(D) \to 0$$
with
$$\delta_0(\nu) = (N\nu, (1-\varphi)\nu),$$
$$\delta_1(\lambda, \mu) = N\mu - (1-p\varphi)\lambda.$$

Hence,

$$H^0(\mathfrak{C}_K(D)) = \mathbf{D}_{\mathrm{st}}^K(D)^{\varphi=1, N=0},$$
$$H^1(\mathfrak{C}_K(D)) = \{(\lambda, \mu) \in \mathbf{D}_{\mathrm{st}}^K(D) \times \mathbf{D}_{\mathrm{st}}^K(D) |\ N\mu = (1 - p\varphi)\lambda\}/\delta_0(\mathbf{D}_{\mathrm{st}}^K(D)),$$
$$H^2(\mathfrak{C}_K(D)) = \mathbf{D}_{\mathrm{st}}^K(D)/(N, 1 - p\varphi)\mathbf{D}_{\mathrm{st}}^K(D).$$

One also checks that

$$
0 \longrightarrow \frac{\mathbf{D}_{\mathrm{st}}^K(D)^{N=0}}{(\varphi-1)\mathbf{D}_{\mathrm{st}}^K(D)^{N=0}} \longrightarrow H^1(\mathfrak{C}_K(D)) \longrightarrow \frac{\mathbf{D}_{\mathrm{st}}^K(D)}{N\mathbf{D}_{\mathrm{st}}^K(D))^{\varphi=p^{-1}}} \longrightarrow 0
$$
$$
\mu \longmapsto (0, \mu)
$$
$$
(\lambda, \mu) \longmapsto \lambda
$$

(12)

furnishes an exact sequence for $H^1(\mathfrak{C}(D))$.

We see that $H^0(\mathfrak{C}(D(k))) = 0$ for $k \gg 0$ resp. $k \ll 0$ since the groups $\mathbf{D}_{\mathrm{st}}(D(k))^{\varphi=1}$ and $(\varphi - 1)\mathbf{D}_{\mathrm{st}}(D(k))$ vanish for those k. Similarly, $H^1(\mathfrak{C}(D(k))) = 0$ for $k \gg 0$ resp. $k \ll 0$.

Now let D be a de Rham (φ, Γ_K)-module and fix a finite extension L/K such that $D|_L$ is semistable with $L_0 = L_0'$.

Lemma 3.29. Let $k \in \mathbb{N}$. Then one has an exact sequence of Γ_K-modules

$$0 \to \bigoplus_{-k \leq i < 0} H^0(\mathfrak{C}(D|_L(-i)))(i) \cap \mathbf{N}_{\mathrm{dR}}(D(k))^{\psi=1}(-k) \to \mathbf{N}_{\mathrm{dR}}(D(k))^{\psi=1}(-k)$$

$$\xrightarrow{\partial^k} \mathbf{N}_{\mathrm{dR}}(D)^{\psi=1} \xrightarrow{\widetilde{\mathcal{R}}_D} \bigoplus_{-k \leq i < 0} H^1(\mathfrak{C}(D|_L(-i)))(i)$$

Proof. The proof may be done in an analogous way as in [27], Lemma 2.2.5. We give a description of the map $\widetilde{\mathcal{R}}_D$ following the definition of a map $\mathcal{R}_D$ (cf. equation (15)) since the constructions which give rise to it will be important later on. We just briefly mention that this map depends on the inclusion $\mathbf{N}_{\mathrm{dR}}(D) \subset \mathbf{N}_{\mathrm{dR}}(D|_L)$ which is induced by the inclusion $D \subset D|_L$. □

From the lemma we see that, by considering the possible eigenvalues for φ,

$$\mathbf{D}_{\infty,e}(D) = \partial^h(1 - p^{-h}\varphi)\mathbf{N}_{\mathrm{dR}}(D(h))^{\psi=1}, \tag{13}$$
$$\mathbf{D}_{\infty,g}(D) = \partial^{-h}(1 - p^h\varphi)\mathbf{N}_{\mathrm{dR}}(D(-h))^{\psi=1} \tag{14}$$

for $h \gg 0$ since the $H^i(\mathfrak{C}(D))$, $i = 0, 1$, vanish in this case. More precisely, for étale (φ, Γ_K)-module one has the following:

Lemma 3.30. Let $D = \mathbf{D}_{\mathrm{rig}}^\dagger(V)$ for a p-adic representation V that is de Rham. Let $h \geq 1$ be such that $\mathrm{Fil}^{-h}\mathbf{D}_{\mathrm{dR}}^K(D) = \mathbf{D}_{\mathrm{dR}}^K(D)$. Then $\mathbf{D}_{\infty,g}(D) = \partial^{-(h+1)}(1 - p^{h+1}\varphi)\mathbf{N}_{\mathrm{dR}}(D(-(h+1)))^{\psi=1}$.

