Non-Vanishing of L-Functions Associated to Cusp Forms of Half-Integral Weight in the Plus Space

Winfried Kohnen and Wissam Raji

Abstract

In this paper, we show a non-vanishing result for L-functions associated to cuspidal Hecke eigenforms of half integral weight in plus space.

1 Introduction

The first named author had shown in [4] that given any real number t_0 and $\epsilon > 0$, then for k large enough, the average of the normalized L-functions $L^*(f, s)$ with f varying over a basis of Hecke eigenforms of weight k on $SL_2(\mathbb{Z})$ does not vanish on the line segment $Ims = t_0$, $(k-1)/2 < Res < k/2-\epsilon, k/2+\epsilon < Res < (k+1)/2$. In [5], the authors extend the result for cuspidal Hecke eigenforms of half integer weight. In what follows, we prove a non-vanishing result for sums of L-functions associated to cuspidal Hecke eigenforms of half-integral weight $k+1/2$ where $k \in 2\mathbb{Z}$ on level 4 in the plus space. For this, we determine the Fourier coefficients of the projected kernel function.

2 Projection onto the Plus Space

If $z \in \mathbb{C} - \{0\}$ and $x \in \mathbb{C}$, let $z^x = e^{x \log z}$ where $-\pi < argz \leq \pi$. Let $q = e^{2\pi i z}$. The symbol $\left(\frac{c}{d}\right)$ for c, $d \neq 0$ integers is given in [6]. In addition, for more details about modular forms of half-integer weight, the reader can refer to [6]. We introduce the group $\mathcal{C}$ consisting of all pairs $(A, \phi(z))$ where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{R})$ and $\phi : \mathbb{H} \to \mathbb{C}$ is a holomorphic function with $|\phi(z)| = (detA)^{-1/4}|cz+d|^{1/2}$ satisfying the following group law $(A, \phi(z))(B, \psi(z)) = (AB, \phi(Bz)\psi(z))$. Throughout if $\zeta = (A, \phi(z))$, k integer and $f : \mathbb{H} \to \mathbb{C}$ is a function, we put $f|_{k+1/2}\zeta = f|\zeta = \phi(z)^{-2k-1}f(Az)$. We have an embedding $\Gamma_0(4) \hookrightarrow \mathcal{C}$ given by $A \to A^* := (A, j(A, z))$ where $j(A, z) = \left(\frac{c}{d}\right)\left(\frac{-4}{d}\right)^{-1/2}(cz+d)^{1/2}$ if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Recall also that $M_{k+1/2}(4)$ consists of complex valued functions f holomorphic on $\mathbb{H}$ which satisfy $f|A^* = f$ for all $A \in \Gamma_0(4)$ and which are holomorphic at the cusps. Also, $S_{k+1/2}(4)$ is the subspace of $M_{k+1/2}(4)$

consisting of f which vanish at the cusps. We also define $M^+_{k+1/2}(4)$ to be the subspace of $M_{k+1/2}(4)$ consisting of functions whose $n-$th Fourier coefficients vanish whenever $(-1)^k n \equiv 2,3 \mod (4)$. We also put $S^+_{k+1/2}(4) = S_{k+1/2}(4) \cap M^+_{k+1/2}(4)$. Let $L(f,s)$ be the $L-$function associated to cusp forms $f \in S^+_{k+1/2}(4)$ defined by $L(f,s) = \sum_{(-1)^k n \equiv 0,1 \ (4)} a_f(n) n^{-s}$ for $\sigma := Re\ s >> 1$ where $a_f(n)$ is the $n-$th Fourier coefficient of f. The completed $L-$function is defined as $L^*(f,s) := (2\pi)^{-s} 2^s \Gamma(s) L(f,s)$.

