ON SPECTRAL SEQUENCES FOR IWASAWA ADJOINTS À LA JANNSEN FOR FAMILIES

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Abstract. In [3] several spectral sequences for (global and local) Iwasawa modules over (possibly noncommutative) Iwasawa algebras (mainly of compact p-adic Lie groups) over $\mathbb{Z}_p$ are established, which are very useful for determining certain properties of such modules in arithmetic applications. Slight generalisations of those can be found in [6] (for abelian groups, but more general coefficient rings) and [9] (for products of possibly non-abelian groups, but with $\mathbb{Z}_p$-coefficients) and [5]. Unfortunately, some of Jannsen's spectral sequences for families of representations as coefficients for (local) Iwasawa cohomology are still missing. We explain and follow the philosophy that all these spectral sequences are consequences or analogues of local cohomology and duality as founded by Grothendieck (and developed for duality groups by Tate).

1. Introduction

Let $\mathcal{O}$ be a complete discrete valuation ring with uniformising element π and finite residue field $\kappa = \mathcal{O}/\mathcal{O}\pi$. By R we denote an $\mathcal{O}$-algebra, which is a complete noetherian local ring with maximal ideal $\mathfrak{m}$, dimension d and finite residue field k.

We are mainly interested in the case of a ring of formal power series $R = \mathcal{O}[[X_1, \ldots, X_t]]$ in t variables, which is a complete regular local ring of Krull/global dimension $d = t + 1$. In particular, such special R satisfies local duality: We set $E^i_R(M) := \mathrm{Ext}^i_R(M, R)$ and fix an injective hull $\mathcal{E} = \mathcal{E}_R$ of $R/\mathfrak{m}$, which defines the exact functor

$$D_R(-) := \mathrm{Hom}_R(-, \mathcal{E}_R)$$

satisfying Matlis duality $D \circ D \cong \mathrm{id}$ (on the full subcategories of (co)finitely generated R-modules). Note that for noetherian (hence compact) or artinian (hence discrete) R-modules M we have $D_R(M) = M^\vee := \mathrm{Hom}_{cts}(M, \mathbb{Q}_p/\mathbb{Z}_p)$ where the latter denotes the Pontriagin dual; in particular, $\mathcal{E} \cong D_R(R) \cong D(R)$. Then local duality states a canonical isomorphism

$$D_R(E^i_R(M)) \cong H^{d-i}_{\mathfrak{m}}(M) \tag{1.1}$$

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for all finitely generated R-modules M. Here $H^i_{\mathfrak{m}}(M)$ denotes local cohomology of M with respect to $\mathfrak{m}$. Furthermore, the adjunction of Tor and Hom

$$\begin{aligned} D_R(M \otimes_R D_R(R)) &= \mathrm{Hom}_R(M \otimes_R D_R(R), D_R(R)) \\ &\cong \mathrm{Hom}_R(M, \mathrm{Hom}_R(D_R(R), D_R(R)) \\ &\cong \mathrm{Hom}_R(M, R) = E^0_R(M). \end{aligned}$$

induces canonical isomorphisms

$$D_R(E^i_R(M)) \cong \mathrm{Tor}^R_i(M, D_R(R)). \tag{1.2}$$

Now let $\Lambda = \Lambda_R(G)$ be the completed group algebra of a profinite group G over R

$$\Lambda_R(G) = R[\![G]\!] = \varprojlim_U R[G/U],$$

where U runs through the open normal subgroups of G. First we consider the case $G = \mathbb{Z}_p^s$. Then there is a well-known isomorphisms

$$\Lambda_R(G) \cong R[\![Y_1, \ldots, Y_s]\!]$$

and, if $R = \mathcal{O}[\![X_1, \ldots, X_t]\!]$, we obtain

$$\Lambda_R(G) \cong \mathcal{O}[\![X_1, \ldots, X_t, Y_1, \ldots, Y_s]\!].$$

As local cohomology can be calculated via Koszul complexes (following Serre) we may consider the regular sequence $(\pi, X_1, \ldots, X_t, Y_1, \ldots, Y_s)$ generating the maximal ideal $\mathfrak{M}$ of Λ and wonder which kind of spectral sequences arise, if we divide this sequence into $\mathbf{g} = (g_i)$, $\mathbf{f} = (f_j)$, and form the local cohomology with regard to these two sequences successively:

$$H^m_{\mathbf{g}}(H^n_{\mathbf{f}}(M)) = \varinjlim_k \mathrm{H}^m(\mathrm{H}^n(\mathbf{f}^k, M)) \Rightarrow H^{m+n}_{\mathfrak{M}}(M) \tag{1.3}$$

for any Λ-module M. Note that the target is the Pontriagin dual of the Iwasawa adjoint $E^{d_\Lambda - (n+m)}_\Lambda(M)$ by local duality (1.1).

For $t = 0$ this should - at least morally - recover the spectral sequences which show up in Jannsen's proof of [3, 2.1,2.2] concerning Iwasawa adjoints:

$$\mathrm{Tor}^{\mathbb{Z}_p}_n(D_m(M^\vee), \mathbb{Q}_p/\mathbb{Z}_p) \Rightarrow E^{n+m}_\Lambda(M)^\vee \tag{1.4}$$

and

$$\varinjlim_k D_n(\mathrm{Tor}^{\mathbb{Z}_p}_m(R\mathbb{Z}_p/p^k\mathbb{Z}_p, M)^\vee) \Rightarrow E^{m+n}_\Lambda(M)^\vee. \tag{1.5}$$

Here the functors $D_n(-)$ (dualizing modules) stem from Tate's spectral sequence and are a building blocks in the theory of duality groups. A comparison of (1.5) and (1.3) suggests that $H^l_{(Y_1,\ldots,Y_d)}(M)$ should be related to Tate's module $D_r(M^\vee)$ (whose definition we recall in subsection 3.1). Indeed, this becomes literally true (see Lemma 3.1) if we extend the setting of local cohomology slightly as done in subsection 2.3. Apart from pointing out this link the motivation for this note is to generalize the spectral sequences (1.4) and (1.5) to the case of more general coefficients, see Theorem 3.4. Although another spectral sequence of Iwasawa adjoints by Jannsen (compare Theorem 3.7) had already been generalised to general coefficients, this generalisation is apparently still missing in the literature for the spectral sequences (1.4) and (1.5). As a corollary we obtain an explicit formula for the Iwasawa adjoint of those Λ-modules M, which are already finitely generated over R, which generalises [3, 2.6] or [7, 5.4.14]. As an applciation we determine the torsionsubmodule of local Iwasawa cohomology generalizing a result of Perrin-Riou in the case $R = \mathbb{Z}_p$, see Theorem 3.9. Finally we generalize in the last section our work [9] in showing that the noncommutative $\Lambda_R(G)$ satisfies local duality, if R does and G is a duality group.

