Multiplicity one for certain paramodular forms of genus two

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We show that certain paramodular cuspidal automorphic irreducible representations of $\mathrm{GSp}(4, \mathbb{A}_\mathbb{Q})$, which are not CAP, are globally generic. Especially, they occur in the cuspidal spectrum with multiplicity one. Our proof relies on a reasonable hypothesis concerning the non-vanishing of central values of automorphic L-series.

1 Introduction

Atkin-Lehner theory defines a one-to-one correspondence between cuspidal automorphic irreducible representations of $\mathrm{GL}(2, \mathbb{A}_\mathbb{Q})$ with archimedean factor in the discrete series and normalized holomorphic elliptic cuspidal newforms on the upper half plane. As an analogue for the symplectic group $\mathrm{GSp}(4, \mathbb{A}_\mathbb{Q})$, a local theory of newforms has been developed by Roberts and Schmidt [RS07] with respect to paramodular groups.

However, still lacking for this theory is the information whether paramodular cuspidal automorphic irreducible representations of $\mathrm{GSp}(4, \mathbb{A}_\mathbb{Q})$ occur in the cuspidal spectrum with multiplicity one. Furthermore, holomorphic paramodular modular cusp forms, i.e. those invariant under some paramodular subgroup of $\mathrm{Sp}(4, \mathbb{Q})$, do not describe all holomorphic Siegel modular cusp forms. Indeed, at least if the weight of the modular forms is high enough, one is lead to conjecture that the paramodular modular holomorphic cusp forms exactly correspond to those holomorphic modular cusp forms for which their local non-archimedean representations, considered from an automorphic point of view, are generic representations. Under certain technical restrictions, we show that this is indeed the case.

To be more precise, suppose $\Pi = \bigotimes_v \Pi_v$ is a paramodular cuspidal automorphic irreducible representation of $\mathrm{GSp}(4, \mathbb{A}_\mathbb{Q})$ of odd paramodular level, which is not CAP and whose archimedean factor Π_∞ is in the discrete series. Then under the assumption of the hypothesis 1.1 below we prove that all local nonarchimedean representations Π_v are generic. Furthermore, we show that the hypothesis implies that Π occurs in the cuspidal spectrum with multiplicity one. The hypothesis imposed concerns the non-vanishing of central L-values and is crucial for our approach.

Hypothesis 1.1. *Suppose $\tilde{\Pi}$ is a unitary cuspidal automorphic irreducible representation of $\mathrm{GL}(4, \mathbb{A}_\mathbb{Q})$ and w is a fixed place of $\mathbb{Q}$. Then there is a finite set S of $\mathbb{Q}$-places and* *an idele class character $\lambda = \otimes_v \lambda_v$ with $\lambda_w = 1$, such that the central value of the partial L-series with Euler factors outside of S is nonzero*

$$L^S(\tfrac{1}{2}, (\lambda \circ \det) \otimes \tilde{\Pi}) \neq 0. \tag{1}$$

The analogous statement is well-known for $\mathrm{GL}(r, \mathbb{A})$, $r = 1, 2, 3$ [HK10], [FH95]. An approximative result for $r = 4$ has been shown by Barthel and Ramakrishnan [BR94], later improved by Luo [L05]. Given a unitary globally generic cuspidal automorphic irreducible representation $\tilde{\Pi}$ of $\mathrm{GL}(4, \mathbb{A}_{\mathbb{Q}})$, a finite set S of $\mathbb{Q}$-places and a complex number s_0 with $\mathrm{Re}(s_0) \neq 1/2$ they prove that there are infinitely many Dirichlet characters λ such that λ_v is unramified for $v \in S$ and the completed L-function $\Lambda(s, \tilde{\Pi})$ satisfies $\Lambda(s_0, (\lambda \circ \det) \otimes \tilde{\Pi}) \neq 0$. We only apply the hypothesis above for the lifts $\tilde{\Pi}$ of the automorphic form Π to the group $\mathrm{GL}(4, \mathbb{A}_{\mathbb{Q}})$ in situations where the Ramanujan conjecture holds. This might allow for improvements on the results mentioned.

We remark, there is good evidence for our result on genericity of paramodular representations. In fact, the generalized strong Ramanujan conjecture for cuspidal automorphic irreducible representations $\Pi = \otimes'_v \Pi_v$ of $\mathrm{GSp}(4, \mathbb{A}_{\mathbb{Q}})$ (not CAP) predicts that every local representation Π_v should be tempered. But paramodular tempered local representations Π_v at non-archimedean places are always generic by Lemma 3.1.