Proof. We may reduce to the case that D is semi-stable with $K_0 = K_0'$ and further by twisting that $h = 1$. We have to check that $\partial : \mathbf{N}_{\mathrm{dR}}(D(-2))^{\psi=1} \to \mathbf{N}_{\mathrm{dR}}(D(-3))^{\psi=1}(1)$ is an isomorphism, i.e., we have to check the vanishing of $H^0(\mathfrak{C}(D(-2)))$ and $H^1(\mathfrak{C}(D(-2)))$. For the first this is obvious since for an admissible filtered (φ, N)-module that is positive the eigenvalues of the Frobenius are positive. Similarly, thanks to the exact sequence (12), we see that the H^1-part vanishes. $\qquad\square$

Remark 3.31. We suspect that in the cases where V is as above and does not contain the subrepresentation $\mathbb{Q}_p(h)$ one actually has $\mathbf{D}_{\infty,g}(D) = \partial^{-h}(1 - p^h \varphi)\mathbf{N}_{\mathrm{dR}}(D(-h))^{\psi=1}$. This would fit in with the characterizing description of the big exponential map in the étale case; cf. also the discussion in [26], section 5.1.

We recall the application $\mathcal{R}_D$. For our purposes (since we may restrict/corestrict) it will be enough for this part to assume that D of PR-type over $\mathbf{B}_{\mathrm{rig},K}^\dagger$.

Definition 3.32. Let $g \in \mathbf{D}_{\infty,g}(D)$ and r be big enough such that $\mathbf{D}_{\infty,g}(D)$ admits the description in (14). A family of elements $(G_k)_{k\in\mathbb{Z}}$ in $\mathbf{B}_{\log,K}^\dagger \otimes_{\mathbf{B}_{\mathrm{rig},K}^\dagger} \mathbf{N}_{\mathrm{dR}}(D)$ is called a **complete solution** for $(1 - \varphi)G = g$ if $\partial(G_k) = G_{k+1}$ (cf. 3.27) and $\partial^r(g) = (1 - p^r\varphi)G_r$ for r big enough.

If $G = (G_k)$ is a complete solution of $g \in \mathbf{D}_{\infty,g}(D)$ we also write $\partial^{-k}(G) = G_k$ by abuse of notation. Let $s \gg 0$ such that $(1 - p^s\varphi)G_s = \partial^s(g)$. Then one sees inductively thanks to Lemma 3.29 that

$$N(G_k) = \sum_{j \geq -k} \lambda_j \frac{t^{j+k}}{(j+k)!} =: L_k, \quad \lambda_j \in \mathbf{D}_{\mathrm{st}}^K(D)$$

$$(\psi \otimes 1 - p^{-k} \otimes \varphi)(G_k) = p^{-k} \sum_{j \geq -k} \mu_j \frac{t^{j+k}}{(j+k)!} =: (\psi \otimes 1)(M_k), \quad \mu_j \in \mathbf{D}_{\mathrm{st}}^K(D),$$

where for almost all j one has $\lambda_j = \mu_j = 0$. On $\mathbf{B}_{\log,K}^\dagger \otimes_{K_0} \mathbf{D}_{\mathrm{st}}^K(D)$, as one checks easily, we have the identity of operators

$$(pN \otimes 1 + 1 \otimes N)(\psi \otimes 1 - p^{-k} \otimes \varphi) = (\psi \otimes 1 - p^{-k+1} \otimes \varphi)(N \otimes 1 + 1 \otimes N) = (\psi \otimes 1 - p^{-k+1} \otimes \varphi)N,$$

hence

$$N((\psi \otimes 1)(M_k)) = (\psi \otimes 1 - p^{-k+1} \otimes \varphi)(L_k),$$

since $N \otimes 1$ vanishes on elements of $\sum t^i \cdot \mathbf{D}_{\mathrm{st}}^K(D)$, hence the relation (by applying $(\psi^{-1} \otimes 1 = \varphi \otimes 1$, which we may since ψ acts invertibly on $\sum t^i \cdot \mathbf{D}_{\mathrm{st}}^K(D)$)

