GENERIC REPRESENTATIONS IN L-PACKETS

MANISH MISHRA

Abstract. We give the details of the construction of a map to restate a conjectural expression about adjoint group action on generic representations in L-packets. We give an application of the construction to give another proof of the classification of the Knapp-Stein R-group associated to a unitary unramified character of a torus. Finally we prove the conjecture for unramified L-packets.

1. Introduction

Let G be a quasi-split connected reductive group defined over a local field k of characteristic zero and let Z be the center of G. Let B be a k-Borel subgroup of G and let T be a maximal k-torus in B. Let U be the unipotent radical of B. A character $\psi : U(k) \to \mathbb{C}^\times$ is called *generic* if the stabilizer of ψ in $T(k)$ is exactly the center $Z(k)$. An irreducible admissible representation π of G is called *generic* (*ψ-generic*) if there exists a generic character ψ of $U(k)$ such that $\mathrm{Hom}_{G(k)}(\pi, \mathrm{Ind}_{U(k)}^{G(k)}\psi) \neq 0$.

The conjectural *local Langlands program* partitions the irreducible admissible representations of G into finite sets known as *L-packets.* Each L-packet is expected to be parametrized by an arithmetic object called the *Langlands parameter,* which is an *admissible homomorphism* from the *Weil-Deligne* group W_k' of k to the *L-group* ${}^LG$ of G. See [Bor79a] for the definitions and statements.

To each Langlands parameter φ, one can associate a finite group $\mathcal{S}_\varphi$ (see [Art06, Section 1, eq. (1.1)]). It is expected that the associated L-packet Π_φ is parametrized by the irreducible representations $\widehat{\mathcal{S}_\varphi}$ of $\mathcal{S}_\varphi$ [Art06, Section 1]. The parametrization will depend on the choice of a Whittaker datum for G, which is a $G(k)$-conjugacy class of pairs (B, ψ), where ψ is a generic character of $U(k)$. When Π_φ is generic, i.e., it has a generic representation, the ψ-generic representation in Π_φ is then required to correspond to the trivial representation of $\mathcal{S}_\varphi$. The parametrization is also expected to satisfy certain conjectural endoscopic character identity [Kal13].

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When φ is a *tempered* parameter, i.e., a parameter whose image projects onto a relatively compact subset of the complex dual $\hat{G}$ of G, Shahidi's *tempered L-packet conjecture* [Sha90, §9] states that Π_φ must be generic.

Let Γ_k be the absolute Galois group of k and write $\mathrm{H}^1(k,-)$ for $\mathrm{H}^1(\Gamma_k,-)$. In Section 3, we construct a map $\gamma_\varphi : R_\varphi := \pi_0(Z_{\hat{G}}(\mathrm{Im}(\varphi))) \to \mathrm{H}^1(k, X(Z))$, where $X(Z)$ is the character lattice of Z and φ is any Langlands parameter. Using Tate duality, we get the dual map $\hat{\gamma}_\varphi : \mathrm{H}^1(k,Z) \to \widehat{R_\varphi}$, where $\widehat{R_\varphi}$ is the set of irreducible representations of R_φ. Let $p : t \in T \to \bar{t} \in T_{\mathrm{ad}} := T/Z$ be the adjoint morphism. The finite abelian group $T_{\mathrm{ad}}(k)/p(T(k)) \hookrightarrow \mathrm{H}^1(k,Z)$ acts simply transitively on the set of $T(k)$-orbits of generic characters [DR10, §3]. The map $\zeta_\varphi := \hat{\gamma}_\varphi | T_{\mathrm{ad}}(k)/p(T(k))$ factors through $\widehat{\mathcal{S}_\varphi}$ (see [GGP12, Sec. 9(4)] , also [Kal13, Sec. 3]).

Now fix a parametrization $\rho \in \widehat{\mathcal{S}_\varphi} \mapsto \pi_\rho \in \Pi_\varphi$ by making the choice of a Whittaker datum. The following is a version of the conjecture in [GGP12, Sec. 9(3)] for generic L-packets.

Conjecture. *A representation $\pi_\rho \in \Pi_\varphi$ is ψ-generic iff $\pi_{t\cdot\rho}$ is $t \cdot \psi$ generic for all $t \in T_{\mathrm{ad}}(k)$, where $t \cdot \rho := \rho \otimes \zeta_\varphi(t)$.*

The map $\hat{\gamma}_\varphi$ was constructed in [Kuo10] in a very special case (G split semisimple and φ is the parameter associated to a unitary character of $T(k)$). For depth zero supercuspidal L-packets, the conjecture follows from [DR10]. When G is semisimple and split and the L-packet is formed by the constituents of a unitary principal series, the conjecture follows from [Kuo02]. In [Kal13], Kaletha gives a proof of the above conjecture for classical groups using very general arguments.