2 Preliminaries

The group of symplectic similitudes $\mathrm{GSp}(4)$ of genus two is defined over $\mathbb{Z}$ by the equation

$$g^t J g = \lambda J$$

for $(g, \lambda) \in \mathrm{GL}(4) \times \mathrm{GL}(1)$ and $J = \left(\begin{smallmatrix} 0 & w \\ -w & 0 \end{smallmatrix}\right)$ with $w = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Since λ is uniquely determind by g, we write g for (g, λ) and $s(g)$ for λ. This defines the similitude character

$$s : \mathrm{GSp}(4) \to \mathrm{GL}(1).$$

Fix a totally real number field $F/\mathbb{Q}$ with integers $\mathfrak{o}$ and adele ring $\mathbb{A}_F = \mathbb{A}_\infty \times \mathbb{A}_{\mathrm{fin}}$. For the profinite completion of $\mathfrak{o}$ we write $\mathfrak{o}_{\mathrm{fin}} \subseteq \mathbb{A}_{\mathrm{fin}}$. The paramodular group $K^{\mathrm{para}}(\mathfrak{a}) \subseteq \mathrm{GSp}(4, \mathbb{A}_{\mathrm{fin}})$ attached to a non-zero ideal $\mathfrak{a} \subseteq \mathfrak{o}$ is the group of all

$$g \in \begin{pmatrix} \mathfrak{o}_{\mathrm{fin}} & \mathfrak{o}_{\mathrm{fin}} & \mathfrak{o}_{\mathrm{fin}} & \mathfrak{a}^{-1}\mathfrak{o}_{\mathrm{fin}} \\ \mathfrak{a}\mathfrak{o}_{\mathrm{fin}} & \mathfrak{o}_{\mathrm{fin}} & \mathfrak{o}_{\mathrm{fin}} & \mathfrak{o}_{\mathrm{fin}} \\ \mathfrak{a}\mathfrak{o}_{\mathrm{fin}} & \mathfrak{o}_{\mathrm{fin}} & \mathfrak{o}_{\mathrm{fin}} & \mathfrak{o}_{\mathrm{fin}} \\ \mathfrak{a}\mathfrak{o}_{\mathrm{fin}} & \mathfrak{a}\mathfrak{o}_{\mathrm{fin}} & \mathfrak{a}\mathfrak{o}_{\mathrm{fin}} & \mathfrak{o}_{\mathrm{fin}} \end{pmatrix} \cap \mathrm{GSp}(4, \mathbb{A}_{\mathrm{fin}}), \qquad s(g) \in \mathfrak{o}_{\mathrm{fin}}^\times.$$

An irreducible smooth representation $\Pi = \Pi_\infty \otimes \Pi_{fin}$ of $\mathrm{GSp}(4, \mathbb{A})$ is called paramodular if Π_{fin} admits non-zero invariants under some paramodular group $K^{\mathrm{para}}(\mathfrak{a})$.

For a non-archimedean field F_v with residue field $\mathfrak{o}_v/\mathfrak{p}_v$, an admissible irreducible representation Π_v of $\mathrm{GSp}(4, F_v)$ for a local non-archimedean place v is called paramodular if it admits non-zero invariants under the local factor $K_v^{\mathrm{para}}(\mathfrak{p}_v^m)$ for some nonnegative integer m.

Lemma 2.1. *A paramodular local irreducible admissible representation Π_v of $\mathrm{GSp}(4,F_v)$ for a non-archimedean field F_v is a twist $\Pi_v \cong (\omega'_v \circ s)\otimes \tilde{\Pi}_v$ for an unramified character ω'_v of $F_v^\times$ and a paramodular irreducible admissible $\tilde{\Pi}_v$ with trivial central character.*

Proof. The central character of Π_v is unramified. □

Lemma 2.2. *Suppose Π is an paramodular unitary automorphic irreducible representation of $\mathrm{GSp}(4,\mathbb{A}_\mathbb{Q})$. Then Π is a twist $\Pi \cong (\omega' \circ s)\otimes\tilde{\Pi}$, where ω' is an unramified idele class character and $\tilde{\Pi}$ is paramodular unitary automorphic with trivial central character.*

