Lifting newforms to vector valued modular forms for the Weil representation

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Abstract

Given a discriminant form D of level N there is a natural lifting mapping elliptic modular forms of level N to vector valued modular forms for the Weil representation associated to D. We show that in some cases the zero component of a lifting of a newform f is just a scalar multiple of f. In order to do so, we split the lifting map into certain partial liftings corresponding to the prime powers exactly dividing N and then proceed to compute the zero components of these partial maps explicitly. As an application we show that the L-function $L_{\mathcal{A}}(f,s)$ of a newform f and an ideal class $\mathcal{A}$ as defined by Gross and Zagier can be written as a certain L-series of the lifting of f.

1 Introduction

As a generalisation of the usual elliptic modular forms one can consider vector valued modular forms. Given an even lattice L of even signature with associated discriminant form $D = L'/L$, a vector valued modular form for the Weil representation ρ_D is a holomorphic function on the complex upper half plane with values in the group algebra $\mathbb{C}[D]$ transforming suitably under $\mathrm{SL}_2(\mathbb{Z})$ and being meromorphic at ∞. Vector valued modular forms are for example important in the theory of Borcherds' automorphic products where they serve as inputs for the singular theta correspondence, see [2].

Let N be the level of D. There is a lifting map

$$\mathcal{L}_D(f) = \sum_{M \in \Gamma_0(N) \backslash \mathrm{SL}_2(\mathbb{Z})} f|_k M \ \rho_D(M)^{-1} \mathfrak{e}_0$$

sending elliptic modular forms f of weight k for $\Gamma_0(N)$ and a certain character χ_D associated to D to vector valued modular forms of weight k. This construction is well-known and has for example been studied by Scheithauer in [10], where one can also find explicit formulas for the component functions of the lifting. As the formulas involve sums over the cusps of $\Gamma_0(N)$ they are useful for squarefree level N but difficult to evaluate for arbitrary N.

The zero component of the above lifting has been computed by Bruinier and Bundschuh in [4], Theorem 5, in the case of N and $|D|$ being an odd prime p, by Bundschuh in his thesis [6], Proposition 4.3.9, in the case of N being squarefree, and by Zhang [11], Theorem 4.16, for special discriminant forms whose level is a positive fundamental discriminant. It was observed that the zero component of a lifted newform f is in some cases just a scalar multiple of f. In the present work we consider the zero component

$$\Phi_D(f) = \langle \mathcal{L}_D(f), \mathfrak{e}_0 \rangle_D$$

of the lifting for arbitrary level N and newforms f. Here $\langle \cdot, \cdot \rangle_D$ denotes the natural inner product on $\mathbb{C}[D]$. We define partial liftings

$$\mathcal{L}_D^q(f) = \sum_{M \in \Gamma_0(N/q) \backslash \operatorname{SL}_2(\mathbb{Z})} f|_k M \ \rho_D(M)^{-1} \mathfrak{e}_0$$

where q is a prime power exactly dividing N, and show that Φ_D splits as a product of the maps $\Phi_D^q(f) = \langle \mathcal{L}_D^q(f), \mathfrak{e}_0 \rangle_D$. The latter are then computed explicitly for newforms f, giving our main result (compare Theorem 8.1):

Theorem 1.1. *Let D be a discriminant form of even signature and level N. Assume that all 2-adic Jordan components of D are even, and that for every odd prime power $q = p^r$ exactly dividing N the exponent n_r in the Jordan decomposition $D_q \cong \bigoplus_{j=1}^{r} (p^j)^{\pm n_j}$ is even. Here D_q is the subgroup of elements of order dividing q. Then*

$$\Phi_D(f) = \prod_{q \| N} \left(1 - \frac{\mathcal{G}_{D_q}(q/p)}{|D_q|}\right) \cdot f$$

for every $f \in S_k^{\text{new}}(N, \chi_D)$ where the product runs over all prime powers $q = p^r$ exactly dividing N, including $p = 2$. Further, $\mathcal{G}_{D_q}(q/p)$ denotes the Gauss sum defined in Section 3 and χ_D is the character given in (3).

As an application, which was in fact the motivation for this work, we apply this result to a rescaled hyperbolic plane in Section 8, and show that the L-series $L_{\mathcal{A}}(f, s)$ associated to a newform f and an ideal class $\mathcal{A}$ defined by Gross and Zagier in [7] can be written as a certain L-series of the vector valued lifting of f. Using results of Bruinier and Yang (see [5]) this may lead to a new proof of the Gross-Zagier formula.

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2 Classical modular forms and Atkin-Lehner operators

Let k be an integer. The group $\mathrm{GL}_2^+(\mathbb{Q})$ acts on the space of functions $f\colon \mathbb{H} \to \mathbb{C}$ via

$$(f|_k\alpha)(\tau) := \det(\alpha)^{k/2} j(\alpha,\tau)^{-k} f(\alpha\tau), \quad \tau \in \mathbb{H},$$

where $j(\alpha,\tau) := c\tau + d$ for $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbb{Q})$. For some positive integer N and some Dirichlet character χ mod N we call a holomorphic function $f\colon \mathbb{H} \to \mathbb{C}$ modular of weight k, level N and character χ if $f|_k\alpha = \chi(\alpha)f$ for all $\alpha \in \Gamma_0(N)$ where $\chi(\alpha) := \chi(d)$ for $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$. If in addition f is meromorphic, holomorphic or vanishes at the cusps of $\Gamma_0(N)$ we call f a weakly holomorphic modular form, a modular form or a cusp form, respectively. The corresponding spaces are denoted by $M_k^!(N,\chi)$, $M_k(N,\chi)$ and $S_k(N,\chi)$.

For a prime p we define the usual Hecke operator T_p by

$$T_p(f) = p^{k/2-1} \sum_{j=0}^{p-1} f\Big|_k \begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix} + p^{k/2-1}\chi(p) f\Big|_k \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$$

for $f \in M_k^!(N,\chi)$ where $\chi(p) = 0$ if p divides N. For $r \geq 2$ and coprime integers n, m we define $T_{p^r} := T_p T_{p^{r-1}} - p^{k-1}\chi(p) T_{p^{r-2}}$ and $T_{nm} := T_n T_m$. This extends the definition of the operator T_p to arbitrary positive integers. Further, we define the usual Fricke involution

$$H_N(f) := f\Big|_k \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$$

which gives a map $M_k^!(N,\chi) \to M_k^!(N,\overline{\chi})$. (This operator is often denoted by W_N. The present notation was adopted from [1] in order to distinguish between the Fricke involution and the Atkin-Lehner operators defined later on.)

Next we quickly recall the notion of newforms. For a positive divisor d of N such that the conductor of χ divides N/d there are two natural embeddings of $S_k(N/d,\chi)$ into $S_k(N,\chi)$, namely the trivial one $f \mapsto f$ and $f \mapsto f(d\tau)$. The space of newforms $S_k^{\mathrm{new}}(N,\chi)$ is the orthogonal complement (with respect to the Petersson inner product) of the subspace of $S_k(N,\chi)$ generated by the images of these embeddings for all divisors $d > 1$ of N. One can check that the Hecke operators T_n and the Fricke involution H_N preserve newforms.

Throughout we write $e(z) := e^{2\pi i z}$ for $z \in \mathbb{C}$. We call $f(\tau) = \sum_{n\geq 1} a_f(n) e(\tau n) \in S_k(N,\chi)$ an eigenform if f is an eigenvector for T_n for all n coprime to N, and we say an eigenform f is normalised if $a_f(1) = 1$. Further, we call f a primitive form if f is a normalised eigenform and a newform. It is well-known that the Fourier coefficients of a primitive form are precisely its T_n eigenvalues and are therefore multiplicative. Moreover, the set of primitive forms of weight k, level N and character χ forms an orthogonal basis of the corresponding space of newforms. We will need the following result:

Proposition 2.1. *Let p be a prime such that p^2 divides N and let χ be a real Dirichlet character mod N of conductor d. Further, let N_p and d_p be the p-components of N and d,* *respectively. If either p is odd, or if $p=2$ and $N_p > d_p$, then every $f \in S_k^{\mathrm{new}}(N,\chi)$ is of the form*

$$f(\tau) = \sum_{\substack{n\geq 1\\(n,p)=1}} a_f(n)e(\tau n), \quad \tau\in\mathbb{H}.$$

So for a positive integer n being divisible by p the Hecke operator T_n vanishes on $S_k^{\mathrm{new}}(N,\chi)$.

Proof. It suffices to prove this for primitive forms. Let $f \in S_k^{\mathrm{new}}(N,\chi)$ be a primitive form. First assume p is odd. Then the p-component of the character χ is either trivial or of the form $(\frac{\cdot}{p})$. In either case we have $N_p \geq p^2 > p \geq d_p$. So the claimed statement follows for any prime p from part (c) of Theorem 4.6.17 in [9]. □

Now we introduce Atkin-Lehner operators following Section 1 and 2 of [1]. For $N = qm$ with q and m being coprime positive integers we define

$$W_q(f) := f\Big|_k \begin{pmatrix} qx & y\\ Nz & qw\end{pmatrix}$$

for $f \in M_k^!(N,\chi)$ where $x,y,z,w\in\mathbb{Z}$ with $y\equiv 1 \bmod q$, $x \equiv 1 \bmod m$ and such that $\det(\left(\begin{smallmatrix} qx & y\\ Nz & qw\end{smallmatrix}\right)) = q$. This gives a well-defined map from $M_k^!(N,\chi_q\chi_m)$ to $M_k^!(N,\chi_q^{-1}\chi_m)$ which preserves the corresponding subspaces of modular forms, cusp forms and newforms. Here χ_q and χ_m denote the q and m components of χ, respectively.

