Non-vanishing of Koecher-Maass series attached to Siegel cusp forms on the real line

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1. Introduction and statement of results

For positive integers k and n we denote by $M_k(\Gamma_n)$ the space of Siegel modular forms of weight k for the Siegel modular group $\Gamma_n := Sp_n(\mathbf{Z}) \subset GL_{2n}(\mathbf{Z})$ of degree n. We let $S_k(\Gamma_n)$ be the subspace of cusp forms.

For $F \in S_k(\Gamma_n)$ we let

$$D_F(s) := \sum_{\{T>0\}/GL_n(\mathbf{Z})} \frac{a(T)}{\epsilon_T} (\det T)^{-s} \qquad (\sigma := \Re(s) > \frac{k+n+1}{2})$$

be the Koecher-Maass L-series attached to F. Here T runs over all positive definite symmetric half-integral (n,n)-matrices, inequivalent under the action $Y \mapsto Y[U] := U'YU$ of $GL_n(\mathbf{Z})$ on $\mathcal{P}_n := \{Y \in M_n(\mathbf{R}) \,|\, Y = Y', Y > 0\}$. Further $\epsilon_T := \#\{U \in GL_n(\mathbf{Z}) \,|\, T[U] = T\}$ is the number of units of T and $a(T)$ denotes the T-th Fourier coefficient of F.

For k even it is well-known (cf. e.g. [5]) that the completed function

$$D_F^*(s) := (2\pi)^{-ns} \prod_{\nu=0}^{n-1} \pi^{\nu/2} \Gamma(s - \frac{\nu}{2}) \cdot D_F(s)$$

has holomorphic continuation to $\mathbf{C}$ and satisfies the functional equation

$$D_F^*(k-s) = (-1)^{\frac{nk}{2}} D_F^*(s).$$

Recently, Das and the author in [1] investigated non-vanishing properties of $D_F^*(s)$ on horizontal line segments inside "critical strips" $\frac{k}{2} + \epsilon \le \sigma \le \frac{k}{2} + \frac{n+1}{4}$ (where $\epsilon > 0$ is small). In particular, it was shown that given $t_0 \in \mathbf{R}$ and $\epsilon > 0$, then there is a positive constant $C_n(t_0, \epsilon)$ depending on n, t_0 and ϵ such that for every even integer $k > C_n(t_0, \epsilon)$ and for each s on the line segment of height t_0 inside the above strip there exists a Hecke eigenform $F \in S_k(\Gamma_n)$ depending on s such that $D_F^*(s) \neq 0$.

The constant $C_n(t_0, \epsilon)$ was not given explicitly (though in principle this seems to be possible) and a priori could be large with respect to n, t_0 and ϵ.

The proof was rather involved and complicated, relying on a careful analysis of the Fourier coefficients of a kernel function for the functional $F \mapsto D_F^*(s)$, for given s.

In this note we would like to give a simple method to prove the existence of Hecke eigenforms in $S_k(\Gamma_n)$ with $k \gg_n 1$ and non-vanishing Koecher-Maass series in a given point σ on the *real axis*, where the constant implied in $k \gg_n 1$ in general is explicit and small.

The basic ingredient is the following

Theorem. *Let k_0 be a positive integer and suppose that $dim S_{k_0}(\Gamma_n) > 0$. Let $\sigma \in \mathbf{R}$. Then there exists a Hecke eigenform $F \in S_{2k_0}(\Gamma_n)$ such that $D_F^*(\sigma) \neq 0$.*

The proof which will be given in the next section is easy and just exploits the explicit representation of $D_F^*(s)$ as a Mellin integral, together with "squaring" in an appropriate way.

Using the existence of the Ikeda lift [3] (Saito-Kurokawa lift in genus 2) we then deduce the following result.

Corollary. *Let n be even. Suppose that k is a positive integer divisible by 4 and that either $k \geq 12 + n$ if $n \equiv 0 \pmod 4$ or $k \geq 18 + n$ if $n \equiv 2 \pmod 4$. Let $\sigma \in \mathbf{R}$. Then there exists a Hecke eigenform $F \in S_k(\Gamma_n)$ such that $D_F^*(\sigma) \neq 0$.*

For the proof see sect. 2. Some further remarks will be given in sect. 3.