$$N(M_k) = (1 - p^{-k+1}\varphi)(L_k).$$

On the coefficients this implies the relation

$$N\mu_j = (1 - p^{-j+1}\varphi)\lambda_j.$$

If $A = \sum_{j \geq -k} \nu_j/(j+k)! \cdot t^{j+k}$ and if one changes G_k to $G'_k = G_k + A$ so that still $\partial^k(G'_k) = \partial^k(G_k)$, then λ_j is changed to $\lambda_j + N(\nu_j)$ and μ_j is changed to $\mu_j + (1 - p^j\varphi)\nu_j$. Hence, L_k is changed to $L_k + N(A)$ and M_k is changed to $M_k + (1 - \varphi)(A)$, so that the class of (λ_i, μ_i) is well-defined in $H^1(\mathfrak{C}(D|_L(-i)))$. The tupel (λ_j, μ_j) may be considered as an element of $H^1(\mathfrak{C}(D|_L(-i)))(i)$, and we denote the collection of these elements element by $\mathcal{R}_D(g)$, i.e. one has a Γ_K-equivariant map

$$\mathcal{R}_D : \mathbf{D}_{\infty,g}(D) \longrightarrow \bigoplus_{i \in \mathbb{Z}} H^1(\mathfrak{C}(D|_L(-i)))(i). \tag{15}$$

We note that the map $\widetilde{\mathcal{R}}_D$ in Lemma 3.29 is the composition of $(1 - \varphi)$ with $\mathcal{R}_D$ and the natural projection to the sum $\bigoplus_{-k \leq i < 0} H^1(\mathfrak{C}(D|_L(-i)))(i)$.

Define for all $k \in \mathbb{Z}$

$$N(G_k) = L_k =: \partial^{-k}(L)$$
$$(\psi \otimes 1 - 1 \otimes \varphi)(G_k) = \psi \otimes 1(M_k) =: \psi \otimes 1(\partial^{-k}(M)).$$

These definitions imply that (calculating again in $\mathbf{B}^\dagger_{\log,K} \otimes_{K_0} \mathbf{D}^K_{\mathrm{st}}(D)$)

$$\psi((1 - \varphi)(G_k) - M_k) = (\psi \otimes 1)((1 - \varphi)(G_k) - M_k) = 0,$$

hence, since ∂ acts invertibly on $(\mathbf{B}^\dagger_{\log,K} \otimes_{K_0} \mathbf{D}^K_{\mathrm{st}}(D))^{\psi=0}$,

$$\partial^k(g) = (1 - p^k\varphi)G_k - M_k.$$

Of course, $M_k = L_k = 0$ for k big enough. We will also refer to the system $H = (L^{[1]}_k, M_k, G_k)$ as a **complete solution** for $g \in \mathbf{D}_{\infty,g}(D)$, where by $L^{[1]}_k$ we mean that the action of φ is multiplied by p. This extra factor is introduced so that the interpolation property holds.

Following Perrin-Riou, we set

$$\mathcal{U}(D) := \bigoplus_{i \in \mathbb{Z}} t^i \cdot \mathbf{D}_{\mathrm{st}}(D)$$

and

$$\mathbf{D}^2_{\infty,g}(D) := \mathcal{U}(D)/(1 - p\varphi, N)\mathcal{U}(D).$$

Proposition 3.33. One has the following exact sequences of Γ_K-modules:

$$0 \longrightarrow \mathbf{D}_{\infty,e}(D) \longrightarrow \mathbf{D}_{\infty,g}(D) \xrightarrow{\mathcal{R}_D} \bigoplus_{i \in \mathbb{Z}} H^1(\mathfrak{C}_L(D|_L(-i)))(i)$$

$$0 \longrightarrow \mathbf{D}_{\infty,f}(D) \longrightarrow \mathbf{D}_{\infty,g}(D) \xrightarrow{\overline{\mathcal{R}}_D} (\mathcal{U}(D|_L)/N\mathcal{U}(D|_L))^{\varphi=p^{-1}}$$

$$0 \longrightarrow \mathbf{D}_{\infty,e}(D) \longrightarrow \mathbf{D}_{\infty,f}(D) \xrightarrow{\mathcal{R}_D} (\mathcal{U}(D|_L))^{N=0}/(1 - \varphi)(\mathcal{U}(D))^{N=0}.$$

Proof. See [27], Proposition 2.3.4. □

We remark that in the case where $K/\mathbb{Q}_p$ is unramified one can show all the right-most maps in the preceeding Proposition are actually surjective. This can be deduced as in [27], Proposition 4.1.1. Additionally, using the preceding Proposition, one can show that $\mathbf{D}_{\infty,f}(\ )$ need not be exact.