Let $q = p^r$ be a prime power. It is shown in [1] that $W_q(f) = \lambda_p(f)f^{(q)}$ for every primitive form $f\in S_k(N,\chi)$ where $|\lambda_p(f)| = 1$ and $f^{(q)} = \sum_{n\geq 1} b(n)e(\tau n)$ is the primitive form defined by

$$b(\ell) = \begin{cases} \overline{\chi_q(\ell)}a_f(\ell), & \text{if } \ell\neq p,\\ \chi_m(p)\overline{a_f(p)}, & \text{if } \ell = p,\end{cases} \tag{1}$$

for primes ℓ. If $a_f(p)\neq 0$ then $\lambda_p(f) = q^{k/2-1}\mathcal{G}(\chi_q)a_f(q)^{-1}$ where $\mathcal{G}(\chi_q)$ denotes the usual Gauss sum of χ_q (this is Theorem 2.1 in [1]).

Proposition 2.2. *Let $N = pm$ with p being prime and $(p,m) = 1$. Further, let χ be a real Dirichlet character such that its p-component χ_p is trivial. Then*

$$(W_p\circ T_p)(f) = (T_p\circ W_p)(f) = -p^{k/2-1}f$$

for $f\in S_k^{\mathrm{new}}(N,\chi)$.

Proof. We may assume that $f \in S_k^{\mathrm{new}}(N,\chi)$ is primitive. Then $a_f(p)^2 = \chi_m(p)p^{k-2}$ by part (2) of Theorem 4.6.17 in [9]. In particular, we have $a_f(p)\neq 0$. Hence the above discussion yields $W_p(f) = \lambda_p(f)f^{(p)}$ where $\lambda_p(f) = -p^{k/2-1}a_f(p)^{-1}$. Since χ_p is trivial by assumption we have $b(\ell) = a_f(\ell)$ for all primes ℓ different to p. Moreover, $\chi_m(p) = \pm 1$ as χ is real. So $a_f(p)^2 = \chi_m(p)p^{k-2}$ gives

$$a_f(p) = \begin{cases} \pm p^{k/2-1}, & \text{if } \chi_m(p) = 1,\\ \pm i p^{k/2-1}, & \text{if } \chi_m(p) = -1.\end{cases}$$

In either case we have $b(p) = \chi_m(p)\overline{a_f(p)} = a_f(p)$. Therefore $f^{(p)} = f$ and thus the claim follows since $T_p(f) = a_f(p)f$. $\square$

3 Discriminant forms and Gauss sums

Let L be an even lattice of signature (b^+, b^-) and let L' be its dual lattice. The quotient $D = L'/L$ is a finite abelian group and the modulo 1 reduction of the bilinear form on L' induces a non-degenerate $\mathbb{Q}/\mathbb{Z}$-valued bilinear form $(\cdot,\cdot)$ on D with corresponding quadratic form $Q(\gamma) = \frac{1}{2}(\gamma,\gamma)$. (Some authors write $\gamma\delta$ for (γ,δ) and $\gamma^2/2$ for $Q(\gamma)$.) Such a group D is called a discriminant form. Its signature $\mathrm{sign}(D)$ is defined as the class of $b^+ - b^-$ mod 8 and the level of D is the smallest integer N such that $NQ(\gamma) = 0 \mod 1$ for all $\gamma \in D$.

Every discriminant form D is isomorphic to an orthogonal direct sum of 'basic' discriminant forms, the so-called Jordan components. We recall their definition from [10]:

- Let $q = p^r$ be a power of an odd prime p. Write $q^{\pm 1}$ for the discriminant form $\mathbb{Z}/q\mathbb{Z}$ where the quadratic form is defined by $Q(\gamma) = a/q$ for a generator γ where $a \in \mathbb{Z}$ with $(\frac{2a}{p}) = \pm 1$. This is called a p-adic Jordan component. The level of $q^{\pm 1}$ is q.
- Let $q = 2^r$ be a power of 2. Write $q_{II}^{\pm 2}$ for the discriminant form $\mathbb{Z}/q\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z}$ where the quadratic form is defined on generators γ, δ by $Q(\gamma) = Q(\delta) = 0, (\gamma,\delta) = 1/q$ mod 1 for q_{II}^{+2}, and $Q(\gamma) = Q(\delta) = 1/q, (\gamma,\delta) = 1/q \mod 1$ for q_{II}^{-2}. This is called an even 2-adic Jordan component. Its level is q.
- Let $q = 2^r$ be a power of 2. Write $q_t^{\pm 1}$ with $t \in \mathbb{Z}/8\mathbb{Z}$ satisfying $(\frac{t}{2}) = \pm 1$ for the discriminant form $\mathbb{Z}/q\mathbb{Z}$ where the quadratic form is defined by $Q(\gamma) = t/2q \mod 1$ for a generator γ. This is called an odd 2-adic Jordan component. Its level is $2q$.

The direct sum of n Jordan components $q^{\pm 1}$, $q_{II}^{\pm 2}$ or $q_{t_j}^{\pm 1}$ with the same prime power q is denoted by $q^{\pm n}$, $q_{II}^{\pm 2n}$ or $q_t^{\pm n}$, respectively, where the signs are multiplied and $t = \sum t_j$. Such sums are also called Jordan components. Note that the level of a discriminant form is the least common multiple of the levels of its Jordan components.

Eventually, we define the oddity of a discriminant form: For $q = 2^r$ set $\mathrm{oddity}(q_{II}^{\pm 2n}) = 4k$ mod 8 and $\mathrm{oddity}(q_t^{\pm n}) = t + 4k$ mod 8 where $k = 1$ if q is not a square and the exponent is $-n$, and $k = 0$ otherwise. The oddity of an arbitrary discriminant form is the sum of the oddities of its 2-adic Jordan components.

For a discriminant form D and $n \in \mathbb{Z}$ the Gauss sum of D is defined by

$$\mathcal{G}_D(n) = \sum_{\gamma \in D} e(nQ(\gamma))$$

where $e(z) := e^{2\pi i z}$ for $z \in \mathbb{C}$ as before. Note that this sum is multiplicative in D in the sense that $\mathcal{G}_D(n) = \mathcal{G}_{D_1}(n)\mathcal{G}_{D_2}(n)$ if $D = D_1 \oplus D_2$ (where a direct sum is always understood to be orthogonal). Therefore it is sufficient to compute Gauss sums of Jordan components. This has been done by Scheithauer in [10]. We use his results to deduce the following formulas:

Proposition 3.1. *Let $q = p^r$ be a prime power and $a, s \in \mathbb{Z}$ with $(a, p) = 1$ and $0 \leq s < r$. For p odd we have*

$$\mathcal{G}_{q^{\pm 1}}(p^s a) = \left(\frac{a}{p^{r-s}}\right) \mathcal{G}_{q^{\pm 1}}(p^s),$$

and for $p = 2$ we have $\mathcal{G}_{q_{II}^{\pm 2}}(2^s a) = \mathcal{G}_{q_{II}^{\pm 2}}(2^s)$ and

$$\mathcal{G}_{q_t^{\pm 1}}(2^s a) = \left(\frac{a}{2^{r-s}}\right) e\left((a-1) \operatorname{oddity}\left((2^{r-s})_t^{\pm 1}\right)/8\right) \mathcal{G}_{q_t^{\pm 1}}(2^s).$$

Proof. Let p be an odd prime. Then Proposition 3.3 from [10] states that for q not dividing c we have

$$\mathcal{G}_{q^{\pm 1}}(c) = \gamma_p((q/q_c)^{\pm 1}) \left(\frac{c/q_c}{q/q_c}\right) \sqrt{q_c q}$$

with $q_c = (c, q)$. Here $\gamma_p((q/q_c)^{\pm 1})$ is a certain 8th root of unity defined in Section 2 of [10]. Comparing the formula for $c = p^s a$ and $c = p^s$ we see that $\mathcal{G}_{q^{\pm 1}}(p^s a)$ and $\mathcal{G}_{q^{\pm 1}}(p^s)$ differ by a factor $(\frac{a}{p^{r-s}})$. For $p = 2$ one uses Proposition 3.5 and 3.6 from [10] and proceeds as above. □

4 Vector valued modular forms

Let D be a discriminant form of level N. From now on we assume that the signature of D is even. The group algebra $\mathbb{C}[D]$ of D is the $\mathbb{C}$-vector space generated by the formal basis vectors $\mathfrak{e}_\gamma$ for $\gamma \in D$ with multiplication defined by $\mathfrak{e}_\gamma \mathfrak{e}_\delta = \mathfrak{e}_{\gamma+\delta}$. There is a natural inner product on $\mathbb{C}[D]$ being antilinear in the second argument, which is defined by $\langle \mathfrak{e}_\gamma, \mathfrak{e}_\delta \rangle_D = 0$ for $\gamma \neq \delta$ and $\langle \mathfrak{e}_\gamma, \mathfrak{e}_\gamma \rangle_D = 1$.