The kernel function of the map $f \to L^*(f,s)$in the case of half integral weight on $\Gamma_0(4)$ is given in [5] by

$$R_{s,k}(z) = \gamma_k(s) \sum_A \left(\frac{c}{d}\right) \left(\frac{-4}{d}\right)^{k+1/2} (cz+d)^{-(k+1/2)} \left(\frac{az+b}{cz+d}\right)^{-s}; \qquad 1 < \sigma < k - 1/2 \tag{1}$$

where

$$\gamma_k(s) = \frac{1}{2} e^{\pi i s/2} \Gamma(s) \Gamma(k + 1/2 - s), \tag{2}$$

and the sum runs over all matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\Gamma_0(4)$.

What we will do is to determine non-vanishing of L-functions for modular forms of half integral weight in the plus space. In order to do so, one needs to define the kernel function that acts on that space.

For the sake of efficiency, we will write

$$R_{s,k}(z) = \gamma_k(s) \sum_A z^{-s}|_{k+1/2} A^* \tag{3}$$

where $A \in \Gamma_0(4)$. In order to determine the kernel function for modular forms in the plus space, one has to use the projection operator pr as given in [2]. For $g \in S_k(\Gamma_0(4))$, we have

$$g|pr = (-1)^{(k+1)/2} \frac{1}{3\sqrt{2}} \left(\sum_{\nu \mod 4} g|\zeta A^*_\nu \right) + \frac{1}{3} g \tag{4}$$

where

$$\zeta = \left(\begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}, e^{(2k+1)\pi i/4} \right), \qquad A^*_\nu = \left(\begin{pmatrix} 1 & 0 \\ 4\nu & 1 \end{pmatrix}, (4\nu z + 1)^{-(k+1/2)} \right).$$

The Fourier expansion of the projected function $g|pr$ (in the more general case of $\Gamma_0(4N)$) is given in [2, Proposition 3] in terms of the Fourier coefficients of g at different cusps. In particular on $\Gamma_0(4)$, we get the following lemma for $R_{s,k}|pr$.

Lemma 1.

$$\begin{aligned} R_{s,k}|pr =& \frac{2}{3} \sum_{n \geq 1, n \equiv 0(4)} \left(a_s(n) + (1 - (-1)^k i) 2^{2k-1} i^{n/4} a_s^1(n/4) \right) q^n \\ &+ \frac{2}{3} \sum_{n \geq 1, (-1)^k n \equiv 1(4)} \left(a_s(n) + 2^{k-1} \left(\frac{(-1)^k n}{2} \right) a_s^{1/2}(n) \right) q^n \end{aligned} \tag{5}$$

where

$$a_s(n) = (2\pi)^s\Gamma(k+1/2-s)n^{s-1} + \frac{1}{2}(2\pi i)^{k+1/2}n^{k-1/2}\frac{\Gamma(s)\Gamma(k+1/2-s)}{\Gamma(k+1/2)}$$
$$\times \sum_{\substack{ac>0\\(a,c)=1,4|c}} \left(\frac{c}{a}\right)\left(\frac{-4}{a}\right)^{k+1/2} c^{-(k+1/2)}\left(\frac{c}{a}\right)^s$$
$$\left(e^{2\pi ina'/c}e^{\pi is/2}{}_1F_1(s,k+1/2;-2\pi in/ac) + e^{-2\pi ina'/c}e^{-\pi is/2}{}_1F_1(s,k+1/2;2\pi in/ac)\right),$$

$$a_s^1(n) = 2\times 4^s(2\pi)^s\Gamma(k+1/2-s)n^{s-1} + 4^s(2\pi i)^{k+1/2}n^{k-1/2}\frac{\Gamma(s)\Gamma(k+1/2-s)}{\Gamma(k+1/2)}$$
$$\times \sum_{\substack{c\geq 1, c\equiv 1 \mod 2\\ d(c)*, ad-bc=1\\ d\equiv -c \mod 4}} \left(\frac{4c}{d}\right)\left(\frac{-4}{-c}\right)^{k+1/2} c^{-(k+1/2)}\left(\frac{c}{a}\right)^s$$
$$\left(e^{2\pi ind/c}e^{\pi is/2}{}_1F_1(s,k+1/2;-2\pi in/ac) + e^{-2\pi ind/c}e^{-\pi is/2}{}_1F_1(s,k+1/2;2\pi in/ac)\right)$$

