Lifting newforms to vector valued modular forms for the Weil representation

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Abstract

Given a discriminant form $D$ of level $N$ there is a natural lifting mapping elliptic modular forms of level $N$ to vector valued modular forms for the Weil representation associated to $D$. We show that in some cases the zero component of a lifting of a newform $f$ is just a scalar multiple of $f$. In order to do so, we split the lifting map into certain partial liftings corresponding to the prime powers exactly dividing $N$ and then proceed to compute the zero components of these partial maps explicitly. As an application we show that the $L$-function $L_A(f, s)$ of a newform $f$ and an ideal class $A$ as defined by Gross and Zagier can be written as a certain $L$-series of the lifting of $f$.

1 Introduction

As a generalisation of the usual elliptic modular forms one can consider vector valued modular forms. Given an even lattice $L$ of even signature with associated discriminant form $D = L'/L$, a vector valued modular form for the Weil representation $\rho_D$ is a holomorphic function on the complex upper half plane with values in the group algebra $\mathbb{C}[D]$ transforming suitably under $\text{SL}_2(\mathbb{Z})$ and being meromorphic at $\infty$. Vector valued modular forms are for example important in the theory of Borcherds’ automorphic products where they serve as inputs for the singular theta correspondence, see [2].

Let $N$ be the level of $D$. There is a lifting map

$$\mathcal{L}_D(f) = \sum_{M \in \Gamma_0(N) \setminus \text{SL}_2(\mathbb{Z})} f|_k M \rho_D(M)^{-1} \chi_D$$

sending elliptic modular forms $f$ of weight $k$ for $\Gamma_0(N)$ and a certain character $\chi_D$ associated to $D$ to vector valued modular forms of weight $k$. This construction is well-known and has for example been studied by Scheithauer in [10], where one can also find explicit formulas for the component functions of the lifting. As the formulas involve sums over the cusps of $\Gamma_0(N)$ they are useful for squarefree level $N$ but difficult to evaluate for arbitrary $N$.

The zero component of the above lifting has been computed by Bruinier and Bundschuh in [4], Theorem 5, in the case of $N$ and $|D|$ being an odd prime $p$, by Bundschuh in his
thesis [6], Proposition 4.3.9, in the case of \( N \) being squarefree, and by Zhang [11], Theorem 4.16, for special discriminant forms whose level is a positive fundamental discriminant. It was observed that the zero component of a lifted newform \( f \) is in some cases just a scalar multiple of \( f \). In the present work we consider the zero component
\[
\Phi_D(f) = \langle L_D(f), e_0 \rangle_D
\]
of the lifting for arbitrary level \( N \) and newforms \( f \). Here \( \langle \cdot, \cdot \rangle_D \) denotes the natural inner product on \( \mathbb{C}[D] \). We define partial liftings
\[
L_D^q(f) = \sum_{M \in \Gamma_0(N/q) \setminus \text{SL}_2(\mathbb{Z})} f|_k M \rho_D(M)^{-1} e_0
\]
where \( q \) is a prime power exactly dividing \( N \), and show that \( \Phi_D \) splits as a product of the maps \( \Phi_D^q(f) = \langle L_D^q(f), e_0 \rangle_D \). The latter are then computed explicitly for newforms \( f \), giving our main result (compare Theorem 8.1):

**Theorem 1.1.** Let \( D \) be a discriminant form of even signature and level \( N \). Assume that all 2-adic Jordan components of \( D \) are even, and that for every odd prime power \( q = p^r \) exactly dividing \( N \) the exponent \( n_r \) in the Jordan decomposition \( D_q \cong \bigoplus_{j=1}^{r} (p^j)^{\pm n_j} \) is even. Here \( D_q \) is the subgroup of elements of order dividing \( q \). Then
\[
\Phi_D(f) = \prod_{q \mid N} \left( 1 - \frac{G_{D_q}(q/p)}{|D_q|} \right) \cdot f
\]
for every \( f \in S^\text{new}_k(N, \chi_D) \) where the product runs over all prime powers \( q = p^r \) exactly dividing \( N \), including \( p = 2 \). Further, \( G_{D_q}(q/p) \) denotes the Gauss sum defined in Section 3 and \( \chi_D \) is the character given in (3).

As an application, which was in fact the motivation for this work, we apply this result to a rescaled hyperbolic plane in Section 8, and show that the \( L \)-series \( L_A(f, s) \) associated to a newform \( f \) and an ideal class \( A \) defined by Gross and Zagier in [7] can be written as a certain \( L \)-series of the vector valued lifting of \( f \). Using results of Bruinier and Yang (see [5]) this may lead to a new proof of the Gross-Zagier formula.

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2 Classical modular forms and Atkin-Lehner operators

Let $k$ be an integer. The group $\text{GL}_2^+(\mathbb{Q})$ acts on the space of functions $f: \mathbb{H} \to \mathbb{C}$ via

$$(f|_k \alpha)(\tau) := \det(\alpha)^{k/2} j(\alpha, \tau)^{-k} f(\alpha \tau), \quad \tau \in \mathbb{H},$$

where $j(\alpha, \tau) := c \tau + d$ for $\alpha = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{GL}_2^+(\mathbb{Q})$. For some positive integer $N$ and some Dirichlet character $\chi$ mod $N$ we call a holomorphic function $f: \mathbb{H} \to \mathbb{C}$ modular of weight $k$, level $N$ and character $\chi$ if $f|_k \alpha = \chi(\alpha) f$ for all $\alpha \in \Gamma_0(N)$ where $\chi(\alpha) := \chi(d)$ for $\alpha = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma_0(N)$. If in addition $f$ is meromorphic, holomorphic or vanishes at the cusps of $\Gamma_0(N)$ we call $f$ a weakly holomorphic modular form, a modular form or a cusp form, respectively. The corresponding spaces are denoted by $M_k^!(N, \chi)$, $M_k(N, \chi)$ and $S_k(N, \chi)$.

For a prime $p$ we define the usual Hecke operator $T_p$ by

$$T_p(f) = p^{k/2-1} \sum_{j=0}^{p-1} \frac{f\left( \left( \begin{array}{cc} 1 & j \\ 0 & p \end{array} \right) \right)}{k} + p^{k/2-1} \chi(p) f\left( \left( \begin{array}{cc} p & 0 \\ 0 & 1 \end{array} \right) \right)$$

for $f \in M_k^!(N, \chi)$ where $\chi(p) = 0$ if $p$ divides $N$. For $r \geq 2$ and coprime integers $n$, $m$ we define $T_{pr} := T_p T_{pr-1} - p^{k-1} \chi(p) T_{pr-2}$ and $T_{nm} := T_n T_m$. This extends the definition of the operator $T_p$ to arbitrary positive integers. Further, we define the usual Fricke involution

$$H_N(f) := f\left( \left( \begin{array}{cc} 0 & -1 \\ N & 0 \end{array} \right) \right)$$

which gives a map $M_k^!(N, \chi) \to M_k^!(N, \chi)$. (This operator is often denoted by $W_N$. The present notation was adopted from [1] in order to distinguish between the Fricke involution and the Atkin-Lehner operators defined later on.)

Next we quickly recall the notion of newforms. For a positive divisor $d$ of $N$ such that the conductor of $\chi$ divides $N/d$ there are two natural embeddings of $S_k(N/d, \chi)$ into $S_k(N, \chi)$, namely the trivial one $f \mapsto f$ and $f \mapsto f(d\tau)$. The space of newforms $S_k^\text{new}(N, \chi)$ is the orthogonal complement (with respect to the Petersson inner product) of the subspace of $S_k(N, \chi)$ generated by the images of these embeddings for all divisors $d > 1$ of $N$. One can check that the Hecke operators $T_n$ and the Fricke involution $H_N$ preserve newforms.

Throughout we write $e(z) := e^{2\pi i z}$ for $z \in \mathbb{C}$. We call $f(\tau) = \sum_{n \geq 1} a_f(n) e(\tau n) \in S_k(N, \chi)$ an eigenform if $f$ is an eigenvector for $T_n$ for all $n$ coprime to $N$, and we say an eigenform $f$ is normalised if $a_f(1) = 1$. Further, we call $f$ a primitive form if $f$ is a normalised eigenform and a newform. It is well-known that the Fourier coefficients of a primitive form are precisely its $T_n$ eigenvalues and are therefore multiplicative. Moreover, the set of primitive forms of weight $k$, level $N$ and character $\chi$ forms an orthogonal basis of the corresponding space of newforms. We will need the following result:

**Proposition 2.1.** Let $p$ be a prime such that $p^2$ divides $N$ and let $\chi$ be a real Dirichlet character mod $N$ of conductor $d$. Further, let $N_p$ and $d_p$ be the $p$-components of $N$ and $d$, respectively. Then $S_k(N, \chi)$ is the orthogonal complement of the space of modular forms $M_k(N, \chi)$.
respectively. If either $p$ is odd, or if $p = 2$ and $N_p > d_p$, then every $f \in S^\text{new}_k(N, \chi)$ is of the form
\[
f(\tau) = \sum_{\substack{n \geq 1 \\ (n, p) = 1}} a_f(n)e(\tau n), \quad \tau \in \mathbb{H}.
\]
So for a positive integer $n$ being divisible by $p$ the Hecke operator $T_n$ vanishes on $S^\text{new}_k(N, \chi)$.