We define an action of the generators $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{C}[D]$ by

$$\rho_D(T)\mathfrak{e}_\gamma = e(Q(\gamma))\mathfrak{e}_\gamma, \qquad \rho_D(S)\mathfrak{e}_\gamma = \frac{e(-\operatorname{sign}(D)/8)}{\sqrt{|D|}} \sum_{\delta \in D} e(-(\gamma, \delta))\mathfrak{e}_\delta.$$

This extends to a unitary representation ρ_D of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{C}[D]$, the so-called Weil representation of D. Here it is crucial that the signature of D is even as otherwise we would have to work with a double cover of $\mathrm{SL}_2(\mathbb{Z})$ (see for example [3], Chapter 1). It is well-known that $\Gamma(N)$ acts trivially in this representation. Moreover, for $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$ we have the formula

$$\rho_D(M)\mathfrak{e}_\gamma = \chi_D(a) e(bdQ(\gamma))\mathfrak{e}_{d\gamma} \tag{2}$$

where

$$\chi_D(a) = \left(\frac{a}{|D|}\right) e((a-1)\operatorname{oddity}(D)/8) \tag{3}$$

denotes the Dirichlet character associated to D (see [10], Proposition 4.5). Note that $\mathrm{oddity}(D)$ is even by the oddity formula, so χ_D is a real character mod N.

Next we consider vector valued functions $F\colon \mathbb{H} \to \mathbb{C}[D]$. Write $F(\tau) = \sum_{\gamma \in D} f_\gamma(\tau)\mathfrak{e}_\gamma$. We call F holomorphic if the component functions f_γ are. Further, we call a holomorphic function $F\colon \mathbb{H} \to \mathbb{C}[D]$ a (weakly holomorphic) vector valued modular form of weight k if it transforms as $F(\tau)|_k M = \rho_D(M)F(\tau)$ for $M \in \mathrm{SL}_2(\mathbb{Z})$ and if every component function f_γ is meromorphic at ∞, that is every f_γ has a Fourier expansion of the form $\sum_{n \in \mathbb{Z}+Q(\gamma)} a(n,\gamma)e(\tau n)$ with $a(n,\gamma) = 0$ for almost all $n < 0$.

Using the transformation behaviour of F and the formula (2) for the action of $\Gamma_0(N)$ on $\mathbb{C}[D]$ it is easy to check that the zero component f_0 of F is a weakly holomorphic modular form of weight k, level N and character χ_D, that is $f_0 \in M_k^!(N,\chi_D)$. Conversely, every scalar valued modular form $f \in M_k^!(N,\chi_D)$ can be lifted to a vector valued modular form of weight k via

$$\mathcal{L}_D(f) := \sum_{M \in \Gamma_0(N) \backslash \mathrm{SL}_2(\mathbb{Z})} f|_k M \ \rho_D(M)^{-1}\mathfrak{e}_0. \tag{4}$$

This lifting has for example been studied in [10], Section 5. Taking the zero component of $\mathcal{L}_D(f)$ for an elliptic modular form f gives a linear map

$$\Phi_D\colon M_k^!(N,\chi_D) \to M_k^!(N,\chi_D),\ f \mapsto \langle \mathcal{L}_D(f), \mathfrak{e}_0 \rangle_D \tag{5}$$

which preserves the corresponding subspaces of modular forms and cusp forms.

5 Splitting of the map Φ_D

As before let D be a discriminant form of even signature and level N. Write $N = mm'$ with $(m,m') = 1$. Then we can decompose $D = D_m \oplus D_{m'}$ as an orthogonal direct sum where D_c denotes the subgroup of elements of order dividing c. Note that D_m and $D_{m'}$ are discriminant forms whose levels are multiples of m and m' with the same prime divisors as m and m', respectively. We define an inner product on $\mathbb{C}[D_m] \otimes \mathbb{C}[D_{m'}]$ by

$$\langle \mathfrak{e}_{\gamma_1} \otimes \mathfrak{e}_{\delta_1}, \mathfrak{e}_{\gamma_2} \otimes \mathfrak{e}_{\delta_2} \rangle_{m \otimes m'} = \langle \mathfrak{e}_{\gamma_1}, \mathfrak{e}_{\gamma_2} \rangle_{D_m} \cdot \langle \mathfrak{e}_{\delta_1}, \mathfrak{e}_{\delta_2} \rangle_{D_{m'}}. \tag{6}$$

One easily checks that the natural map $\mathbb{C}[D_m] \otimes \mathbb{C}[D_{m'}] \to \mathbb{C}[D]$ sending $\mathfrak{e}_\gamma \otimes \mathfrak{e}_\delta$ to $\mathfrak{e}_\gamma \mathfrak{e}_\delta$ is an isometry. Moreover, it is an isomorphism of representations, namely of the tensor product representation $\rho_{D_m} \otimes \rho_{D_{m'}}$ on $\mathbb{C}[D_m] \otimes \mathbb{C}[D_{m'}]$ and the Weil representation ρ_D on $\mathbb{C}[D]$. Thereby one obtains the useful formulas

$$\rho_D(M)\mathfrak{e}_{\gamma+\delta} = \rho_{D_m}(M)\mathfrak{e}_\gamma \ \cdot \ \rho_{D_{m'}}(M)\mathfrak{e}_\delta, \tag{7}$$

$$\langle \rho_D(M)\mathfrak{e}_{\gamma_1+\delta_1}, \mathfrak{e}_{\gamma_2+\delta_2} \rangle_D = \langle \rho_{D_m}(M)\mathfrak{e}_{\gamma_1}, \mathfrak{e}_{\gamma_2} \rangle_{D_m} \cdot \langle \rho_{D_{m'}}(M)\mathfrak{e}_{\delta_1}, \mathfrak{e}_{\delta_2} \rangle_{D_{m'}} \tag{8}$$

for $\gamma, \gamma_1, \gamma_2 \in D_m$, $\delta, \delta_1, \delta_2 \in D_{m'}$ and $M \in \mathrm{SL}_2(\mathbb{Z})$. Since the inner products of D and D_m (resp. $D_{m'}$) agree on D_m (resp. $D_{m'}$) we will in the following simply write $\langle \cdot, \cdot \rangle$ for all of these.

Lemma 5.1. *Let D be a discriminant form of even signature and level $N = mm'$ with $(m, m') = 1$. Then for $M \in \Gamma_0(m)$ and $x \in \mathbb{C}[D_m]$ we have*

$$\langle \rho_D(M)x, \mathfrak{e}_0 \rangle = \langle x, \mathfrak{e}_0 \rangle \cdot \langle \rho_D(M)\mathfrak{e}_0, \mathfrak{e}_0 \rangle$$

with $\rho_D(M)\mathfrak{e}_0 = \chi_{D_m}(M)\rho_{D_{m'}}(M)\mathfrak{e}_0 \in \mathbb{C}[D_{m'}]$.

Proof. Writing $x = \sum_{\gamma \in D_m} \langle x, \mathfrak{e}_\gamma \rangle \mathfrak{e}_\gamma$ we see that it is enough to prove the lemma for the basis vectors $\mathfrak{e}_\gamma$ in $\mathbb{C}[D_m]$. The first statement is trivial for $x = \mathfrak{e}_0$. For $\gamma \in D_m$ with $\gamma \neq 0$ we have by (8) that

$$\langle \rho_D(M)\mathfrak{e}_\gamma, \mathfrak{e}_0 \rangle = \langle \rho_{D_m}(M)\mathfrak{e}_\gamma, \mathfrak{e}_0 \rangle \cdot \left\langle \rho_{D_{m'}}(M)\mathfrak{e}_0, \mathfrak{e}_0 \right\rangle .$$

Let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(m)$. Since d is coprime to m and the level of D_m is a multiple of m we have $d\gamma \neq 0$. Using (2) we thus obtain

$$\langle \rho_{D_m}(M)\mathfrak{e}_\gamma, \mathfrak{e}_0 \rangle = \langle \chi_{D_m}(M)e(bdQ(\gamma))\mathfrak{e}_{d\gamma}, \mathfrak{e}_0 \rangle = 0.$$

It follows from (7) and the action of $\Gamma_0(m)$ in ρ_{D_m} that

$$\rho_D(M)\mathfrak{e}_0 = \rho_{D_m}(M)\mathfrak{e}_0 \cdot \rho_{D_{m'}}(M)\mathfrak{e}_0 = \chi_{D_m}(M)\rho_{D_{m'}}(M)\mathfrak{e}_0 \in \mathbb{C}[D_{m'}].$$

This proves the lemma. □

We now consider the partial lifting

$$\mathcal{L}_D^m(f) := \sum_{M \in \Gamma_0(N)\backslash\Gamma_0(N/m)} f|_k M \ \rho_D(M)^{-1}\mathfrak{e}_0. \tag{9}$$

This gives a well-defined linear map sending modular forms $f \in M_k^!(N, \chi_D)$ to holomorphic vector valued functions $F : \mathbb{H} \to \mathbb{C}[D]$ with $F|_k(M\tau) = \rho_D(M)F(\tau)$ for $M \in \Gamma_0(N/m)$. The component functions of $\mathcal{L}_D^m(f)$ are linear combinations of $f|_k M$ for certain $M \in \Gamma_0(N/m)$ and are therefore meromorphic at the cusps. The zero component

$$\Phi_D^m(f) := \langle \mathcal{L}_D^m(f), \mathfrak{e}_0 \rangle \tag{10}$$

is again a modular form in $M_k^!(N, \chi_D)$.