2. Local cohomology revisited

In this section R denotes any commutative noetherian ring. We write R-Mod and R-mod for the category of all and finitely generated R-modules, respectively.

2.1. **Koszul complexes.** For $x \in R$ we let $K(x)$ denote the chain complex

$$0 \longrightarrow R \xrightarrow{\quad x \quad} R \longrightarrow 0$$

concentrated in degrees 1 and 0. If $\mathbf{x} = (x_1, \dots, x_r)$ we define the Koszul complex $K(\mathbf{x})$ to be the total tensor product complex

$$K(x_1) \otimes_R K(x_2) \otimes_R \cdots \otimes_R K(x_r).$$

If M is an R-module, we define

$$\mathrm{H}^n(\mathbf{x}, M) = \mathrm{H}^n(\mathrm{Hom}_R(K(\mathbf{x}), M)) \text{ and } \mathrm{H}_n(\mathbf{x}, M) = \mathrm{H}_n(K(\mathbf{x}) \otimes_R M).$$

Note that there are isomorphisms

$$\mathrm{H}_n(\mathbf{x}, M) \cong \mathrm{H}^{r-n}(\mathbf{x}, M) \tag{2.6}$$

and

$$D(\mathrm{H}^n(\mathbf{x}, M)) \cong \mathrm{H}_n(\mathbf{x}, D(M)) \cong \mathrm{H}^{r-n}(\mathbf{x}, D(M)). \tag{2.7}$$

Denoting by $(\mathbf{x})$ the ideal of R generated by $x_1, \dots, x_r$, we have that

$$\mathrm{H}_0(\mathbf{x}, M) = M_{(\mathbf{x})} := M/(\mathbf{x})M \text{ and } \mathrm{H}^0(\mathbf{x}, M) =_{(\mathbf{x})} M := \{m \in M | (\mathbf{x})m = 0\}.$$

Remark 2.1. Let $\phi : S \to R$ be a (injective) ring homomorphism and and let M be an R-module which we also consider as S-module via ϕ. Then for a sequence $\mathbf{x}$ in S, which we also consider as sequence in R, we have canonical isomorphisms $K_R(\mathbf{x}) \otimes_R M \cong K_S(\mathbf{x}) \otimes_S M$ - at least as S-modules. Hence we obtain isomorphisms $\mathrm{H}^R_s(\mathbf{x}, M) \cong \mathrm{H}^S_s(\mathbf{x}, M)$ and similar for cohomology $\mathrm{H}^s_R(\mathbf{x}, M) \cong \mathrm{H}^s_S(\mathbf{x}, M)$ for all $s \geq 0$. Therefore we often do not take care about the ring of definition henceforth.

Henceforth we will assume that R is a local commutative ring with maximal ideal $\mathfrak{m}$ and denote by $\mathbf{x} = (\mathbf{x}(i))_{i \geq 0}$ a system of r-tuples $\mathbf{x}(i) = (x(i)_1, \dots, x(i)_r)$ such that $x(i+1)_k = x(i)_k t(i)_k$ for some $t(i)_k \in \mathfrak{m}$ for all $i \geq 0$ and $1 \leq k \leq r$. The map of complexes $K(x(i+1)_k) \to K(x(i)_k)$ given as follows

$$\begin{array}{ccccccc} 0 & \longrightarrow & R & \xrightarrow{x(i+1)_k} & R & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle t(i)_k} & & \| & & \\ 0 & \longrightarrow & R & \xrightarrow{x(i)_k} & R & \longrightarrow & 0 \end{array}$$

induce a map $K(\mathbf{x}(i+1)) \to K(\mathbf{x}(i))$ on the total tensor product and hence induces maps $\mathrm{H}_n(\mathbf{x}(i+1), M) \to \mathrm{H}_n(\mathbf{x}(i), M)$ on homology and $\mathrm{H}^n(\mathbf{x}(i), M) \to \mathrm{H}^n(\mathbf{x}(i+1), M)$ on cohomology. We define

$$H^n_{\mathbf{x}}(M) = \varinjlim_i \mathrm{H}^n(\mathbf{x}(i), M) \text{ and } \mathrm{H}_n(\mathbf{x}, M) = \varprojlim_i \mathrm{H}_n(\mathbf{x}(i), M).$$

Note that the usual definition of these (co)homology groups corresponds just to the special case that $x(i)_k = x^i_k$ for a fixed sequence $(x_1, \ldots, x_r)$ of elements in R.

Question: Do the $\mathrm{H}^*_{\mathbf{x}}(-)$ form an universal δ-functor?

2.2. **A spectral sequence for Koszul complexes.** For two sequences $y = (y_1, \ldots, y_s)$ and $z = (z_1, \ldots, z_t)$ we can build the joined one $x := y \sharp z := (y_1, \ldots, y_s, z_1, \ldots, z_t)$ and we obtain $K(x)$ as the total tensor product

$$K(x) = K(y) \otimes_R K(z)$$

of $K(y)$ and $K(z)$. In particular, for any R-module M the complex $K(x) \otimes_R M$ is the total complex of the double complex $K_\bullet(y) \otimes_R (K_\bullet(z) \otimes_R M)$ whence we obtain as one of the two associated spectral sequences the following converging homological spectral sequence

$$\mathrm{H}_m(y, \mathrm{H}_n(z, M)) \Rightarrow \mathrm{H}_{m+n}(x, M). \tag{2.8}$$

