SIMULTANEOUS SIGN CHANGE OF FOURIER-COEFFICIENTS OF TWO CUSP FORMS

SANOLI GUN, WINFRIED KOHNEN AND PURUSOTTAM RATH

ABSTRACT. We consider the simultaneous sign change of Fourier coefficients of two modular forms with real Fourier coefficients. In an earlier work, the second author with Sengupta proved that two cusp forms of different (integral) weights with real algebraic Fourier coefficients have infinitely many Fourier coefficients of the same as well as opposite sign, up to the action of a Galois automorphism. In the first part, we strengthen their result by doing away with the dependency on the Galois conjugacy. In fact, we extend their result to cusp forms with arbitrary real Fourier coefficients. Next we consider simultaneous sign change at prime powers of Fourier coefficients of two integral weight Hecke eigenforms which are newforms. Finally, we consider an analogous question for Fourier coefficients of two half-integral weight Hecke eigenforms.

1. Introduction and statements of the Theorems

Throughout the paper, let p be a prime number, $z \in \mathfrak{H}$ be an element of the Poincaré upper-half plane and $q = e^{2\pi i z}$. Also let $\mathbb{D}$ be the set of square-free positive integers.

The theme of sign change of Fourier coefficients of modular forms constitutes an interesting active area of research. In a recent work [5], the second author and Sengupta consider the question of simultaneous sign change of Fourier coefficients of two cusp forms of different weights with real algebraic Fourier coefficients. They proved that given two normalized cusp forms f and g of same level and different weights with totally real algebraic Fourier coefficients, there exists a Galois automorphism σ such that f^σ and g^σ have infinitely many Fourier coefficients of the opposite sign. The proof uses Rankin - Selberg theory, a classical theorem of Landau, and finally the bounded denominator principle.

In the first part of this paper, we dispense with the bounded denominator argument by appealing to an elementary observation about real zeros of Dirichlet series. This allows us to strengthen their results and remove the dependency on the action of the absolute Galois group. In fact, removing the arithmetic component allows us to work with cusp forms having real, not necessarily algebraic Fourier coefficients. More precisely, we prove the following:

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Theorem 1. *Let*

$$f(z) := \sum_{n\geq 1} a(n)q^n \qquad \textit{and} \qquad g(z) := \sum_{n\geq 1} b(n)q^n$$

be non-zero cusp forms of level N and weights $1 < k_1 < k_2$ respectively. Further, let $a(n), b(n)$ be real numbers. If $a(1)b(1) \neq 0$, then there exists infinitely many n such that $a(n)b(n) > 0$ and infinitely many n such that $a(n)b(n) < 0$.

As a consequence of Theorem 1, we have the following corollary.

Corollary 2. *Let f, g be non-zero cusp forms of level N and weights $k_1 \neq k_2$. Suppose that*

$$f(z) := \sum_{n\geq 1} a(n)q^n \qquad \textit{and} \qquad g(z) := \sum_{n\geq 1} b(n)q^n,$$

where $a(n), b(n)$ are complex numbers. If the sequences $\{\Re(a(n))\}_n$ and $\{\Re(b(n))\}_n$ are not identically zero sequences, then there exists infinitely many n with $\Re\{a(n)\}\Re\{b(n)\} > 0$ and there exists infinitely many n such that $\Re\{a(n)\}\Re\{b(n)\} < 0$. An analogous result holds for the imaginary parts of the Fourier coefficients of f and g.

Remark 1.1. *We note that the second author and Sengupta* [5] *proved that if f and g have totally real algebraic Fourier coefficients $\{a(n)\}$ and $\{b(n)\}$ for $n \geq 1$ with $a(1) = 1 = b(1)$, then there exists an element σ of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ such that $a(n)^\sigma b(n)^\sigma \gtrless 0$ for infinitely many n. The proof in fact is carried out in full detail only in the case $a(n)^\sigma b(n)^\sigma < 0$, the case $a(n)^\sigma b(n)^\sigma > 0$ requires an easy additional argument, cf. e.g. the proof of our Theorem 1.*

If we restrict ourselves to normalised Hecke eigenforms which are newforms, we have the following stronger theorem.