Proof. Since $\Pi \cong \otimes_v \Pi_v$ is paramodular, the local central character ω_v of Π_v at the non-archimedean places v is unramified. The global central character $\omega = \prod_v \omega_v$ is invariant under $\mathbb{Q}$, the archimedean factor ω_∞ satisfies $\omega_\infty(-1)=1$, so it factors over the archimedean valuation $|\cdot|_\infty$. Now there is $t\in\mathbb{R}$ with $\omega = |\cdot|_\mathbb{A}^{it}$ because $\omega(p)=1$ for every prime p. Finally, let $\omega' = |\cdot|_\mathbb{A}^{it/2}$ and $\tilde{\Pi} = (\omega'^{-1}\circ s)\otimes\Pi$. □

We fix standard parabolic subgroups such that the unipotent radicals have non-zero entries in the upper right corner. For example, the Siegel parabolic subgroup is

$$P = M \rtimes S = \begin{pmatrix} * & * \\ & * \end{pmatrix} \cap \mathrm{GSp}(4)\ ,$$

where M is the Levi subgroup of blockdiagonal matrices and $S = \{\begin{pmatrix} I_2 & s \\ & I_2\end{pmatrix} \mid s^t = wsw\}$ is the unipotent radical.

3 Local spinor factors

Fix a local nonarchimedean field F_v of characteristic $\neq 2$ with valuation character $\nu(x) = |x|_v$, residue field $\mathfrak{o}_v/\mathfrak{p}_v$ of order q and uniformizer $\varpi \in \mathfrak{p}_v$. Fix a non-trivial additive character $\psi : (F_v,+)\to\mathbb{C}^\times$. Attached to every smooth character $\chi : F_v^\times \to \mathbb{C}^\times$ is the Tate L-factor

$$L(s,\chi) = \begin{cases} (1-\chi(\varpi)q^{-s})^{-1} & \chi \text{ unramified},\\ 1 & \chi \text{ ramified}.\end{cases}$$

In this section we consider preunitary irreducible admissible representations Π_v of the group $\mathrm{GSp}(4,F_v)$. The non-cuspidal Π_v have been classified by Sally and Tadic [ST94] and we use their notation. Roberts and Schmidt [RS07] have designated them with roman numerals.

Lemma 3.1. *For tempered preunitary irreducible admissible representations Π_v the following assertions are equivalent:*

- Π_v *is generic and has unramified central character,*
- Π_v *is paramodular.*

Especially, this holds for cuspidal Π_v.

Proof. By Lemma 2.1, we can assume that Π_v has trivial central character. Then this is a result of Roberts and Schmidt [RS07, 7.5.8]. □

For generic Π_v and a smooth character[1] $\mu : F_v^\times \to \mathbb{C}^\times$, Novodvorsky [N79] has defined a local degree four L-factor $L_{\mathrm{Nvd}}(s, \Pi_v, \mu)$. Later, Piatetski-Shapiro [PS97] has constructed a local degree four L-factor $L_{\mathrm{PSS}}(s, \Pi_v, \mu)$ for infinite-dimensional irreducible admissible representations of $\mathrm{GSp}(4, F_v)$. We briefly review the construction:

Fix a matrix $b \in \mathrm{GL}(2, F_v)$ with $b' = wbw$ and the attached linear form

$$l : S(F_v) \to F_v, \quad \left(\begin{smallmatrix} 1 & s \\ & 1 \end{smallmatrix}\right) \mapsto \mathrm{tr}(bs) \ .$$

Let D be the connected component of the stabilizer of l in $M(F_v)$. We assume that $D \cong K^\times$ for a quadratic field extension K of F_v. An arbitrary choice of a character of D and the fixed character $\psi \circ l$ of S define a character α of $R = D \ltimes S(F_v) \subseteq P(F_v)$; we fix α once and for all. For every infinite-dimensional irreducible admissible representation Π_v of $\mathrm{GSp}(4, F_v)$ there is a non-trivial functional $\ell : \mathrm{GSp}(4, F_v) \to \mathbb{C}$ with $\ell(\Pi(r)u) = \alpha(r)\ell(u)$ for $r \in R$ and $u \in \Pi$, unique up to scalar multiples. The Bessel model of Π is the right action of $\mathrm{GSp}(4, F_v)$ on the space of generalized Whittaker functions