Theorem 5.2. *Let D be a discriminant form of even signature and level $N = q_1 \cdots q_l$ with pairwise coprime prime powers $q_i = p_i^{r_i}$. Then*

$$\Phi_D = \Phi_D^N = \Phi_D^{q_1} \circ \cdots \circ \Phi_D^{q_l}$$

on $M_k^!(N, \chi_D)$.

Proof. It is sufficient to prove $\Phi_D^{rs} = \Phi_D^s \circ \Phi_D^r$ for all pairwise coprime integers r, s, t with $N = rst$.

Let $m := N/r$ and $n := N/s$. Let $\mathcal{A}$ and $\mathcal{B}$ be systems of coset representatives of $\Gamma_0(N)\backslash\Gamma_0(m)$ and $\Gamma_0(N)\backslash\Gamma_0(n)$, respectively. One easily verifies that $\mathcal{B}$ is also a system of coset representatives of $\Gamma_0(m)\backslash\Gamma_0(t)$. Hence $\mathcal{AB}$ is a system of coset representatives of $\Gamma_0(N)\backslash\Gamma_0(t)$. Writing out the definition of $\Phi_D^{rs}(f)$ with $f \in M_k^!(N, \chi_D)$ gives

$$\Phi_D^{rs}(f) = \sum_{\alpha\in\mathcal{A}}\sum_{\beta\in\mathcal{B}} f|_k\alpha|_k\beta \left\langle \rho_D(\beta^{-1})(\rho_D(\alpha^{-1})\mathfrak{e}_0), \mathfrak{e}_0\right\rangle .$$

The second statement of Lemma 5.1 shows that $\rho_D(\alpha^{-1})\mathfrak{e}_0 \in \mathbb{C}[D_{N/m}]$ as $\alpha^{-1} \in \Gamma_0(m)$. Note that $\beta^{-1} \in \Gamma_0(n) \subseteq \Gamma_0(N/m)$, so the first part of Lemma 5.1 applied to $x = \rho_D(\alpha)^{-1}\mathfrak{e}_0$ gives

$$\left\langle \rho_D(\beta^{-1})(\rho_D(\alpha^{-1})\mathfrak{e}_0), \mathfrak{e}_0\right\rangle = \left\langle \rho_D(\alpha^{-1})\mathfrak{e}_0, \mathfrak{e}_0\right\rangle \cdot \left\langle \rho_D(\beta)^{-1}\mathfrak{e}_0, \mathfrak{e}_0\right\rangle .$$

Thus Φ_D^{rs} becomes

$$\begin{aligned}\Phi_D^{rs}(f) &= \left\langle \sum_{\beta\in\mathcal{B}} \left\langle \sum_{\alpha\in\mathcal{A}} f|_k\alpha\ \rho_D(\alpha^{-1})\mathfrak{e}_0, \mathfrak{e}_0 \right\rangle\Big|_k \beta\ \rho_D(\beta^{-1})\mathfrak{e}_0, \mathfrak{e}_0\right\rangle \\ &= \left\langle \mathcal{L}_D^{N/s}\left\langle \mathcal{L}_D^{N/r}(f), \mathfrak{e}_0\right\rangle, \mathfrak{e}_0\right\rangle = (\Phi_D^s \circ \Phi_D^r)(f).\end{aligned}$$

To rewrite the sum over $\mathcal{B}$ into $\mathcal{L}^{N/s}$ we had to use that $\mathcal{B}$ is a system of coset representatives of $\Gamma_0(N)\backslash\Gamma_0(n)$. □

6 Computation of partial liftings for prime powers

Let D be a discriminant form of even signature and level N. Write $N = qm$ with $q = p^r$ being some prime power and $(p, m) = 1$. Then $D = D_q \oplus D_m$.

Lemma 6.1. *A set of coset representatives for the quotient $\Gamma_0(N)\backslash\Gamma_0(m)$ is given by the elements $\alpha_0, \ldots, \alpha_{q/p-1}, \beta_0, \ldots, \beta_{q-1}$ where*

$$\alpha_j = -ST^{-pjm}S = \begin{pmatrix} 1 & 0 \\ pjm & 1\end{pmatrix} \quad \textit{and} \quad \beta_l = -ST^{-m}ST^l = \begin{pmatrix} 1 & l \\ m & ml+1\end{pmatrix}.$$

Proof. It is well-known that $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(M)] = M\prod_{\ell|M}(1 + 1/\ell)$ where the product runs over all primes ℓ dividing M. Hence $[\Gamma_0(m) : \Gamma_0(N)] = q(1 + 1/p)$ which agrees with the number of given representatives. Thus it remains to check that these matrices indeed represent different cosets which is an easy calculation. □

Lemma 6.2. *Let $\alpha_0, \ldots, \alpha_{q/p-1}$ and $\beta_0, \ldots, \beta_{q-1}$ be the representatives given in the previous lemma. Then*

$$\left\langle \rho_D(\alpha_j)^{-1}\mathfrak{e}_0, \mathfrak{e}_0\right\rangle = \frac{\mathcal{G}_{D_q}(pjm)}{|D_q|} \quad \textit{and} \quad \left\langle \rho_D(\beta_l)^{-1}\mathfrak{e}_0, \mathfrak{e}_0\right\rangle = \frac{\mathcal{G}_{D_q}(m)}{|D_q|}$$

where $\mathcal{G}_{D_q}$ is the Gauss sum of D_q defined in Section 3.

Proof. A straightforward computation using the definitions of $\rho_D(S)$ and $\rho_D(T)$ and the usual orthogonality relations for characters shows

$$\left\langle \rho_D(\alpha_j)^{-1}\mathfrak{e}_0, \mathfrak{e}_0 \right\rangle = \frac{\mathcal{G}_D(pjm)}{|D|} \quad \text{and} \quad \left\langle \rho_D(\beta_l)^{-1}\mathfrak{e}_0, \mathfrak{e}_0 \right\rangle = \frac{\mathcal{G}_D(m)}{|D|}.$$

Since $|D| = |D_q| \cdot |D_m|$ and $\mathcal{G}_D(cm) = \mathcal{G}_{D_q}(cm) \cdot \mathcal{G}_{D_m}(cm) = \mathcal{G}_{D_q}(cm) \cdot |D_m|$ for $c = pj$ and $c = 1$ we obtain the claimed formulas. □

Using the previous two lemmas we see for $f \in M^!_k(N, \chi_D)$ that

$$\Phi^q_D(f) \;=\; \frac{1}{|D_q|} \sum_{j=0}^{q/p-1} \mathcal{G}_{D_q}(pjm) \cdot f|_k\alpha_j \;+\; \frac{\mathcal{G}_{D_q}(m)}{|D_q|} \sum_{l=0}^{q-1} f|_k\beta_l. \tag{11}$$

Proposition 6.3. *Let $\beta_0, \ldots, \beta_q$ be as in Lemma 6.1 and let $f \in M^!_k(N, \chi_D)$. Then*

$$\sum_{l=0}^{q-1} f|_k\beta_l = \chi_q(-m)q^{1-k/2}(T_q \circ W_q)(f) \quad \in M^!_k(N, \chi_D).$$

Proof. Fix a choice of integers z, w with $qw - mz = 1$ and define $w_q := \left(\begin{smallmatrix} q & 1 \\ Nz & qw \end{smallmatrix}\right)$ as in Section 2. A direct computation shows that

$$M_l := \beta_l T^z \begin{pmatrix} 1 & l \\ 0 & q \end{pmatrix}^{-1} \omega_q^{-1} \begin{pmatrix} 1/q & 0 \\ 0 & 1/q \end{pmatrix}^{-1} = \begin{pmatrix} * & * \\ * & qw - m \end{pmatrix} \in \Gamma_0(N).$$

Moreover,

$$\chi_D(M_l) = \chi_D(qw - m) = \chi_q(qw - m)\chi_m(qw - m) = \chi_q(-m)$$

since $qw - m \equiv 1 \bmod m$ and $qw - m \equiv -m \bmod q$. Let $f \in M^!_k(N, \chi_D)$. Then

$$\begin{aligned}\sum_{l=0}^{q-1} f|_k\beta_l &= \left(\sum_{l=0}^{q-1} \chi_D(M_l) \cdot W_q(f)|_k \begin{pmatrix} 1 & l \\ 0 & q \end{pmatrix} \right) \Bigg|_k T^{-z} \\ &= \chi_q(-m)q^{1-k/2}(T_q \circ W_q)(f)|_k T^{-z}\end{aligned}$$

as $\left(\begin{smallmatrix} q & 0 \\ 0 & q \end{smallmatrix}\right)$ acts trivially and $T_q(g) = q^{k/2-1} \sum_{l=0}^{q-1} g|_k \left(\begin{smallmatrix} 1 & l \\ 0 & q \end{smallmatrix}\right)$ by definition. This proves the proposition since T acts as the identity on $(T_q \circ W_q)(f)$. □