Now let G be unramified and let φ be the parameter associated to a unitary unramified character λ of $T(k)$. The construction of the map γ_φ allows one to obtain a nice description of the group R_φ as a certain subgroup of an extended affine Weyl group (Proposition 10). Using this, in Theorem 11, we obtain in a conceptual and uniform way, the classification of the Knapp-Stein R-group associated to λ. This kind of classification was obtained by Keys [Key82, §3] in a case by case manner. For split groups, using different methods, another way of getting the classification obtained by Keys was recently given by Kamran and Plymen [KP13]. Our situation is more general and we also describe the isomorphism, which has a simple description.

Finally in Theorem 12, we prove the conjecture for *unramified L-packets* (see Sec. 5). We do not assume the packet to be tempered.

2. Preliminaries

2.1. **Group Cohomology.** For details about this subsection, see [Ser97, Ch. 5].

Let Γ be a topological group and let

$$1 \to A \to B \to C \to 1 \tag{2.1}$$

be a short exact sequence of Γ-groups. Assume that A is central subgroup of B. Then C acts on B by inner automorphisms and it acts trivially on A. Let $\gamma : \Gamma \to C$ be a co-cycle in C, i.e., it satisfies the relation $\gamma(ab) = \gamma(a)^a\gamma(b)$ for all $a, b \in \Gamma$. By twisting the short exact sequence in (2.1) by γ, we get another short exact sequence

$$1 \to A \to {}_\gamma B \to {}_\gamma C \to 1$$

From this we get a long exact cohomology sequence

$$\begin{aligned} 1 \quad \to & \quad \mathrm{H}^0(\Gamma, A) \to \mathrm{H}^0(\Gamma, {}_\gamma B) \to \mathrm{H}^0(\Gamma, {}_\gamma C) \to \mathrm{H}^1(\Gamma, A) \to \\ \to & \quad \mathrm{H}^1(\Gamma, {}_\gamma B) \to \mathrm{H}^1(\Gamma, {}_\gamma C). \end{aligned}$$

2.2. **Affine roots and affine transformations.**

2.2.1. *The group* Ω. Let $\Psi = (X, R, \Delta, \check{X}, \check{R}, \check{\Delta})$ be a based root datum in the sense of [Spr79, 1.9]. So X and $\check{X}$ are free abelian groups in duality by a pairing $X \times \check{X} \to \mathbb{Z}$, R is a root system in the vector space $Q \otimes \mathbb{R}$, where Q is the root lattice, $\check{R}$ is the set of co-roots, $\Delta \subset R$ is a basis and $\check{\Delta}$ is the dual basis. Let $W = W(\Psi)$ be the Weyl group. The set Δ determines an alcove C in $V := X \otimes \mathbb{R}$ in the following way. Let $\check{\Delta} = \{\check{\alpha}_1, \ldots, \check{\alpha}_l\}$ and let $\check{\beta} = \sum_{i=1}^{l} n_i \check{\alpha}_i$ be the highest co-root. Then C is the alcove in V defined by $C = \{x \in V : \check{\alpha}_0(x) \geq 0, \ldots, \check{\alpha}_l(x) \geq 0\}$, where $\check{\alpha}_0 = 1 - \check{\beta}$. Let $\tilde{W} = W \ltimes X$ and $\tilde{W}^\circ = W \ltimes Q$. Let Ω be the stabilizer of C in $\tilde{W}$. Then $\tilde{W} = \Omega \ltimes \tilde{W}^\circ$.

Now assume that Ψ is *semisimple* [Spr79, 1.1] and R is an irreducible root system in V. Let c_0 be the *weighted barycenter* of C, characterized by the equations $\check{\alpha}_i(c_0) = 1/h$ for $i = 0, ., l$, where h is the Coxeter number. For any $w \in W$, let $\tilde{w}$ be the affine map $x \in V \mapsto w(x - c_0) + c_0$. It is the unique affine map fixing c_0 with tangent part w. The following lemmas follow from [AYY13, Lemma 6.2].

Lemma 1. *For any $w \in W$, the following are equivalent:*

(1) $\tilde{w} \in \tilde{W}$.
(2) $\tilde{w} \in \Omega$.

Lemma 2. *There is an isomorphism $\iota : \Omega \to X/Q$ defined by any of the following ways*

(1) $\iota(\tilde{w}) = (w^{-1} - 1)c_0 + Q$.
(2) The natural projection $\tilde{W} \to \tilde{W}/\tilde{W}^\circ = X/Q$ restricted to Ω.

