Noname manuscript No.
(will be inserted by the editor)

Characterizations of Jacobi cusp forms and cusp forms of Maass Spezialschar

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Received: 3 October 2013 / Accepted: date

Abstract In this paper, we give characterizations of Jacobi cusp forms of weight k and index 1 on a congruence subgroup $\Gamma_0(N)$ and cusp forms of weight k on the full Siegel modular group $\mathrm{Sp}_4(\mathbb{Z})$ in Maass Spezialschar for $k \geq 4$ even and $N \geq 1$ odd and squarefree.

Keywords Jacobi forms · Maass Spezialschar · Fourier coefficients of automorphic forms

Mathematics Subject Classification (2000) 11F50 · 11F30 · 11F46

1 Introduction

In [3, 10] a characterization of elliptic cusp forms of even integral weight $k \geq 2$ on a congruence subgroup $\Gamma_0(N)$ with $N \geq 1$ was given with regard to the growth conditions of their Fourier coefficients. The similar results in the cases of Siegel modular forms of genus 2 on the full Siegel modular group and Jacobi forms on the full Jacobi modular group were given in [4]. They made use of the structure of Jacobi Eisenstein space and the growth conditions of the Fourier coefficients of Jacobi Eisenstein series in order to prove a characterization of Jacobi cusp forms. Using these and the Fourier-Jacobi expansion of a given Siegel modular form of genus 2, they also proved a characterization of Siegel cusp forms.

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In [1] a sufficient condition for $h \in M^+_{k+1/2}(\Gamma_0(4N))$ with $k \geq 2$ and $N \geq 1$ odd and squarefree to be a cusp form was given in a similar manner. Here, the plus space $M^+_{k+1/2}(\Gamma_0(4N))$ is defined by the space of modular forms $h \in M_{k+1/2}(\Gamma_0(4N))$ whose Fourier coefficients $a(n)(n \geq 0)$ vanish unless $(-1)^k n \equiv 0, 1 \pmod 4$. The authors used the complete description of the subspace $E^+_{k+1/2}(\Gamma_0(4N))$ generated by Cohen-type Eisenstein series whose Fourier coefficients were given explicitly in terms of special values of Dirichlet L-functions and modified divisors sums. They proved that no non-zero element of $E^+_{k+1/2}(\Gamma_0(4N))$ satisfies the given growth conditions on the Fourier coefficients. This result is distinguished from other results because only the growth conditions of the Fourier coefficients $a(|D|)$ for all fundamental discriminants D such that $(-1)^k D > 0$ were needed.

On the other hand, Kramer [5] gave an explicit isomorphism between the space $J_{k,1}(\Gamma_0(N))$ of Jacobi forms of weight k and index 1 on $\Gamma_0(N)$ and $M^+_{k-1/2}(\Gamma_0(4N))$ preserving cusp forms, for $k \geq 2$ even and $N \geq 1$ odd. Also, Eichler and Zagier [2] gave an explicit isomorphism between $J_{k,1}(\Gamma(1))$ and $M^*_k(\Gamma_2)$ preserving cusp forms. Here, $M^*_k(\Gamma_2)$ is the space of Siegel modular forms of weight k on $\Gamma_2 = \mathrm{Sp}_4(\mathbb{Z})$ whose Fourier coefficients satisfy a certain identity which will be described later. Maass [6–9] on his part studied $M^*_k(\Gamma_2)$ in connection with Saito-Kurokawa conjecture. A precise statement of those theorems will be given later.

In this paper, we will give characterizations of cusp forms in $J_{k,1}(\Gamma_0(N))$ and $M^*_k(\Gamma_2)$ for $k \geq 4$ even and $N \geq 1$ odd and squarefree. The results can be easily obtained by using the above isomorphisms and the main theorem of [1]. Our work is meant as an addition and complement to the result of [1].