Here we used the fact that $K(z)$ consists of free, hence flat R-modules in order to identify

$$H_m(K(y) \otimes_R (K_\bullet(z) \otimes_R M)) = K(y) \otimes_R H_m(K_\bullet(z) \otimes_R M) = K(y) \otimes_R \mathrm{H}_m(z, M).$$

Using (2.6) or the adjointness of Hom and tensor products, i.e., the double complex $\mathrm{Hom}_R(K(y), \mathrm{Hom}_R(K(z), M))$ whose total complex is $\mathrm{Hom}_R(K(x), M)$ one obtains similarly the convergent cohomological spectral sequence

$$\mathrm{H}^m(y, \mathrm{H}^n(z, M)) \Rightarrow \mathrm{H}^{m+n}(x, M). \tag{2.9}$$

Now we consider again sequences of families $\mathbf{y}$, $\mathbf{z}$ and define similarly as above $\mathbf{x}$. Taking direct limits in (2.9) gives

$$\varinjlim_j \mathrm{H}^m_{\mathbf{y}}(\mathrm{H}^n(\mathbf{z}(j), M)) \cong \varinjlim_{i,j} \mathrm{H}^m(\mathbf{y}(i), \mathrm{H}^n(\mathbf{z}(j), M)) \Rightarrow \mathrm{H}^{m+n}_{\mathbf{x}}(M). \tag{2.10}$$

2.3. **Local cohomology.** Let $\mathbf{I} = (I_j)_j$ be a decreasing family of finitely generated ideals of R indexed over the natural numbers. Then we define for an R-module M the submodule

$$\mathrm{H}^0_{\mathbf{I}}(M) := \{m \in M | I_j m = 0 \text{ for some } j\} = \varinjlim_j \mathrm{Hom}_R(R/I_j, M).$$

The right derived functor will be denoted by

$$\mathrm{H}^q_{\mathbf{I}}(M) \cong \varinjlim_j \mathrm{Ext}^q_R(R/I_j, M).$$

Note that the usual definition is jus the special case $I_i = I^j$ for a fixed finitely generated ideal I of R. But all the following properties of this slight generalisation can be proved in the same way, see [10, §4.6], [1] for the standard definitions and proofs. We write

$$R\text{-Mod}_{\mathbf{I}},\ R\text{-mod}_{\mathbf{I}}$$

for the full subcategory of R-Mod , R-mod respectively, consisting of those modules M, for which $\mathrm{H}^0_{\mathbf{I}}(M) = M$ holds.

Remark 2.2.
 (i) If $\mathbf{J} = (J_j)$ denotes another decreasing family satisfying that for each k we have $I_{j(k)} \subseteq J_k$ and $J_{j(k)} \subseteq I_k$ for some $j(k)$, then $\mathrm{H}^q_{\mathbf{I}}(M) = \mathrm{H}^q_{\mathbf{J}}(M)$ for all $q \geq 0$.
 (ii) If each R/I_j has a finite resolution by finitely generated projective R-modules, then $\mathrm{H}^q_{\mathbf{I}}(-)$ commutes with direct limits.
 (iii) $H^q_{\mathbf{I}}(M)$ belongs to R-Mod$_{\mathbf{I}}$ for all M in R-Mod and $q \geq 0$.
 (iv) $H^0_{\mathbf{I}} : R\text{-Mod} \to R\text{-Mod}_{\mathbf{I}}$ preserves injectives.

Proof. Compare (i), (iii), (iv) with [11, Exer. 4.6.1, 4.6.3]. For (ii) choose a resolution $P_\bullet$ of R/I_j by finitely generated projectives in order to calculate $\mathrm{Ext}^l_\Lambda(R/I_j, M)$. Since $\mathrm{Hom}_R(P_l, -)$ commutes with direct limits (as P_l is finitely generated, i.e. any homomorphism $\phi : P_l \to \varinjlim_i M_i$ factors over some M_i), $\mathrm{Ext}^l_R(R/I_j, -)$ does also and the claim follows. □

Now let $\mathbf{x}$ be a family of sequences as at the end of section 2.1 and define the family of ideals $\mathbf{I} = (I_i)_i$ by setting $I_i = \sum_k Rx(i)_k$.

Proposition 2.3. *In the above situation assume that each $\mathbf{x}(i)$ forms a regular sequence for R, then we have a canonical isomorphism*

$$\mathrm{H}^n_{\mathbf{x}}(M) \cong \mathrm{H}^n_{\mathbf{I}}(M)$$

for every finitely generated R-module M and $n \geq 0$.

Proof. If r is the length of $\mathbf{x}$, then we have

$$\mathrm{H}^r_{\mathbf{x}}(M) = \varinjlim_i \mathrm{H}^r(\mathbf{x}(i), M) = \varinjlim_i \mathrm{Ext}^r_R(R/I_i, M) = \mathrm{H}^r_{\mathbf{I}}(M)$$

by [10, Cor. 4.5.5] and due to the regularity of $\mathbf{x}(i)$. For finitely generated free modules the functors $\{\mathrm{H}^{r-i}_{\mathbf{x}}(-)\}_i$ and $\{\mathrm{H}^{r-i}_{\mathbf{I}}(-)\}_i$ both vanish for $i \neq 0$ by Cor. 4.5.4 (Acyclicity) of (loc. cit.) plus (2.6) and hence they are universal δ-functors from R-mod to R-Mod. □

2.4. **A spectral sequence for local cohomology.** For decreasing families $\mathbf{I}$ and $\mathbf{J}$ we set $\mathbf{K} := (I_i + J_i)_i$. We have that $\mathrm{H}^0_{\mathbf{I}}(\mathrm{H}^0_{\mathbf{J}}(M)) = \mathrm{H}^0_{\mathbf{K}}(M)$.