2.2.2. *Based root datum.* Let $\Psi = (X, R, \Delta, \check{X}, \check{R}, \check{\Delta})$ be a reduced based root datum. Let θ be a finite group acting on Ψ. In [Yu], Jiu-Kang Yu defines the following 6-touple $\underline{\Psi} = (\underline{X}, \underline{R}, \underline{\Delta}, \underline{\check{X}}, \underline{\check{R}}, \underline{\check{\Delta}})$:

$$\begin{aligned}
\underline{X} &= X_\theta/\text{torsion}, \\
\underline{\check{X}} &= \check{X}^\theta, \\
\underline{R} &= \{\underline{a} : a \in R\}, \qquad \text{where } \underline{a} := a|_{\underline{\check{X}}} \\
\underline{\check{R}} &= \{\breve{\alpha} : \alpha \in \underline{R}\}, \\
\underline{\Delta} &= \{\underline{a} : a \in \Delta\}, \\
\underline{\check{\Delta}} &= \{\breve{\alpha} : \alpha \in \underline{\Delta}\}.
\end{aligned}$$

The explanation for the defining formulas is as follows. We first note that $\underline{X}$ and $\underline{\check{X}}$ are free abelian groups, dual to each other under the canonical pairing $(\underline{x}, y) \mapsto < x, y >$, for $\underline{x} \in \underline{X}$, $y \in \underline{\check{X}} \subset \check{X}$, where x is any preimage of $\underline{x}$ in X. Define $\breve{\alpha}$ for $\alpha \in \underline{R}$ as follows:

$$\breve{\alpha} = \begin{cases} \sum\limits_{a \in R : a|_{\underline{\check{X}}} = \alpha} \check{a}, & \text{if } 2\alpha \notin \underline{R} \\ 2 \sum\limits_{a \in R : a|_{\underline{\check{X}}} = \alpha} \check{a}, & \text{if } 2\alpha \in \underline{R} \end{cases} \tag{2.2}$$

In [Yu], Jiu-Kang Yu proves the following.

Theorem 3. [Jiu-Kang Yu] *The 6-touple $\underline{\Psi} = (\underline{X}, \underline{R}, \underline{\Delta}, \underline{\check{X}}, \underline{\check{R}}, \underline{\check{\Delta}})$, with the canonical pairing between $\underline{X}$ and $\underline{\check{X}}$ and the correspondence $\underline{R} \to \underline{\check{R}}$, $\alpha \mapsto \breve{\alpha}$, is a based root datum. Moreover, the homomorphism $W(\Psi)^\sigma \to \mathbf{GL}(\underline{\check{X}})$, $w \mapsto w|_{\underline{\check{X}}}$ is injective and the image is $W(\underline{\Psi})$.*

The above Theorem for simply connected groups is proved in [Ree10, Sec. 3.3].

3. A construction and a conjecture

3.1. **Construction.** Let G be a quasi-split group defined over a local field k of characteristic zero. Let T be a maximal k-torus of G which is contained in a k-Borel subgroup B. Let $\hat{G}_{\text{sc}}$ be the simply connected cover of the derived group $\hat{G}_{\text{der}}$ of $\hat{G}$, where $\hat{G}$ is the complex dual of G. Let $\hat{T} \subset \hat{G}$ be the torus dual to T and $\hat{T}_{\text{sc}}$ be the pull back of $(\hat{T} \cap \hat{G}_{\text{der}})^\circ$ via $\hat{G}_{\text{sc}} \to \hat{G}_{\text{der}}$. Let $X = X(T)$ (resp. $\check{X} = \check{X}(T)$) denote the group of characters (resp. co-characters) of T. Let Z be the center of G and let $\hat{\mathfrak{z}}$ be the Lie algebra of the center $\hat{Z}$ of $\hat{G}$. Then $\tilde{G} := \hat{G}_{\text{sc}} \times \hat{\mathfrak{z}}$ is the topological universal cover of $\hat{G}$. We have a short exact sequence

$$1 \to \pi_1(\hat{G}) \to \tilde{G} \to \hat{G} \to 1, \tag{3.1}$$

where $\pi_1(\hat{G})$ is the topological fundamental group of $\hat{G}$. Let Q denote the root lattice. Then from [Spr79, 2.15],

$$X(Z) \cong X/Q. \tag{3.2}$$

The algebraic fundamental group of $\hat{G}$ is $\check{X}(\hat{T})/\check{X}(\hat{T}_{\mathrm{sc}}) = X/Q$. Since $\hat{G}$ is a complex algebraic group, its algebraic fundamental group is the same as its topological fundamental group (see [BGA14]). Therefore

$$X/Q \cong \pi_1(\hat{G}). \tag{3.3}$$

Let W_k (resp. Γ_k) denote the Weil group (resp. absolute Galois group) of k. Define $W_k' := W_k$ if k is archimedean and $W_k' := W_k \times \mathrm{SL}(2,\mathbb{C})$ if k is non-archimedean. W_k' is called the Weil-Deligne group of k. Let $\varphi : W_k' \to {}^L G$ be a Langlands parameter (see [Bor79b, Sec. 8.2]). View φ as an admissible homomorphism. Then φ determines a co-cycle $\phi|_{W_k} : W_k \to {}^L G \to \hat{G}$. We can twist the exact sequence (3.1) by the co-cycle ϕ (see Section 2.1). Then using the isomorphism $X(Z) \cong \pi_1(\hat{G})$, we get

$$\tilde{\gamma} : \mathrm{H}^0(W_k, {}_\phi\hat{G}) \to \mathrm{H}^1(W_k, X(Z)).$$