**Proof.** It suffices to prove this for primitive forms. Let $f \in S^\text{new}_k(N, \chi)$ be a primitive form. First assume $p$ is odd. Then the $p$-component of the character $\chi$ is either trivial or of the form $(\frac{\cdot}{p})$. In either case we have $N_p \geq p^2 > p > d_p$. So the claimed statement follows for any prime $p$ from part (c) of Theorem 4.6.17 in [9].

Now we introduce Atkin-Lehner operators following Section 1 and 2 of [1]. For $N = qm$ with $q$ and $m$ being coprime positive integers we define
\[
W_q(f) := f\Bigg|_k \begin{pmatrix} qx & y \\ Nz & qw \end{pmatrix}
\]
for $f \in M^1_k(N, \chi)$ where $x, y, z, w \in \mathbb{Z}$ with $y \equiv 1 \mod q$, $x \equiv 1 \mod m$ and such that $
\det\left(\begin{pmatrix} qx & y \\ Nz & qw \end{pmatrix}\right) = q$. This gives a well-defined map from $M^1_k(N, \chi_m)$ to $M^1_k(N, \chi_m^{-1})$ which preserves the corresponding subspaces of modular forms, cusp forms and newforms. Here $\chi_q$ and $\chi_m$ denote the $q$ and $m$ components of $\chi$, respectively.

Let $q = p^r$ be a prime power. It is shown in [1] that $W_q(f) = \lambda_p(f) f^{(q)}$ for every primitive form $f \in S_k(N, \chi)$ where $|\lambda_p(f)| = 1$ and $f^{(q)} = \sum_{n \geq 1} b(n)e(\tau n)$ is the primitive form defined by
\[
b(\ell) = \begin{cases} 
\chi_q(\ell)a_f(\ell), & \text{if } \ell \neq p, \\
\chi_m(p)a_f(p), & \text{if } \ell = p,
\end{cases}
\]
for primes $\ell$. If $a_f(p) \neq 0$ then $\lambda_p(f) = q^{k/2-1}G(\chi_q)a_f(q)^{-1}$ where $G(\chi_q)$ denotes the usual Gauss sum of $\chi_q$ (this is Theorem 2.1 in [1]).

**Proposition 2.2.** Let $N = pm$ with $p$ being prime and $(p, m) = 1$. Further, let $\chi$ be a real Dirichlet character such that its $p$-component $\chi_p$ is trivial. Then
\[
(W_p \circ T_p)(f) = (T_p \circ W_p)(f) = -p^{k/2-1} f
\]
for $f \in S^\text{new}_k(N, \chi)$.

**Proof.** We may assume that $f \in S^\text{new}_k(N, \chi)$ is primitive. Then $a_f(p)^2 = \chi_m(p)p^{k-2}$ by part (2) of Theorem 4.6.17 in [9]. In particular, we have $a_f(p) \neq 0$. Hence the above discussion yields $W_p(f) = \lambda_p(f)f^{(p)}$ where $\lambda_p(f) = -p^{k/2-1}a_f(p)^{-1}$. Since $\chi_p$ is trivial by assumption we have $b(\ell) = a_f(\ell)$ for all primes $\ell$ different to $p$. Moreover, $\chi_m(p) = \pm 1$ as $\chi$ is real. So $a_f(p)^2 = \chi_m(p)p^{k-2}$ gives
\[
a_f(p) = \begin{cases} 
\pm p^{k/2-1}, & \text{if } \chi_m(p) = 1, \\
\pm ip^{k/2-1}, & \text{if } \chi_m(p) = -1.
\end{cases}
\]
In either case we have $b(p) = \chi_m(p)a_f(p) = a_f(p)$. Therefore $f(p) = f$ and thus the claim follows since $T_p(f) = a_f(p)f$. \hfill\Box

3 Discriminant forms and Gauss sums

Let $L$ be an even lattice of signature $(b^+, b^-)$ and let $L'$ be its dual lattice. The quotient $D = L'/L$ is a finite abelian group and the modulo 1 reduction of the bilinear form on $L'$ induces a non-degenerate $\mathbb{Q}/\mathbb{Z}$-valued bilinear form $\langle \cdot, \cdot \rangle$ on $D$ with corresponding quadratic form $Q(\gamma) = \frac{1}{2}(\langle \gamma, \gamma \rangle)$. (Some authors write $\gamma\delta$ for $(\gamma, \delta)$ and $\gamma^2/2$ for $Q(\gamma)$.) Such a group $D$ is called a discriminant form. Its signature sign$(D)$ is defined as the class of $b^+ - b^-$ mod 8 and the level of $D$ is the smallest integer $N$ such that $NQ(\gamma) = 0 \mod 1$ for all $\gamma \in D$.

Every discriminant form $D$ is isomorphic to an orthogonal direct sum of ‘basic’ discriminant forms, the so-called Jordan components. We recall their definition from [10]:

- Let $q = p^r$ be a power of an odd prime $p$. Write $q^{\pm 1}$ for the discriminant form $\mathbb{Z}/q\mathbb{Z}$ where the quadratic form is defined by $Q(\gamma) = a/q$ for $a \in \mathbb{Z}$ with $(\frac{a}{p}) = \pm 1$. This is called a $p$-adic Jordan component. The level of $q^{\pm 1}$ is $q$.
- Let $q = 2^r$ be a power of 2. Write $q^{\pm 2}_t$ for the discriminant form $\mathbb{Z}/q\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z}$ where the quadratic form is defined on generators $\gamma, \delta$ by $Q(\gamma) = Q(\delta) = 0, (\gamma, \delta) = 1/q$ mod 1 for $q^{\pm 2}_t$, and $Q(\gamma) = Q(\delta) = 1/q, (\gamma, \delta) = 1/q$ mod 1 for $q^{\pm 2}$. This is called an even $2$-adic Jordan component. Its level is $q$.
- Let $q = 2^r$ be a power of 2. Write $q^{\pm 1}_t$ with $t \in \mathbb{Z}/8\mathbb{Z}$ satisfying $(\frac{t}{2}) = \pm 1$ for the discriminant form $\mathbb{Z}/q\mathbb{Z}$ where the quadratic form is defined by $Q(\gamma) = t/2q$ mod 1 for a generator $\gamma$. This is called an odd $2$-adic Jordan component. Its level is $2q$.

The direct sum of $n$ Jordan components $q^{\pm 1}_t, q^{\pm 2}_t$ or $q^{\pm 1}_i$ with the same prime power $q$ is denoted by $q^{\pm n}, q^{\pm 2n}_t$ or $q^{\pm n}_t$, respectively, where the signs are multiplied and $t = \sum t_j$. Such sums are also called Jordan components. Note that the level of a discriminant form is the least common multiple of the levels of its Jordan components.

Eventually, we define the oddity of a discriminant form: For $q = 2^r$ set oddity$(q^{\pm 2n}_t) = 4k$ mod 8 and oddity$(q^{\pm n}_t) = t + 4k$ mod 8 where $k = 1$ if $q$ is not a square and the exponent is $-n$, and $k = 0$ otherwise. The oddity of an arbitrary discriminant form is the sum of the oddities of its 2-adic Jordan components.

For a discriminant form $D$ and $n \in \mathbb{Z}$ the Gauss sum of $D$ is defined by

$$\mathcal{G}_D(n) = \sum_{\gamma \in D} e(nQ(\gamma))$$

where $e(z) := e^{2\pi i z}$ for $z \in \mathbb{C}$ as before. Note that this sum is multiplicative in $D$ in the sense that $\mathcal{G}_D(n) = \mathcal{G}_{D_1}(n)\mathcal{G}_{D_2}(n)$ if $D = D_1 \oplus D_2$ (where a direct sum is always understood to be orthogonal). Therefore it is sufficient to compute Gauss sums of Jordan components. This has been done by Scheithauer in [10]. We use his results to deduce the following formulas:

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Proposition 3.1. Let $q = p^s$ be a prime power and $a, s \in \mathbb{Z}$ with $(a, p) = 1$ and $0 \leq s < r$. For $p$ odd we have

$$G_{q^a}(p^s a) = \left( \frac{a}{p^r-s} \right) G_{q^a}(p^s),$$

and for $p = 2$ we have $G_{q^a}(2^s a) = G_{q^a}(2^s)$ and

$$G_{q^a}(2^s a) = \left( \frac{a}{2^r-s} \right) e \left( ((a-1) \text{ oddity} (2^{r-s}+1) / 8 \right) G_{q^a}(2^s).$$

Proof. Let $p$ be an odd prime. Then Proposition 3.3 from [10] states that for $q$ not dividing $c$ we have

$$G_{q^a}(c) = \gamma_p((q/q_c)^{\pm 1}) \left( \frac{c/q_c}{q/q_c} \right) \sqrt{q/q_c}$$

with $q_c = (c, q)$. Here $\gamma_p((q/q_c)^{\pm 1})$ is a certain 8th root of unity defined in Section 2 of [10]. Comparing the formula for $c = p^s a$ and $c = p^s$ we see that $G_{q^a}(p^s a)$ and $G_{q^a}(p^s)$ differ by a factor $\left( \frac{a}{p^r-s} \right)$. For $p = 2$ one uses Proposition 3.5 and 3.6 from [10] and proceeds as above.