Theorem 3. *Let*

$$f(z) := \sum_{n\geq 1} a(n)q^n \qquad \textit{and} \qquad g(z) := \sum_{n\geq 1} b(n)q^n$$

be two distinct newforms which are normalized Hecke eigenforms of level N_1, N_2 and weights k_1, k_2 respectively. Then there exists an infinite set S of primes such that the following holds: For every $p \in S$, the sets

$$\{m \in \mathbb{N} \mid a(p^m)b(p^m) > 0\} \quad \textit{and} \quad \{m \in \mathbb{N} \mid a(p^m)b(p^m) < 0\}$$

are infinite.

A crucial ingredient in the proof of the above theorem is a result of D. Ramakrishnan which asserts that a normalised Hecke eigenform f which is a newform of weight k, level N and trivial character with Hecke eigenvalues $a(p)$ for $(p, N) = 1$, is determined up to a quadratic twist by the knowledge of $a(p)^2$ for all primes p in a set of sufficiently large density (see section 2 for the exact statement).

In a recent work [9], Kowalski, Lau, Soundararajan and Wu have shown that a newform f is uniquely determined by the signs of the sequence $\{a(p)\}$ as p varies over any set of prime numbers of Dirichlet density one. Recall a set A of primes has Dirichlet density a real number κ if and only if

$$\frac{\sum_{p\in A}\frac{1}{p^s}}{\log\frac{1}{s-1}} \to \kappa$$

when $s \to 1$. In this context, see also the papers by Matomäki [12] and Pribitkin [16].

Finally, we consider the simultaneous sign change of Fourier coefficients of two half-integral weight newforms. The question of sign change of Fourier coefficients of a single half-integral weight newform has already been studied in [1] (see also [4, 8]). In order to state our result, we shall need to introduce notations and definitions.

Let $N, k \geq 1$ be integers and ψ be a Dirichlet character modulo $4N$. Then the space of cusp forms of weight $k+1/2$ for the congruence subgroup $\Gamma_0(4N)$ with character ψ is denoted by $S_{k+1/2}(4N,\psi)$. When $k=1$, we shall work only with the orthogonal complement (with respect to the Petersson scalar product) of the subspace of $S_{3/2}(4N,\psi)$ generated by the unary theta functions. The space $S_{k+1/2}(4N,\psi)$ is mapped to the space of integer weight cusp forms $S_{2k}(2N,\psi^2)$ under the Shimura liftings (see [19, 14] for further details). More precisely, for any $f \in S_{k+1/2}(4N,\psi)$ with a Fourier expansion

$$f(z) = \sum_{n\geq 1} a(n)q^n, \qquad q = e^{2\pi i z}$$

and any $t \in \mathbb{D}$, let

$$A(n) := \sum_{d|n} \psi_{t,N}(d) d^{k-1} a\left(\frac{n^2}{d^2}t\right)$$

where $\psi_{t,N}$ denotes the character

$$\psi_{t,N}(d) := \psi(d)\left(\frac{(-1)^k t}{d}\right).$$

Then by the works of Shimura [19] and Niwa [14] it is known that the series

$$F(z) = \sum_{n\geq 1} A(n)q^n$$

is an element in $S_{2k}(2N,\psi^2)$. It is also known that when N is odd and square-free and $\psi^2 = 1$, there is a Hecke invariant subspace $S^{\mathrm{new}}_{k+1/2}(4N,\psi) \subset S_{k+1/2}(4N,\psi)$ consisting of "newforms" which the Hecke operators isomorphically map onto the space of newforms $S^{\mathrm{new}}_{2k}(2N) \subset S_{2k}(2N)$ under a suitable linear combination of the Shimura lifts (see [6, 7, 11]). In this set-up, we have the following:

Theorem 4. *Suppose that $k_1, k_2 > 1$ are distinct natural numbers, N_1, N_2 are odd square-free natural numbers and ψ_1, ψ_2 are real characters modulo $4N_1$ and $4N_2$ respectively. Suppose that*

$$\begin{aligned} f(z) &:= \sum_{n\geq 1} a(n)q^n \in S^{new}_{k_1+1/2}(4N_1, \psi_1) \\ \text{and} \qquad g(z) &:= \sum_{n\geq 1} b(n)q^n \in S^{new}_{k_2+1/2}(4N_2, \psi_2) \end{aligned}$$

are Hecke eigenforms. If the Fourier coefficients $a(n), b(n) \in \mathbb{R}$ for all $n \geq 1$ and there exists a natural number $t \in \mathbb{D}$ with $a(t)b(t) \neq 0$, then there exists an infinite set S of primes such that for any $p \in S$, the sequence $\{a(tp^{2m})b(tp^{2m})\}_{m\in\mathbb{N}}$ has both positive and negative sign infinitely often.