Definition 3.34. a) For a torsion free element γ of Γ_K and $i \in \mathbb{Z}$ Perrin-Riou's differential operator $\nabla_i = l_i$ is defined as

$$\nabla_i = \frac{\log(\gamma)}{\log_p(\chi(\gamma))} - i = \nabla_0 - i$$

b) The operator $\nabla_0/(\gamma_n - 1)$ for n such that Γ_n is cyclic is defined as

$$\frac{\nabla_0}{\gamma_n - 1} := \frac{\log(\gamma_n)}{\log_p(\chi(\gamma))(\gamma_n - 1)} := \frac{1}{\log_p(\chi(\gamma_n))} \sum_{i=1}^{\infty} \frac{(1-\gamma_n)^{i-1}}{i}.$$

First, we remark that the second operator is *not* a quotient of two operators, although it behaves as one would like. To clarify we observe that the first definition is independent of the choice of γ since $\log(\gamma^m)/\log_p(\chi(\gamma^m)) = m/m \cdot \log(\gamma)/\log_p(\chi(\gamma))$. Hence, if $\nabla_0(y)$ for some $y \in D$ (for instance, $y \in D^{\psi=0}$) is such that $\gamma_n - 1$ acts invertibly on it we see that $(\gamma_n - 1)^{-1}\nabla_0(y) = \frac{\nabla_0}{\gamma_n - 1}(y)$. From this it also follows that $(\gamma_n - 1)\frac{\nabla_0}{\gamma_n - 1} = \nabla_0$. Secondly we observe that

$$\nabla_i = \frac{\log(\chi(\gamma)^{-i} \cdot \gamma)}{\log_p(\chi(\gamma))} = \mathrm{Tw}^{-i}\left(\frac{\log(\gamma)}{\log_p(\chi(\gamma))}\right)$$

where Tw^i is the operator on $\mathcal{B}(\Gamma_K)$ which sends γ to $\chi(\gamma)^k\gamma$.

Definition 3.35. If $h \geq 1$ we define $\Omega_h := \nabla_{h-1} \cdot \ldots \cdot \nabla_0 \in \mathcal{H}(\Gamma_K)$.

Lemma 3.36. Let D be a de Rham (φ, Γ_K)-module over $\mathbf{B}_{\mathrm{rig},K}^{\dagger}$ and let $h \in \mathbb{N}$ such that $\mathrm{Fil}^{-h}\mathbf{D}_{\mathrm{dR}}^K(D) = \mathbf{D}_{\mathrm{dR}}^K(D)$. Then $\Omega_h(\mathbf{N}_{\mathrm{dR}}(D)) \subset D$.

Proof. Since $\Omega_h = \nabla_{h-1} \circ \nabla_{h-2} \circ \ldots \circ \nabla_0 = t^h\partial^h$ it suffices to show that $t^h\mathbf{N}_{\mathrm{dR}}(D) \subset D$. First assume that D is semi-stable. We know from Proposition 2.18 that if D is positive, then $\mathbf{D}_{\mathrm{st}}^K(D) = (\mathbf{B}_{\mathrm{log},K}^{\dagger}[1/t] \otimes D)^{\Gamma_K} \subset \mathbf{B}_{\mathrm{log},K}^{\dagger} \otimes_{\mathbf{B}_{\mathrm{rig},K}^{\dagger}} D$, so that $\mathbf{N}_{\mathrm{dR}}(D) = (\mathbf{B}_{\mathrm{log},K}^{\dagger} \otimes_{\mathbf{B}_{\mathrm{rig},K}^{\dagger}} \mathbf{D}_{\mathrm{st}}^K(D))^{N=0} \subset D$. For general D if $h \geq 1$ is as in the statement then $D(-h)$ is positive, so that $t^h\mathbf{N}_{\mathrm{dR}}(D) \subset D$. Now if D is de Rham and L/K a finite extension such that $D|_L$ is semi-stable, then we have that $t^h\mathbf{N}_{\mathrm{dR}}(D) \subset t^h\mathbf{N}_{\mathrm{dR}}(D|_L) \subset D|_L$ and $t^h\mathbf{N}_{\mathrm{dR}}(D) \subset D[1/t]$, so that $t^h\mathbf{N}_{\mathrm{dR}}(D) \subset D$ as required. □