$$W_u(g) = \ell(\Pi(g)u), \quad u \in \Pi_v,\ g \in \mathrm{GSp}(4, F_v) \ .$$

Fix some W_u, a Schwartz-Bruhat function $\Phi : K^2 \to \mathbb{C}$ and a character $\mu : F_v^\times \to \mathbb{C}$. Piateskii-Shapiro has attached to these data a zeta integral $L(W, \Phi, \mu, s)$ as a meromorphic function of s. The L-factor $L_{\mathrm{PSS}}(s, \Pi, \mu)$ of Piateskii-Shapiro and Soudry is then defined as a the inverse of a normalized polynomial in q^{-s} such that for every W, Φ the function $L(W, \Phi, \mu, s)/L_{\mathrm{PSS}}(s, \Pi, \mu)$ is holomorphic, see [PS97, PSS81]. It is a product of Tate L-factors, so it is uniquely determined by its poles.

Lemma 3.2. *For a generic irreducible admissible representation* Π_v *the local L-factors* $L_{\mathrm{Nvd}}(\Pi_v, s)$ *and* $L_{\mathrm{Nvd}}(\Pi_v, s)$ *coincide. If* Π_v *is also tempered, but not a twist of the Steinberg representation, then the poles of the associated local L-factor are contained the strip* $-1/2 \le \mathrm{Re}(s) \le 0$.

Proof. The local L-factors are uniquely determined by their poles. For $L_{\mathrm{PSS}}(\Pi_v, s)$, see [PS97, Thm. 4.4], [D14], [D15b] for non-cuspidal Π_v and [D15a] for cuspidal Π_v. For $L_{\mathrm{Nvd}}(\Pi_v, s)$, see Takloo-Bighash [TB94, Thms. 4.1,5.1]. □

A pole of $L_{\mathrm{PSS}}(s, \Pi_v, \mu)$ is called regular if there is a semisimple twodimensional F_v-algebra K such that the pole appears in the local zeta integral of a Schwarz-Bruhat function $\phi : K \times K \to \mathbb{C}$ with $\phi(0, 0) = 0$ in the sense of Piatetskii-Shapiro and Soudry [PSS81, §2]. Otherwise the pole is exceptional.

[1] We write $L_{\mathrm{PSS}}(\mu\Pi, s)$ for $L_{\mathrm{PSS}}(\Pi, \mu, s) = L_{\mathrm{PSS}}(\mu\Pi, 1, s)$.

Lemma 3.3. *Let Π_v be an irreducible admissible representation of* $\mathrm{GSp}(4, F_v)$*, that is not one-dimensional.*

1. *Regular poles of $L_{\mathrm{PSS}}(s, \Pi_v, \mu)$ can only occur if Π_v is a constituent of a Siegel induced representation.*
2. *Exceptional poles of $L_{\mathrm{PSS}}(s, \Pi_v, \mu)$ can only occur if Π_v is non-generic.*

Proof. See Soudry and Piatetskii-Shapiro [PSS81, Thm. 2.3] and Danisman [D14, §5.4], [D15a, Prop. 18]. □

This implies that for generic cuspidal irreducible Π_v, the L-factor $L_{\mathrm{PSS}}(s, \Pi_v, \mu)$ is trivial.

Lemma 3.4. *Let Π_v be a preunitary nongeneric irreducible admissible representations of* $\mathrm{GSp}(4, F_v)$*, that is not one-dimensional. Then $L_{\mathrm{PSS}}(\Pi_v, s)$ has a regular pole on the line* $\mathrm{Re}(s) = \frac{1}{2}$ *exactly in the following cases:*

IIb $\Pi_v \cong (\chi \circ \det) \rtimes \sigma$ *for a pair of characters χ, σ that are either both unitary or satisfy $\chi^2 = \nu^{2\beta}$ for $0 < \beta < \frac{1}{2}$ with unitary $\chi\sigma$. $L_{\mathrm{PSS}}(s, \Pi_v)$ contains the Tate factor $L(s, \nu^{-1/2}\chi\sigma)$, so poles with* $\mathrm{Re}(s) = 1/2$ *occur if and only if $\chi\sigma$ is unramified.*