Proposition 6.4. *Let $\alpha_0, \ldots, \alpha_{q/p-1}$ be as in Lemma 6.1 and let $f \in M^!_k(N, \chi_D)$. Then*

$$\sum_{j=0}^{q/p-1} \mathcal{G}_{D_q}(pjm) f|_k\alpha_j = (-1)^k H_N \left(\sum_{n\in\mathbb{Z}} a_g(n)\mu(n; D_q, m)e(\tau n) \right) \quad \in M^!_k(N, \chi_D)$$

where $g := H_N(f) = \sum_{n\in\mathbb{Z}} a_g(n)e(\tau n)$ is the Fricke involution of f and

$$\mu(n; D_q, m) = \sum_{j=0}^{q/p-1} \mathcal{G}_{D_q}(pjm)e(-pjn/q).$$

Proof. Let $h_N = \left(\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right)$ and $g := H_N(f) = f|_k h_N = \sum_{n\in\mathbb{Z}} a_g(n)e(\tau n)$. Then $g \in M_k^!(N, \chi_D)$ since χ_D is real. Further, a direct computation shows that

$$h_N \alpha_j h_N = \begin{pmatrix} -N & 0 \\ 0 & -N \end{pmatrix} \begin{pmatrix} 1 & -pj/q \\ 0 & 1 \end{pmatrix}.$$

As $h_N^2 = \left(\begin{smallmatrix} -N & 0 \\ 0 & -N \end{smallmatrix}\right)$ acts as a multiplication by $(-1)^k$ we may write $f|_k\alpha_j = f|_k h_N^2 \alpha_j h_N^2$. Hence we obtain

$$\begin{aligned}\sum_{j=0}^{q/p-1} \mathcal{G}_{D_q}(pjm) f|_k\alpha_j &= \sum_{j=0}^{q/p-1} \mathcal{G}_{D_q}(pjm)\, H_N\left(g \Big|_k \begin{pmatrix} -N & 0 \\ 0 & -N \end{pmatrix} \begin{pmatrix} 1 & -pj/q \\ 0 & 1 \end{pmatrix}\right) \\ &= (-1)^k H_N \left(\sum_{n\in\mathbb{Z}} a_g(n) \left(\sum_{j=0}^{q/p-1} \mathcal{G}_{D_q}(pjm) e(-pjn/q) \right) e(\tau n) \right).\end{aligned}$$

This gives the stated formula. □

7 Partial liftings of newforms

As before let D be a discriminant form of even signature and level $N = qm$ with $q = p^r$, p prime and $(p, m) = 1$. From now on we will focus on the action of Φ_D^q on the space of newforms $S_k^{\text{new}}(N, \chi_D)$. It turns out to be reasonable to consider the cases $p \neq 2$ and $p = 2$ separately.

7.1 Powers of odd primes

Let $q = p^r$ be a power of an odd prime p. Then $\chi_q = \left(\frac{\cdot}{|D_q|}\right)$. Let

$$D_q \cong \bigoplus_{j=1}^{r} (p^j)^{\pm n_j} \tag{12}$$

with non-negative integers n_j be a Jordan decomposition of D_q. In the special case $q = p$ the discriminant form D_q is of the form $p^{\pm n_r}$ and we have $\chi_p = \left(\frac{\cdot}{p}\right)^{n_r}$.

Theorem 7.1. *Let $f \in S_k^{\text{new}}(N, \chi_D)$ and define $\psi := \left(\frac{\cdot}{p}\right)^{n_r}$.*

1. *For $q = p$ we have*

$$\Phi_D^p(f) = f + \frac{\mathcal{G}_{D_p}(1)}{|D_p|} \psi(-1) p^{1-k/2} (T_p \circ W_p)(f).$$

2. For $q = p^r$ with $r \geq 2$ we have

$$\Phi_D^q(f) = f + \frac{\mathcal{G}_{D_q}(p^{r-1})}{|D_q|}\psi(-m)\mathcal{G}(\psi)(-1)^k H_N\big((H_N(f))_\psi\big)$$

where $\mathcal{G}(\psi) = \sum_{j=1}^{p-1}\psi(j)e(j/p)$ denotes the usual Gauss sum of ψ, and g_ψ denotes the twist of g by ψ, that is $g_\psi = \sum_{n\geq 1} a_g(n)\psi(n)e(\tau n)$.

Proof. First, let $q = p$. The contribution to Φ_D^p coming from Proposition 6.4 is just f since $\mathcal{G}_{D_p}(0) = |D_p|$. Adding the expression given in Proposition 6.3 we obtain the first formula. Note that $\mathcal{G}_{D_p}(m) = (\frac{m}{p})^{n_r}\mathcal{G}_{D_p}(1)$ by Proposition 3.1 and $\psi = \chi_p = (\frac{\cdot}{p})^{n_r}$.

Now let $q = p^r$ with $r \geq 2$. Write $g = \sum_{n\geq 1} a_g(n)e(\tau n) = H_N(f)$. Proposition 2.1 states that $a_g(n) = 0$ for all n being divisible by p, and that $(T_q \circ W_q)(f) = 0$ as $W_q(f)$ is a newform of level N and character χ_D. So applying Proposition 6.3 and Proposition 6.4 to Eq. (11) yields

$$\Phi_D^q(f) = \frac{(-1)^k}{|D_q|}H_N\left(\sum_{\substack{n\geq 1\\(n,p)=1}} a_g(n)\mu(n;D_q,m)e(\tau n)\right). \tag{13}$$

Let $n \geq 1$ with $(n,p) = 1$. Then

$$\mu(n;D_q,m) = \sum_{j=0}^{q/p-1}\mathcal{G}_{D_q}(pjm)e(-pjn/q) = |D_q| + \sum_{s=1}^{r-1}\sum_{\substack{\ell=0\\(\ell,p)=1}}^{p^{r-s}-1}\mathcal{G}_{D_q}(p^s\ell m)e(-\ell n/p^{r-s}).$$

By Proposition 3.1 we can write $\mathcal{G}_{D_q}(p^s\ell m) = \psi_s(\ell m)\mathcal{G}_{D_q}(p^s)$ with

$$\psi_s(\ell) := \prod_{j=s+1}^{r}\left(\frac{\ell}{p^{j-s}}\right)^{n_j},$$

thus giving

$$\mu(n;D_q,m) = |D_q| + \sum_{s=1}^{r-1}\psi_s(m)\mathcal{G}_{D_q}(p^s)\sum_{\ell=0}^{p^{r-s}-1}\psi_s(\ell)e(-\ell n/p^{r-s}). \tag{14}$$

Using that ψ_s has conductor 1 or p one can easily see that the inner sum vanishes unless $s = r-1$, in which case it equals the usual Gauss sum

$$\sum_{\ell=1}^{p-1}\psi_{r-1}(\ell)e(-\ell n/p) = \psi(-n)\mathcal{G}(\psi)$$

of $\psi = \psi_{r-1} = \left(\frac{\cdot}{p}\right)^{n_r}$. Therefore we finally obtain

$$\mu(n;D_q,m) = |D_q| + \psi(-m)\mathcal{G}_{D_q}(p^{r-1})\psi(n)\mathcal{G}(\psi).$$

If we insert this expression into (13) we are done. □

Corollary 7.2. *If n_r is even then*

$$\Phi_D^q(f) = \left(1 - \frac{\mathcal{G}_{D_q}(p^{r-1})}{|D_q|}\right) f$$

for every $f \in S_k^{\mathrm{new}}(N, \chi_D)$. In particular, Φ_D^q defines an isomorphism of $S_k(N, \chi_D)$.

Proof. Let n_r be even. Then $\psi = (\frac{\cdot}{p})^{n_r}$ is trivial. If $q = p$ then the p-component $\chi_p = \psi$ of χ_D is trivial and thus we have $(T_p \circ W_p)(f) = -p^{k/2-1} f$ for $f \in S_k^{\mathrm{new}}(N, \chi_D)$ by Proposition 2.2. This gives the claimed formula.

If $q = p^r$ with $r \geq 2$ then $(H_N(f))_\psi = H_N(f)$ since ψ is trivial and since Proposition 2.1 implies that $a_{H_N(f)}(n) = 0$ for all n being divisible by p. So the formula follows as $\mathcal{G}(\psi) = -1$ and $(-1)^k H_N^2$ acts as the identity.

It remains to note that we have $\left|\mathcal{G}_{D_q}(p^{r-1})\right| / |D_q| = p^{-n_r/2}$ in both cases, that is the single eigenvalue of Φ_D^q is non-zero. So Φ_D^q is an isomorphism of $S_k(N, \chi_D)$. □

Proposition 7.3. *1. If n_r is odd and either $q = p$ or $q = p^r$ with $r \geq 3$, then Φ_D^q still defines an endomorphism of $S_k^{\mathrm{new}}(N, \chi_D)$.*

2. If n_r is odd with $n_r \neq 1$ then Φ_D^q is injective on $S_k^{\mathrm{new}}(N, \chi_D)$.

Proof. Let n_r be odd and $\psi = (\frac{\cdot}{p})$. For (1) we only have to note that the operators T_p, W_p and H_N preserve newforms, and that if $q = p^r$ with $r \geq 3$ the twist of a newform by ψ is still a newform. (This follows from Theorem 3.12 in [8] as $e(\chi) = 1$, $e(\omega) = 0, 1$ and $v \geq 3$ in the situation of the theorem.)