2 Main results

Before we state the first result, we need to introduce the basic ingredients. It is well known that a Jacobi form $\phi = \sum c(n,r) q^n \zeta^r \in J_{k,1}(\Gamma_0(N))$ has a theta decomposition, namely, $\phi(\tau, z) = \sum_{\mu=0}^{1} \varphi_\mu(\tau)\, \vartheta_{\mu,1}(\tau, z)$ where

$$\varphi_\mu(\tau) := \sum_{\substack{n \in \mathbb{N}_0 \\ 4n-\mu^2 \geq 0}} c(n,\mu) q^{(4n-\mu^2)/4} \quad \text{and} \quad \vartheta_{\mu,1}(\tau, z) := \sum_{n \in \mathbb{Z}} q^{(2n-\mu)^2/4} \zeta^{2n-\mu} \tag{1}$$

for $\mu = 0, 1$. Using this theta decomposition, Kramer [5] gave an explicit isomorphism between $J_{k,1}(\Gamma_0(N))$ and $M^+_{k-1/2}(\Gamma_0(4N))$. The details are as follows:

Theorem 1 (([5])) *Let $k \in 2\mathbb{N}$ and N an odd natural number. Then there is an isomorphism between $J_{k,1}(\Gamma_0(N))$ and $M^+_{k-1/2}(\Gamma_0(4N))$ defined by*

$$\phi(\tau, z) \mapsto h(\tau) := \varphi_0(4\tau) + \varphi_1(4\tau). \tag{2}$$

The inverse is given by

$$h(\tau) \mapsto \phi'(\tau, z) := \sum_{\mu=0}^{1} \varphi'_\mu(\tau)\, \vartheta_{\mu,1}(\tau, z) \tag{3}$$

where $h(\tau) = \varphi'_0(4\tau) + \varphi'_1(4\tau)$ *with*

$$\varphi'_\mu(4\tau) = \sum_{\substack{n\in\mathbb{N}_0 \\ n\equiv -\mu^2 \pmod 4}} a(n)\, q^n \text{ for } \mu = 0, 1.$$

Moreover, there is an induced isomorphism between $J^{\mathrm{cusp}}_{k,1}(\Gamma_0(N))$ *and* $S^+_{k-1/2}(\Gamma_0(4N))$.

On the other hand, a sufficient condition for $h \in M^+_{k+1/2}(\Gamma_0(4N))$ to be a cusp form was given in terms of the growth conditions of the Fourier coefficients in [1].

Theorem 2 (([1])) *Let* $h \in M^+_{k+1/2}(\Gamma_0(4N))$ *with* $k \in \mathbb{N}$, $k \geq 2$ *and* $N \geq 1$ *odd and squarefree. Write* $a(n)$ $(n \geq 0)$ *for the Fourier coefficients of* h *and suppose that*

$$a(|D|) \ll_h |D|^c \tag{4}$$

for all fundamental discriminants D *such that* $(-1)^k D > 0$, *where* $c > 0$ *is any number strictly less than* $k - 1/2$. *Then* h *is a cusp form.*

By combining the above two theorems, we obtain the following theorem.

Theorem 3 *Let* $k \geq 4$ *be even and* $N \geq 1$ *odd and squarefree. Let* $\phi \in J_{k,1}(\Gamma_0(N))$ *and write* $\phi = \sum c(n, r)\, q^n \zeta^r$. *Assume that there is a number* $c \in (0, k - 3/2)$ *such that*

$$c(n, r) \ll_\phi |D|^c \tag{5}$$

for all fundamental discriminants $D < 0$ *and unique* $n \in \mathbb{N}$, $r \in \{0, 1\}$ *satisfying* $D = r^2 - 4n$. *Then* ϕ *must be a Jacobi cusp form.*

Proof By Theorem 1, it suffices to show that the corresponding $h(\tau)$ under isomorphism is a cusp form. Recall that

$$\varphi_\mu(\tau) = \sum_{\substack{n\in\mathbb{N}_0 \\ 4n-\mu^2\geq 0}} c(n, \mu) q^{(4n-\mu^2)/4}$$

for $\mu = 0, 1$. Write $h(\tau) = \varphi_0(4\tau) + \varphi_1(4\tau) = \sum a(l) q^l$. Then the l-th Fourier coefficient $a(l)$ is equal to

$$\begin{cases} c(n, \mu) & \text{if there exist } n \in \mathbb{N}_0, \mu \in \{0, 1\} \text{ satisfying } l = 4n - \mu^2, \\ 0 & \text{otherwise.} \end{cases}$$

Now by the assumption on $c(n, r)$ in (5), we know that

$$a(|D|) \ll_h |D|^c$$

for all fundamental discriminants $D < 0$. Note that $0 < c < k - 3/2$. Hence by Theorem 2, $h(\tau)$ is a cusp form. This proves the theorem.