IIIb $\Pi_v \cong \chi \rtimes (\sigma \circ \det)$ *for unitary characters σ and χ with $\chi \neq 1$. The regular poles with* $\mathrm{Re}(s) = 1/2$ *come from the Tate factors $L(s, \nu^{-1/2}\sigma)$ and $L(s, \nu^{-1/2}\sigma\chi)$ in $L_{\mathrm{PSS}}(s, \Pi_v)$, so they occur for unramified σ or $\sigma\chi$, respectively.*

Vb,c $\Pi_v \cong L(\nu^{1/2}\xi \,\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$ *for unitary characters σ and ξ with $\xi^2 = 1 \neq \xi$. The regular poles with* $\mathrm{Re}(s) = 1/2$ *come from the Tate factor $L(s, \nu^{-1/2}\sigma)$ and appear for unramified σ.*

Vd $\Pi_v \cong L(\nu\xi, \xi \rtimes \nu^{-1/2}\sigma)$ *for unitary characters σ and ξ with $\xi^2 = 1 \neq \xi$. The regular poles with* $\mathrm{Re}(s) = 1/2$ *come from the Tate factors $L(s, \nu^{-1/2}\sigma)$ and $L(s, \nu^{-1/2}\xi\sigma)$, and occur for unramified σ or $\xi\sigma$, respectively.*

VIc $\Pi_v \cong L(\nu^{1/2}\,\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$ *for a unitary character σ. The Tate factor $L(s, \nu^{-1/2}\sigma)$ gives rise to regular poles with* $\mathrm{Re}(s) = 1/2$ *when σ is unramified.*

VId $\Pi_v \cong L(\nu, 1 \rtimes \nu^{-1/2}\sigma)$ *for unitary σ. The Tate factor $L(s, \nu^{-1/2}\sigma)^2$ gives rise to double regular poles with* $\mathrm{Re}(s) = 1/2$ *when σ is unramified.*

XIb $\Pi_v \cong L(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$*, where π is a preunitary cuspidal irreducible admissible representation of* $\mathrm{GL}(2, F_v)$ *with trivial central character and σ is a unitary character. The regular poles with* $\mathrm{Re}(s) = 1/2$ *occur with the Tate factor $L(s, \nu^{-1/2}\sigma)$ when σ is unramified.*

Proof. Supercuspidal representations do not admit regular poles by Lemma 3.3. For the non-cuspidal representations, see Danisman[2] [D14, Table A.1], [D15b, Table A.1]. □

[2]Danisman [D14, D15b] only considers local function fields of odd characteristic, but the same proof holds for local non-archimedean number fields with odd residue characteristic.

Lemma 3.5 ([RS07, Table A.12]). *For non-generic preunitary irreducible admissible representations* Π_v *of* $\mathrm{GSp}(4, F_v)$ *with central character* ω*, the following assertions are equivalent:*

- Π_v *is paramodular,*
- Π_v *is isomorphic to one of the following Langlands quotients*

  IIb $(\chi \circ \det) \rtimes \sigma$ *for characters* χ, σ *such that* $\chi\sigma$ *is unramified and either both are unitary or* $\chi^2 = \nu^{2\beta}$ *for* $0 < \beta < \frac{1}{2}$ *with unitary* $\chi\sigma$*,*

  IIIb $\chi \rtimes (\sigma \circ \det)$ *for unramified unitary* χ, σ *with* $\chi \neq 1$*,*

  IVd $\sigma \circ s$ *for unramified unitary* σ*,*

  Vb,c $L(\nu^{1/2}\xi\,\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$ *for* ξ *with* $\xi^2 = 1 \neq \xi$ *and unramified unitary* σ*,*

  Vd $L(\nu\xi, \xi \rtimes \nu^{-1/2}\sigma)$ *for unramified unitary* σ, ξ *with* $\xi^2 = 1 \neq \xi$*,*

  VIc $L(\nu^{1/2}\,\mathrm{St}, \nu^{-1/2}\sigma)$ *for unramified unitary* σ*,*

  VId $L(\nu, 1 \rtimes \nu^{-1/2}\sigma)$ *for unramified unitary* σ*,*

  XIb $L(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$ *for a cuspidal preunitary irreducible admissible representation* π *of* $\mathrm{GL}(2, F_v)$ *with trivial central character and an unramified unitary character* σ*.*

Proof. By Lemma 2.1, we can assume that the central character is trivial $\omega = 1$. For non-cuspidal Π_v, see [RS07, Table A.12]. Cuspidal non-generic preunitary representations are never paramodular by Lemma 3.1. □

Proposition 3.6. *Let* Π_v *be a paramodular preunitary irreducible admissible representation of* $\mathrm{GSp}(4, F_v)$*, that is not one-dimensional. Then the following assertions are equivalent:*

- *The representation* Π_v *is non-generic,*
- *The local* L*-factor* $L_{\mathrm{PSS}}(\Pi_v, s)$ *of Piatetskii-Shapiro and Soudry has at least one regular pole on the line* $\mathrm{Re}(s) = 1/2$*.*

Proof. For non-generic representations, this is implied by combining Lemma 3.5 and Lemma 3.4. The conditions for preunitarity are given by [ST94].