It remains to prove (2): If $q = p^r$ with $r \geq 2$ and $f \in S_k^{\mathrm{new}}(N, \chi_D)$ with $\Phi_D^q(f) = 0$ then part (2) of Theorem 7.1 implies $g = -\lambda g_\psi$ where

$$\lambda := \frac{\mathcal{G}_{D_q}(p^{r-1})}{|D_q|} \psi(-m) \mathcal{G}(\psi)$$

and $g := H_N(f)$. Suppose that $g \neq 0$. Comparing the first non-zero Fourier coefficient on both sides of $g = -\lambda g_\psi$ we obtain $|\lambda| = 1$. On the other hand, a direct computation using Proposition 3.3 in [10] to evaluate the Gauss sum $\mathcal{G}_{D_q}(p^{r-1})$ and the well-known fact that $|\mathcal{G}(\psi)| = p^{1/2}$ yields $|\lambda| = p^{(1-n_r)/2}$. If $n_r \neq 1$ this is a contradiction. So we have $g = 0$ and thus $f = 0$. Hence Φ_D^q is injective.

For the case $q = p$ let $f \in S_k^{\mathrm{new}}(N, \chi_D)$ be primitive. Recall from Section 2 that $W_p(f) = \lambda_p(f) f^{(p)}$ where $f^{(p)} = \sum_{n \geq 1} b(n) e(\tau n)$ is defined as in (1) and $\lambda_p(f) = p^{k/2-1} \mathcal{G}(\psi) a_f(p)^{-1}$ since $a_f(p) \neq 0$ by part (1) of Theorem 4.6.17 in [9]. Hence we have

$$\Phi_D^p(f) = f + \lambda \frac{b(p)}{a_f(p)} f^{(p)}$$

where $\lambda := \mathcal{G}_{D_p}(1) |D_p|^{-1} \psi(-1) \mathcal{G}(\psi)$. It is easy to see that $(f^{(p)})^{(p)} = f$. Therefore we obtain

$$\Phi_D^p\left(f^{(p)}\right) = f^{(p)} + \lambda \frac{a_f(p)}{b(p)} f.$$

So the subspace U_f spanned by f and $f^{(p)}$ is invariant under Φ_D^p. We distinguish two cases: If $f = f^{(p)}$ then $\Phi_D^p(f) = (1+\lambda)f$. If $f \neq f^{(p)}$ then one easily checks that the two forms $f \pm b(p)a_f(p)^{-1}f^{(p)}$ are eigenvectors of Φ_D^p with corresponding eigenvalues $1 \pm \lambda$. In order two show that Φ_D^p is injective on U_f it therefore suffices to note that $1 \pm \lambda \neq 0$ which is true if $n_r \neq 1$ as $|\lambda| = p^{(1-n_r)/2}$. Now Φ_D^p is also injective on $S_k^{\mathrm{new}}(N, \chi_D)$ since the space of newforms may be written as a direct sum of U_f's. □

Corollary 7.4. *If either n_r is even, or if n_r is odd with $n_r \neq 1$ and $q = p^r$ with $r \neq 2$, then Φ_D^q defines an isomorphism of $S_k^{\mathrm{new}}(N, \chi_D)$.*

In the following we give examples which show that part (1) and (2) of Proposition 7.3 are sharp to some degree:

Example 7.5. Let $D = 9^{+1}$ be a Jordan component. Then $\mathrm{sign}(D)$ is even and χ_D is trivial. Put $N = 9$. The space $S_6^{\mathrm{new}}(N, \chi_D)$ is generated by the primitive form $f = q + 6q^2 + O(q^4)$ where $q = e^{2\pi i\tau}$. We have shown that

$$\Phi_D^9(f) = f + \lambda\, H_N((H_N(f))_\psi)$$

for some non-zero constant $\lambda \in \mathbb{C}^*$ where $\psi = \left(\frac{\cdot}{3}\right)$. As $S_6^{\mathrm{new}}(N, \chi_D)$ is 1-dimensional we have $H_N(f) = \mu f$ for some $\mu \in \mathbb{C}^*$. So $(H_N(f))_\psi = \mu(q - 6q^2 + O(q^4))$ is not a multiple of f, hence not a newform. Therefore $H_N((H_N(f))_\psi)$ and $\Phi_D^9(f)$ cannot be newforms either. So Φ_D^9 is not an endomorphism of $S_6^{\mathrm{new}}(N, \chi_D)$.

Example 7.6. Let $D = 3^{+1}$ be a Jordan component. Using the notation of [10], Section 2, we have $3\text{-excess}(D) = 2 \bmod 8$, $p\text{-excess}(D) = 0 \bmod 8$ for all primes $p > 3$ and $\mathrm{oddity}(D) = 0 \bmod 8$. So the oddity-formula yields $\mathrm{sign}(D) \equiv -(3\text{-excess}(D)) \equiv 6 \bmod 8$. Further, we have $\chi_D = \left(\frac{\cdot}{3}\right)$. Put $N = 3$. By Theorem 7.1 we have

$$\Phi_D^3(f) = f + i3^{(1-k)/2}(T_3 \circ W_3)(f)$$

as $\mathcal{G}_D(1) = e(\mathrm{sign}(D)/8)\sqrt{|D|}$ by Milgram's formula. Let $k = 7$. The space $S_7^{\mathrm{new}}(N, \chi_D)$ is generated by the primitive form $f = q - 27q^3 + O(q^4)$. Hence $(T_3 \circ W_3)(f) = i3^{(k-1)/2}f$ since $\mathcal{G}(\chi_D) = i\sqrt{3}$ and $f^{(3)} = f$ as $S_7^{\mathrm{new}}(N, \chi_D)$ is 1-dimensional. So $\Phi_D^3(f) = 0$ and thus $\Phi_D^3(f)$ vanishes on $S_7^{\mathrm{new}}(N, \chi_D)$.

7.2 Powers of 2

Let now $q = 2^r$ be a power of $p = 2$. If $q = 2$ then D does not contain any odd 2-adic Jordan components and we may proceed as in the previous subsection. Moreover, the 2-component $\chi_2 = \left(\frac{\cdot}{|D_2|}\right)$ of χ_D will always be trivial since $|D_2| = 2^{2n}$ for some positive integer n. So Proposition 2.2 applies and thus we obtain:

Corollary 7.7. *If $N = 2m$ with $(2, m) = 1$ then*

$$\Phi_D^2(f) = \left(1 - \frac{\mathcal{G}_{D_2}(1)}{|D_2|}\right) f$$

for every $f \in S_k^{\mathrm{new}}(N, \chi_D)$. *In particular,* $|\mathcal{G}_{D_2}(1)| / |D_2| = 2^{-n}$ *implies that* Φ_D^2 *defines an isomorphism of* $S_k^{\mathrm{new}}(N, \chi_D)$.

Next let $q = 2^r$ with $r \geq 2$. Using the formula given in (3) it is not difficult to see that $\chi_q = (\frac{\cdot}{|D_q|})\varepsilon_D$ where ε_D is the trivial character if $\mathrm{oddity}(D) \equiv 0 \bmod 4$, and ε_D is the (unique) primitive character mod 4 if $\mathrm{oddity}(D) \equiv 2 \bmod 4$. Let

$$D_q \cong \bigoplus_{j=1}^{r_1} (2^j)_{II}^{\pm 2n_j} \oplus \bigoplus_{j=1}^{r_2} (2^j)_{t_j}^{\pm m_j} \tag{15}$$

be a Jordan decomposition of D_q with non-negative integers n_j, m_j and $n_{r_1}, m_{r_2} \geq 1$. Note that $r = \max(r_1, r_2 + 1)$. As in the proof of Theorem 7.1 we want to apply Proposition 2.1. This is possible if the conductor d_q of χ_q is smaller than q. On the other hand, if $d_q \geq q$ we may generalise part (1) of Theorem 7.1:

Proposition 7.8. *Let* $d_q \geq q = 2^r$ *with* $r \geq 2$. *Then either* $q = d_q = 4$ *or* $q = d_q = 8$, *and in both cases we have*

$$\Phi_D^q(f) = f + \frac{\mathcal{G}_{D_q}(1)}{|D_q|} \chi_q(-1) q^{1-k/2} (T_q \circ W_q)(f)$$

for every $f \in S_k^{\mathrm{new}}(N, \chi_D)$. *In particular,* Φ_D^q *defines an endomorphism of* $S_k^{\mathrm{new}}(N, \chi_D)$.

Proof. The conductor d_q of χ_q is either 1, 4 or 8. Since $q \geq 4$ we have $d_q \neq 1$. Let $d_q = 4$. Then $\chi_q = \varepsilon_D$ is primitive mod 4 and $q = 4$. Hence there is exactly one odd Jordan component $2_{t_1}^{\pm m_1}$. Moreover, we have $\mathrm{oddity}(D) \equiv m_1 \equiv t_1 \equiv 2 \bmod 4$ as $\varepsilon_D(a) = e((a-1)\,\mathrm{oddity}(D)/8)$ is non-trivial.