Since $\mathrm{H}^0(W_k, {}_\phi\hat{G}) \supset Z_{\hat{G}}(\mathrm{Im}(\varphi))$, by restriction this induces

$$\tilde{\gamma}' : Z_{\hat{G}}(\mathrm{Im}(\varphi)) \to \mathrm{H}^1(W_k, X(Z)).$$

Since this map is continuous and $\mathrm{H}^1(W_k, X(Z))$ is discrete, $\ker(\tilde{\gamma}') \supset (Z_{\hat{G}}(\mathrm{Im}(\varphi)))^\circ$. Thus we get a map

$$\gamma_\varphi' : R_\varphi := \pi_0(Z_{\hat{G}}(\mathrm{Im}(\varphi))) \to \mathrm{H}^1(W_k, X(Z)). \tag{3.4}$$

Since R_φ is finite, γ_φ' induces

$$\gamma_\varphi'' : R_\varphi \to H^1(W_k, X(Z))^{\mathrm{tor}}.$$

By [Kar11, Theorem 4.1.3 (ii)], we have a functorial isomorphism

$$\mathrm{H}^1(W_k, X(Z))^{\mathrm{tor}} = \mathrm{H}^1(k, X(Z)).$$

Here we are abbreviating $\mathrm{H}^1(\Gamma_k, -)$ by the notation $\mathrm{H}^1(k, -)$. We thus get a map

$$\gamma_\varphi : R_\varphi \to \mathrm{H}^1(k, X(Z)). \tag{3.5}$$

By Tate Duality ([Mil06, Corr. 2.4]), we have an isomorphism

$$\mathrm{H}^1(k, X(Z)) \cong \mathrm{Hom}(\mathrm{H}^1(k, Z), \mathbb{C}^\times). \tag{3.6}$$

Using the isomorphism (3.6) in (3.5), we get a map

$$\hat{\gamma}_\varphi : \mathrm{H}^1(k, Z) \to \widehat{R_\varphi}, \tag{3.7}$$

where $\widehat{R_\varphi}$ is the set of irreducible representations of R_φ. Since $\mathrm{H}^1(k, X(Z))$ is abelian, the image of $\hat{\gamma}_\varphi$ lies in the group of one dimensional representations of R_φ.

3.2. **Statement of a conjecture.** Let U be the unipotent radical of B and let $p : G \to G_{\mathrm{ad}} := G/Z$ be the adjoint morphism. We denote by the same symbol, the induced map $p : T \to T_{\mathrm{ad}} := T/Z$.

Definition 4. A character $\psi : U(k) \to \mathbb{C}^\times$ is *generic* if its stabilizer in $T_{\mathrm{ad}}(k)$ is trivial.

The group $T_{\mathrm{ad}}(k)$ acts simply transitively on the set of generic characters of $U(k)$. Hence the finite abelian group $T_{\mathrm{ad}}(k)/p(T(k))$ acts simply transitively on the set of $T(k)$-orbits of generic characters.

Definition 5. The *pure inner forms* of G are the groups G' over k which are obtained by inner twisting by elements in the pointed set $\mathrm{H}^1(k, G)$.

All pure inner forms have the same center Z over k. Let G' be a pure inner form of G. Denote the maximal torus of G' (resp. G'_{ad}) corresponding to T (resp. T_{ad}) by T' (resp. T'_{ad}). We will denote the adjoint morphism for all inner forms by the same symbol p.

We have a canonical inclusion $T'_{\mathrm{ad}}(k)/p(T'(k)) \hookrightarrow \mathrm{H}^1(k, Z)$ and a canonical isomorphism $T'_{\mathrm{ad}}(k)/p(T(k)) \cong G'_{\mathrm{ad}}(k)/p(G'(k))$ (Lemma 5.1 [DR10]). Equation (3.7) thus induces

$$\zeta'_\varphi : G'_{\mathrm{ad}}(k)/p(G'(k)) \to \widehat{R_\varphi}.$$

Let $\tilde{\Pi}_\varphi$ denote the *Vogan L-packet* associate to φ. It is the union of the standard L-packets associated to φ of G and all its pure inner forms. By standard, we mean L-packets as defined in [Bor79a]. Let $\rho \in \widehat{R_\varphi} \mapsto \pi_\rho \in \tilde{\Pi}_\varphi$ be the parametrization defined after the choice of a Whittaker datum. Assume that this parametrization is compatible with Deligne's normalization of the local Artin map (see [GGP12, Sec. 3]). Let Π'_φ be the standard L-packet of G' contained in $\tilde{\Pi}_\varphi$. The following is a conjecture in [GGP12, Sec. 9 (3)].