Question: Does there always exist a (convergent) spectral sequence

$$\mathrm{H}^n_{\mathbf{I}}(\mathrm{H}^m_{\mathbf{J}}(M)) \Rightarrow H^{n+m}_{\mathbf{K}}(M)?$$

Using the results of subsection 2.2 we can at least obtain a spectral sequence in the following situation:

Proposition 2.4. *Assume that* $\mathbf{I}, \mathbf{J}$ *and* $\mathbf{K}$ *are associated with regular sequences* $\mathbf{y}, \mathbf{z}$ *and* $\mathbf{x} = \mathbf{y} \sharp \mathbf{z}$ *in* R*, respectively. Then, for every* $M \in R$*-Mod there are cohomological spectral sequences*

$$\varinjlim_k \mathrm{H}^n_{\mathbf{I}}(\mathrm{Ext}^m_R(R/J_k, M)) = \varinjlim_k \mathrm{H}^n_{\mathbf{I}}(\mathrm{H}^m(\mathbf{z}(k), M)) \Rightarrow \mathrm{H}^{n+m}_{\mathbf{K}}(M).$$

and

$$\mathrm{H}^n_{\mathbf{I}}(\mathrm{H}^m_{\mathbf{J}}(M)) \Rightarrow \mathrm{H}^{n+m}_{\mathbf{K}}(M).$$

Proof. Combine the spectral sequence (2.10) with proposition 2.3. For the second one take Remark 2.2 (ii) and the regularity assumption into account. □

3. Iwasawa Adjoints

3.1. **Tate's dualzing modules as local cohomology groups.** Let G be a profinite group and A a discrete G-module. We shall write $\mathcal{D}(G)$ and $\mathcal{C}(G)$ for the categories of discrete and compact Λ-modules, respectively, whereas we denote the full subcategories of cofinitely and finitely generated modules by $\mathcal{D}_{cfg}(G)$ and $\mathcal{C}_{fg}(G)$, respectively. Then Tate defines functors D_i

$$D_i(A) = \varinjlim_U H^i(U, A)^*$$

where U runs through all open subgroups of G, $N^* = \mathrm{Hom}(N, \mathbb{Q}/\mathbb{Z})$ (considered as discrete G-module) for any discrete G-module N and the transition maps are the duals cor^* of the corestriction maps. Since D_i is the ith derived functor of D_0 the system (D_i) forms a universal δ-functor. They show up in Tate's spectral sequence [7, Thm. 2.5.3].

We now concentrate first on the group $G = \mathbb{Z}_p^s$ with open subgroups $G_i = p^i\mathbb{Z}_p^s$. We choose topological generators $g_1, \ldots, g_s$ and define $\mathbf{y}$ by $y(i)_k = g_k^{p^i} - 1$. The corresponding $\mathbf{I}$ as before Proposition 2.3 satisfies $I_i = I(G_i) := \ker\left(\Lambda_R(G) \to R[G/G_i]\right)$.

Lemma 3.1. *For every finitely generated* $\Lambda_R(G)$*-module* M *we have canonical isomorphisms*

$$D_r(M^\vee) \cong \mathrm{H}^{s-r}_{\mathbf{y}}(M).$$

Proof. We have

$$\begin{aligned} \mathrm{H}^s_{\mathbf{y}}(M) &= \varinjlim_i H^s_{\mathbf{y}(i)}(M) \\ &= \varinjlim_i D(H^0_{\mathbf{y}(i)}(M^\vee)) \\ &= \varinjlim_i \mathrm{Hom}_\Lambda(\Lambda/I(G_i), M^\vee)^* \\ &= \varinjlim_i ((M^\vee)^{G_i})^* = D_0(M^\vee) \end{aligned}$$

where we use (2.7) for the second isomorphism. Both systems of functors are universal δ-functors from the category $\Lambda-\mathrm{mod}$ to $\Lambda-\mathrm{Mod}$ whence the result follows. Indeed, note that $\mathbf{y}(i)$ is regular for each i and compare the proof of Proposition 2.3. □

3.2. **Spectral sequences for $G = \mathbb{Z}_p^s$.** Let $R = \mathcal{O}[[X_1, \ldots, X_t]]$ and set $\Lambda := \Lambda_R(G)$ with $G \cong \mathbb{Z}_p^s$ and maximal ideal $\mathfrak{M}$. Using local duality (1.1) for $\Lambda \cong \mathcal{O}[[X_1, \ldots, X_t, Y_1, \ldots, Y_s]]$, Lemma 3.1 and Proposition 2.4 we obtain

Proposition 3.2. *We have the following convergent homological spectral sequence:*

$$\varinjlim_k D_n(\mathrm{Ext}_R^{t+1-m}(R/\mathfrak{m}^k, M)^\vee) \Rightarrow E_\Lambda^{m+n}(M)^\vee.$$

Proof. We choose the regular sequences $\mathbf{y}$ as before Lemma 3.1 and $\mathbf{z}$ induced by $(\pi, X_1, \ldots, X_t)$. Then $\mathrm{H}^n_{\mathfrak{M}}(M) = \mathrm{H}^n_{\mathbf{K}}(M)$ since the systems $(\mathfrak{M}^k)$ and (K_k) are cofinal in each other. □

For $R = \mathbb{Z}_p$ this is one of Jannsen's spectral sequences specialized to $G = \mathbb{Z}_p^s$.

Remark 3.3. If M is finitely generated over R, then the groups $\mathrm{Ext}_R^{t+1-m}(R/\mathfrak{m}^k, M)$ are all finite, i.e., $D_n(\mathrm{Ext}_R^{t+1-m}(R/\mathfrak{m}^k, M)^\vee) = 0$ for $n \neq s$ since G is a duality group of dimension s. Hence $E^n_\Lambda(M)^\vee \cong \varinjlim_k D_s(\mathrm{Ext}_R^{s+t+1-n}(R/\mathfrak{m}^k, M)^\vee) = 0$ for $n < s$.