The factor $\mu(n; D_4, m)$ from Proposition 6.4 simplifies to

$$\mu(n; D_4, m) = |D_4| + \mathcal{G}_{D_4}(2m) e(-n/2).$$

By [10], Proposition 3.5, the Gauss sum of $2_{t_1}^{\pm m_1}$ vanishes for $2m$, so $\mathcal{G}_{D_4}(2m) = 0$. In order to obtain the claimed formula from (11) it remains to note that we still have $\mathcal{G}_{D_4}(m) = \chi_4(m)\mathcal{G}_{D_4}(1)$ by Proposition 3.1.

Next let $d_q = 8$. If $q = 4$ there can be at most one odd Jordan component $2_{t_1}^{\pm m_1}$. If there is no such component then χ_q is trivial, and if there is one then m_1 needs to be even as the signature of D is even by assumption, and thus χ_q is of conductor at most 4. So let $q = 8$. Then we have exactly two odd Jordan components, namely $2_{t_1}^{\pm m_1}$ and $4_{t_2}^{\pm m_2}$, since otherwise d_q is at most 4 as before. Remark that m_1 and thus m_2 need to be odd in order to give a character χ_8 of conductor 8. Next we have

$$\mu(n; D_8, m) = |D_8| + \mathcal{G}_{D_8}(4m) e(-n/2) + \mathcal{G}_{D_8}(2m) e(-n/4) + \mathcal{G}_{D_8}(6m) e(-3n/4).$$

Again all Gauss sums vanish by [10], Proposition 3.5, and $\mathcal{G}_{D_8}(m) = \chi_8(m)\mathcal{G}_{D_8}(1)$. Therefore the claimed formula holds. □

Using exactly the same arguments as in Proposition 7.3 together with Proposition 3.5 and 3.6 in [10] one obtains the following statement:

Proposition 7.9. *1. Let $q = d_q = 4$. If the number of even and odd 2-adic Jordan components of D_4 is not minimal (that is we do not have $m_1 = 2$ and $n_1 = n_2 = 0$) then Φ_D^4 defines an isomorphism of $S_k^{\mathrm{new}}(N, \chi_D)$.*

2. Let $q = d_q = 8$. If the number of even and odd 2-adic Jordan components of D_8 is not minimal (that is we do not have $m_1 = m_2 = 1$ and $n_1 = n_2 = n_3 = 0$) then Φ_D^8 defines an isomorphism of $S_k^{\mathrm{new}}(N, \chi_D)$.

Next we consider the cases in which Proposition 2.1 applies. Since the signature of an odd 2-adic Jordan component does not need to be even, the general case is quite messy. Therefore we make further assumptions in order to obtain a nice statement.

Theorem 7.10. *Let $d_q < q = 2^r$ with $r \geq 2$ and $f \in S_k^{\mathrm{new}}(N, \chi_D)$. Further, we assume that the signature of all odd 2-adic Jordan components of D_q is even, which is equivalent to $m_1, \dots, m_{r_2}$ being even by the oddity formula.*

1. If $r = r_1 > r_2 + 1$, or if $m_{r_2} \equiv 0 \bmod 4$ then

$$\Phi_D^q(f) = \left(1 - \frac{\mathcal{G}_{D_q}(2^{r-1})}{|D_q|}\right) f$$

where $\mathcal{G}_{D_q}(p^{r-1}) = 0$ if $r = r_2 + 1$. In particular, Φ_D^q defines an isomorphism of $S_k^{\mathrm{new}}(N, \chi_D)$.

2. If $r = r_2 + 1 \geq r_1$ and $m_{r_2} \equiv 2 \bmod 4$ then

$$\Phi_D^q(f) = f + \frac{\mathcal{G}_{D_q}(2^{r-2})}{|D_q|} \psi(-m)\mathcal{G}(\psi)(-1)^k H_N\big((H_N(f))_\psi\big)$$

where ψ is primitive mod 4 and $\mathcal{G}(\psi) = \sum_{j=0}^{3} \psi(j)e(j/p) = 2i$. If $r \geq 5$ then Φ_D^q is an endomorphism of $S_k^{\mathrm{new}}(N, \chi_D)$, and if we do not have $r_2 + 1 = r_1$ and $m_{r_2} = 2$ then Φ_D^q is injective.

Proof. Since all m_j are even, $|D_q|$ is a square and thus $\chi_q = \varepsilon_D$ is of conductor 1 or 4. Further, we may apply Proposition 2.1 as $q > d_q$ by assumption. So there is no contribution of the β_l's in (11) as $T_q(W_q(f)) = 0$, and we only have to consider factors $\mu(n; D_q, m)$ for odd n. By Proposition 3.1 we have $\mathcal{G}_{D_q}(2^s lm) = \psi_s(km)\mathcal{G}_{D_q}(2^s)$ for odd l and $1 \leq s \leq r-1$ with

$$\psi_s(a) := \prod_{j=s+1}^{r_2} \chi_{(2^{j-s})_{t_j}^{\pm m_j}}(a). \tag{16}$$

Note that $(2^{j-s})_{t_j}^{\pm m_j}$ is a discriminant form of even signature as we assume that m_j is even. Moreover, the character ψ_s is of conductor 1 or 4 for the same reason as χ_q is. Next we observe that ψ_s is trivial for $r_2 \leq s \leq r-1$ and $\mathcal{G}_{D_q}(2^{r_2}) = 0$ by [10], Proposition 3.5. Thus Eq. (14) becomes

$$\mu(n; D_q, m) = |D_q| + \sum_{s=1}^{r_2-1} \psi_s(m)\mathcal{G}_{D_q}(2^s) \sum_{\substack{\ell=0 \\ (l,2)=1}}^{2^{r-s}-1} \psi_s(\ell)e(-\ell n/2^{r-s}) \tag{17}$$

$$+ \sum_{s=r_2+1}^{r-1} \mathcal{G}_{D_q}(2^s) \sum_{\substack{\ell=0 \\ (l,2)=1}}^{2^{r-s}-1} e(-\ell n/2^{r-s}).$$

The first inner sum vanishes for $r - s \geq 3$ as the conductor of ψ_s is at most 4, and the second inner sum vanishes for $r - s \geq 2$. We distinguish the following two cases:

1. Let $r = r_1 > r_2 + 1$. Then the first sum vanishes completely and we are left with the last summand of the second sum: $\mu(n; D_q, m) = |D_q| - \mathcal{G}_{D_q}(2^{r-1})$. So we are indeed in the first case of the theorem.

2. Suppose that $r = r_2 + 1 \geq r_1$. Then the secon sum vanishes and only the last summand of the first sum remains:

$$\mu(n; D_q, m) = |D_q| + \psi_{r-2}(m)\mathcal{G}_{D_q}(2^{r-2}) \left[e(-n/4) + \psi_{r-2}(-1)e(n/4)\right].$$

   By (16) we know that ψ_{r-2} is trivial if $m_{r_2} \equiv 0 \bmod 4$ and primitive mod 4 if $m_{r_2} \equiv 2 \bmod 4$. In the latter case we obtain part (2) of the theorem, and otherwise it remains to note that $\mathcal{G}_{D_q}(2^{r-1}) = 0$ as mentioned above in order to obtain part (1) again.

In order to show that Φ_D^q is an endomorphism or injective we may argue as in Proposition 7.3, now using Proposition 3.5 and 3.6 in [10] instead of Proposition 3.3. □

If we omit the assumption that $m_1, \ldots, m_{r_2}$ are even integers, the functions ψ_s defined above might not be characters anymore. However, if we assume that the even 2-adic Jordan components "dominate" the odd 2-adic Jordan components in the following sense we recover part (1) of the previous theorem:

Proposition 7.11. *Let $d_q < q = 2^r$ with $r \geq 2$. If $r = r_1 > r_2 + 2$ then*

$$\Phi_D^q(f) = \left(1 - \frac{\mathcal{G}_{D_q}(2^{r-1})}{|D_q|}\right) f.$$

for every $f \in S_k^{\text{new}}(N, \chi_D)$. In particular, Φ_D^q defines an isomorphism of $S_k^{\text{new}}(N, \chi_D)$.

Proof. We may almost completely follow the proof of the previous theorem. Note that for $1 \leq s \leq r-1$ the function

$$\psi_s(a) := \prod_{j=s+1}^{r_2} \left(\frac{a}{2^{j-s}}\right)^{m_j} e\left((a-1)\,\text{oddity}\left((2^{j-s})_{t_j}^{\pm m_j}\right)/8\right)$$

will in general not be a character. Nevertheless ψ_s is well-defined modulo 8 and thus the inner sum in (17) still vanishes for $r - s \geq 4$. If $r = r_1 > r_2 + 2$ this is the case for $s = 1, \ldots, r_2 - 1$. So we may proceed as before.

It remains to note that $|\mathcal{G}_{D_q}(p^{r-1})|/|D_q| = 2^{-n_r} \neq 1$. So Φ_D^q indeed defines an isomorphism. □

8 Summary and an application

Given a particular nice situation we may summarise our results using Theorem 5.2:

Theorem 8.1. *Let D be a discriminant form of even signature and level N. Assume that for every odd prime power q exactly dividing N the exponent n_r in the Jordan decomposition (12) of D_q is even. If N is even we further assume that either all 2-adic Jordan components of D are even, or that the even 2-adic Jordan components 'dominate' the odd ones as in Proposition 7.11. Then*

$$\Phi_D(f) = \prod_{q \| N} \left(1 - \frac{\mathcal{G}_{D_q}(q/p)}{|D_q|}\right) \cdot f$$

for every $f \in S_k^{\mathrm{new}}(N, \chi_D)$ where the product runs over all prime powers $q = p^r$ exactly dividing N, including $p = 2$.