Definition 3.37. Let D be a de Rham (φ, Γ_K)-module over $\mathbf{B}_{\mathrm{rig},K}^{\dagger}$ and $h \geq 1$ be such that $\mathrm{Fil}^{-h}\mathbf{D}_{\mathrm{dR}}^K(D) = \mathbf{D}_{\mathrm{dR}}^K(D)$. We define Perrin-Riou's **big exponential map** by

$$\Omega_{D,h} : \mathbf{D}_{\infty,g}(D) \longrightarrow D^{\psi=0}$$
$$g \longmapsto \nabla_{h-1} \circ \ldots \circ \nabla_0(g)$$

Lemma 3.38. One has the following commutative diagram:

$$
\begin{array}{ccc}
\mathbf{D}_{\infty,g}(D) & \xrightarrow{\partial^{-k}} & \mathbf{D}_{\infty,g}(D(k)) \\
\downarrow{\scriptstyle\Omega_h} & & \downarrow{\scriptstyle\Omega_{h+k}} \\
D^{\psi=0} & \xrightarrow{\quad t^k \quad} & D(k)^{\psi=0}
\end{array}
$$

Proof. This is clear from the fact that $\Omega_h = t^h \partial^h$. $\qquad\square$

Lemma 3.39. Let D be as before and assume that K is such that Γ_K is torsion free. Then one has a canonical map $h_{K,D}^1 : (\varphi - 1)D^{\psi=1} \to H^1(K,D)/(D^{\varphi=1}/(\gamma_K - 1))$ such that the diagram

$$
\begin{array}{ccc}
(\varphi - 1)D^{\psi=1} & \xleftarrow{\quad\quad \varphi-1 \quad\quad} & D^{\psi=1} \\
\downarrow{\scriptstyle\tilde{h}_{K,D}^1} & & \downarrow{\scriptstyle h_{K,D}^1} \\
H^1(K,D)/(D^{\varphi=1}/(\gamma_K - 1)) & \xleftarrow{\quad\quad\quad} & H^1(K,D)
\end{array}
$$

is commutative.

Proof. Obviously $D^{\psi=1}/D^{\varphi=1} \cong (\varphi - 1)D^{\psi=1}$. It is clear that the map $h_{K_n,D}^1$ factorizes over $D_{\Gamma_K}^{\psi=1}$. The claim follows. $\qquad\square$

Remark 3.40. If D is of PR-type and let h be such that (14) is satisfied. If $g \in \mathbf{D}_{\infty,g}(V)$ and $k \geq 1 - h$ we actually have $\Omega_h(g) \otimes e_k \in (1 - \varphi)D(k)^{\psi=1}$.

Proof. Let $\partial^{-k}(g) = (1 - \varphi)\partial^{-k}(G) - \partial^{-k}(M)$. Then

$$
\partial^{-k}(M) = \sum_{j \geq 0}^{h+k-1} \mu_{j-k} \frac{t^j}{j!} \in \mathcal{H} \otimes \mathbf{D}_{\mathrm{st}}(V(k)).
$$

Since $\nabla_{h+k-1} \circ \ldots \circ \nabla_0 = t^{h+k}\partial^{h+k}$ the $\partial^{-k}(M)$-part of $\partial^{-k}(g)$ is killed by Ω_h. $\qquad\square$

Hence, we see that if h is such that (14) is satisfied and $h - r > 0$ the diagram

$$
\begin{array}{ccc}
(\mathbf{B}_{\log,K}^\dagger \otimes_F \mathbf{D}_{\mathrm{st}}^K(D(-r)))^{N=0,\psi=1} & \xrightarrow{\quad\Omega_{h-r}\quad} & D(-r)^{\psi=1} \\
\downarrow{\scriptstyle 1-p^r\varphi} & & \downarrow{\scriptstyle 1-p^r\varphi} \\
(1 - p^r\varphi)(\mathbf{B}_{\log,K}^\dagger \otimes_F \mathbf{D}_{\mathrm{st}}^K(D(-r)))^{N=0,\psi=1} & \xrightarrow{\Omega_{h-r}} & (1 - p^r\varphi)D(r)^{\psi=1} \\
\downarrow{\scriptstyle\partial^{-r}} & & \downarrow{\scriptstyle\mathrm{Tw}^r} \\
\mathbf{D}_{\infty,g}(D) & \xrightarrow{\quad\quad\Omega_h\quad\quad} & (1 - \varphi)D^{\psi=1}
\end{array}
$$

commutes.