3.3. **Spectral sequences for compact p-adic Lie groups.** Let R be as in the introduction, i.e., a complete noetherian local ring with maximal ideal $\mathfrak{m}$ and (finite) residue field $\kappa = R/\mathfrak{m}$. For a compact p-adic Lie group G we write $\Lambda = \Lambda_R(G)$ for its Iwasawa algebra with coefficients in R. In the following we denote by G_n any basis of neighbourhoods of the identity in G consisting of open normal subgroups; e.g. one can choose for G_i the lower central p-series $P_i(G)$ given by

$$P_1(G) = G,\ P_{i+1}(G) = P_i(G)^p[P_i(G), G],\ i \geq 1.$$

Consider the following descriptions of $\mathrm{E}^0_\Lambda(M) := \mathrm{Hom}_\Lambda(M,\Lambda)$[1] for a finitely generated (hence compact) Λ-module M :

$$\begin{aligned}\mathrm{E}^0_\Lambda(M) &= \mathrm{Hom}_\Lambda(M, R[\![G]\!])\\ &= \varprojlim_n \mathrm{Hom}_{R[G/G_n]}(M_{G_n}, R[G/G_n])\\ &= \varprojlim_n \mathrm{Hom}_R(M_{G_n}, R).\end{aligned}$$

where the last map is induced by the trace map. We firstly obtain

$$\begin{aligned}\mathrm{E}^0_\Lambda(M)^\vee &= \varinjlim_n \mathrm{Hom}_R(M_{G_n}, R)^\vee\\ &= \varinjlim_n M_{G_n} \otimes_R R^\vee\\ &= D_0(M^\vee) \otimes_R R^\vee,\end{aligned}$$

secondly

$$\begin{aligned}\mathrm{E}^0_\Lambda(M)^\vee &= \varinjlim_n M_{G_n} \otimes_R R^\vee\\ &= \varinjlim_n \varinjlim_k M_{G_n} \otimes_R R/\mathfrak{m}^k\\ &= \varinjlim_n \varinjlim_k (M/\mathfrak{m}^k M)_{G_n}\\ &= \varinjlim_k D_0((M/\mathfrak{m}^k M)^\vee),\end{aligned}$$

and thirdly

$$\begin{aligned}\mathrm{E}^0_\Lambda(M)^\vee &= \varinjlim_n \varinjlim_k M_{G_n}/\mathfrak{m}^k M_{G_n}\\ &= \varinjlim_n H^{t+1}_\mathfrak{m}(M_{G_n}).\end{aligned}$$

Here for the second identity in the second displayed formula we used that by local duality (1.1) we have an isomorphism

$$R^\vee = H^{t+1}_\mathfrak{m}(R) = \varinjlim_k H^{t+1}(\mathbf{x}(k), R) = \varinjlim_k H_0(\mathbf{x}(k), R) = \varinjlim_k R/\mathfrak{m}^k.$$

In particular, we assume for the last two formulas, that R is regular of dimension $t+1$.

Next we are going to express these three different presentations as composites $F_i \circ G_i\dagger$ of each time two functors F_i and G_i, $i = 1, 2, 3$:

$$G_1 = D_0(-^\vee) : \Lambda\text{-mod} \to \Lambda\text{-Mod and } F_1 = \mathrm{Tor}^R_0(-, R^\vee) : \Lambda\text{-Mod} \to \Lambda\text{-Mod}.$$

Let $(\Lambda\text{-mod}))^\mathbb{N}$ be the category of inverse systems in Λ-mod and consider the right exact functor

$$G_2 : \Lambda\text{-mod} \to (\Lambda\text{-mod})^\mathbb{N},$$

[1]Here the action of G on $f \,\epsilon\, \mathrm{Hom}_\Lambda(M,\Lambda)$ for $M \,\epsilon\, \Lambda$-mod is defined by the rule $(gf)(M) := f(m)g^{-1}$ as usual.

which sends M to the inverse system $\{\mathrm{H}_0(\mathbf{x}(n+1), M) \to \mathrm{H}_0(\mathbf{x}(n), M)\}$, and the left exact contravariant functor

$$F_2 : \varinjlim_n D_0(-^\vee) : (\Lambda\text{-mod})^{\mathbb{N}} \to \Lambda\text{-Mod}.$$

We also have the functor

$$G_3 : \Lambda\text{-mod} \to (\Lambda\text{-mod})^{\mathbb{N}},$$

which sends M to the inverse system $\{\mathrm{H}_0(G_{n+1}, M) \xrightarrow{cor} \mathrm{H}_0(G_n, M)\}$, and the left exact contravariant functor

$$F_3 : \varinjlim_n H^{t+1}_{\mathfrak{m}}(-) : (\Lambda\text{-mod})^{\mathbb{N}} \to \Lambda\text{-Mod}.$$

If, for a subring S of Λ, Λ-mod$_S$ denotes the full subcategory of Λ-mod consisting of those modules which are finitely generated over S, then the image of G_2 lands in Λ-mod$_{\mathcal{O}[[G]]}$ while the image of G_3 is contained in Λ-mod$_R$. Note that the category $(\Lambda\text{-mod})^{\mathbb{N}}$ has enough projectives, because Λ-mod has ([2], Prop. 1.1), hence we can form the left derived functor of F_2, F_3. One immediately checks that the functors G_i send projectives to F_i-acyclics. Thus the Grothendieck spectral sequence for the composition of the above functors gives

Theorem 3.4. *With notation as above, for every $M \in \Lambda -$ Mod, there are a convergent homological spectral sequences*

(i)

$$\mathrm{Tor}^R_n(D_m(M^\vee), R^\vee) \Rightarrow E^{n+m}_\Lambda(M)^\vee$$

and, if R is regular of dimension $t+1$,

(ii)

$$\varinjlim_k D_n(\mathrm{Ext}^{t+1-m}_R(R/\mathfrak{m}^k, M)^\vee) = \varinjlim_k D_n(\mathrm{Tor}^R_m(R/\mathfrak{m}^k, M)^\vee) \Rightarrow E^{m+n}_\Lambda(M)^\vee.$$

(iii)

$$\varinjlim_k \mathrm{E}^n_R(\mathrm{H}^m(G_k, M^\vee)^\vee)^\vee = \mathrm{E}^n_R(\mathrm{H}_m(G_k, M))^\vee \Rightarrow \mathrm{E}^{m+n}_{\Lambda(G)}(M)^\vee.$$

If R is not regular in (ii) or (iii) one may replace Ext or Tor by $H^?(\mathbf{x}(k), M)$.

Remark 3.5. If M is finitely genrated as an R-module, then the groups $\mathrm{Ext}^{t+1-m}_R(R/\mathfrak{m}^k, M)$ are all finite. In case G is a duality group of dimension s the groups $D_n(\mathrm{Ext}^{t+1-m}_R(R/\mathfrak{m}^k, M)^\vee)$ will vanish for all $n \neq s$ whence the spectral sequence degenerates and gives

$$E^m(M)^\vee \cong \varinjlim_k D_s(\mathrm{Ext}^{t+1-(m-s)}_R(R/\mathfrak{m}^k, M)^\vee)$$

which is zero whenever $s > m$.