4 Vector valued modular forms

Let $D$ be a discriminant form of level $N$. From now on we assume that the signature of $D$ is even. The group algebra $\mathbb{C}[D]$ of $D$ is the $\mathbb{C}$-vector space generated by the formal basis vectors $e_\gamma$ for $\gamma \in D$ with multiplication defined by $e_\gamma e_\delta = e_{\gamma+\delta}$. There is a natural inner product on $\mathbb{C}[D]$ being antilinear in the second argument, which is defined by $\langle e_\gamma, e_\delta \rangle_D = 0$ for $\gamma \neq \delta$ and $\langle e_\gamma, e_\gamma \rangle_D = 1$.

We define an action of the generators $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ of $\text{SL}_2(\mathbb{Z})$ on $\mathbb{C}[D]$ by

$$\rho_D(T)e_\gamma = e(Q(\gamma))e_\gamma, \quad \rho_D(S)e_\gamma = \frac{e(-\text{sign}(D)/8)}{\sqrt{|D|}} \sum_{\delta \in D} e(-\langle \gamma, \delta \rangle)e_\delta.$$

This extends to a unitary representation $\rho_D$ of $\text{SL}_2(\mathbb{Z})$ on $\mathbb{C}[D]$, the so-called Weil representation of $D$. Here it is crucial that the signature of $D$ is even as otherwise we would have to work with a double cover of $\text{SL}_2(\mathbb{Z})$ (see for example [3], Chapter 1). It is well-known that $\Gamma(N)$ acts trivially in this representation. Moreover, for $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ we have the formula

$$\rho_D(M)e_\gamma = \chi_D(a)e(bdQ(\gamma))e_\delta, \quad (2)$$

where

$$\chi_D(a) = \left( \frac{a}{|D|} \right) e((a-1) \text{ oddity}(D)/8) \quad (3)$$

$$6$$
denotes the Dirichlet character associated to \( D \) (see [10], Proposition 4.5). Note that \( \text{oddity}(D) \) is even by the oddity formula, so \( \chi_D \) is a real character mod \( N \).

Next we consider vector valued functions \( F: \mathbb{H} \to \mathbb{C}[D] \). Write \( F(\tau) = \sum_{\gamma \in D} f_\gamma(\tau) \mathbf{c}_\gamma \). We call \( F \) holomorphic if the component functions \( f_\gamma \) are. Further, we call a holomorphic function \( F: \mathbb{H} \to \mathbb{C}[D] \) a (weakly holomorphic) vector valued modular form of weight \( k \) if it transforms as \( F(\tau)|_k M = \rho_D(M)F(\tau) \) for \( M \in \text{SL}_2(\mathbb{Z}) \) and if every component function \( f_\gamma \) is meromorphic at \( \infty \), that is every \( f_\gamma \) has a Fourier expansion of the form \( \sum_{n \in \mathbb{Z} + Q(\gamma)} a(n, \gamma)e(\tau n) \) with \( a(n, \gamma) = 0 \) for almost all \( n < 0 \).

Using the transformation behaviour of \( F \) and the formula (2) for the action of \( \Gamma_0(N) \) on \( \mathbb{C}[D] \) it is easy to check that the zero component \( f_0 \) of \( F \) is a weakly holomorphic modular form of weight \( k \), level \( N \) and character \( \chi_D \), that is \( f_0 \in M_k^! (N, \chi_D) \). Conversely, every scalar valued modular form \( f \in M_k^! (N, \chi_D) \) can be lifted to a vector valued modular form of weight \( k \) via

\[
\mathcal{L}_D(f) := \sum_{M \in \Gamma_0(N) \setminus \text{SL}_2(\mathbb{Z})} f|_k M \rho_D(M)^{-1} \mathbf{c}_0.
\]

This lifting has for example been studied in [10], Section 5. Taking the zero component of \( \mathcal{L}_D(f) \) for an elliptic modular form \( f \) gives a linear map

\[
\Phi_D: M_k^!(N, \chi_D) \to M_k^!(N, \chi_D), \quad f \mapsto \langle \mathcal{L}_D(f), \mathbf{c}_0 \rangle_D
\]

which preserves the corresponding subspaces of modular forms and cusp forms.

5 Splitting of the map \( \Phi_D \)

As before let \( D \) be a discriminant form of even signature and level \( N \). Write \( N = mm' \) with \( (m, m') = 1 \). Then we can decompose \( D = D_m \oplus D_{m'} \) as an orthogonal direct sum where \( D_c \) denotes the subgroup of elements of order dividing \( c \). Note that \( D_m \) and \( D_{m'} \) are discriminant forms whose levels are multiples of \( m \) and \( m' \) with the same prime divisors as \( m \) and \( m' \), respectively. We define an inner product on \( \mathbb{C}[D_m] \otimes \mathbb{C}[D_{m'}] \) by

\[
\langle \mathbf{c}_{\gamma_1} \otimes \mathbf{c}_{\delta_2}, \mathbf{c}_{\gamma_2} \otimes \mathbf{c}_{\delta_2} \rangle_{m \otimes m'} = \langle \mathbf{c}_{\gamma_1}, \mathbf{c}_{\gamma_2} \rangle_{D_m} \cdot \langle \mathbf{c}_{\delta_1}, \mathbf{c}_{\delta_2} \rangle_{D_{m'}}.
\]

One easily checks that the natural map \( \mathbb{C}[D_m] \otimes \mathbb{C}[D_{m'}] \to \mathbb{C}[D] \) sending \( \mathbf{c}_\gamma \otimes \mathbf{c}_\delta \) to \( \mathbf{c}_\gamma \mathbf{c}_\delta \) is an isometry. Moreover, it is an isomorphism of representations, namely of the tensor product representation \( \rho_{D_m} \otimes \rho_{D_{m'}} \) on \( \mathbb{C}[D_m] \otimes \mathbb{C}[D_{m'}] \) and the Weil representation \( \rho_D \) on \( \mathbb{C}[D] \). Thereby one obtains the useful formulas

\[
\rho_D(M)\mathbf{c}_{\gamma + \delta} = \rho_{D_m}(M)\mathbf{c}_\gamma \cdot \rho_{D_{m'}}(M)\mathbf{c}_\delta, \quad (7)
\]

\[
\langle \rho_D(M)\mathbf{c}_{\gamma_1 + \delta_1}, \mathbf{c}_{\gamma_2 + \delta_2} \rangle_D = \langle \rho_{D_m}(M)\mathbf{c}_{\gamma_1}, \mathbf{c}_{\gamma_2} \rangle_{D_m} \cdot \langle \rho_{D_{m'}}(M)\mathbf{c}_{\delta_1}, \mathbf{c}_{\delta_2} \rangle_{D_{m'}}, \quad (8)
\]

for \( \gamma, \gamma_1, \gamma_2 \in D_m, \delta, \delta_1, \delta_2 \in D_{m'} \) and \( M \in \text{SL}_2(\mathbb{Z}) \). Since the inner products of \( D \) and \( D_m \) (resp. \( D_{m'} \)) agree on \( D_m \) (resp. \( D_{m'} \)) we will in the following simply write \( \langle \cdot, \cdot \rangle \) for all of these.
Lemma 5.1. Let \( D \) be a discriminant form of even signature and level \( N = mm' \) with \((m, m') = 1\). Then for \( M \in \Gamma_0(m) \) and \( x \in \mathbb{C}[D_m] \) we have

\[
\langle \rho_D(M)x, e_0 \rangle = \langle x, e_0 \rangle \cdot \langle \rho_D(M)e_0, e_0 \rangle
\]

with \( \rho_D(M)e_0 = \chi_{D_m}(M)\rho_{D_m'}(M)e_0 \in \mathbb{C}[D_m'] \).