Remark 1.2. *One can prove similar results in the context of the plus space* [7, 8]*. We leave this to the reader.*

2. Intermediate Lemmas and Theorem

In order to prove the theorems, we need the following lemmas.

Lemma 5. *Let $s \in \mathbb{C}$ and*

$$R(s) := \sum_{n\geq 1} \frac{a(n)}{n^s}$$

be a Dirichlet series with real coefficients. Assume further that $a_n \geq 0$ or $a_n \leq 0$ for all n. If R has a real zero α in the region of convergence, then R is identically zero.

Proof of Lemma 5. The lemma follows from noting that

$$a(n) \geq 0 \quad \text{or} \quad a(n) \leq 0 \quad \text{and} \quad \sum_{n\geq 1} \frac{a(n)}{n^\alpha} = 0 \quad \text{implies that} \quad a(n) = 0.$$

Lemma 6. *Let $s \in \mathbb{C}$ and $a(n) \in \mathbb{R}$. For $m \geq 1$, consider the Dirichlet polynomial*

$$R(s) := \sum_{1\leq n\leq m} \frac{a(n)}{n^s}.$$

If $R(s)$ has infinitely many real zeros, then R is identically zero.

Proof of Lemma 6. Suppose that R is not identically zero. Since R has infinitely many zeros, we may assume that $a(m) \neq 0$ with $m \geq 2$. Further the set of real zeros of R cannot be bounded by the identity theorem for holomorphic functions. For any real zero r of $R(s)$, one has

$$-a(1) = a(2)2^{-r} + \cdots + a(m)m^{-r}.$$

We let $|r| \to \infty$. By hypothesis, $a(1), \cdots, a(m)$ are real numbers and hence if $r \to -\infty$, then

$$-a(1) = m^{-r}\left(a(2)(\frac{2}{m})^{-r} + ... + a(m)\right) \to \pm\infty$$

depending on the sign of $a(m)$, a contradiction. On the other hand, if $r \to +\infty$, $a(1) = 0$ and then we can work with $a(2)$ and so on.

Before we state the next theorem which plays an important role throughout the paper, we need to introduce more notations. For every pair of natural numbers $N, k \geq 2$, the space of newforms of weight k, level N is denoted by $S_k^{\text{new}}(N)$. For a normalised Hecke eigenform

$$f(z) := \sum_{n\geq 1} a(n)q^n \in S_k^{\text{new}}(N),$$

and for $s \in \mathbb{C}$ with $\Re(s) \gg 1$, let us set for $p \nmid N$

$$L_p(\text{Ad}(f),\ s) := (1 - \frac{\alpha_p}{\bar{\alpha_p}}p^{-s})^{-1}(1 - \frac{\bar{\alpha_p}}{\alpha_p}p^{-s})^{-1}(1 - p^{-s})^{-2},$$

where α_p and $\bar{\alpha_p}$ are non-zero algebraic integers satisfying $a(p) = \alpha_p + \bar{\alpha_p}$ and $|\alpha_p| = p^{(k-1)/2}$. We now state the following theorem of D. Ramakrishnan (see [17]) which will play an important role in proving Theorem 3 and Theorem 4.

Theorem 7. (*D. Ramakrishnan*, [17]) *Let*

$$f(z) := \sum_{n\geq 1} a(n)q^n \in S_{k_1}^{new}(N_1) \quad \text{and} \quad g(z) := \sum_{n\geq 1} b(n)q^n \in S_{k_2}^{new}(N_2)$$

be normalised Hecke eigenform such that for all primes p outside a set M of Dirichlet density $\delta(M) < \frac{1}{18}$, we have

$$L_p(\text{Ad}(f),\ s) = L_p(\text{Ad}(g),\ s).$$

Then $k_1 = k_2$ and for all primes p co-prime to N_1N_2, we have $a(p) = \chi(p)b(p)$, where χ is a Dirichlet character of conductor N dividing N_1N_2.

Finally, we shall be frequently using a classical theorem of Landau which asserts that a Dirichlet series with non-negative coefficients has a singularity on the real line at its abscissa of convergence (see page 16 of [13], for instance).