Conjecture 6. *For $g \in G'_{\mathrm{ad}}(k)$, $\pi_\rho \circ \mathrm{Ad}(g) = \pi_{g\cdot\rho}$, where $g \cdot \rho = \rho \otimes \zeta'_\varphi(g)$ and $\pi_\rho \in \Pi'_\varphi$. Thus π_ρ is ψ-generic iff $\pi_{g\cdot\rho}$ is $g \cdot \psi$ generic.*

We have a natural inclusion $\pi_0(\hat{Z}^{\Gamma_k}) \subset \widehat{R_\varphi}$. Let $\tau \in \widehat{R_\varphi}$. In [GGP12, Sec. 9(4)], it is explained that the pure inner form of G which acts on the representation corresponding to the parameter (φ, τ) is determined by the character $\tau|\pi_0(\hat{Z}^{\Gamma_k})$. Thus the standard L-packet $\Pi_\varphi \subset \tilde{\Pi}_\varphi$ of G is parametrized by $\tau \in \widehat{R_\varphi}$ whose restriction to $\pi_0(\hat{Z}^{\Gamma_k})$ is trivial. In other words, the standard L-packet is parametrized by the irreducible representations $\widehat{\mathcal{S}_\varphi} \hookrightarrow \widehat{R_\varphi}$ of the group $\mathcal{S}_\varphi := \pi_0(Z_{\hat{G}}(\mathrm{Im}(\varphi))/\hat{Z}^{\Gamma_k})$. The map $\zeta_\varphi : G_{\mathrm{ad}}(k)/p(G(k)) \to \widehat{R_\varphi}$ thus must factor through $\widehat{\mathcal{S}_\varphi}$. Conjecture 6 for standard generic L-packets can be stated as:

Conjecture 6′. *$\pi_\rho \in \Pi_\varphi$ is ψ-generic iff $\pi_{g\cdot\rho}$ is $g\cdot\psi$ generic, where $\rho \in \widehat{R_\varphi}$ $g \in G_{\mathrm{ad}}(k)$, and where $g\cdot\rho = \rho \otimes \zeta_\varphi(g)$.*

Remark 7. In [Kal13, Sec. 3], Kaletha constructs a map $\zeta_\varphi : G_{\mathrm{ad}}(k)/p(G(k)) \to \widehat{\mathcal{S}_\varphi}$. In [Kal13, Sec. 1, eq. (1.1)], he states the above conjecture in a more precise manner by comparing the parametrization of a tempered L-packet for different choices of Whittaker data. He also points out that the action of $g \in G_{\mathrm{ad}}(k)$, should send $\rho \in \widehat{\mathcal{S}_\varphi}$ to $\rho \otimes \zeta_\varphi(g)$ or $\rho \otimes \zeta_\varphi^{-1}(g)$ depending on which of the two possible normalizations of the local Artin map one chooses. The normalization in Conjecture 6 uses Deligne's normalization [GGP12, Sec. 3].

4. Description of R-group

Let the notations be as in Section 3. Assume that G is unramified, i.e., it is quasi-split and split over an unramified extension of k. We also assume k to be non-archimedean. Let I be the inertia subgroup of W_k and let σ be the Frobenious element in W_k/I. Throughout this section, we will abbreviate $\mathrm{H}^1(W_k/I, -)$ by the notation $\mathrm{H}^1(\sigma, -)$.

4.1. **Case of an unramified parameter.** Let $\bar{s} \in \hat{T}$ and let φ be the Langlands parameter determined by the map $\sigma \mapsto \bar{s}$. Let s be a lift of $\bar{s}$ in $\hat{T}_{\mathrm{sc}} \times \hat{\mathfrak{z}}$.