Proof. Writing \( x = \sum_{\gamma \in D_m} \langle x, e_\gamma \rangle e_\gamma \) we see that it is enough to prove the lemma for the basis vectors \( e_\gamma \) in \( \mathbb{C}[D_m] \). The first statement is trivial for \( x = e_0 \). For \( \gamma \in D_m \) with \( \gamma \neq 0 \) we have by (8) that

\[
\langle \rho_D(M)e_\gamma, e_0 \rangle = \langle \rho_{D_m}(M)e_\gamma, e_0 \rangle \cdot \langle \rho_{D_m'}(M)e_0, e_0 \rangle.
\]

Let \( M = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma_0(m) \). Since \( d \) is coprime to \( m \) and the level of \( D_m \) is a multiple of \( m \) we have \( d\gamma \neq 0 \). Using (2) we thus obtain

\[
\langle \rho_{D_m}(M)e_\gamma, e_0 \rangle = \langle \chi_{D_m}(M)e(bdQ(\gamma))e_{d\gamma}, e_0 \rangle = 0.
\]

It follows from (7) and the action of \( \Gamma_0(m) \) in \( \rho_{D_m} \) that

\[
\rho_D(M)e_0 = \rho_{D_m}(M)e_0 \cdot \rho_{D_m'}(M)e_0 = \chi_{D_m}(M)\rho_{D_m'}(M)e_0 \in \mathbb{C}[D_m'].
\]

This proves the lemma. \(\square\)

We now consider the partial lifting

\[
L_m^D(f) := \sum_{M \in \Gamma_0(N)\setminus \Gamma_0(N/m)} f|kM \rho_D(M)^{-1}e_0. \tag{9}
\]

This gives a well-defined linear map sending modular forms \( f \in M^!_{k}(N, \chi_D) \) to holomorphic vector valued functions \( F : \mathbb{H} \to \mathbb{C}[D] \) with \( F|k(M\tau) = \rho_D(M)F(\tau) \) for \( M \in \Gamma_0(N/m) \).

The component functions of \( L_m^D(f) \) are linear combinations of \( f|kM \) for certain \( M \in \Gamma_0(N/m) \) and are therefore meromorphic at the cusps. The zero component

\[
\Phi_m^D(f) := \langle L_m^D(f), e_0 \rangle \tag{10}
\]

is again a modular form in \( M^!_{k}(N, \chi_D) \).

Theorem 5.2. Let \( D \) be a discriminant form of even signature and level \( N = q_1 \cdots q_l \) with pairwise coprime prime powers \( q_i = p_i^{r_i} \). Then

\[
\Phi_D = \Phi_D^N = \Phi_D^{q_1} \circ \cdots \circ \Phi_D^{q_l}
\]

on \( M^!_{k}(N, \chi_D) \).
Proof. It is sufficient to prove \( \Phi_D^{rs} = \Phi_D^r \circ \Phi_D^s \) for all pairwise coprime integers \( r, s, t \) with \( N = rst \).

Let \( m := N/r \) and \( n := N/s \). Let \( A \) and \( B \) be systems of coset representatives of \( \Gamma_0(N) \backslash \Gamma_0(m) \) and \( \Gamma_0(N) \backslash \Gamma_0(n) \), respectively. One easily verifies that \( B \) is also a system of coset representatives of \( \Gamma_0(m) \backslash \Gamma_0(t) \). Hence \( AB \) is a system of coset representatives of \( \Gamma_0(N) \backslash \Gamma_0(t) \). Writing out the definition of \( \Phi_D^{rs}(f) \) with \( f \in M_k^0(N, \chi_D) \) gives

\[
\Phi_D^{rs}(f) = \sum_{\alpha \in A} \sum_{\beta \in B} f_{k|\alpha|_{k}\beta} \langle \rho_D(\beta^{-1})(\rho_D^{-1}(\alpha)_0), \epsilon_0 \rangle .
\]

The second statement of Lemma 5.1 shows that \( \rho_D(\alpha^{-1})_0 \in \mathbb{C}[D_{N/m}] \) as \( \alpha^{-1} \in \Gamma_0(m) \).

Note that \( \beta^{-1} \in \Gamma_0(n) \subseteq \Gamma_0(N/m) \), so the first part of Lemma 5.1 applied to \( x = \rho_D(\alpha)^{-1}_0 \) gives

\[
\langle \rho_D(\beta^{-1})(\rho_D(\alpha^{-1})_0), \epsilon_0 \rangle = \langle \rho_D(\alpha^{-1})_0, \epsilon_0 \rangle \cdot \langle \rho_D(\beta^{-1})_0, \epsilon_0 \rangle.
\]

Thus \( \Phi_D^{rs} \) becomes

\[
\Phi_D^{rs}(f) = \left\langle \sum_{\beta \in B} \left\langle \sum_{\alpha \in A} f_{k|\alpha|_{k}\beta} \rho_D(\alpha^{-1})_0, \epsilon_0 \right\rangle , \beta \rho_D(\beta^{-1})_0, \epsilon_0 \right\rangle
= \langle \mathcal{L}_D^{N/s}(f), \epsilon_0 \rangle = (\Phi_D^r \circ \Phi_D^s)(f).
\]

To rewrite the sum over \( B \) into \( \mathcal{L}^{N/s} \) we had to use that \( B \) is a system of coset representatives of \( \Gamma_0(N) \backslash \Gamma_0(n) \).

6 Computation of partial liftings for prime powers

Let \( D \) be a discriminant form of even signature and level \( N \). Write \( N = qm \) with \( q = p^r \) being some prime power and \( (p, m) = 1 \). Then \( D = D_q \oplus D_m \).

Lemma 6.1. A set of coset representatives for the quotient \( \Gamma_0(N) \backslash \Gamma_0(m) \) is given by the elements \( \alpha_0, \ldots, \alpha_{q/p-1}, \beta_0, \ldots, \beta_{q-1} \) where

\[
\alpha_j = -ST^{-pjm}S = \begin{pmatrix} 1 & 0 \\ pjm & 1 \end{pmatrix} \quad \text{and} \quad \beta_l = -ST^{-m}ST^l = \begin{pmatrix} 1 & l \\ m & ml + 1 \end{pmatrix} .
\]

Proof. It is well-known that \( [\text{SL}_2(\mathbb{Z}) : \Gamma_0(M)] = M \prod_{\ell|M}(1 + 1/\ell) \) where the product runs over all primes \( \ell \) dividing \( M \). Hence \( [\Gamma_0(m) : \Gamma_0(N)] = q(1 + 1/p) \) which agrees with the number of given representatives. Thus it remains to check that these matrices indeed represent different cosets which is an easy calculation.

Lemma 6.2. Let \( \alpha_0, \ldots, \alpha_{q/p-1} \) and \( \beta_0, \ldots, \beta_{q-1} \) be the representatives given in the previous lemma. Then

\[
\langle \rho_D(\alpha_j)^{-1}\epsilon_0, \epsilon_0 \rangle = \frac{G_{D_q}(pjm)}{|D_q|} \quad \text{and} \quad \langle \rho_D(\beta_l)^{-1}\epsilon_0, \epsilon_0 \rangle = \frac{G_{D_q}(m)}{|D_q|} .
\]

where \( G_{D_q} \) is the Gauss sum of \( D_q \) defined in Section 3.
Proof. A straightforward computation using the definitions of $\rho_D(S)$ and $\rho_D(T)$ and the usual orthogonality relations for characters shows
\[
\langle \rho_D(\alpha_j)^{-1}e_0, e_0 \rangle = \frac{G_D(pjm)}{|D|} \quad \text{and} \quad \langle \rho_D(\beta_l)^{-1}e_0, e_0 \rangle = \frac{G_D(m)}{|D|}.
\]
Since $|D| = |D_q| \cdot |D_m|$ and $G_D(cm) = G_{D_q}(cm) \cdot G_{D_m}(cm) = G_{D_q}(cm) \cdot |D_m|$ for $c = pj$ and $c = 1$ we obtain the claimed formulas.

Using the previous two lemmas we see for $f \in M_k^1(N, \chi_D)$ that
\[
\Phi'_{\beta}(f) = \frac{1}{|D_q|} \sum_{j=0}^{q/p-1} G_{D_q}(pjm) \cdot f|\alpha_j + \frac{G_{D_q}(m)}{|D_q|} \sum_{l=0}^{q-1} f|\beta_l. \tag{11}
\]

**Proposition 6.3.** Let $\beta_0, \ldots, \beta_q$ be as in Lemma 6.1 and let $f \in M_k^1(N, \chi_D)$. Then
\[
\sum_{l=0}^{q-1} f|\beta_l = \chi_q(-m)q^{1-k/2}(T_q \circ W_q)(f) \in M_k^1(N, \chi_D).
\]

**Proof.** Fix a choice of integers $z, w$ with $qw - mz = 1$ and define $w_0 := \left(\begin{smallmatrix} q & 1 \\ 1 & 0 \end{smallmatrix}\right)$ as in Section 2. A direct computation shows that
\[
M_l := \beta_l T^z \left(\begin{smallmatrix} 1 & l \\ q & 0 \end{smallmatrix}\right)^{-1} \omega_q^{-1} \left(\begin{smallmatrix} 1/q & 0 \\ 0 & 1/q \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} * & * \\ * & qw-m \end{smallmatrix}\right) \in \Gamma_0(N).
\]