For generic representations, every pole is regular by Lemma 3.3. The L-factor $L_{\mathrm{PSS}}(\Pi_v, s)$ coincides with Novodvorsky's L-factor by Lemma 3.2. But for preunitary generic irreducible admissible Π_v, the factor $L_{\mathrm{Nvd}}(\Pi_v, s)$ does not admit poles on the line $\mathrm{Re}(s) = 1/2$ [RS07]. □

4 Global genericity

Let $F = \mathbb{Q}$ be the field of rational numbers and $\mathbb{A} = \mathbb{R} \times \mathbb{A}_{\mathrm{fin}}$ its adele ring. We want to show that a paramodular cuspidal automorphic irreducible representation $\Pi = \bigotimes_v \Pi_v$ of

$\mathrm{GSp}(4,\mathbb{A})$, which is not CAP, is locally generic at every nonarchimedean place.

Recall that for a cuspidal automorphic irreducible representation Π of $\mathrm{GSp}(4,\mathbb{A})$ with central character ω_Π, the product $L_{\mathrm{PSS}}(\Pi,s)=\prod_v L_{\mathrm{PSS}}(\Pi_v,s)$ converges for s in a right half plane and admits a meromorphic continuation to $\mathbb{C}$ [PS97, Thm. 5.3]. This is the global degree four L-series of Piatetskii-Shapiro and Soudry. It satisfies a functional equation

$$L_{\mathrm{PSS}}(\Pi,s)=\epsilon(\Pi,s)L_{\mathrm{PSS}}(\Pi^\vee,1-s) \tag{2}$$

where $\Pi^\vee\cong\Pi\otimes(\omega_\Pi^{-1}\circ s)$ is the contragredient [W05b, §1.1].

Proposition 4.1 (Generalized Ramanujan)**.** *Suppose Π is a cuspidal automorphic irreducible representation of $\mathrm{GSp}(4,\mathbb{A})$, not CAP, such that Π_∞ belongs to the discrete series. For every local non-archimedean place v where the local factor Π_v is spherical, the Satake parameters of Π_v are complex numbers with absolute value one. Especially, Π_v is isomorphic to the irreducible tempered principal series representation $\chi_1\times\chi_2\rtimes\sigma$ for unitary unramified characters χ_1,χ_2,σ of $\mathbb{Q}_v$.*

Proof. See [W09, Thm. 3.3]. □

Proposition 4.2. *Suppose Π is a cuspidal irreducible automorphic representation of $\mathrm{GSp}(4,\mathbb{A})$, not CAP, such that Π_∞ belongs to the discrete series. Then Π is weakly equivalent to a unique globally generic automorphic irreducible representation Π_{gen} of $\mathrm{GSp}(4,\mathbb{A})$ whose archimedean local component $\Pi_{\mathrm{gen},\infty}$ is in the local archimedean L-packet of Π_∞. This lift $\Pi\mapsto\Pi_{\mathrm{gen}}$ commutes with character twists by unitary idele class characters. The central characters of Π_{gen} and Π coincide.*

Proof. See [W05a, Thm. 1]. The proof in loc. cit. relies on certain Hypotheses A and B shown in [W09]. The lift commutes with twists because Π_{gen} is unique. The central characters are weakly equivalent, so they coincide globally by strong multiplicity one for $\mathrm{GL}(1,\mathbb{A})$. □

Proposition 4.3. *Suppose $\Pi=\Pi_\infty\otimes\Pi_{\mathrm{fin}}$ is an irreducible cuspidal automorphic representation, not CAP nor a weak endoscopic lift, so that Π_∞ belongs to the discrete series. (Suppose Π is weakly equivalent to a multiplicity one representation Π'). Then Π_∞ is contained in an archimedean local L-packet $\{\Pi_\infty^+,\Pi_\infty^-\}$ such that the cuspidal multiplicities $m(\Pi_\infty^+\otimes\Pi_{\mathrm{fin}})$ and $m(\Pi_\infty^-\otimes\Pi_{\mathrm{fin}})$ coincide.*