3. Proof of Theorem 1 and Corollary 2

Proof of Theorem 1. By given hypothesis, we have $a(1)b(1) \neq 0$. First we will show that there exists infinitely many n such that

$$\frac{a(n)b(n)}{a(1)b(1)} < 0. \tag{1}$$

Without loss of generality, we can assume that

$$a(1)b(1) > 0 \tag{2}$$

as otherwise we can replace f by $-f$. If equation (1) is not true, then there exists an $n_0 \in \mathbb{N}$ such that

$$a(n)b(n) \geq 0 \tag{3}$$

for all $n \geq n_0$. Set $M := \prod_{p \leq n_0} p$,

$$f_1(z) := \sum_{\substack{n \geq 1 \\ (n,M)=1}} a(n)q^n \quad \text{and} \quad g_1(z) := \sum_{\substack{n \geq 1 \\ (n,M)=1}} b(n)q^n.$$

Then f_1 and g_1 are cusp forms of level NM^2 and weights k_1 and k_2 respectively. For $s \in \mathbb{C}$ with $\Re(s) \gg 1$, the Rankin-Selberg L-function of f_1 and g_1 is defined by

$$R_{f_1,g_1}(s) := \sum_{\substack{n \geq 1 \\ (n,M)=1}} \frac{a(n)b(n)}{n^s}.$$

For $\Re(s) \gg 1$, set

$$L_{f_1,g_1}(s) := \zeta_{NM^2}(2s - (k_1 + k_2) + 2) R_{f_1,g_1}(s) := \sum_{n \geq 1} c(n) n^{-s},$$

where

$$\zeta_{NM^2}(s) := \prod_{p \mid NM^2} (1 - p^{-s}) \zeta(s).$$

It follows from (2) and (3) that $c(n) \geq 0$ for all $n \geq 1$. It is known (see page 144 of [10], also page 3565 of [5]) that

$$(2\pi)^{-2s} \Gamma(s) \Gamma(s - k_1 + 1) L_{f_1,g_1}(s)$$

is entire, hence $L_{f_1,g_1}(s)$ is also entire. By Landau's Theorem it therefore follows that the Dirichlet series $L_{f_1,g_1}(s)$ converges everywhere. Since $L_{f_1,g_1}(s)$ has real zeros (coming from the poles of the Γ-factors), by Lemma 5, we have that $c(n) = 0$ for all $n \geq 1$. This contradicts the assumption that $a(1)b(1) \neq 0$ and hence completes the proof of (1).

In order to complete the proof of the theorem, we need to show that there exists infinitely many n such that

$$\frac{a(n)b(n)}{a(1)b(1)} > 0. \tag{4}$$

It is sufficient to assume that $a(1)b(1) > 0$. We then have to show that there exists infinitely many n such that $a(n)b(n) > 0$. If not, then $a(n)b(n) \leq 0$ for all n large. Note that $a(n)b(n)$ can not be equal to zero for almost all n. For in this case $\sum_n a(n)b(n)n^{-s}$ is a Dirichlet polynomial and the function

$$(2\pi)^{-2s}\Gamma(s)\Gamma(s-k_1+1)\zeta_N(2s-(k_1+k_2)+2)\sum_{n=1}^{\infty}\frac{a(n)b(n)}{n^s}$$

is entire. Presence of the double Γ-factors ensures that $\sum_n a(n)b(n)n^{-s}$ has infinitely many real zeros, and hence by Lemma 6, we find that $a(1)b(1)=0$, a contradiction. Hence we can choose a natural number d such that $a(d)b(d)<0$. Consider

$$\begin{aligned} f_2(z) &:= \sum_{n\geq 1} a(nd)q^n := \sum_{n\geq 1} A(n)q^n \\ \text{and} \quad g_2(z) &:= \sum_{n\geq 1} b(nd)q^n := \sum_{n\geq 1} B(n)q^n. \end{aligned}$$

Then f_2 and g_2 are non-zero cusp forms of level dN and weights k_1 and k_2 respectively (see page 28 of [15]). Since

$$A(1)B(1)=a(d)b(d)<0,$$

by (1), we have $A(n)B(n)>0$ for infinitely many n. This proves our claim.

Proof of Corollary 2. For a cusp form $h(z):=\sum_{n\geq 1} c(n)q^n$, let h^τ denote the cusp form $h^\tau(z):=\sum_{n\geq 1}\overline{c(n)}q^n$. Consider the non-zero cusp forms

$$F:=\frac{f+f^\tau}{2} \quad \text{and} \quad G:=\frac{g+g^\tau}{2}$$

of level N and weights k_1 and k_2 respectively with real Fourier coefficients. Now if we apply Theorem 1 to F and G, we see that there exists infinitely many n such that $\Re(a(n))\Re(b(n)) \gtrless 0$.