Remark 1. The statement of the previous theorem also holds for various other discriminant forms whose Jordan decomposition involve odd 2-adic Jordan components (compare for example part (1) of Theorem 7.10).

Example 8.2. Let N be a positive integer and let $L = II_{1,1}(N)$ denote a rescaled hyperbolic plane, that is a 2-dimensional lattice having basis vectors e, f with $\langle e, f \rangle = N$ and norm $q(e) = q(f) = 0$. The corresponding discriminant form $D = L'/L$ has order $|D| = N^2$, signature 0 mod 8 and level N. In particular, the character χ_D is trivial.

If 2 divides N then the only 2-adic Jordan component of D is $(2^r)_{II}^{+2}$ where 2^r exactly divides N. The other Jordan components of D are $q^{\pm 2}$ with $(\frac{-1}{p}) = \pm 1$ for odd prime powers $q = p^r$ exactly dividing N. Hence Theorem 8.1 is applicable, that is we have

$$\Phi_D(f) = \prod_{q \| N} \left(1 - \frac{\mathcal{G}_{D_q}(q/p)}{|D_q|}\right) \cdot f \tag{18}$$

for every $f \in S_k^{\mathrm{new}}(N, \chi_D)$. The Gauss sums can be computed with the formulas from Propositions 3.3 and 3.6 in [10] giving $\mathcal{G}_{D_q}(q/p) = q^2/p$. As $|D_q| = q^2$ we obtain

$$\Phi_D(f) = \prod_{p | N} \left(1 - \frac{1}{p}\right) \cdot f = \frac{\varphi(N)}{N} f \tag{19}$$

where the product runs over all primes dividing N and φ denotes Euler's φ function.

The following application was proposed by J. H. Bruinier, and was in fact one of the main motivations for the considerations of this work (compare [5], Sections 4 and 8):

Let N be an arbitrary positive integer and let $L = II_{1,1}(1) \oplus II_{1,1}(N)$ be a lattice of signature $(2,2)$ where $II_{1,1}(N)$ denotes a rescaled hyperbolic plane as in the previous example. Then the discriminant form $D = L'/L$ is isomorphic to the one given in the previous example and thus we obtain $\Phi_D(f) = \frac{\varphi(N)}{N} f$ for every $f \in S_k^{\mathrm{new}}(N, \chi_D)$.

Let V_+ be a 2-dimensional positive definite subspace of $V := L \otimes \mathbb{Q}$ with negative definite complement $U := V_+^\perp$, and let $\mathcal{P} := L \cap V_+$ and $\mathcal{N} := L \cap U$ be definite sublattices of L. Put $M := \mathcal{P} \oplus \mathcal{N}$. In general M is only a sublattice of L of finite index. The inclusions $M \subseteq L \subseteq L' \subseteq M'$ show $L/M \subseteq L'/M \subseteq M'/M$. Further, we have a natural map

$$L'/M \to L'/L, \ \gamma = x + M \mapsto \overline{\gamma} = x + L \tag{20}$$

which gives rise to a map sending modular forms F for the Weil representation $\rho_{L'/L}$ to modular forms F_M for $\rho_{M'/M}$ defined by

$$(F_M)_\gamma = \begin{cases} F_{\overline{\gamma}}, & \text{if } \gamma \in L'/M, \\ 0, & \text{if } \gamma \notin L'/M, \end{cases} \tag{21}$$

for $\gamma \in M'/M$, see Lemma 3.1 in [5].

Let F be a vector valued cusp form for $\rho_{L'/L}$ of weight $1 + b^-/2 = 2$, and let $b(n,\gamma)$ with $\gamma \in M'/M$ and $Q(\gamma) \equiv n \bmod 1$ denote the coefficients of F_M. Identifying M'/M with $\mathcal{P}'/\mathcal{P} \oplus \mathcal{N}'/\mathcal{N}$ we may view $\mathcal{P}'/\mathcal{P}$ as a subgroup of M'/M. In [5], Eq. (4.24), Bruinier and Yang define an L-function of F_M by

$$L(F_M, U, s) = (4\pi)^{-(s+2)/2}\Gamma\left(\frac{s+2}{2}\right) \sum_{n \geq 1} \sum_{\gamma \in \mathcal{P}'/\mathcal{P}} r(n,\gamma)\overline{b(n,\gamma)} n^{-(s+2)/2} \tag{22}$$

where $r(n,\gamma)$ are the coefficients of the vector valued theta series of $\mathcal{P}$, that is

$$\Theta_{\mathcal{P}}(\tau) = \sum_{x \in \mathcal{P}'} e(\tau q(x)) \mathfrak{e}_{x+\mathcal{P}} = \sum_{\gamma \in \mathcal{P}'/\mathcal{P}} \sum_{\substack{n \in \mathbb{Z}+Q(\gamma) \\ n \geq 0}} r(n,\gamma) e(\tau n) \mathfrak{e}_\gamma. \tag{23}$$

Note that the zero component of $\Theta_{\mathcal{P}}$ equals the usual scalar valued theta function

$$\theta_{\mathcal{P}}(\tau) = \sum_{x \in \mathcal{P}} e(\tau q(x)) = \sum_{n \geq 0} r(n) e(\tau n). \tag{24}$$

For simplicity, we drop the subscript M in $L(F_M, U, s)$, writing $L(F, U, s)$ instead.

Let now $F = \mathcal{L}_D(f)$ be the lifting of a primitive form $f \in S_2^{\mathrm{new}}(N, \chi_D)$. Further, we assume that the discriminant d of $\mathcal{P}$ is an odd (and therefore squarefree) negative fundamental discriminant which is coprime to N. Then every element in $\mathcal{P}'/\mathcal{P} \cap L'/M$ has order dividing $(d, N) = 1$, and thus the intersection equals 0. So in (22) only the summand for $\gamma = 0$ remains, giving

$$L(\mathcal{L}_D(f), U, s) = (4\pi)^{-(s+2)/2}\Gamma\left(\frac{s+2}{2}\right)\frac{\varphi(N)}{N}\sum_{n\geq 1} r(n)a_f(n)n^{-(s+2)/2}. \tag{25}$$

(Note that the Fourier coefficients $a_f(n)$ of f are real since the character χ_D is trivial.) As in [7] we assume that there are Heegner points of discriminant d on the curve $X_0(N)$ which is equivalent to saying that every prime divisor of N splits in $K := \mathbb{Q}(\sqrt{d})$. So $(\frac{d}{p}) = 1$ for every prime p dividing N. Then

$$L\left(\left(\frac{d}{\cdot}\right), s\right) = \prod_{p|N}\left(1 - p^{-s}\right)^{-1} \cdot \sum_{\substack{n\geq 1 \\ (n,dN)=1}} \left(\frac{d}{n}\right) n^{-s}.$$

On the other hand the Dirichlet class number formula states that

$$L\left(\left(\frac{d}{\cdot}\right), 1\right) = \frac{2\pi\, h(d)}{\omega(d)\sqrt{|d|}}$$

where $h(d)$ is the class number of K and $\omega(d)$ is the order of the unit group of $\mathcal{O}_K$. Moreover, it is not difficult to see that

$$\theta_{\mathcal{A}}(\tau) = \frac{1}{\omega(d)}\theta_{\mathcal{P}}(\tau)$$

where $\mathcal{A}$ is the ideal class associated to $\mathcal{P}$ and $\theta_{\mathcal{A}}$ is the theta function given in [7], equation (5.2). Taking derivatives on both sides of (25) and plugging in $s = 0$ we obtain

$$L'(\mathcal{L}_D(f), U, 0) = \frac{\omega(d)^2\sqrt{|d|}}{8\pi^2 h(d)} L'_{\mathcal{A}}(f, 1) \tag{26}$$

where $L_{\mathcal{A}}(f, s)$ is the L-function associated to the newform f and the ideal class $\mathcal{A}$ as in [7], equation (5.3). Here we used that $L(\mathcal{L}_D(f), U, 0) = 0$ and $L_{\mathcal{A}}(f, 1) = 0$.

Finally, we choose some harmonic weak Maass form $g \in H_{0,\overline{\rho}_D}$ of weight 0 and dual representation $\overline{\rho}_D$ such that $\xi(g) = \mathcal{L}_D(f)$ where $\xi\colon H_{0,\overline{\rho}_D} \to S_{2,\rho_D}$ is the differential operator defined in Section 3.1 of [5]. Then equation (26) determines the second summand $L'(\xi(g), U, 0)$ in Theorem 4.7 of [5]. On the other hand the value of the automorphic Green function on the left-hand side of the cited formula is essentially the archimedian part of a certain height pairing (compare [5], equation (5.1)). In order to obtain a formula for $L'_{\mathcal{A}}(f, 1)$ in the spirit of Gross and Zagier one could try to relate the first summand of the right-hand side of [5], Theorem 4.7, to the finite part of this height pairing.

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