and

$$a_s^{1/2}(n) = 2\times 4^s(2\pi)^s\Gamma(k+1/2-s)n^{s-1} + 4^s(2\pi i)^{k+1/2}n^{k-1/2}\frac{\Gamma(s)\Gamma(k+1/2-s)}{\Gamma(k+1/2)}$$
$$\times \sum_{\substack{c\geq 1, c\equiv 1 \mod 2\\ d(c)*, ad-2bc=1\\ d\equiv -c \mod 4}} \left(\frac{4c}{d}\right)\left(\frac{-4}{-c}\right)^{k+1/2} c^{-(k+1/2)}\left(\frac{c}{a}\right)^s$$
$$\left(e^{2\pi ind/c}e^{\pi is/2}{}_1F_1(s,k+1/2;-2\pi in/ac) + e^{-2\pi ind/c}e^{-\pi is/2}{}_1F_1(s,k+1/2;2\pi in/ac)\right)$$

Proof. As for the expansion at the infinite cusp, we follow similar steps as in [4] with minor changes due to the fact of considering real weight. We consider first the case $ac = 0$ where we get the elements $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ in $\Gamma_0(4)$. The contribution of the terms with $ac = 0$ to the sum thus is given by

$$(2\pi)^s\Gamma(k+1/2-s)\sum_{n\geq 1} n^{s-1}e^{2\pi inz}. \tag{6}$$

To determine the contribution of the terms with $ac \neq 0$, we consider

$$\int_{iC}^{iC+1} \sum_{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\Gamma_0(4); ac\neq 0} \left(\frac{c}{d}\right)\left(\frac{-4}{d}\right)^{k+1/2}(cz+d)^{-(k+1/2)}\left(\frac{az+b}{cz+d}\right)^{-s} e^{-2\pi inz}dz, \quad C>0. \tag{7}$$

We get

$$R_{s,k}(z) = \sum_{n\geq 1} a_s(n) e^{2\pi inz} \tag{8}$$

where the Fourier coefficients $a_s(n)$ are given by

$$a_s(n) = (2\pi)^s \Gamma(k+1/2-s) n^{s-1} + \frac{1}{2}(2\pi i)^{k+1/2} n^{k-1/2} \frac{\Gamma(s)\Gamma(k+1/2-s)}{\Gamma(k+1/2)}$$
$$\times \sum_{\substack{ac>0 \\ (a,c)=1, 4|c}} \left(\frac{c}{a}\right)\left(\frac{-4}{a}\right)^{k+1/2} c^{-(k+1/2)} \left(\frac{c}{a}\right)^s$$
$$\left(e^{2\pi ina'/c} e^{\pi is/2} {}_1F_1(s, k+1/2; -2\pi in/ac) + e^{-2\pi ina'/c} e^{-\pi is/2} {}_1F_1(s, k+1/2; 2\pi in/ac)\right).$$

Here a' is an inverse of a modulo c, and ${}_1F_1(\alpha, \beta; z)$ is Kummer's degenerate hypergeometric function.

As for the expansion at the cusp 1, we have to determine the expansion of

$$R_{s,k}|\eta^1(z) \tag{9}$$

where $\eta^1(z) = \left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, (z+1)^{k+1/2}\right)$.

We have to determine $a_s^1(n)$ such that

$$R_{s,k}|\eta^1(z) = (z+1)^{-(k+1/2)} R_{s,k}\left(\frac{z}{z+1}\right) = \sum_{n\geq 1} a_s^1(n) e^{2\pi inz/4} \tag{10}$$

We have

$$\begin{aligned} R_{s,k}|\eta^1(z) &= \gamma_k(s)(z+1)^{-(k+1/2)} \sum_{n\in\mathbb{Z}} \left(\frac{z}{z+1}+n\right)^{-s} \\ &+ 2\gamma_k(s) \sum_{\substack{c>0, d \\ (d,c)=1, ad-bc=1}} \left(\frac{4}{d}\right)\left(\frac{c+d}{d}\right)\left(\frac{-4}{d}\right)^{k+1/2} ((c+d)z+d)^{-(k+1/2)} \left(\frac{(a+b)z+b}{(c+d)z+d}\right)^{-s} \end{aligned} \tag{11}$$

where a and b chosen such that $ad-bc=1$.