Similarly, one can apply Theorem 1 to

$$F:=\frac{f-f^\tau}{2i} \quad \text{and} \quad G:=\frac{g-g^\tau}{2i}$$

to get the desired result for the imaginary part of the Fourier coefficients of f and g.

4. Simultaneous sign change at prime powers

In this section, we discuss simultaneous sign change of Fourier coefficients of Hecke eigenforms which are newforms at prime powers.

Proof of Theorem 3. For any prime $p \nmid N_1N_2$, we define

$$F_p(s):=\sum_{m=0}^{\infty}\frac{a(p^m)b(p^m)}{p^{ms}}.$$

Since f and g are normalised Hecke eigenforms, we have

$$\sum_{m=0}^{\infty} \frac{a(p^m)}{p^{ms}} = \frac{1}{(1-\alpha_p p^{-s})(1-\bar{\alpha_p} p^{-s})}$$

$$\text{and} \quad \sum_{m=0}^{\infty} \frac{b(p^m)}{p^{ms}} = \frac{1}{(1-\beta_p p^{-s})(1-\bar{\beta}_p p^{-s})},$$

where $\alpha_p, \bar{\alpha}_p, \beta_p, \bar{\beta}_p$ are non-zero algebraic integers such that $a(p) = \alpha_p + \bar{\alpha}_p, |\alpha_p| = p^{(k_1-1)/2}$ and $b(p) = \beta_p + \bar{\beta}_p, |\beta_p| = p^{(k_2-1)/2}$. Then $F_p(s)$ can be written as (see [2], page 73 for instance)

$$F_p(s) = \frac{1 - p^{k_1+k_2-2}p^{-2s}}{(1-\alpha_p\beta_p p^{-s})(1-\bar{\alpha_p}\beta_p p^{-s})(1-\alpha_p\bar{\beta}_p p^{-s})(1-\bar{\alpha}_p\bar{\beta}_p p^{-s})}.$$

Now consider the following set

$$A := \{\, p \mid \alpha_p\beta_p \notin \mathbb{R} \text{ and } \bar{\alpha_p}\beta_p \notin \mathbb{R}\}\,.$$

For every $p \in A, p \nmid N_1N_2$, $F_p(s)$ has no real poles. However $F_p(s)$ has complex poles and hence is not entire. Indeed, the degree of the numerator as a polynomial in p^{-s} is 2 and that of the denominator is 4. Then by Landau's Theorem, for each such prime p, there are infinitely many m_1 such that $a(p^{m_1})b(p^{m_1}) > 0$ and infinitely many m_2 such that $a(p^{m_2})b(p^{m_2}) < 0$. Thus if A is infinite, we are done.

Suppose now that A is finite. Then $F_p(s)$ has real poles for almost all primes. If $F_p(s)$ has a real pole, then either $\alpha_p\beta_p \in \mathbb{R}$ or $\bar{\alpha_p}\beta_p \in \mathbb{R}$, that is,

(5)
$$\frac{\alpha_p}{\bar{\alpha_p}} = \frac{\beta_p}{\bar{\beta}_p}$$
$$\text{or} \quad \frac{\alpha_p}{\bar{\alpha_p}} = \frac{\bar{\beta}_p}{\beta_p}.$$

Recall

$$L_p(Ad(f),\ s) := (1 - \frac{\alpha_p}{\bar{\alpha_p}}p^{-s})^{-1}(1 - \frac{\bar{\alpha_p}}{\alpha_p}p^{-s})^{-1}(1-p^{-s})^{-2}$$
$$L_p(Ad(g),\ s) := (1 - \frac{\beta_p}{\bar{\beta}_p}p^{-s})^{-1}(1 - \frac{\bar{\beta}_p}{\beta_p}p^{-s})^{-1}(1-p^{-s})^{-2}.$$

Using (5), we see that for almost all primes, we have

$$L_p(Ad(f),\ s) = L_p(Ad(g),\ s).$$

Then by Theorem 7, we have $k := k_1 = k_2$ and $a(p) = \chi(p)b(p)$ for all primes $(p, N_1N_2) = 1$. Here χ is a character of conductor N dividing N_1N_2. Note that we have $\chi^2 = 1$ as $a(p), b(p)$ are real numbers. Further, by the multiplicity one theorem (see page 30 of [15]), one has

$$a(p) = \chi(p)b(p) \neq 0$$

for a set T of primes of positive density.