Let $\mathrm{H}^1(\sigma, \tilde{G})_{\mathrm{ss}} \subset \mathrm{H}^1(\sigma, \tilde{G})$ denote the σ-conjugacy classes of the semisimple elements of $\tilde{G}$, where $\tilde{G} = \hat{G}_{\mathrm{sc}} \times \hat{\mathfrak{z}}$ as in Section 3. Denote by $[t]$, the class of $t \in \tilde{G}_{\mathrm{ss}}$ in $\mathrm{H}^1(\sigma, \tilde{G})_{\mathrm{ss}}$. Let $A := \pi_1(\hat{G})$. Let $\underline{A}$ denote A_σ, the co-invariant of A with respect to σ. We have $\mathrm{H}^1(\sigma, A) \cong \underline{A}$. Let $\underline{x}$ denote the image of $x \in A$ in $\underline{A}$. Then there is an action of $\mathrm{H}^1(\sigma, A)$ on $\mathrm{H}^1(\sigma, \tilde{G})_{\mathrm{ss}}$ given by

$$\underline{x} \cdot [t] := [xt] \qquad \text{for } x \in A, t \in \tilde{G}_{\mathrm{ss}}.$$

Denote by $\underline{A}_\varphi$ the stabilizer of $[s]$ in $\underline{A}$.

Lemma 8. *The map γ_φ in equation* (3.5) *induces an isomorphism $R_\varphi \cong \underline{A}_\varphi$.*

Proof. We have

$$\begin{aligned} R_\varphi \cong \ker(H^1(\sigma, A) &\to \mathrm{H}^1(\sigma, \tilde{G})) \\ &= \{\underline{x} \in \underline{A} | g^{-1} x s ({}^\sigma g) s^{-1} = 1 \qquad \text{for some } g \in \tilde{G}\} \\ &= \{\underline{x} \in \underline{A} | \underline{x} \cdot [s] = [s]\} \\ &= \underline{A}_\varphi. \end{aligned}$$

□

Remark. The above Lemma is also proved in [Yu].

4.2. **Action of Ω.** Let $\Psi = (X, R, \Delta, \check{X}, \check{R}, \check{\Delta})$ be the based root datum of (G, B, T). So X (resp. $\check{X}$) is the group of characters (resp. co-characters) of T, R (resp. $\check{R}$) is the set of roots (resp. co-roots) of T in the Lie algebra of G and Δ (resp. $\check{\Delta}$) is a basis in R (resp. $\check{R}$) determined by B. Let $\underline{\Psi} = (\underline{X}, \underline{R}, \underline{\Delta}, \underline{\check{X}}, \underline{\check{R}}, \underline{\check{\Delta}})$ be the based root datum obtained from $\Psi = (X, R, \Delta, \check{X}, \check{R}, \check{\Delta})$ by the construction given in 2.2.2. Let $\underline{Q}$ be the lattice generated by $\underline{R}$. Let $\underline{C}$ be the alcove in $\underline{V} := \underline{X} \otimes \mathbb{R}$ determined by $\underline{\Delta}$. Let $W = W(\underline{\Psi})$ be the Weyl group of of the based root datum $\underline{\Psi}$. By Theorem 3, it is the relative Weyl group of G. Let $\underline{\Omega} \cong \underline{X}/\underline{Q}$ be the stabilizer in $W \ltimes \underline{X}$ of $\underline{C}$ (see Section 2.2.1).

By [Bor79a, Lemma 6.5] (or more directly by [Mis15, Prop. 11]), we have

$$\hat{T}_\sigma/W \cong (\hat{G} \rtimes \sigma)_{\mathrm{ss}}/\mathrm{Int}(\hat{G}), \tag{4.1}$$

where $(\hat{G} \rtimes \sigma)_{\mathrm{ss}}$ is the set of semisimple elements in $\hat{G} \rtimes \sigma$ and $\mathrm{Int}(\hat{G})$ denotes the group of inner automorphisms of $\hat{G}$.

Let $\hat{T}^{\mathrm{cpt}}$ be the maximal compact subtorus in $\hat{T}$. Write $\hat{T} = X \otimes \mathbb{C}^\times$. Under this identification, $\hat{T}^{\mathrm{cpt}} = X \otimes (\mathbb{R}/\mathbb{Z}) \cong X \otimes \mathbb{R}/X$. Let $\hat{G}^{\mathrm{cpt}}$ be the set of those semi-simple elements of $\hat{G}$ which lie in some maximal compact subtorus of $\hat{G}$. The isomorphism in (4.1) induces an isomorphism

$$\begin{aligned} \hat{G}^{\mathrm{cpt}} \rtimes \sigma/\mathrm{Int}(\hat{G}) &\cong (\hat{T}^{\mathrm{cpt}})_\sigma/W \\ &\cong \underline{X} \otimes \mathbb{R}/W \ltimes \underline{X} \\ &= \underline{X} \otimes \mathbb{R}/((W \ltimes \underline{Q}) \rtimes \underline{\Omega}) && (4.2) \\ &\longleftrightarrow \underline{\bar{C}}/\underline{\Omega}, && (4.3) \end{aligned}$$

where $\underline{\bar{C}}$ is the closure of the alcove $\underline{C}$ determined by $\underline{\Delta}$.