Replace $c+d$ by c and $a+b$ by a to get

$$
\begin{aligned}
R_{s,k}|\eta^1(z) &= \gamma_k(s)(z+1)^{-(k+1/2)} \sum_{n\in\mathbb{Z}} \left(\frac{z(n+1)+n}{z+1}\right)^{-s} \\
&+ 2\gamma_k(s) \sum_{\substack{c>0,d \\ (d,c)=1, ad-bc=1 \\ c\equiv d \ mod\ 4}} \left(\frac{4c}{d}\right)\left(\frac{-4}{d}\right)^{k+1/2} (cz+d)^{-(k+1/2)} \left(\frac{az+b}{cz+d}\right)^{-s} \\
&= 2\gamma_k(s) \sum_{\substack{c>0,d \\ (d,c)=1, ad-bc=1 \\ d\equiv -c \ mod\ 4, c\equiv 1 \ mod\ 2}} \left(\frac{4c}{d}\right)\left(\frac{-4}{d}\right)^{k+1/2} (cz+d)^{-(k+1/2)} \left(\frac{az+b}{cz+d}\right)^{-s}.
\end{aligned} \tag{12}
$$

Thus we get

$$
\begin{aligned}
R_{s,k}|\eta^1(z) = 2\times 4^s\gamma_k(s) &\sum_{c\geq 1, c\equiv 1\ mod\ 2} \left(\frac{-4}{-c}\right)^{k+1/2} \sum_{\substack{d(c)^*, ad-bc=1 \\ d\equiv -c\ mod\ 4}} \left(\frac{4c}{d}\right) \\
&.\sum_{r\in\mathbb{Z}} (4c(z/4+r)+d)^{-k-1/2} \left(\frac{a(z/4+r)+b}{c(z/4+r)+d}\right)^{-s}
\end{aligned} \tag{13}
$$

where $d(c)^*$ means that d runs through a primitive residue system modulo c. We proceed with finding $a_s^1(n)$ similarly as we determined the expansion at infinity to get

$$
\begin{aligned}
&a_s^1(n) = 2\times 4^s(2\pi)^s\Gamma(k+1/2-s)n^{s-1} + 4^s(2\pi i)^{k+1/2}n^{k-1/2}\frac{\Gamma(s)\Gamma(k+1/2-s)}{\Gamma(k+1/2)} \\
&\times \sum_{\substack{c\geq 1, c\equiv 1\ mod\ 2 \\ d(c)^*, ad-bc=1 \\ d\equiv -c\ mod\ 4}} \left(\frac{4c}{d}\right)\left(\frac{-4}{-c}\right)^{k+1/2} c^{-(k+1/2)} \left(\frac{c}{a}\right)^s \\
&\left(e^{2\pi ind/c}e^{\pi is/2}{}_1F_1(s,k+1/2;-2\pi in/ac) + e^{-2\pi ind/c}e^{-\pi is/2}{}_1F_1(s,k+1/2;2\pi in/ac)\right).
\end{aligned}
$$

What remains is to determine the expansion at the cusp 1/2. We have

$$
R_{s,k}|\eta^{1/2}(z) \tag{14}
$$

where $\eta^{1/2}(z) = \left(\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, (2z+1)^{k+1/2}\right)$.

We have to determine $a_s^{1/2}(n)$ such that

$$
R_{s,k}|\eta^{1/2}(z) = (2z+1)^{-(k+1/2)}R_{s,k}\left(\frac{z}{2z+1}\right) = \sum_{n\geq 1, n\equiv 1\ mod\ 4} a_s^1(n)e^{2\pi inz/4}. \tag{15}
$$