Moreover,
\[
\chi_D(M_l) = \chi_D(qw-m) = \chi_q(qw-m)\chi_m(qw-m) = \chi_q(-m)
\]
since $qw-m \equiv 1 \mod m$ and $qw-m \equiv -m \mod q$. Let $f \in M_k^1(N, \chi_D)$. Then
\[
\sum_{l=0}^{q-1} f|\beta_l = \left(\sum_{l=0}^{q-1} \chi_D(M_l) \cdot W_q(f)|_k \left(\begin{smallmatrix} 1 & l \\ 0 & q \end{smallmatrix}\right)\right)|_{T^z} = \chi_q(-m)q^{1-k/2}(T_q \circ W_q)(f)|_{T^z}
\]
as $\left(\begin{smallmatrix} q & 0 \\ 0 & q \end{smallmatrix}\right)$ acts trivially and $T_q(g) = q^{k/2-1} \sum_{l=0}^{q-1} g|_k \left(\begin{smallmatrix} 1 & l \\ 0 & q \end{smallmatrix}\right)$ by definition. This proves the proposition since $T$ acts as the identity on $(T_q \circ W_q)(f)$. \hfill $\square$

**Proposition 6.4.** Let $\alpha_0, \ldots, \alpha_{q/p-1}$ be as in Lemma 6.1 and let $f \in M_k^1(N, \chi_D)$. Then
\[
\sum_{j=0}^{q/p-1} G_{D_q}(pjm) f|\alpha_j = (-1)^k H_N \left(\sum_{n \in \mathbb{Z}} a_q(n) \mu(n; D_q, m)e(\tau n)\right) \in M_k^1(N, \chi_D)
\]
where $g := H_N(f) = \sum_{n \in \mathbb{Z}} a_q(n)e(\tau n)$ is the Fricke involution of $f$ and
\[
\mu(n; D_q, m) = \sum_{j=0}^{q/p-1} G_{D_q}(pjm)e(-pjn/q).
\]
Proof. Let \( h_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix} \) and \( g := H_N(f) = f |_k h_N = \sum_{n \in \mathbb{Z}} a_g(n) e(\tau n) \). Then \( g \in M^1_k(N, \chi_D) \) since \( \chi_D \) is real. Further, a direct computation shows that

\[
 h_N \alpha_j h_N = \begin{pmatrix} -N & 0 \\ 0 & -N \end{pmatrix} \begin{pmatrix} 1 -pj/q \\ 0 & 1 \end{pmatrix}.
\]

As \( h_N^2 = \begin{pmatrix} -N & 0 \\ 0 & -N \end{pmatrix} \) acts as a multiplication by \((-1)^k\) we may write \( f |_k \alpha_j = f |_k h_N^2 \alpha_j h_N^2 \).

This gives the stated formula. \( \Box \)

7 Partial liftings of newforms

As before let \( D \) be a discriminant form of even signature and level \( N = qm \) with \( q = p^r \), \( p \) prime and \((p, m) = 1\). From now on we will focus on the action of \( \Phi^p_D \) on the space of newforms \( S^\text{new}_k(N, \chi_D) \). It turns out to be reasonable to consider the cases \( p \neq 2 \) and \( p = 2 \) separately.

7.1 Powers of odd primes

Let \( q = p^r \) be a power of an odd prime \( p \). Then \( \chi_q = \left( \frac{\cdot}{D_q} \right) \). Let

\[
 D_q \cong \bigoplus_{j=1}^{r} (p^j)^{\pm n_j}
\]

with non-negative integers \( n_j \) be a Jordan decomposition of \( D_q \). In the special case \( q = p \) the discriminant form \( D_q \) is of the form \( p^{\pm n} \) and we have \( \chi_p = \left( \frac{\cdot}{p} \right)^{n_r} \).

Theorem 7.1. Let \( f \in S^\text{new}_k(N, \chi_D) \) and define \( \psi := \left( \frac{\cdot}{p} \right)^{n_r} \).

1. For \( q = p \) we have

\[
 \Phi^p_D(f) = f + \frac{\mathcal{G}_{D_q}(1)}{|D_p|} \psi(-1)p^{1-k/2}(T_p \circ W_p)(f).
\]
2. For \( q = p^r \) with \( r \geq 2 \) we have

\[
\Phi_D^q(f) = f + \frac{G_{D_q}(p^{r-1})}{|D_q|} \psi(-m)G(\psi)(-1)^k H_N((H_N(f))_\psi)
\]

where \( G(\psi) = \sum_{j=1}^{p-1} \psi(j)e(j/p) \) denotes the usual Gauss sum of \( \psi \), and \( g_\psi \) denotes the twist of \( g \) by \( \psi \), that is \( g_\psi = \sum_{n \geq 1} a_g(n)\psi(n)e(\tau n) \).

**Proof.** First, let \( q = p \). The contribution to \( \Phi_D^q \) coming from Proposition 6.4 is just \( f \) since \( G_{D_p}(0) = |D_p| \). Adding the expression given in Proposition 6.3 we obtain the first formula. Note that \( G_{D_p}(m) = (\frac{m}{p})^n G_{D_p}(1) \) by Proposition 3.1 and \( \psi = \chi_D = (\frac{\cdot}{p})^n \).

Now let \( q = p^r \) with \( r \geq 2 \). Write \( g = \sum_{n \geq 1} a_g(n)e(\tau n) = H_N(f) \). Proposition 2.1 states that \( a_g(n) = 0 \) for all \( n \) being divisible by \( p \), and that \( (T_q \circ W_q)(f) = 0 \) as \( W_q(f) \) is a newform of level \( N \) and character \( \chi_D \). So applying Proposition 6.3 and Proposition 6.4 to Eq. (11) yields

\[
\Phi_D^q(f) = \frac{(-1)^k}{|D_q|} H_N \left( \sum_{n \geq 1} a_g(n)\mu(n; D_q, m)e(\tau n) \right). \tag{13}
\]

Let \( n \geq 1 \) with \( (n, p) = 1 \). Then

\[
\mu(n; D_q, m) = \sum_{j=0}^{q/p-1} G_{D_q}(pjn)e(-pjn/q) = |D_q| + \sum_{s=1}^{r-1} \sum_{\ell=0}^{p-1} G_{D_q}(p^s \ell m)e(-\ell n/p^{r-s}).
\]

By Proposition 3.1 we can write \( G_{D_q}(p^s \ell m) = \psi_s(\ell m) G_{D_q}(p^s) \) with

\[
\psi_s(\ell) := \prod_{j=s+1} \left( \frac{\ell}{p^j} \right)^{n_j},
\]

thus giving

\[
\mu(n; D_q, m) = |D_q| + \sum_{s=1}^{r-1} \psi_s(m) G_{D_q}(p^s) \sum_{\ell=0}^{p^{r-s}-1} \psi_s(\ell)e(-\ell n/p^{r-s}). \tag{14}
\]

Using that \( \psi_s \) has conductor 1 or \( p \) one can easily see that the inner sum vanishes unless \( s = r - 1 \), in which case it equals the usual Gauss sum

\[
\sum_{\ell=1}^{p-1} \psi_{r-1}(\ell)e(-\ell n/p) = \psi(-n) G(\psi)
\]

of \( \psi = \psi_{r-1} = (\frac{\cdot}{p})^n \). Therefore we finally obtain

\[
\mu(n; D_q, m) = |D_q| + \psi(-m) G_{D_q}(p^{r-1}) \psi(n) G(\psi).
\]

If we insert this expression into (13) we are done. \( \square \)
Corollary 7.2. If \( n_r \) is even then

\[
\Phi_D^g(f) = \left(1 - \frac{G_{D_q}(p^r - 1)}{|D_q|}\right)f
\]

for every \( f \in S_k^{\text{new}}(N, \chi_D) \). In particular, \( \Phi_D^g \) defines an isomorphism of \( S_k(N, \chi_D) \).

Proof. Let \( n_r \) be even. Then \( \psi = (\frac{\cdot}{p})^{n_r} \) is trivial. If \( q = p \) then the \( p \)-component \( \chi_p = \psi \) of \( \chi_D \) is trivial and thus we have \( (T_p \circ W_p)(f) = -p^{k/2-1}f \) for \( f \in S_k^{\text{new}}(N, \chi_D) \) by Proposition 2.2. This gives the claimed formula.

If \( q = p^r \) with \( r \geq 2 \) then \( (H_N(f))_\psi = H_N(f) \) since \( \psi \) is trivial and since Proposition 2.1 implies that \( a_{H_N(f)}(n) = 0 \) for all \( n \) being divisible by \( p \). So the formula follows as \( G(\psi) = -1 \) and \((-1)^k H_N^2 \) acts as the identity.

It remains to note that we have \( |G_{D_q}(p^r - 1)| / |D_q| = p^{-nr/2} \) in both cases, that is the single eigenvalue of \( \Phi_D^g \) is non-zero. So \( \Phi_D^g \) is an isomorphism of \( S_k(N, \chi_D) \).

Proposition 7.3. 1. If \( n_r \) is odd and either \( q = p \) or \( q = p^r \) with \( r \geq 3 \), then \( \Phi_D^g \) still defines an endomorphism of \( S_k^{\text{new}}(N, \chi_D) \).