If χ is trivial, that is, $a(p) = b(p)$ for almost all primes p, then again by the multiplicity one theorem, we have $f = g$, a contradiction. Now suppose that χ is quadratic. Then $a(p) = \pm b(p)$ for almost all primes p. Further, by the multiplicity one theorem, there is an infinite subset $S \subset T$ of primes such that $a(p) = -b(p) \neq 0$.

Fix a prime $p \in S$. Then $a(p) = -b(p) \neq 0$ and

$$\begin{aligned}\sum_{m\geq 0} a(p^m)X^m &= \frac{1}{1 - a(p)X + p^{k-1}X^2} \\ &= \frac{1}{1 + b(p)X + p^{k-1}X^2} \\ &= \sum_{m\geq 0} b(p^m)(-X)^m \\ &= \sum_{m\geq 0} (-1)^m b(p^m)X^m.\end{aligned} \tag{6}$$

Hence $a(p^m) = b(p^m)$ for all even natural numbers $m \geq 1$ and $a(p^m) = -b(p^m)$ for all odd natural numbers $m \geq 1$. If

$$a(p^m) = -b(p^m) = 0$$

for all but finitely many odd $m \geq 1$, then

$$\sum_{m\geq 0} a(p^m)X^m - \sum_{m\geq 0} a(p^m)(-X)^m = \sum_{\substack{m\geq 0 \\ m \text{ odd}}} a(p^m)X^m \tag{7}$$

is a polynomial. But the left hand side of (7) is equal to

$$\frac{1}{1 - a(p)X + p^{k-1}X^2} - \frac{1}{1 + a(p)X + p^{k-1}X^2}.$$

This forces that $1 - a(p)X + p^{k-1}X^2 = 1 + a(p)X + p^{k-1}X^2$ which implies that $a(p) = 0$, a contradiction. Hence for any $p \in S$, there exists infinitely many odd natural numbers $m \geq 1$ such that

$$a(p^m) = -b(p^m) \neq 0, \quad \text{that is,} \quad a(p^m)b(p^m) < 0.$$

In a similar way, if we consider $\sum_{m\geq 0} a(p^m)X^m + \sum_{m\geq 0} a(p^m)(-X)^m$ and argue as before, we can show that for any $p \in S$, we have $a(p^m) = b(p^m) \neq 0$, that is, $a(p^m)b(p^m) > 0$ for infinitely many even natural numbers $m \geq 1$. This completes the proof of the theorem.

5. Simultaneous sign changes for half-integral weight modular forms

Throughout the section, we assume that N_1, N_2 are odd, square-free natural numbers and ψ_1, ψ_2 are real Dirichlet characters modulo $4N_1$ and $4N_2$ respectively. Suppose that

$$f \in S^{\text{new}}_{k_1+1/2}(4N_1, \psi_1) \quad \text{and} \quad g \in S^{\text{new}}_{k_2+1/2}(4N_2, \psi_2)$$

are newforms with real Fourier coefficients with $a(t)b(t) \neq 0$ for some $t \in \mathbb{D}$. Here we study simultaneous sign changes of Fourier coefficients of these forms. As mentioned in the introduction, these newforms f and g correspond to newforms

$$F(z) = \sum_{n=1}^{\infty} A(n)q^n \quad \text{and} \quad G(z) = \sum_{n=1}^{\infty} B(n)q^n$$

respectively in the spaces $S_{2k_1}^{\text{new}}(2N_1)$ and $S_{2k_2}^{\text{new}}(2N_2)$. Now we proceed to the proof of Theorem 4.

Proof of Theorem 4. For odd primes p with $(p, N_1N_2) = 1$, the newforms f and g are eigenfunctions of $T(p^2)$ with eigenvalues λ_p and γ_p respectively. Hence the corresponding integral weight newforms F and G are eigenfunctions of the Hecke operators $T(p)$ with eigenvalues λ_p and γ_p respectively. Since $\psi_1^2 = 1 = \psi_2^2$, the eigenvalues λ_p and γ_p are real. By hypothesis, there exists a natural number $t \in \mathbb{D}$ such that $a(t)b(t) \neq 0$.