We argue in the same way as at the cusp 1 and we obtain

$$a_s^{1/2}(n) = 2\times 4^s(2\pi)^s\Gamma(k+1/2-s)n^{s-1} + 4^s(2\pi i)^{k+1/2}n^{k-1/2}\frac{\Gamma(s)\Gamma(k+1/2-s)}{\Gamma(k+1/2)}$$
$$\times \sum_{\substack{c\geq 1, c\equiv 1 \mod 2\\ d(c)^*, ad-2bc=1\\ d\equiv -c \ mod\ 4}} \left(\frac{4c}{d}\right)\left(\frac{-4}{-c}\right)^{k+1/2} c^{-(k+1/2)}\left(\frac{c}{a}\right)^s$$
$$\left(e^{2\pi i n d/c}e^{\pi i s/2}{}_1F_1(s,k+1/2;-2\pi i n/ac) + e^{-2\pi i n d/c}e^{-\pi i s/2}{}_1F_1(s,k+1/2;2\pi i n/ac)\right).$$

□

3 Statement of Results

Throughout this section, we will assume that $k\in 2\mathbb{Z}$. If $f=\sum_{n\equiv 0,1(4)} a_f(n)e^{2\pi i n z}\in S^+_{k+1/2}(4)$, the L-function associated to f is defined by

$$L(f,s) = \sum_{n\equiv 0,1(4)} \frac{a_f(n)}{n^s}, \qquad \sigma >> 1.$$

The completed $L-$function defined as $L^*(f,s)$ has the following functional equation

$$L^*(f|W_4, k+1/2-s) = L^*(f,s) \tag{16}$$

where W_4 is the Fricke involution on $S_{k+1/2}(4)$ defined by

$$f|W_4(z) = (-2\pi i z)^{-k-1/2}f(-1/4z), \tag{17}$$

cf [6]. Using [1], we know that $S^+_{k+1/2}(4)$ has an orthogonal basis of Hecke eigenforms with respect to all Hecke operators $T^+(p^2)$, p prime. Let $\{f_{k,1}, f_{k,2}, ...f_{k,d}\}$ be such a basis of Hecke eigenforms, where d is the dimension of the space $S^+_{k+1/2}(4)$. We know as well that we can choose the coefficients $a(n)$ of the eigenforms to be real.

Since our projection operator is hermitian, we have for $f\in S^+_{k+1/2}(4)$,

$$< f, R_{\bar{s},k} >=< f|pr, R_{\bar{s},k} >=< f, R_{\bar{s},k}|pr >= c_k L^*(f, k+1/2-s) \tag{18}$$

where c_k is a non-zero constant depending only on k. We therefore deduce that

$$< f, R_{\bar{s},k}|pr >= c_k L^*(f,k+1/2-s) \tag{19}$$

for all $f\in S^+_{k+1/2}(4)$. As a result, we have

$$R_{s,k}|pr = c_k^{'}\sum_{j=1}^{d}\frac{1}{< f_{k,j}, f_{k,j} >}L^*(f_{k,j}, k+1/2-s)f_{k,j} \tag{20}$$

(where $c_k^{'}$ = constant) and we get

$$R_{s,k}|pr = c_k^{'} \sum_{j=1}^{d} \frac{1}{< f_{k,j}, f_{k,j} >} L^*(f_{k,j}|W_4, s) f_{k,j}. \tag{21}$$

Theorem 1. *Let $t_0 \in \mathbb{R}$. Then there exists a constant $C(t_0, \epsilon) > 0$ depending only on t_0 and ϵ such that for $k > C(t_0, \epsilon)$ and $k \in 2\mathbb{Z}$, the function*

$$\sum_{j=1}^{d} \frac{1}{< f_{k,j}, f_{k,j} >} L^*(f_{k,j}|W_4, s) \tag{22}$$

does not vanish at any point $s = \sigma + it_0$ with $t = t_0, k/2 - 1/4 < \sigma < k/2 + 3/4$.