Let D be of PR-type, $g \in \mathbf{D}_{\infty,g}(D)$ and $G = (L_k, M_k, G_k)$ be a complete solution for g. Then for each k and $n \gg 0$ one has that the element

$$\Xi_{n,k}(G) := p^{n(k-1)}\varphi^{-n}\partial^{-k}(H)(0) := p^{n(k-1)}(p^{-n}\varphi^{-n}\partial^{-k}(L)(0), \varphi^{-n}\partial^{-k}(M)(0), \varphi^{-n}\partial^{-k}(G)(0))$$

may be viewed as an element in $H^1(\mathfrak{C}_{\mathrm{st}}(K, D(k)))$ (see (10)).

Theorem 3.41. Let D be a de Rham (φ, Γ_K)-module over $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$, $g \in \mathbf{D}_{\infty,g}(D)$ and G a complete solution for g in L. Let h be such that (14) is satisfied. Then for $k \geq 1 - h$ and $n \gg 1$ one has

$$h^1_{K_n,D(k)}(\nabla_{h-1} \circ \ldots \circ \nabla_0(g) \otimes e_k)$$
$$= p^{-n(K_n)}(-1)^{h+k-1}(h+1-k)! \frac{1}{[L_n : K_n]} \mathrm{Cor}_{L_n/K_n} \exp_{K_n,D(k)}(\Xi_{n,k}(G)),$$

where we consider the elements on both sides in $H^1(K_n, D)/(D^{\varphi=1}/(\gamma_{K_n} - 1))$.

Proof. The proof is divided into several parts. The first general assumption is that D is of PR-type.

Let D be pure of slope ≤ 0. Then the exponential map has the description given in Proposition 3.14. We may assume n big enough so that Γ_K^n is torsion free. Recall the relation

$$\Omega_{D(k),h+k}(\partial^{-k}(G)) = \Omega_{D,h}(G) \otimes e_k$$

Hence, for the $k \geq 1 - h$ we have

$$h^1_{K_n,D(k)}(\nabla_{h-1} \circ \ldots \circ \nabla_0(G) \otimes e_k) = h^1_{K_n,D(k)}(\nabla_{h+k-1} \circ \ldots \circ \nabla_0(\partial^{-k}(G))).$$

Let $y_h = \nabla_{h+k-1} \circ \ldots \nabla_0(\partial^{-k}(G))$ and $w_{n,h} = \nabla_{h+k-1} \circ \ldots \frac{\nabla_0}{\gamma_n - 1}(\partial^{-k}(G))$. Then in this case

$$h^1_{K_n,D(k)}(y_h)(\sigma) = \frac{\sigma - 1}{\gamma_n - 1} y_h - (\sigma - 1)b_{n,h} \in H^1(K_n, D(k)),$$

where $b_{n,h} \in \widetilde{D}$ is such that $(\gamma_n - 1)(\varphi - 1)b_{n,h} = (\varphi - 1)y_h$. Recall that $\partial^{-k}(g) = (1 - \varphi)\partial^{-k}(G) - \partial^{-k}(M)$ and $\Omega_{D(k),h+k}(\partial^{-k}(g)) = (1 - \varphi)\Omega_{D(k),h+k}(\partial^{-k}(G))$, hence

$$\nabla_{h+k-1} \circ \ldots \circ \frac{\nabla_0}{\gamma_n - 1}(\partial^{-k}(g)) = \nabla_{h+k-1} \circ \ldots \circ \frac{\nabla_0}{\gamma_n - 1}((1-\varphi)G_{-k}) - \nabla_{h+k-1} \circ \ldots \circ \frac{\nabla_0}{\gamma_n - 1}(M_{-k}).$$