Let $\hat{\mathfrak{z}}^{\mathrm{cpt}} := X/Q \otimes \mathbb{R}$. It is the Lie algebra of the maximal compact subtorus of $\hat{Z}$. Let $\tilde{G}^{\mathrm{cpt}} = \hat{G}^{\mathrm{cpt}}_{\mathrm{sc}} \times \hat{\mathfrak{z}}^{\mathrm{cpt}}$. Then

$$\begin{aligned} \tilde{G}^{\mathrm{cpt}} \rtimes \sigma/\mathrm{Int}(\tilde{G}) &\cong \tilde{T}^{\mathrm{cpt}}_\sigma/W \\ &\cong ((\underline{X_{\mathrm{sc}}} \otimes (\mathbb{R}/\mathbb{Z})) \times (\underline{X}/\underline{Q} \otimes \mathbb{R}))/W && (4.4) \\ &\cong \underline{X} \otimes R/(\underline{Q} \rtimes W) \qquad \text{since } \underline{X_{\mathrm{sc}}} = \underline{Q} \\ &\longleftrightarrow \underline{\bar{C}}. && (4.5) \end{aligned}$$

We have $\underline{A} \cong (X/Q)_\sigma \twoheadrightarrow \underline{X}/\underline{Q} \cong \underline{\Omega}$. In Lemma 9 below, we will show that the action of $\underline{A}$ on $\tilde{G}^{\mathrm{cpt}} \rtimes \sigma/\mathrm{Inn}(\tilde{G}) \subset (\tilde{G} \rtimes \sigma)_{\mathrm{ss}}/\mathrm{Inn}(\tilde{G})$ is compatible with the action of $\underline{\Omega}$ on $\underline{\bar{C}}$. Now G is isogenous to $Z^\circ \times (G_{\mathrm{sc}})_{\mathrm{der}}$, where $(G_{\mathrm{sc}})_{\mathrm{der}}$ is the simply connected cover of the derived group of G and Z° is the identity component of the center of G. Since any simply connected semisimple group is the direct product of almost simple groups, it suffices to prove the compatibility in the case when G is almost simple.

Let $\underline{a} \in \underline{A}$ and let a be a lift of $\underline{a}$ in A. Let $\underline{c_0}$ be the weighted barycenter of $\underline{C}$ and let $\underline{a} \mapsto \tilde{\omega}_a$ under the surjection $\underline{A} \twoheadrightarrow \underline{\Omega}$, where $\omega_a \in W$ and $\tilde{\omega}_a$ is the affine transformation $x \in \underline{X} \otimes \mathbb{R} \mapsto \omega_a(x - \underline{c_0}) + \underline{c_0}$ (see Section 2.2.1).

Let $[s] \mapsto x_{[s]}$ under the bijection $\tilde{G}^{\mathrm{cpt}} \rtimes \sigma/\mathrm{Int}(\tilde{G}) \leftrightarrow \bar{\underline{C}}$, where $[s]$ denotes the class of $s \in \tilde{G}^{\mathrm{cpt}}$. Without loss of generality, we can assume that $s \in \tilde{T}^{\mathrm{cpt}}$.

Lemma 9. *We have* $\tilde{\omega}_a \cdot x_{[s]} = x_{[as]}$.

Proof. Let $\tilde{W}^\circ = W \ltimes \underline{Q}$. We have

$$\begin{aligned}\tilde{\omega}_a \cdot x_{[s]} &= \omega_a(x_{[s]} - \underline{c_0}) + \underline{c_0} \\ &= \omega_a \cdot x_{[s]} + (1 - \omega_a)\underline{c_0} \\ &= \omega_a(x_{[s]} + (\omega_a^{-1} - 1)\underline{c_0}).\end{aligned}$$

By Lemma 2, $\tilde{\omega}_a \mapsto (\omega_a^{-1} - 1)\underline{c_0} + \underline{Q}$ under the isomorphism $\underline{\Omega} \cong \underline{X}/\underline{Q}$. Using this we get that $x_{[a]} \equiv (\omega_a^{-1} - 1)\underline{c_0} \mod \tilde{W}^\circ$. Thus

$$\begin{aligned}\tilde{\omega}_a \cdot x_{[s]} &\equiv x_{[s]} + x_{[a]} \mod \tilde{W}^\circ \\ &\equiv x_{[as]} \mod \tilde{W}^\circ.\end{aligned}$$

Since $\tilde{\omega}_a \cdot x_{[s]} \in \bar{\underline{C}}$ and $x_{[as]} \in \bar{\underline{C}}$, we conclude that

$$\tilde{\omega}_a \cdot x_{[s]} = x_{[as]}.$$

□

4.3. **Tempered parameter.** Let λ be a unitary unramified character of $T(k)$. Let $\lambda \mapsto [\bar{s}]$ under the bijection

$$\mathrm{Hom}(T(k), \mathbb{S}^1)/W \cong \hat{G}^{\mathrm{cpt}} \rtimes \sigma/\mathrm{Int}(\hat{G}),$$

where $\bar{s}$ can be chosen to be in $\hat{T}$. Here $\mathbb{S}^1$ denotes the unit circle in $\mathbb{C}$. Let φ be the Langlands parameter determined by the map $\sigma \mapsto \bar{s}$. Let s be a lift of $\bar{s}$ in $\hat{T}_{\mathrm{sc}} \times \hat{\mathfrak{z}}$.