2. If \( n_r \) is odd with \( n_r \neq 1 \) then \( \Phi_D^g \) is injective on \( S_k^{\text{new}}(N, \chi_D) \).

Proof. Let \( n_r \) be odd and \( \psi = (\frac{\cdot}{p}) \). For (1) we only have to note that the operators \( T_p, W_p \) and \( H_N \) preserve newforms, and that if \( q = p^r \) with \( r \geq 3 \) the twist of a newform by \( \psi \) is still a newform. (This follows from Theorem 3.12 in [8] as \( e(\chi) = 1, e(\omega) = 0, 1 \) and \( v \geq 3 \) in the situation of the theorem.)

It remains to prove (2): If \( q = p^r \) with \( r \geq 2 \) and \( f \in S_k^{\text{new}}(N, \chi_D) \) with \( \Phi_D^q(f) = 0 \) then part (2) of Theorem 7.1 implies \( g = -\lambda g_\psi \) where

\[
\lambda := \frac{G_{D_q}(p^r - 1)}{|D_q|} \psi(-m)G(\psi)
\]

and \( g := H_N(f) \). Suppose that \( g \neq 0 \). Comparing the first non-zero Fourier coefficient on both sides of \( g = -\lambda g_\psi \) we obtain \( |\lambda| = 1 \). On the other hand, a direct computation using Proposition 3.3 in [10] to evaluate the Gauss sum \( G_{D_q}(p^r - 1) \) and the well-known fact that \( |G(\psi)| = p^{1/2} \) yields \( |\lambda| = p^{(1-nr)/2} \). If \( n_r \neq 1 \) this is a contradiction. So we have \( g = 0 \) and thus \( f = 0 \). Hence \( \Phi_D^g \) is injective.

For the case \( q = p \) let \( f \in S_k^{\text{new}}(N, \chi_D) \) be primitive. Recall from Section 2 that \( W_p(f) = a_p(f) f^{(p)} \) where \( f^{(p)} = \sum_{n \geq 1} b(n) \tau(n) \) is defined as in (1) and \( a_p(f) = p^{k/2-1} G(\psi) a_f(p)^{-1} \) since \( a_f(p) \neq 0 \) by part (1) of Theorem 4.6.17 in [9]. Hence we have

\[
\Phi_D^g(f) = f + \lambda \frac{b(p)}{a_f(p)} f^{(p)}
\]

and \( \lambda := G_{D_p}(1) |D_p|^{-1} \psi(-1) G(\psi) \). It is easy to see that \( (f^{(p)})^{(p)} = f \). Therefore we obtain

\[
\Phi_D^g(f^{(p)}) = f^{(p)} + \lambda \frac{a_f(p)}{b(p)} f.
\]
Example 7.5. Let \( D = 9^{+1} \) be a Jordan component. Then \( \text{sign}(D) \) is even and \( \chi_D \) is trivial. Put \( N = 9 \). The space \( S_6^{\text{new}}(N, \chi_D) \) is generated by the primitive form \( f = q + 6q^2 + O(q^4) \) where \( q = e^{2\pi i r} \). We have shown that
\[
\Phi_D^6(f) = f + \lambda H_N((H_N(f))_\psi)
\]
for some non-zero constant \( \lambda \in \mathbb{C}^* \) where \( \psi = (\frac{3}{7}) \). As \( S_6^{\text{new}}(N, \chi_D) \) is 1-dimensional we have \( H_N(f) = \mu f \) for some \( \mu \in \mathbb{C}^* \). So \( (H_N(f))_\psi = \mu(q - 6q^2 + O(q^4)) \) is not a multiple of \( f \), hence not a newform. Therefore \( H_N((H_N(f))_\psi) \) and \( \Phi_D^6(f) \) cannot be newforms either. So \( \Phi_D^6 \) is not an endomorphism of \( S_6^{\text{new}}(N, \chi_D) \).

Example 7.6. Let \( D = 3^{+1} \) be a Jordan component. Using the notation of \([10]\), Section 2, we have 3-excess\((D) = 2 \mod 8\), \( p\)-excess\((D) = 0 \mod 8 \) for all primes \( p > 3 \) and oddity\((D) = 0 \mod 8 \). So the oddity-formula yields \( \text{sign}(D) \equiv -(3\text{-excess}(D)) \equiv 6 \mod 8 \). Further, we have \( \chi_D = (\frac{3}{7}) \). Put \( N = 3 \). By Theorem 7.1 we have
\[
\Phi_D^3(f) = f + i3^{(k-1)/2}(T_3 \circ W_3)(f)
\]
as \( G_D(1) = e(\text{sign}(D)/8)\sqrt{|D|} \) by Milgram’s formula. Let \( k = 7 \). The space \( S_7^{\text{new}}(N, \chi_D) \) is generated by the primitive form \( f = q - 27q^3 + O(q^4) \). Hence \( (T_3 \circ W_3)(f) = i3^{(k-1)/2}f \) since \( G(\chi_D) = i\sqrt{3} \) and \( f^{(3)} = f \) as \( S_7^{\text{new}}(N, \chi_D) \) is 1-dimensional. So \( \Phi_D^3(f) = 0 \) and thus \( \Phi_D^3(f) \) vanishes on \( S_7^{\text{new}}(N, \chi_D) \).

7.2 Powers of 2

Let now \( q = 2^r \) be a power of \( p = 2 \). If \( q = 2 \) then \( D \) does not contain any odd 2-adic Jordan components and we may proceed as in the previous subsection. Moreover, the 2-component \( \chi_2 = (\frac{1}{2^n}) \) of \( \chi_D \) will always be trivial since \( |D_2| = 2^{2n} \) for some positive integer \( n \). So Proposition 2.2 applies and thus we obtain:

Corollary 7.7. If \( N = 2m \) with \((2, m) = 1\) then
\[
\Phi_D^2(f) = \left(1 - \frac{G_D(1)}{|D_2|}\right) f
\]
for every \( f \in S_k^{\text{new}}(N, \chi_D) \). In particular, \(|G_{D_2}(1)|/|D_2| = 2^{-n}\) implies that \( \Phi_D^2 \) defines an isomorphism of \( S_k^{\text{new}}(N, \chi_D) \).

Next let \( q = 2^r \) with \( r \geq 2 \). Using the formula given in (3) it is not difficult to see that \( \chi_q = (\frac{\cdot}{q}) \varepsilon_D \) where \( \varepsilon_D \) is the trivial character if \( \text{oddity}(D) \equiv 0 \mod 4 \), and \( \varepsilon_D \) is the (unique) primitive character mod 4 if \( \text{oddity}(D) \equiv 2 \mod 4 \). Let

\[
D_q \cong \bigoplus_{j=1}^{r_1} (2^j)^{+2n_j} \oplus \bigoplus_{j=1}^{r_2} (2^j)^{1+m_j},
\]

be a Jordan decomposition of \( D_q \) with non-negative integers \( n_j, m_j \) and \( n_{r_1}, m_{r_2} \geq 1 \). Note that \( r = \max(r_1, r_2 + 1) \). As in the proof of Theorem 7.1 we want to apply Proposition 2.1. This is possible if the conductor \( d_q \) of \( \chi_q \) is smaller than \( q \). On the other hand, if \( d_q \geq q \) we may generalise part (1) of Theorem 7.1:

**Proposition 7.8.** Let \( d_q \geq q = 2^r \) with \( r \geq 2 \). Then either \( q = d_q = 4 \) or \( q = d_q = 8 \), and in both cases we have

\[
\Phi_D^2(f) = f + \frac{G_{D_q}(1)}{|D_q|} \chi_q(-1)q^{1-k/2}(T_q \circ W_q)(f)
\]

for every \( f \in S_k^{\text{new}}(N, \chi_D) \). In particular, \( \Phi_D^2 \) defines an endomorphism of \( S_k^{\text{new}}(N, \chi_D) \).

**Proof.** The conductor \( d_q \) of \( \chi_q \) is either 1, 4 or 8. Since \( q \geq 4 \) we have \( d_q \neq 1 \). Let \( d_q = 4 \). Then \( \chi_q = \varepsilon_D \) is primitive mod 4 and \( q = 4 \). Hence there is exactly one odd Jordan component \( 2_{t_1}^{+m_1} \). Moreover, we have oddity\((D) \equiv m_1 \equiv t_1 \equiv 2 \mod 4 \) as \( \varepsilon_D(a) = e((a-1) \text{ oddity}(D)/8) \) is non-trivial.

The factor \( \mu(n; D_q, m) \) from Proposition 6.4 simplifies to

\[
\mu(n; D_q, m) = |D_q| + G_{D_q}(2m)e(-n/2).
\]

By [10], Proposition 3.5, the Gauss sum of \( 2_{t_1}^{+m_1} \) vanishes for \( 2m \), so \( G_{D_q}(2m) = 0 \). In order to obtain the claimed formula from (11) it remains to note that we still have \( G_{D_q}(m) = \chi_q(m)G_{D_q}(1) \) by Proposition 3.1.