Proof. See [W05b, Prop. 1.5]. □

Proposition 4.4. *Suppose $\Pi=\otimes_v\Pi_v$ is a globally generic irreducible cuspidal automorphic representation of $\mathrm{GSp}(4,\mathbb{A})$. Then there is a unique globally generic automorphic representation $\tilde\Pi=\bigotimes_v\tilde\Pi_v$ of $\mathrm{GL}(4,\mathbb{A})$ with partial Rankin-Selberg L-function*

$$L^S(\tilde\Pi,s)=L^S_{\mathrm{PPS}}(\Pi,s)$$

for a sufficiently large set S of $\mathbb{Q}$-places. This lift is local in the sense that $\tilde\Pi_v$ only depends on Π_v. It commutes with character twists by global idele class characters.

Proof. See [AS06]. For non-archimedean places v, where the local representation is spherical, suppose that $\Pi_v \cong \chi_1 \times \chi_2 \rtimes \sigma$ is a spherical principal series. The unramified characters χ_1, χ_2, σ are unitary by Prop. 4.1. The local lift of Π_v is the $\mathrm{GL}(4, \mathbb{A})$-representation

$$\tilde{\Pi}_v \quad \cong \quad \chi_1\chi_2\sigma \times \chi_1\sigma \times \chi_2\sigma \times \sigma$$

parabolically induced from the Levi quotient of the standard Borel [AS06, Prop. 2.5]. Especially, the lift commutes with local character twists at non-archimedean places where Π_v is unramified. The rest is clear by the strong multiplicity one theorem for GL(4). □

We now come to the main theorem of this section:

Theorem 4.5. *Suppose* $\Pi = \Pi_\infty \otimes \bigotimes_{v\neq\infty} \Pi_v$ *of* $\mathrm{GSp}(4, \mathbb{A}_\mathbb{Q})$ *is a paramodular unitary cuspidal irreducible automorphic representation with* Π_∞ *in the discrete series such that* Π *is not CAP nor weak endoscopic. We assume the paramodular level* N *is odd and that Hypothesis 1.1 holds. Then* Π_v *is locally generic at all nonarchimedean places* v.

The converse direction is also true: A generic non-archimedean local irreducible admissible representation Π_v is always paramodular [RS07].

Proof. Denote by Π_∞^W the generic constituent of the archimedean L-packet attached to Π_∞. By Prop. 4.3 the irreducible representation $\Pi' = \Pi_\infty^W \otimes \Pi_{\mathrm{fin}}$ is cuspidal automorphic and occurs in the cuspidal spectrum with the same multiplicity as Π.

By Prop. 4.2 there is a globally generic automorphic irreducible representation Π_{gen}, weakly equivalent to Π, such that the archimedean local factor $\Pi_{\mathrm{gen},\infty}$ is isomorphic to Π_∞^W. Since Π is not CAP, Π_{gen} is cuspidal again.

Since Π' and Π_{gen} are weakly equivalent, there is a finite set S of non-archimedean places such that $\Pi_{\mathrm{gen},v}$ is isomorphic to Π_v for every place $v \notin S$. This implies

$$\frac{L_{\mathrm{PSS}}(\Pi', s)}{L_{\mathrm{PSS}}(\Pi_{gen}, s)} = \prod_{v\in S} \frac{L_{\mathrm{PSS}}(\Pi_v, s)}{L_{\mathrm{PSS}}(\Pi_{gen,v}, s)} \ . \tag{3}$$

At the local non-archimedean place v with even residue characteristic, the local representation Π_v is spherical by assumption and therefore generic by Prop. 4.1. Now suppose there is at least one non-archimedean place w of odd residue characteristic where the unitary local irreducible admissible representation Π_w is non-generic. Then the right hand side of (3) admits a pole with $\mathrm{Re}(s) = 1/2$ by Prop. 3.6. By a suitable twist with a power of the adelic valuation character $|\cdot|_\mathbb{A}^{it} = \prod_v |\cdot|_v^{it}$ for $t \in \mathbb{R}$, we can assume without loss of generality that the pole occurs at $s = 1/2$. Since Π' is not CAP, $L_{\mathrm{PSS}}(\Pi', s)$ is holomorphic and this implies $L_{\mathrm{PSS}}(\Pi_{gen}, 1/2) = 0$ by (3). Hence, for the globally generic irreducible automorphic representation $\tilde{\Pi}$ of $\mathrm{GL}(4, \mathbb{A})$ attached to Π_{gen} by Prop. 4.4, the central value of the partial L-function $L^S(\tilde{\Pi}, 1/2) = 0$ vanishes for sufficiently large S.