With this we may choose

$$b_{n,h} = (\varphi - 1)^{-1}\left(\frac{\Omega_{D(k),h+k}}{\gamma_n - 1}((1 - \varphi)G_{-k}) - \frac{\Omega_{D(k),h+k}}{\gamma_n - 1}(M_{-k})\right) \in \widetilde{D}.$$

Now for $n \gg 0$ we have $g \in \mathbf{B}^{\dagger,n}_{\log,K} \otimes \mathbf{D}^K_{\mathrm{st}}(D)$, hence the cocycle $h^1_{K_n,V(k)}(y_h)(\sigma) = (\sigma - 1)(w_{n,h} - b_{n,h})$ is cohomologuous to

$$h^1_{K_n,V(k)}(y_h)(\sigma) = (\sigma - 1)\left(\varphi^{-n}(w_{n,h}) - \varphi^{-n}(b_{n,h})\right)$$

since $(\varphi - 1)(w_{n,h} - b_{n,h}) \in \mathbf{D}_{\mathrm{st}}^K(D(k))$ so that G_K acts trivially (and φ acts as usual invertibly on $\mathbf{D}_{\mathrm{st}}^K(D(k))$). We use the exact sequences from the generalized Bloch-Kato map from Proposition 3.14. By the general properties of the connecting homomorphism for continuous cohomology we have the following: if $(x, y, z) \in H^1(\mathfrak{C}_{\mathrm{st}}(K, D(k)))$ and $\tilde{x} \in \widetilde{D}_{\log}[1/t]$ is such that $g(\tilde{x}) = (x, y, z)$ then $\exp_{K_n, D(k)}((x, y, z))(\sigma) = (\sigma - 1)\tilde{x}$. First one has

$$\varphi^{-n}(y) - \varphi^{-n}(y)(0) \in tK_0[[t]] \otimes_{K_0} \mathbf{D}_{\mathrm{st}}^K(D),$$

hence

$$\frac{\nabla_0}{\gamma_n - 1} \varphi^{-n}(y) = p^{-n}\varphi^{-n}(y)(0) + tz_1.$$

The same recursion as in [3], Theorem II.3 shows that

$$\varphi^{-n}(w_{n,h}) - (-1)^{h-1}(h-1)!p^{-n}\varphi^{-n}(y)(0) \in \mathbf{B}_{\mathrm{dR}}^+ \otimes D.$$

Next we have

$$N(\varphi^{-n}(w_{n,h}) - \varphi^{-n}(b_{n,h})) = p^{-n}\varphi^{-n}\left(\nabla_{h+k-1} \circ \ldots \frac{\nabla_0}{\gamma_n - 1}(N\partial^{-k}(G))\right).$$

Again we see by recursion with our choice of h that since $N\partial^{-k}(G) = L_{-k}$ and

$$L_{-k} = \sum_{i=0}^{h-1} \lambda_i \cdot t^i/i!,$$

that we obtain an equality

$$p^{-n}\varphi^{-n}\left(\nabla_{h+k-1} \circ \ldots \frac{\nabla_0}{\gamma_n - 1}(L_{-k})\right) = (-1)^{h-1}(h-1)!p^{-2n}\varphi^{-n}(L_{-k})(0).$$

Finally one has

$$(\varphi - 1)(\varphi^{-n}(w_{n,h}) - \varphi^{-n}(b_{n,h})) = \varphi^{-n}\left(\nabla_{h+k-1} \circ \ldots \frac{\nabla_0}{\gamma_n - 1}(M_{-k})\right).$$

Similarly, as before we have

$$M_{-k} = \sum_{i=0}^{h-1} \mu_i \cdot t^i/i!,$$

so that the recursion shows

$$\varphi^{-n}\left(\nabla_{h+k-1} \circ \ldots \frac{\nabla_0}{\gamma_n - 1}(L_{-k})\right) = (-1)^{h-1}(h-1)!p^{-n}\varphi^{-n}(M_{-k})(0).$$

Altogether this shows that

$$(-1)^{h-1}(h-1)!p^{-n}\exp_{K_n, D(k)}(\Xi_{n,k}(G))(\sigma) = (\sigma - 1)(\varphi^{-n}(w_{n,h}) - \varphi^{-n}(b_{n,h})),$$

which is the claim in this case.