Further, for $s \in \mathbb{C}$ with $\Re(s) \gg 1$, one has

$$\begin{aligned} \sum_{m\geq 0} \frac{a(tp^{2m})}{p^{ms}} &= a(t)\frac{1-\psi_{t,N_1}(p)p^{k_1-1-s}}{1-\lambda_p p^{-s} + p^{2k_1-1-2s}} \\ \text{and} \quad \sum_{m\geq 0} \frac{b(tp^{2m})}{p^{ms}} &= b(t)\frac{1-\psi_{t,N_2}(p)p^{k_2-1-s}}{1-\gamma_p p^{-s} + p^{2k_2-1-2s}}, \end{aligned}$$

where ψ_{t,N_i} denotes the character

$$\psi_{t,N_i}(d) := \psi_i(d)\left(\frac{(-1)^{k_i}t}{d}\right).$$

Write

$$\begin{aligned} 1-\lambda_p p^{-s} + p^{2k_1-1-2s} &= (1-\alpha_p p^{-s})(1-\bar{\alpha}_p p^{-s}) \\ \text{and} \quad 1-\gamma_p p^{-s} + p^{2k_2-1-2s} &= (1-\beta_p p^{-s})(1-\bar{\beta}_p p^{-s}), \end{aligned}$$

where $\alpha_p + \bar{\alpha}_p = \lambda_p,\ \beta_p + \bar{\beta}_p = \gamma_p$ and $\alpha_p\bar{\alpha}_p = p^{2k_1-1},\ \beta_p\bar{\beta}_p = p^{2k_2-1}$. Now consider

$$T_p(s) := \sum_{m\geq 0} \frac{a(tp^{2m})b(tp^{2m})}{p^{ms}}$$

Using partial fractions, we see that

$$T_p(s) = \frac{a(t)b(t)H(p^{-s})}{(1-\alpha_p\beta_p p^{-s})(1-\bar{\alpha_p}\beta_p p^{-s})(1-\alpha_p\bar{\beta}_p p^{-s})(1-\bar{\alpha}_p\bar{\beta}_p p^{-s})},$$

where H is a polynomial of degree ≤ 3. Again $T_p(s)$ has poles and hence is not entire. Now consider the set

$$X := \{\, p \mid \alpha_p\beta_p \notin \mathbb{R} \text{ and } \bar{\alpha_p}\beta_p \notin \mathbb{R}\}\,.$$

For every $p \in X$, $T_p(s)$ has no real poles and hence by Landau's Theorem, the sequence $\{a(tp^{2m})b(tp^{2m})\}_m$ has infinitely many sign changes. Thus, if X is infinite, we are done. Suppose not. Now for any $p \notin X, p \nmid 2N_1N_2$ and $s \in \mathbb{C}$, we have $L_p(\mathrm{Ad}(F),\ s) = L_p(\mathrm{Ad}(G),\ s)$. Recall

$$\begin{aligned} L_p(\mathrm{Ad}(F),\ s) &:= (1 - \frac{\alpha_p}{\bar{\alpha_p}}p^{-s})^{-1}(1 - \frac{\bar{\alpha_p}}{\alpha_p}p^{-s})^{-1}(1-p^{-s})^{-2} \\ L_p(\mathrm{Ad}(G),\ s) &:= (1 - \frac{\beta_p}{\bar{\beta_p}}p^{-s})^{-1}(1 - \frac{\bar{\beta_p}}{\beta_p}p^{-s})^{-1}(1-p^{-s})^{-2}. \end{aligned}$$

But then by Theorem 7, we have $2k_1 = 2k_2$, a contradiction. This completes the proof of Theorem 4.

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(Sanoli Gun) INSTITUTE OF MATHEMATICAL SCIENCES, C.I.T CAMPUS, TARAMANI, CHENNAI 600 113, INDIA.
E-mail address: sanoli@imsc.res.in

(Winfried Kohnen) MATHEMATISCHES INSTITUT DER UNIVERSITÄT, INF 288, D-69120 HEIDELBERG, GERMANY
E-mail address: winfried@mathi.uni-heidelberg.de

(Purusottam Rath) CHENNAI MATHEMATICAL INSTITUTE, PLOT NO H1, SIPCOT IT PARK, PADUR PO, SIRUSERI 603103, TAMIL NADU, INDIA.
E-mail address: rath@cmi.res.in