Corollary 3.6. *Assume that G is a duality group at p of dimension s with dualizing module $D^{(p)}_s = \varinjlim_m D_s(\mathbf{Z}/p^m\mathbf{Z}) \cong \mathbb{Q}_p/\mathbb{Z}_p(\chi_G)$ for the dualizing character $\chi_G : G \to \mathbb{Z}_p^\times$. Then, if M is finitely generated over R,*

$$\mathrm{E}^i(M)^\vee \cong H^{t+1-(i-s)}_{\mathbf{z}}(M)(\chi_G),$$

where $\mathbf{z}$ *is induced from a regular sequence of* R *of length* $t+1$*, and, in particular, the following holds:*

(i) *If* M *is* Λ*-module which is free of finite rank as* R*-module, then*

$$\mathrm{E}^i(M)^\vee \cong \begin{cases} M \otimes_R R^\vee(\chi_G) & \text{if } i = s, \\ 0 & \text{otherwise.} \end{cases}$$

(ii) *If* M *is* Λ*-module which is torsion as* R*-module, then*

$$\mathrm{E}^i(M)^\vee = 0$$

for $i \leq s$*. More generally, if* M *is of codimension* j *as* R*-module, then* $\mathrm{E}^i(M)^\vee = 0$ *for* $i < s+j$*. If* M *is pure of codimension* j *as* R*-module, then* $\mathrm{E}^i(M)^\vee = 0$ *for* $i \neq s+j$*.*

(iii) *If* N *is a finite* π*-primary* Λ*-module, then*

$$\mathrm{E}^i(N)^\vee \cong \begin{cases} N(\chi_G) \cong \mathrm{Hom}_{\mathbb{Z}_p}(N^\vee, D_s^{(p)}) & \text{if } i = t+s+1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. (compare with [3] 2.6 or [7] 5.4.14.) For any finite G-module A we have

$$\begin{aligned} D_s(A) &= \varinjlim_n H^s(G_n, A)^* \\ &= \varinjlim_n H^0(G_n, \mathrm{Hom}_{\mathbb{Z}_p}(A, D_s^{(p)})) \\ &= \mathrm{Hom}_{\mathbb{Z}_p}(A, D_s^{(p)}), \end{aligned}$$

whence we obtain from Remark 3.5

$$\begin{aligned} E^i(M)^\vee &\cong \varinjlim_k D_s(\mathrm{Ext}_R^{t+1-(i-s)}(R/\mathfrak{m}^k, M)^\vee) \\ &\cong \varinjlim_k \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{Ext}_R^{t+1-(i-s)}(R/\mathfrak{m}^k, M)^\vee, D_s^{(p)})) \\ &\cong \varinjlim_k \mathrm{Ext}_R^{t+1-(i-s)}(R/\mathfrak{m}^k, M)(\chi_G) \\ &\cong H_{\mathbf{z}}^{t+1-(i-s)}(M)(\chi_G). \end{aligned}$$

If M is free as an R-module the latter group equals $M \otimes_R R^\vee(\chi_G)$ for $i = s$ and zero otherwise. If N is finite, then it becomes $N(\chi_G) \cong \mathrm{Hom}_{\mathbb{Z}_p}(N^\vee, D_s^{(p)})$ for $i = t+s+1$ and zero otherwise. □

Jannsen ($\mathbb{Z}_p$-coefficients) and Nekovar (general coeefficients, but abelian G) also developed another spectral sequence, which has been generalized by Lim and Sharifi to (rather) general coefficients and general G as we will recall now.

Let T be a finitely generated free R-module with continuous G_K-action $A := T \otimes_R R^\vee$ and $T^* := \mathrm{Hom}_R(T, R)$. Then $A^* := T^* \otimes_R R^\vee \cong T^\vee$ by adjointness of Hom and Tor as well as the canonical isomorphism $T \cong \mathrm{E}_R^0\mathrm{E}_R^0(T)$. Furthermore, we have $A^\vee \cong T^*$, because $A^\vee \cong \mathrm{Hom}_R(T \otimes R^\vee, R^\vee) \cong \mathrm{Hom}_R(T, \mathrm{Hom}_R(R^\vee, R^\vee)) = T^*$.

Theorem 3.7 (Jannsen, Nekovar, Lim/Sharifi)**.**

$$\mathrm{E}^i_\Lambda(\mathrm{H}^j(K_\infty, A)^\vee) = \mathrm{E}^i(\mathrm{H}^{2-j}_{Iw}(K_\infty, T^*(1)))^\iota ?? \Rightarrow \mathrm{H}^{i+j}_{Iw}(K_\infty, T).$$

Proof. The spectral sequence follows from the first isomorphism of [5, Thm. 4.2.2] combined with [4, Thm. 5.4.2]. □

Corollary 3.8. *Assume* $\mathrm{cd}_p(G) \le 2$*. Then the exact sequence of low degrees degenerates to*

$$\begin{aligned} 0 \longrightarrow \mathrm{E}^1(A(k_\infty)^\vee) \longrightarrow \mathrm{H}^1_{Iw}(K_\infty, T) \longrightarrow \mathrm{E}^0(\mathrm{H}^1(K_\infty, A)^\vee) \longrightarrow \\ \mathrm{E}^2(A(k_\infty)^\vee) \longrightarrow \ker(\ \mathrm{H}^1_{Iw}(K_\infty, T) \longrightarrow \mathrm{E}^0(\mathrm{H}^2(K_\infty, A)^\vee)) \longrightarrow \\ \mathrm{E}^1(\mathrm{H}^1(K_\infty, A)^\vee) \longrightarrow \mathrm{E}^3(A(k_\infty)^\vee) \longrightarrow 0. \end{aligned}$$