Next let \( d_q = 8 \). If \( q = 4 \) there can be at most one odd Jordan component \( 2_{t_1}^{+m_1} \). If there is no such component then \( \chi_q \) is trivial, and if there is one then \( m_1 \) needs to be even as the signature of \( D \) is even by assumption, and thus \( \chi_q \) is of conductor at most 4. So let \( q = 8 \). Then we have exactly two odd Jordan components, namely \( 2_{t_1}^{+m_1} \) and \( 4_{t_2}^{+m_2} \), since otherwise \( d_q \) is at most 4 as before. Remark that \( m_1 \) and thus \( m_2 \) need to be odd in order to give a character \( \chi_8 \) of conductor 8. Next we have

\[
\mu(n; D_q, m) = |D_q| + G_{D_q}(4m)e(-n/2) + G_{D_q}(2m)e(-n/4) + G_{D_q}(6m)e(-3n/4).
\]

Again all Gauss sums vanish by [10], Proposition 3.5, and \( G_{D_q}(m) = \chi_8(m)G_{D_q}(1) \). Therefore the claimed formula holds.
Using exactly the same arguments as in Proposition 7.3 together with Proposition 3.5 and 3.6 in [10] one obtains the following statement:

**Proposition 7.9.**

1. Let $q = d_q = 4$. If the number of even and odd 2-adic Jordan components of $D_4$ is not minimal (that is we do not have $m_1 = 2$ and $n_1 = n_2 = 0$) then $\Phi_D^4$ defines an isomorphism of $S_{k}^{\text{new}}(N, \chi_D)$.

2. Let $q = d_q = 8$. If the number of even and odd 2-adic Jordan components of $D_8$ is not minimal (that is we do not have $m_1 = m_2 = 1$ and $n_1 = n_2 = n_3 = 0$) then $\Phi_D^8$ defines an isomorphism of $S_{k}^{\text{new}}(N, \chi_D)$.

Next we consider the cases in which Proposition 2.1 applies. Since the signature of an odd 2-adic Jordan component does not need to be even, the general case is quite messy. Therefore we make further assumptions in order to obtain a nice statement.

**Theorem 7.10.** Let $d_q < q = 2^r$ with $r \geq 2$ and $f \in S_{k}^{\text{new}}(N, \chi_D)$. Further, we assume that the signature of all odd 2-adic Jordan components of $D_q$ is even, which is equivalent to $m_1, \ldots, m_{r_2}$ being even by the oddity formula.

1. If $r = r_1 > r_2 + 1$, or if $m_{r_2} \equiv 0$ mod 4 then

$$\Phi_D^r(f) = \left(1 - \frac{G_{D_q}(2^{r-1})}{|D_q|}\right)f$$

where $G_{D_q}(p^{r-1}) = 0$ if $r = r_2 + 1$. In particular, $\Phi_D^r$ defines an isomorphism of $S_{k}^{\text{new}}(N, \chi_D)$.

2. If $r = r_2 + 1 \geq r_1$ and $m_{r_2} \equiv 2$ mod 4 then

$$\Phi_D^r(f) = f + \frac{G_{D_q}(2^{r-2})}{|D_q|} \psi(-m)G(\psi)(-1)^k H_N((H_N(f))_\psi)$$

where $\psi$ is primitive mod 4 and $G(\psi) = \sum_{j=0}^{3} \psi(j)e(j/p) = 2i$. If $r \geq 5$ then $\Phi_D^r$ is an endomorphism of $S_{k}^{\text{new}}(N, \chi_D)$, and if we do not have $r_2 + 1 = r_1$ and $m_{r_2} = 2$ then $\Phi_D^r$ is injective.

**Proof.** Since all $m_j$ are even, $|D_q|$ is a square and thus $\chi_q = \varepsilon_D$ is of conductor 1 or 4. Further, we may apply Proposition 2.1 as $q > d_q$ by assumption. So there is no contribution of the $\beta_i$’s in (11) as $T_q(W_q(f)) = 0$, and we only have to consider factors $\mu(n; D_q, m)$ for odd $n$. By Proposition 3.1 we have $G_{D_q}(2^s lm) = \psi_s(km)G_{D_q}(2^s)$ for odd $l$ and $1 \leq s \leq r - 1$ with

$$\psi_s(a) := \prod_{j=s+1}^{r_2} \chi_{(2j-s)l_j}^{\pm m_j}(a).$$ (16)

Note that $(2j-s)l_j^{\pm m_j}$ is a discriminant form of even signature as we assume that $m_j$ is even. Moreover, the character $\psi_s$ is of conductor 1 or 4 for the same reason as $\chi_q$ is. Next we
observe that \( \psi_s \) is trivial for \( r_2 \leq s \leq r - 1 \) and \( \mathcal{G}_{D_q}(2^{r_2}) = 0 \) by [10], Proposition 3.5. Thus Eq. (14) becomes

\[
\mu(n; D_q, m) = |D_q| + \sum_{s=1}^{r_2-1} \psi_s(m) \mathcal{G}_{D_q}(2^s) \sum_{\ell=0}^{2^{r-s}-1} \psi_s(\ell) e(-\ell n/2^{r-s}) \tag{17}
\]

\[
+ \sum_{s=r_2+1}^{r-1} \mathcal{G}_{D_q}(2^s) \sum_{\ell=0}^{2^{r-s}-1} e(-\ell n/2^{r-s}).
\]

The first inner sum vanishes for \( r - s \geq 3 \) as the conductor of \( \psi_s \) is at most 4, and the second inner sum vanishes for \( r - s \geq 2 \). We distinguish the following two cases:

1. Let \( r = r_1 > r_2 + 1 \). Then the first sum vanishes completely and we are left with the last summand of the second sum: \( \mu(n; D_q, m) = |D_q| - \mathcal{G}_{D_q}(2^{r-1}) \). So we are indeed in the first case of the theorem.

2. Suppose that \( r = r_2 + 1 \geq r_1 \). Then the second sum vanishes and only the last summand of the first sum remains:

\[
\mu(n; D_q, m) = |D_q| + \psi_{r-2}(m) \mathcal{G}_{D_q}(2^{r-2}) [e(-n/4) + \psi_{r-2}(-1)e(n/4)].
\]

By (16) we know that \( \psi_{r-2} \) is trivial if \( m_{r_2} \equiv 0 \text{ mod } 4 \) and primitive mod 4 if \( m_{r_2} \equiv 2 \text{ mod } 4 \). In the latter case we obtain part (2) of the theorem, and otherwise it remains to note that \( \mathcal{G}_{D_q}(2^{r-1}) = 0 \) as mentioned above in order to obtain part (1) again.

In order to show that \( \Phi^f_{D} \) is an endomorphism or injective we may argue as in Proposition 7.3, now using Proposition 3.5 and 3.6 in [10] instead of Proposition 3.3.

If we omit the assumption that \( m_1, \ldots, m_{r_2} \) are even integers, the functions \( \psi_s \) defined above might not be characters anymore. However, if we assume that the even 2-adic Jordan components “dominate” the odd 2-adic Jordan components in the following sense we recover part (1) of the previous theorem:

**Proposition 7.11.** Let \( d_q < q = 2^r \) with \( r \geq 2 \). If \( r = r_1 > r_2 + 2 \) then

\[
\Phi^f_{D}(f) = \left( 1 - \frac{\mathcal{G}_{D_q}(2^{r-1})}{|D_q|} \right) f.
\]

for every \( f \in S^\text{new}_k(N, \chi_D) \). In particular, \( \Phi^f_{D} \) defines an isomorphism of \( S^\text{new}_k(N, \chi_D) \).

**Proof.** We may almost completely follow the proof of the previous theorem. Note that for \( 1 \leq s \leq r - 1 \) the function

\[
\psi_s(a) := \prod_{j=s+1}^{r_2} \left( \frac{a}{2^{j-s}} \right)^{m_j} e\left( (a - 1) \text{ oddity} \left( \frac{(2^{j-s})^{\pm m_j}}{2} \right) /8 \right)
\]

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will in general not be a character. Nevertheless $\psi_s$ is well-defined modulo 8 and thus the inner sum in (17) still vanishes for $r - s \geq 4$. If $r = r_1 > r_2 + 2$ this is the case for $s = 1, \ldots, r_2 - 1$. So we may proceed as before.

It remains to note that $|G_{Dq}(p^{r-1})|/|D_q| = 2^{-n_r} \neq 1$. So $\Phi^q_D$ indeed defines an isomorphism. \hfill $\square$

8 Summary and an application

Given a particular nice situation we may summarise our results using Theorem 5.2:

**Theorem 8.1.** Let $D$ be a discriminant form of even signature and level $N$. Assume that for every odd prime power $q$ exactly dividing $N$ the exponent $n_r$ in the Jordan decomposition (12) of $D_q$ is even. If $N$ is even we further assume that either all 2-adic Jordan components of $D$ are even, or that the even 2-adic Jordan components ‘dominate’ the odd ones as in Proposition 7.11. Then

$$\Phi_D(f) = \prod_{q\|N} \left(1 - \frac{G_{Dq}(q/p)}{|D_q|}\right) \cdot f$$

for every $f \in S^\text{new}_k(N, \chi_D)$ where the product runs over all prime powers $q = p^r$ exactly dividing $N$, including $p = 2$.