Proof. Assume that $\sum_{j=1}^{d} \frac{1}{<f_{k,j}, f_{k,j}>} L^*(f_{k,j}|W_4, s) = 0$, for s as given in the theorem. This implies in particular that the first Fourier coefficient of $R_{s,k}|pr$ is 0. Thus dividing by $\frac{2}{3} 4^s (2\pi)^s \Gamma(k + 1/2 - s)$, we have for $k \in 2\mathbb{Z}$,

$$\begin{aligned}
&4^{-s} + \frac{(2\pi i)^{k+1/2-s}}{2 \times 4^s \Gamma(k + 1/2 - s)} \sum_{\substack{ac>0 \\ (a,c)=1, 4|c}} \left(\frac{c}{a}\right) \left(\frac{-4}{a}\right)^{k+1/2} c^{-(k+1/2)} (c/a)^s \\
&\left(e^{2\pi i a'/c} e^{\pi i s/2} {}_1f_1(s, k + 1/2; -2\pi i/ac) + e^{-2\pi i a'/c} e^{-\pi i s/2} {}_1f_1(s, k + 1/2; 2\pi i/ac)\right) \\
&+ 2 + \frac{(2\pi i)^{k+1/2-s}}{\Gamma(k + 1/2 - s)} \sum_{\substack{c \geq 1, c \equiv 1 \ mod\ 2 \\ d(c)^*, ad-2bc=1 \\ d \equiv -c \ mod\ 4}} \left(\frac{4c}{d}\right) \left(\frac{-4}{-c}\right)^{k+1/2} c^{-(k+1/2)} (c/a)^s \\
&\left(e^{2\pi i d/c} e^{\pi i s/2} {}_1f_1(s, k + 1/2; -2\pi i/ac) + e^{-2\pi i d/c} e^{-\pi i s/2} {}_1f_1(s, k + 1/2; 2\pi i/ac)\right) = 0,
\end{aligned} \tag{23}$$

where

$${}_1f_1(\alpha, \beta; z) = \frac{\Gamma(\alpha)\Gamma(\beta - \alpha)}{\Gamma(\beta)} {}_1F_1(\alpha, \beta; z). \tag{24}$$

Let $s = k/2 + 1/4 - \delta + it_0$, where $0 \leq \delta \leq 1/2$. We have in [4] that

$$|{}_1f_1(s, k + 1/2; 2\pi i/ac)| \leq 1. \tag{25}$$

As a result, we have

$$\begin{aligned}
2 - 4^{-k/2-1/4+\delta} \leq |2 + 4^{-s}| \leq &\frac{(2\pi)^{k/2+1/4+\delta}}{2 \times 4^{k/2+1/4-\delta} |\Gamma(\frac{k}{2} + \frac{1}{4} + \delta - it_0)|} \sum_{\substack{ac>0 \\ (a,c)=1, 4|c}} \frac{1}{c^{k+1/2}} (e^{-\pi t_0/2} + e^{\pi t_0/2}) \\
&+ \frac{(2\pi)^{k/2+1/4+\delta}}{|\Gamma(\frac{k}{2} + \frac{1}{4} + \delta - it_0)|} \sum_{\substack{c \geq 1, c \equiv 1 \ mod\ 2 \\ d(c)^*, ad-2bc=1 \\ d \equiv -c \ mod\ 4}} \frac{1}{c^{k+1/2}} (e^{-\pi t_0/2} + e^{\pi t_0/2})
\end{aligned} \tag{26}$$

We now take the limit of both sides as $k \to \infty$ to get $2 \leq 0$ and hence a contradiction. □

It is important to note that the above result translates into the same non-vanishing result for $L^*(f_{k,j}, s)$ and this due to the fact that the strip we are considering is invariant under $s \to k + 1/2 - s$.

Corollary 1. *Let $s_0 = \sigma + it_0$ with $k/2 - 1/4 < \sigma < k/2 + 3/4$, then for $k > C(t_0, \epsilon)$, k even, there exists a Hecke eigenform $f \in S^+_{k+1/2}(4)$ whose L-value does not vanish at s_0.*

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Winfried Kohnen, University of Heidelberg, Im Neuenheimer Feld 205, 69120 Heidelberg, Germany
E-mail address: **winfried@mathi.uni-heidelberg.de**
Wissam Raji, American University of Beirut, Beirut, Lebanon
E-mail address: **wr07@aub.edu.lb**