The 5-term exact sequence of lower degrees starts as follow
(3.11)

$$0 \to \mathrm{E}^1_\Lambda(\mathrm{H}^0(K_\infty, A)^\vee) \longrightarrow \mathrm{H}^1_{Iw}(K_\infty, T) \longrightarrow \mathrm{E}^0_\Lambda(\mathrm{H}^1(K_\infty, A)^\vee) \longrightarrow \mathrm{E}^2_\Lambda(\mathrm{H}^0(K_\infty, A)^\vee),$$

in which the last term is pseudo-null. Since the third non-trivial term is Λ-torsionfree, the first one - being Λ-torsion - is actually the Λ-torsion submodule of the second one:

$$\mathrm{E}^1_\Lambda(\mathrm{H}^0(K_\infty, A)^\vee) \cong \mathrm{E}^1_\Lambda((T^*)_{G_{K_\infty}}) \cong tor_\Lambda \mathrm{H}^1_{Iw}(K_\infty, T). \tag{3.12}$$

Since T^* is finitely generated over R the Remark 3.5 implies:

Theorem 3.9. *If G is a duality group of dimension $s \ge 2$ and R regular, then the Λ-module $\mathrm{H}^1_{Iw}(K_\infty, T)$ is torsionfree. For $s = 1$ and we have*

$$tor_\Lambda \mathrm{H}^1_{Iw}(K_\infty, T) \cong ((T^*)_{G_{K_\infty}} \otimes R^\vee(\chi_G))^\vee \cong \mathrm{Hom}_R((T^*)_{G_{K_\infty}}, R)(\chi_G^{-1})$$

where χ denotes the character of the dualizing module for G.

Proof. Use the corollary for $i = s = 1$ and the fact that $\mathrm{H}^{t+1}_{\mathbf{z}}(M)^\vee \cong \mathrm{E}^0_R(M)$. □

4. Local Duality

In this section let $\Lambda = \Lambda_R(G) = R[\![G]\!]$ be the completed group algebra over R, *where G is a pro-p Poincaré group, such that Λ is Noetherian,* $\mathfrak{M}$ the maximal ideal of Λ and $k = \Lambda/\mathfrak{M}$ its finite residue class field. It is well known that the global homological dimension of Λ is $d := gl(\Lambda_R(G)) = \mathrm{cd}(G) + gl(R)$. By Λ-Mod we denote the category of (abstract) modules over the (abstract) ring Λ and we write Λ-mod for the full subcategory of finitely generated modules. In the sequel we will use frequently the equivalence of the latter category with the category of finitely generated compact modules.

Definition 4.1. For a finitely generated Λ-module M, we define the depth by

$$\mathrm{depth}(M) := \min\{i \mid \mathrm{Ext}^i_\Lambda(k, M) \neq 0\}.$$

Recall that for a commutative Noetherian ring Λ the I-depth $depth_I(M)$ of a finitely generated Λ-module M with respect to an ideal I is the maximal length of a M-regular sequence in I. For a local ring the depth(M) is $depth_{\mathfrak{M}}(M)$, while the grade is $j(M) = depth_{ann(M)}(\Lambda)$, where $ann(M)$ is the annihilator of M in Λ.

We consider the additive functor $\Gamma_{\mathfrak{M}}(-) : \Lambda\text{-Mod} \to \Lambda\text{-Mod}$ defined by $\Gamma_{\mathfrak{M}}(M) := \{x \in M \mid \mathfrak{M}^l x = 0 \text{ for some l }\}$ and state some basic properties:

Lemma 4.2. (i) $\Gamma_{\mathfrak{M}}(M) = \varinjlim_l \mathrm{Hom}_\Lambda(\Lambda/\mathfrak{M}^l, M)$, *in particular, the functor* $\Gamma_{\mathfrak{M}}(-)$ *is left exact.*

(ii) *The restriction of* $\Gamma_{\mathfrak{M}}$ *to* Λ*-mod equals* T_0*, i.e.* $\Gamma_{\mathfrak{M}}(M)$ *is the maximal finite submodule of* M*, if the latter module is finitely generated.*

Proof. Since $\mathrm{Hom}_\Lambda(\Lambda/\mathfrak{M}^l, M) = \{x \in M \mid \mathfrak{M}^l x = 0\}$, the first statement is obvious. If M is finitely generated, there is some l such that $\mathfrak{M}^l \mathrm{T}_0(M) = 0$, i.e. $\mathrm{T}_0(M) \subseteq \Gamma_{\mathfrak{M}}(M)$. On the other hand $\Lambda/\mathfrak{M}^l$ is a finite ring. Therefore $\Lambda x \subseteq \mathrm{T}_0(M)$ holds for any $x \in \Gamma_{\mathfrak{M}}(M)$. □

Let $\mathcal{D}(\Lambda\text{-Mod})$ (resp. $\mathcal{C}(\Lambda\text{-Mod})$) mean the category of discrete (resp. compact) Λ-modules, where Λ is endowed with its canonical $(\mathfrak{m}, I)$-topology.

Proof. □

Proposition 4.3. *The forgetful functor defines an equivalence of categories*

$$\mathcal{D}(\Lambda\text{-}Mod) \cong \Lambda\text{-}Mod_{\mathfrak{M}}.$$

Proof. Both categories consists exactly of direct limits of finite modules (cf. [7, Prop. (5.2.4)] for $\mathcal{D}(\Lambda\text{-Mod})$). □

Lemma 4.4. (i) $\mathrm{H}^i_{\mathfrak{M}}(\Lambda\text{-}Mod) \subseteq \Lambda\text{-}Mod_{\mathfrak{M}} \cong \mathcal{D}(\Lambda\text{-}Mod)$ *for all* $i \geq 0$.

(ii) *For any* $M \in \Lambda$*-mod, it holds* $\mathrm{depth}(M) = \min\{i \mid \mathrm{H}^i_{\mathfrak{M}}(M) \neq 0\}$.

(iii) $\mathrm{depth}(\Lambda) = d$ *and* $\mathrm{H}^d_{\mathfrak{M}}(\Lambda) = \Lambda^\vee$.

(iv) $\mathrm{Hom}_\Lambda(M, \mathrm{H}^d_{\mathfrak{M}}(\Lambda)) \cong M^\vee$ *for all* M *in* Λ*-*$Mod_{\mathfrak{M}}$ *or in* Λ*-mod , in particular,* $\mathrm{H}^d_{\mathfrak{M}}(\Lambda)$ *is an injective* Λ*-module.*

Proof. Since $\mathrm{H}^i_{\mathfrak{M}}(-)$ are the derived functors of $\mathrm{H}^0_{\mathfrak{M}}(-)$, it suffices to prove (i) for the latter functor. But in this case the statement holds just by definition.