Now we turn our attention to the second result. Let F be a Siegel modular form of weight k on the full Siegel modular group $\Gamma_2 = \mathrm{Sp}_4(\mathbb{Z})$. Then F can be written as

$$F(\tau, z, \tau') = \sum_{\substack{n,r,m \in \mathbb{Z} \\ n,m,4nm-r^2 \geq 0}} A(n,r,m)\, e(n\tau + rz + m\tau') = \sum_{m=0}^{\infty} \phi_m(\tau, z)\, e(m\tau'). \tag{6}$$

We denote by $M_k^*(\Gamma_2)$ the space of Siegel modular forms of weight k on Γ_2 whose Fourier coefficients satisfy the following identity

$$A(n,r,m) = \sum_{d|(n,r,m)} d^{k-1} A\Big(\frac{nm}{d^2}, \frac{r}{d}, 1\Big) \tag{7}$$

for all n, r, m. This space was studied by Maass [6–9] and it is now so-called Maass Spezialschar. We denote by $S_k^*(\Gamma_2) \subset M_k^*(\Gamma_2)$ the space of Siegel cusp forms.

Define a Hecke operator V_m on $J_{k,1}(\Gamma(1))$ by

$$\phi|\mathrm{V}_m = \sum_{n,r} \Bigg(\sum_{d|(n,r,m)} d^{k-1} c\Big(\frac{nm}{d^2}, \frac{r}{d}\Big)\Bigg) q^n \zeta^r \quad \text{for } m > 0, \tag{8}$$

$$\phi|\mathrm{V}_0 = c(0,0)\Bigg[-\frac{\mathrm{B}_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \Bigg] \qquad \text{for } m = 0, \tag{9}$$

where $\phi = \sum c(n,r) q^n \zeta^r \in J_{k,1}(\Gamma(1))$ and B_k is a Bernoulli number. Then it is known that V_m maps $J_{k,1}(\Gamma(1))$ and $J_{k,1}^{\mathrm{cusp}}(\Gamma(1))$ to $J_{k,m}(\Gamma(1))$ and $J_{k,m}^{\mathrm{cusp}}(\Gamma(1))$, respectively.

Theorem 4 (([2])) *There is an isomorphism between $J_{k,1}(\Gamma(1))$ and $M_k^*(\Gamma_2)$, given by*

$$\phi \mapsto \sum_{m=0}^{\infty} (\phi|\mathrm{V}_m)(\tau, z)\, e(m\tau'). \tag{10}$$

The inverse isomorphism is defined by

$$F \mapsto \phi_1 \tag{11}$$

where F and ϕ_1 are given by equation (6). Furthermore, this isomorphism preserves cusp forms. In other words, this induces an isomorphism between $J_{k,1}^{\mathrm{cusp}}(\Gamma(1))$ and $S_k^*(\Gamma_2)$.

Using this isomorphism, we obtain the following theorem.

Theorem 5 *Let $k \geq 4$ be even and $F \in M_k^*(\Gamma_2)$ given by equation (6). Assume that there is a number $c \in (0, k - 3/2)$ such that*

$$A(n,r,1) \ll_F |D|^c \tag{12}$$

for all fundamental discriminants $D < 0$ and unique $n \in \mathbb{N}$, $r \in \{0,1\}$ satisfying $D = r^2 - 4n$. Then F must be a Siegel cusp form.

Proof Due to Theorem 4, it is sufficient to show that ϕ_1 is a Jacobi cusp form. Write

$$\phi_1(\tau, z) = \sum_{\substack{n,r\in\mathbb{Z}\\4n-r^2\geq 0}} c(n,r)\, q^n \zeta^r.$$

By equation (6), $c(n,r) = A(n,r,1)$. Therefore, by using the assumption in (12),

$$c(n,r) \ll_{\phi_1} |D|^c$$

for all fundamental discriminants $D < 0$ and unique $n \in \mathbb{N}$, $r \in \{0,1\}$ satisfying $D = r^2 - 4n$. Note that $0 < c < k - 3/2$. Hence ϕ_1 is a Jacobi cusp form by Theorem 3. This completes the proof of the theorem.

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