Assuming Hypothesis 1.1 is true, there is a unitary idele class character λ of $\mathbb{Q}^\times \backslash \mathbb{A}^\times$ such that

1. $\lambda_w \equiv 1$ at the fixed place w,
2. $L^S((\lambda \circ s) \otimes \tilde{\Pi}, 1/2) \neq 0$ for the partial L-factor outside of S.

The twist $(\lambda \circ s) \otimes \Pi$ is again cuspidal automorphic. The above argument applied to $(\lambda \circ s) \otimes \Pi$ instead of Π gives $L^S((\lambda \circ s) \otimes \tilde{\Pi}, 1/2) = 0$. Contradiction.

Therefore Π_v is locally generic at the every non-archimedean place v. □

Corollary 4.6. *The representation* Π' *is globally generic and isomorphic to* Π_{gen}.

Proof. This is clear by the corollary of Jiang and Soudry [JS07]. □

4.1 Weak endoscopic lift

A cuspidal automorphic irreducible representation Π of $\mathrm{GSp}(4, \mathbb{A})$ is a weak endoscopic lift if there is a pair of cuspidal automorphic irreducible representations σ_1, σ_2 of $\mathrm{GL}(2, \mathbb{A})$ with the same central character, such that the local L-factors at almost every place are

$$L_{\mathrm{PSS}}(s, \Pi_v) = L(s, \sigma_{1,v}) L(s, \sigma_{2,v}) \ .$$

Proposition 4.7. *Suppose* $\Pi \cong \otimes_v \Pi_v$ *is a paramodular weak endoscopic lift, such that* Π_∞ *is in the discrete series and the paramodular level is odd. Then* Π *is globally generic.*

Proof. The automorphic representations σ_1 and σ_2 are locally tempered (Ramanujan), so the local endoscopic lift Π_v is also tempered [W09, §4.11]. At every non-archimedean place v with odd residue characteristic, the local representation Π_v is paramodular, hence locally generic by Lemma 3.1. By [W09, Thm. 5.2], the archimedean factor Π_∞ must also be generic. But then Π_v is locally generic at every place, so Π is globally generic by [W09, Thm. 4.1]. □

5 Multiplicity one

We now show that under the above restrictions, paramodular Π occur in the cuspidal spectrum with multiplicity one.

Theorem 5.1. *Suppose* Π *is a cuspidal irreducible automorphic representation* $\Pi = \Pi_\infty \otimes \Pi_{\mathrm{fin}}$ *of* $\mathrm{GSp}(4, \mathbb{A})$ *with* Π_∞ *in the discrete series, not CAP, with odd paramodular level* N*. Assume that Hypothesis 1.1 holds. Then* Π *occurs in the cuspidal spectrum with multiplicity one.*

Proof. If Π is weak endoscopic, it is globally generic by Prop. 4.7 and occurs in the cuspidal spectrum with multiplicity one [W09, Thm. 5.2]. Otherwise let $\Pi' = \Pi_\infty^W \otimes \Pi_{\mathrm{fin}}$ as above, then Π' is globally generic by Cor. 4.6 and hence satisfies $m(\Pi') = 1$ [JS07]. By Prop. 4.3 the multiplicity of Π in the cuspidal spectrum is $m(\Pi) = m(\Pi') = 1$. □

The cuspidal automorphic irreducible representation of $\mathrm{GSp}(4, \Pi)$, which are CAP, are non-generic at every local place. These are the Saito-Kurokawa lifts and the Howe-Piatetskii-Shapiro lifts constructed by Piatetskii-Shapiro [PS83b], and the Soudry lifts [S88]. They all occur via theta-liftings, so the multiplicity one property should follow from the global Howe duality conjecture and multiplicity preservation [R84]. For Saito-Kurokawa lifts, see Gan [G08, Cor. 5.10].

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