Now we will prove (ii) and set $k = \min\{i \mid \mathrm{H}^i_{\mathfrak{M}}(M) \neq 0\}$. Since $\mathrm{Ext}^i_\Lambda(\Lambda/\mathfrak{M}^l, M) = 0$ for all $i < \mathrm{depth}(M)$ (note that $\Lambda/\mathfrak{M}^l$ has a finite composition series with subquotients isomorphic to k), it holds $\mathrm{depth}(M) \leq k$. So we only have to prove that $\mathrm{H}^j_{\mathfrak{M}}(M) \neq 0$ for $j = \mathrm{depth}(M) < \infty$. But the short exact sequences

$$0 \longrightarrow \mathfrak{M}/\mathfrak{M}^l \longrightarrow \Lambda/\mathfrak{M}^l \longrightarrow k \longrightarrow 0$$

induce the long exact sequences

$$0 = \mathrm{Ext}^{j-1}_\Lambda(\mathfrak{M}/\mathfrak{M}^l, M) \longrightarrow \mathrm{Ext}^j_\Lambda(k, M) \longrightarrow \mathrm{Ext}^j_\Lambda(\Lambda/\mathfrak{M}^l, M) \longrightarrow \cdots,$$

i.e. $0 \neq \mathrm{Ext}^j_\Lambda(k, M) \subseteq \mathrm{H}^j_{\mathfrak{M}}(M)$.

Using 3.6 and denoting the character of the dualizing module by χ, we calculate

$$\mathrm{H}^i_{\mathfrak{M}}(\Lambda) = \varinjlim_l \mathrm{E}^i(\Lambda/\mathfrak{M}^l) = \begin{cases} \varinjlim_l (\Lambda/\mathfrak{M}^l(\chi))^\vee = \Lambda^\vee & \text{if } i = d \\ 0 & \text{otherwise,} \end{cases}$$

whence (iii) follows. In order to prove (iv) first let M be in Λ-$\mathrm{Mod}_{\mathfrak{M}}$, i.e. $M = \varinjlim_i M_i$ for some finite Λ-modules M_i. Then, noting that M_i is a $\Lambda/\mathfrak{M}^{l(i)}$-module for some $l(i)$ and using the adjunction of "Hom and $\otimes$",

$$\begin{aligned} \mathrm{Hom}_\Lambda(M, \mathrm{H}^d_{\mathfrak{M}}(\Lambda)) &= \mathrm{Hom}_\Lambda(\varinjlim_i M_i, \varinjlim_l (\Lambda/\mathfrak{M}^l)^\vee) \\ &= \varprojlim_i \mathrm{Hom}_\Lambda(M_i, \varinjlim_l (\Lambda/\mathfrak{M}^l)^\vee) \\ &= \varprojlim_i \mathrm{Hom}_\Lambda(M_i, \mathrm{Hom}_{\mathbb{Z}_p}(\Lambda/\mathfrak{M}^{l(i)}, \mathbb{Q}_p/\mathbb{Z}_p)) \\ &= \varprojlim_i \mathrm{Hom}_{\mathbb{Z}_p}(M_i, \mathbb{Q}_p/\mathbb{Z}_p) \\ &= M^\vee. \end{aligned}$$

Now let M be in Λ-mod . Then, noting that $\mathrm{Hom}_\Lambda(M, -)$ commutes with direct limits, because M is finitely generated,

$$\begin{aligned} \mathrm{Hom}_\Lambda(M, \mathrm{H}^d_{\mathfrak{M}}(\Lambda)) &= \mathrm{Hom}_\Lambda(M, \varinjlim_l (\Lambda/\mathfrak{M}^l)^\vee) \\ &= \varinjlim_l \mathrm{Hom}_\Lambda(M, (\Lambda/\mathfrak{M}^l)^\vee) \\ &= \varinjlim_l \mathrm{Hom}_\Lambda(M/\mathfrak{M}^l, \mathrm{Hom}_{\mathbb{Z}_p}(\Lambda/\mathfrak{M}^l, \mathbb{Q}_p/\mathbb{Z}_p)) \\ &= \varinjlim_l \mathrm{Hom}_{\mathbb{Z}_p}(M/\mathfrak{M}^l, \mathbb{Q}_p/\mathbb{Z}_p) \\ &= M^\vee. \end{aligned}$$

□

After this technical preparations we are able to prove the following

Theorem 4.5. *Let G be a pro-p Poincaré group with $d := \mathrm{cd}(G) + gl(R) < \infty$ and such that $\Lambda = \Lambda(G)$ is Noetherian. Then, for any $M \epsilon \Lambda$-mod,*

$$\mathrm{E}^i(M) \cong \mathrm{Hom}_\Lambda(\mathrm{H}^{d-i}_{\mathfrak{M}}(M), \mathrm{H}^d_{\mathfrak{M}}(\Lambda)) \cong \mathrm{H}^{d-i}_{\mathfrak{M}}(M)^\vee =: T^i(M).$$

Proof. Consider the right exact contravariant additive functor $T^0(-) = \mathrm{H}^d_{\mathfrak{M}}(M)^\vee$ on Λ-mod (note that $\mathrm{H}^i_{\mathfrak{M}}(M) = 0$ for all $i > d$ as Λ has global dimension d). By [8, Thm. 3.36 and Remarks] there is a natural equivalence of functors

$$T^0(-) \cong \mathrm{Hom}_\Lambda(-, T^0(\Lambda)) \cong \mathrm{Hom}_\Lambda(-, \Lambda)$$

on Λ-mod. Therefore, it suffices to show that the functors $T^i(-)$ are the left derived functors of $T^0(-)$. But $\{T^i(-)\}_{i \geq 0}$ is a universal δ-functor because they are effaceable by projectives in Λ-mod (Since T^0 is additive, it is sufficient to verify that $\mathrm{H}^i_{\mathfrak{M}}(\Lambda) = 0$ for all $i < d$, which is done by lemma 4.4 (iii)). □

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