**Remark 1.** The statement of the previous theorem also holds for various other discriminant forms whose Jordan decomposition involve odd 2-adic Jordan components (compare for example part (1) of Theorem 7.10).

**Example 8.2.** Let $N$ be a positive integer and let $L = H_{1,1}(N)$ denote a rescaled hyperbolic plane, that is a 2-dimensional lattice having basis vectors $e, f$ with $\langle e, f \rangle = N$ and norm $q(e) = q(f) = 0$. The corresponding discriminant form $D = L'/L$ has order $|D| = N^2$, signature 0 mod 8 and level $N$. In particular, the character $\chi_D$ is trivial.

If 2 divides $N$ then the only 2-adic Jordan component of $D$ is $(2^r)\sqrt{N}$ where $2^r$ exactly divides $N$. The other Jordan components of $D$ are $q^\pm 2$ with $(\frac{-1}{p}) = \pm 1$ for odd prime powers $q = p^r$ exactly dividing $N$. Hence Theorem 8.1 is applicable, that is we have

$$\Phi_D(f) = \prod_{q\|N} \left(1 - \frac{G_{Dq}(q/p)}{|D_q|}\right) \cdot f$$

for every $f \in S^\text{new}_k(N, \chi_D)$. The Gauss sums can be computed with the formulas from Propositions 3.3 and 3.6 in [10] giving $G_{Dq}(q/p) = q^2/p$. As $|D_q| = q^2$ we obtain

$$\Phi_D(f) = \prod_{p|N} \left(1 - \frac{1}{p}\right) \cdot f = \frac{\varphi(N)}{N} f$$

where the product runs over all primes dividing $N$ and $\varphi$ denotes Euler’s $\varphi$ function.
The following application was proposed by J. H. Bruinier, and was in fact one of the main motivations for the considerations of this work (compare [5], Sections 4 and 8):

Let \( N \) be an arbitrary positive integer and let \( L = \mathcal{H}_{1,1}(1) \oplus \mathcal{H}_{1,1}(N) \) be a lattice of signature \((2,2)\) where \( \mathcal{H}_{1,1}(N) \) denotes a rescaled hyperbolic plane as in the previous example. Then the discriminant form \( D = L'/L \) is isomorphic to the one given in the previous example and thus we obtain \( \Phi_D(f) = \frac{\varphi(N)}{N} f \) for every \( f \in S_{k}^{\text{new}}(N, \chi_D) \).

Let \( V_+ \) be a 2-dimensional positive definite definite subspace of \( V := \mathbb{L} \otimes \mathbb{Q} \) with negative definite complement \( U := V_+^\perp \), and let \( \mathcal{P} := L \cap V_+ \) and \( \mathcal{N} := L \cap U \) be definite sublattices of \( L \). Put \( M := \mathcal{P} \oplus \mathcal{N} \). In general \( M \) is only a sublattice of \( L \) of finite index. The inclusions \( M \subseteq L \subseteq L' \subseteq M' \) show \( L/M \subseteq L'/M \subseteq M'/M \). Further, we have a natural map

\[
L'/M \to L'/L, \quad \gamma = x + M \mapsto \overline{\gamma} = x + L
\]

which gives rise to a map sending modular forms \( F \) for the Weil representation \( \rho_{L'/L} \) to modular forms \( F_M \) for \( \rho_{M'/M} \) defined by

\[
(F_M)_\gamma = \begin{cases} 
F_{\overline{\gamma}}, & \text{if } \gamma \in L'/M, \\
0, & \text{if } \gamma \notin L'/M,
\end{cases}
\]

for \( \gamma \in M'/M \), see Lemma 3.1 in [5].

Let \( F \) be a vector valued cusp form for \( \rho_{L'/L} \) of weight \( 1 + b^-/2 = 2 \), and let \( b(n, \gamma) \) with \( \gamma \in M'/M \) and \( Q(\gamma) \equiv n \mod 1 \) denote the coefficients of \( F_M \). Identifying \( M'/M \) with \( \mathcal{P}'/\mathcal{P} \oplus \mathcal{N}'/\mathcal{N} \) we may view \( \mathcal{P}'/\mathcal{P} \) as a subgroup of \( M'/M \). In [5], Eq. (4.24), Bruinier and Yang define an \( L \)-function of \( F_M \) by

\[
L(F_M, U, s) = (4\pi)^{-(s+2)/2} \Gamma\left(\frac{s+2}{2}\right) \sum_{n \geq 1} \sum_{\gamma \in \mathcal{P}'/\mathcal{P}} r(n, \gamma) b(n, \gamma) n^{-(s+2)/2}
\]

where \( r(n, \gamma) \) are the coefficients of the vector valued theta series of \( \mathcal{P} \), that is

\[
\Theta_\mathcal{P}(\tau) = \sum_{x \in \mathcal{P}'} e(\tau q(x)) \epsilon_{\mathcal{P}'} = \sum_{\gamma \in \mathcal{P}'/\mathcal{P}} \sum_{n \geq 0} r(n, \gamma) e(\tau n) \epsilon_{\gamma}.
\]

Note that the zero component of \( \Theta_\mathcal{P} \) equals the usual scalar valued theta function

\[
\theta_\mathcal{P}(\tau) = \sum_{x \in \mathcal{P}} e(\tau q(x)) = \sum_{n \geq 0} r(n) e(\tau n).
\]

For simplicity, we drop the subscript \( M \) in \( L(F_M, U, s) \), writing \( L(F, U, s) \) instead.

Let now \( F = L_D(f) \) be the lifting of a primitive form \( f \in S_2^{\text{new}}(N, \chi_D) \). Further, we assume that the discriminant \( d \) of \( \mathcal{P} \) is an odd (and therefore squarefree) negative fundamental discriminant which is coprime to \( N \). Then every element in \( \mathcal{P}'/\mathcal{P} \cap L'/M \) has

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order dividing \((d, N) = 1\), and thus the intersection equals 0. So in (22) only the summand for \(\gamma = 0\) remains, giving

\[
L(\mathcal{L}_D(f), U, s) = (4\pi)^{-s/2} \Gamma\left(\frac{s + 2}{2}\right) \frac{\varphi(N)}{N} \sum_{n \geq 1} r(n)a_f(n)n^{-(s+2)/2}.
\] (25)

(Note that the Fourier coefficients \(a_f(n)\) of \(f\) are real since the character \(\chi_D\) is trivial.) As in [7] we assume that there are Heegner points of discriminant \(d\) on the curve \(X_0(N)\) which is equivalent to saying that every prime divisor of \(N\) splits in \(K := \mathbb{Q}(\sqrt{d})\). So \((\frac{d}{p}) = 1\) for every prime \(p\) dividing \(N\). Then

\[
L\left(\left(\frac{d}{\cdot}\right), s\right) = \prod_{p | N} (1 - p^{-s})^{-1} \sum_{n \geq 1 \atop (n, dN) = 1} \left(\frac{d}{n}\right) n^{-s}.
\]

On the other hand the Dirichlet class number formula states that

\[
L\left(\left(\frac{d}{\cdot}\right), 1\right) = \frac{2\pi h(d)}{\omega(d)\sqrt{|d|}}
\]

where \(h(d)\) is the class number of \(K\) and \(\omega(d)\) is the order of the unit group of \(\mathcal{O}_K\). Moreover, it is not difficult to see that

\[
\theta_A(\tau) = \frac{1}{\omega(d)} \theta_P(\tau)
\]

where \(A\) is the ideal class associated to \(P\) and \(\theta_A\) is the theta function given in [7], equation (5.2). Taking derivatives on both sides of (25) and plugging in \(s = 0\) we obtain

\[
L'(\mathcal{L}_D(f), U, 0) = \frac{\omega(d)^2 \sqrt{|d|}}{8\pi^2 h(d)} L'_A(f, 1)
\] (26)

where \(L_A(f, s)\) is the \(L\)-function associated to the newform \(f\) and the ideal class \(A\) as in [7], equation (5.3). Here we used that \(L(\mathcal{L}_D(f), U, 0) = 0\) and \(L_A(f, 1) = 0\).

Finally, we choose some harmonic weak Maass form \(g \in H_{0, \rho_D}\) of weight 0 and dual representation \(\rho_D\) such that \(\xi(g) = \mathcal{L}_D(f)\) where \(\xi : H_{0, \rho_D} \to S_{2, \rho_D}\) is the differential operator defined in Section 3.1 of [5]. Then equation (26) determines the second summand \(L'(\xi(g), U, 0)\) in Theorem 4.7 of [5]. On the other hand the value of the automorphic Green function on the left-hand side of the cited formula is essentially the archimedian part of a certain height pairing (compare [5], equation (5.1)). In order to obtain a formula for \(L'_A(f, 1)\) in the spirit of Gross and Zagier one could try to relate the first summand of the right-hand side of [5], Theorem 4.7, to the finite part of this height pairing.

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References