2. Proofs

We start with giving the *proof of the Theorem.* We first recall the Mellin integral formula for the Koecher-Maass L-series. Let $F \in S_k(\Gamma_n)$. Then according to [5, p. 209] the equality

$$D_F^*(s) = \int_{\mathcal{R}_n, detY \geq 1} F(iY)\Big((\det Y)^s + (-1)^{\frac{nk}{2}}(\det Y)^{k-s}\Big)d\mu \tag{1}$$

holds for all $s \in \mathbf{C}$. Here $\mathcal{R}_n$ is a fundamental domain for the action $Y \mapsto Y[U]$ of $GL_n(\mathbf{Z})$ on $\mathcal{P}_n$ (e.g. Minkowski's fundamental domain of reduced matrices) and $d\mu = (\det Y)^{-\frac{n+1}{2}}dY$ is the invariant measure.

Since by hypothesis $S_{k_0}(\Gamma_n)$ (and so also $M_{k_0}(\Gamma_n)$) is not the zero space it follows that $nk_0 \equiv 0 \pmod 2$, as one sees by acting with the negative identity matrix.

Now choose a non-zero function $F_0 \in S_{k_0}(\Gamma_n)$. Since as is easy to see complex conjugation acts on $S_{k_0}(\Gamma_n)$ through the corresponding action on Fourier coefficients, we may assume that F_0 has real Fourier coefficients and hence $F_0(iY)\,(Y > 0)$ is real. Therefore $F_0^2 \in S_{2k_0}(\Gamma_n) \setminus \{0\}$ takes real non-negative values for $Y > 0$, and from (1) applied with $F = F_0^2$ we obtain

$$D_{F_0^2}^*(\sigma) = \int_{\mathcal{R}_n, \det Y \geq 1} F_0^2(iY)\Big((\det Y)^\sigma + (\det Y)^{2k_0-\sigma}\Big)d\mu$$

for all $\sigma \in \mathbf{R}$. Using the identity theorem for holomorphic functions in one variable and induction, we see that $F_0^2(iY)\,(Y > 0)$ is not identically zero and thus conclude that $D^*_{F_0^2}(\sigma)$ is a positive, hence non-zero real number for each $\sigma \in \mathbf{R}$.

Let $\{F_1, \dots, F_g\}$ be a basis of $S_{2k_0}(\Gamma_n)$ consisting of Hecke eigenforms. Writing F_0^2 in terms of this basis and taking L-values our claim follows.

We shall now give the *proof of the Corollary.* Let ℓ and m be positive integers and suppose that $\ell \equiv m \pmod 2$. Then for each Hecke eigenform $f \in S_{2\ell}(\Gamma_1)$ Ikeda [3] has constructed a Hecke eigenform $F \in S_{\ell+m}(\Gamma_{2n})$. (If $m = 1$ this is just the Saito-Kurokawa lift.) Note that $dim S_{2\ell}(\Gamma_1) > 0$ if and only if $\ell \geq 6, \ell \neq 7$.

Distinguishing the cases m even (with $\ell = 6, 8, 10, \dots$) and m odd (with $\ell = 9, 11, 13, \dots$) and applying the Theorem with $n = 2m$ and $k_0 = \ell + m$ we obtain our assertion.

Remark. We would like to point out that the function F_0^2 given above does not depend on the real number σ and its completed Koecher-Maass series is a positive real number for each σ.

3. Further remarks

One can prove similar results as given in the Corollary for weights $k \equiv 2 \pmod 4$, under suitable conditions, too. For example, if $n = 2$ one starts with the unique (up to non-zero scalar multiples) non-zero cusp form ϕ_{35} of (smallest) odd weight 35 for Γ_2 [2] and multiplies say with non-zero cusp forms given by the Saito-Kurokawa lift. One then applies the Theorem.

We also want to recall that one can often prove that $dim S_{n+1}(\Gamma_n) > 0$. (In geometric terms this means that the field $K(\Gamma_n)$ of meromorphic Siegel modular functions for Γ_n is not a rational function field.) This, for example, is true for $n \equiv 0 \pmod{24}$ [4, p. 163]. One can then start with a non-zero cusp form of weight $n + 1$ and obtain Hecke eigenforms with non-vanishing Koecher-Maass series at points of the real line in a similar way as before.

Finally, let us mention that some of our results, under suitable conditions can be generalized to congruence subgroups $\Gamma \subset \Gamma_n$.

References

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