Next assume D is pure of slope > 0. Then the exponential map has the description given in Proposition 3.16. First we note that $h^1_{K_n,D(k)}(\Omega_{D,h}(g) \otimes e_k) = (x,y)$ with

$$y = \Omega_{D(k),h+k}(G_{-k}), \quad x = \nabla_{h+k-1} \circ \ldots \circ \frac{\nabla_0}{\gamma_K - 1}((\varphi - 1)(G_{-k})).$$

The exponential map sends $\Xi_{n,k}(G)$ to $\varphi^{-n}(G_{-k})(0) \in X^1(\widetilde{D})^{G_K}$. The identification $\widetilde{D}/(\varphi - 1) \overset{\sim}{\to} X^1(\widetilde{D})$ is given by the following construction (see [5], Remark 3.4): If $x \in \widetilde{D}/(\varphi - 1)$ and $y \in \widetilde{D}[1/t]$ is chosen so that $(\varphi - 1)y = x$ then for $n \gg 0$ the image of x is $\varphi^{-n}(y)$. With this we see that under these identifications the class of $h^1_{K_n,D(k)}(\Omega_{D,h}(g) \otimes e_k)$ is send to

$$\varphi^{-n}(\nabla_{h+k-1} \circ \ldots \circ \frac{\nabla_0}{\gamma_K - 1}(G_{-k})) \equiv (-1)^{h-1}(h-1)! p^{-n} \varphi^{-n}(G_{-k})(0) \mod \mathbf{B}^+_{\mathrm{dR}} \otimes D$$

where we use the same recursion as before, hence the claim in this case.

In the general case of semistable a D of PR-type one may use the exact $0 \to D_{\leq 0} \to D \to D_{>0} \to 0$, where $\mathbf{D}_{\leq 0}$ is the biggest submodule of D with slopes ≤ 0, and $D_{>0} = D/D_{\leq 0}$, which is a (φ, Γ_K)-module with slopes > 0. By using the description of the isomorphism (9) and the explicit description of the transition morphism for the cone one is reduced, since all maps are compatible with exact sequences, to the case of a module with all slopes ≤ 0 or all slopes > 0. But in these cases we have just verified that the statement holds.

Now assume D is de Rham and let L/K be a finite extension such that D is of PR-type over L. Then for $y \in \mathbf{D}_{\infty,g}(D)$ one has, if we consider $y \in \mathbf{D}_{\infty,g}(D|_L)$

$$\mathrm{Res}_{L_n/K_n}(h^1_{K_n,D(k)}(\Omega_{D,h}(y))) = h^1_{L_n,D|_L(k)}(\Omega_{D,h}(y)),$$

so that the claim follows from Proposition 2.13. $\qquad\qquad\square$

For the record we state the next proposition in case D is semi-stable. As before, let $h \geq 1$ be such that (14) is satisfied for D, and dually let $h^* \geq 1$ be such that (14) is satisfied for $D^*(1)$

Proposition 3.42. a) If $k \geq 1 - h$ and $n \geq 1$ then

$$h^1_{K_n,D(k)}(\nabla_{h-1} \circ \ldots \circ \nabla_0(g) \otimes e_k) = p^{-n(K_n)}(-1)^{h+k-1}(h+1-k)! \exp_{K_n,D(k)}(\Xi_{n,k}(G))$$

b) If $k \leq -h^*$ and $n \geq 1$ then

$$\exp^*_{K_n,D^*(1)}(h^1_{K_n,D(k)}(\nabla_{h-1} \circ \ldots \circ \nabla_0(g) \otimes e_k)) = p^{-n(K_n)}\frac{1}{(-h-k)!}\varphi^{-n}(\partial^{-k} g \otimes t^{-j} e_j)(0)$$

Proof. The first part is just the preceding theorem. For the second observe that due to Proposition 3.21 one has

$$\exp^*_{K_n,D^*(1)}(h^1_{K_n,D(k)}(\nabla_{h-1} \circ \ldots \circ \nabla_0(g) \otimes e_k)) = p^{-n(K_n)}\varphi^{-n}(\nabla_{h-1} \circ \ldots \circ \nabla_0(g) \otimes e_k)(0).$$

A computation with the Taylor series shows that

$$p^{-n(K_n)}\varphi^{-n}(\nabla_{h-1} \circ \ldots \circ \nabla_0(g) \otimes e_k)(0) = p^{-n(K_n)}\frac{1}{(-h-j)!}\varphi^{-n}(\partial^{-k}g \otimes t^{-k}e_k)(0),$$

hence the claim. □

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