Let $\underline{\Omega}_\varphi$ be the stabilizer of $x_{[s]} \in \bar{\underline{C}}$ in $\underline{\Omega}$. We have

Proposition 10. $\mathcal{S}_\varphi \cong \underline{\Omega}_\varphi$.

Proof. By [Key87, Lem. 2.5(iii)], $\pi_0(\hat{Z}^\sigma) = \pi_0(\hat{T}^\sigma)$. But $\pi_0(\hat{T}^\sigma) \cong (X_\sigma)^{\mathrm{tor}}$. Since $\mathcal{S}_\varphi \cong R_\varphi/\pi_0(\hat{Z}^\sigma)$, by Lemma 8 we get that $\mathcal{S}_\varphi \cong \underline{A}_\varphi/(X_\sigma)^{\mathrm{tor}} \cong \underline{\Omega}_\varphi$. Lemma 9 shows that $\mathcal{S}_\varphi$ and $\underline{\Omega}_\varphi$ have compatible actions on $\tilde{G}^{\mathrm{cpt}} \rtimes \sigma/\mathrm{Int}(\tilde{G})$ and $\bar{\underline{C}}$ respectively. □

When G is almost simple and simply connected, the non-trivial $\underline{\Omega}$ are given by the table below (see [Kan01, Sec. 9-4] and [Ree10, Table-1]).

	$\underline{\Omega}$
A_n	$\mathbb{Z}/(n+1)\mathbb{Z}$
B_n	$\mathbb{Z}/2\mathbb{Z}$
C_n	$\mathbb{Z}/2\mathbb{Z}$
D_n (n even)	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$
D_n (n odd)	$\mathbb{Z}/2\mathbb{Z}$
E_6	$\mathbb{Z}/3\mathbb{Z}$
E_7	$\mathbb{Z}/2\mathbb{Z}$
${}^2A_{2n-1}$ ($n\geq 3$)	$\mathbb{Z}/2\mathbb{Z}$
${}^2D_{n+1}$ ($n\geq 2$)	$\mathbb{Z}/2\mathbb{Z}$

TABLE 1

Let R_λ be the Knapp-Stein R-group associated to λ (see [Key87, §2] for definition). By [Key87, Prop. 2.6], $R_\lambda \cong \mathcal{S}_\varphi$. Using Proposition 10, we obtain $\underline{\Omega}_\varphi \cong R_\lambda$. In fact, the isomorphism is given by the restriction of the natural projection $W \ltimes \underline{X} \to W$ to $\underline{\Omega}_\varphi$. We get

Theorem 11. *Let G be an almost simple, simply connected, unramified group defined over a non-archimedean local field k. The non-trivial R_λ that can appear are precisely the subgroups of $\underline{\Omega}$ in table* 1*.*

(see also [KP13]).

This gives the classification obtained by Keys in [Key82, §3] in the case of unramified groups.

5. Unramified L-packet

Let the notations be as in Section 3. Assume further that G is unramified and that k is non-archimedean. As in Section 4, let I be the inertia subgroup of W_k and let σ be the Frobenious element in W_k/I.

An *unramified L-packet* consists of those irreducible subquotients of an unramified principal series representation of $G(k)$ which have a non-zero vector fixed by some hyperspecial subgroup of $G(k)$. Unramified L-packets are in bijective correspondence with $(\hat{G}\rtimes\sigma)_{\mathrm{ss}}/\mathrm{Int}(\hat{G})$. Let φ be a Langlands parameter determined by the σ-conjugacy class of a semi-simple element and let Π_φ be the associated unramified L-packet. The L-packet Π_φ is parametrized by $\widehat{\mathcal{S}_\varphi}$, where $\mathcal{S}_\varphi := \pi_0(Z_{\hat{G}}(\mathrm{Im}(\varphi))/\hat{Z}^{\Gamma_k})$ as in Section 3, after making the choice of a hyperspecial point. We denote the parametrization by $\rho\in\widehat{\mathcal{S}_\varphi}\mapsto \pi_\rho\in\Pi_\varphi$.

Let K be a compact subgroup of $G(k)$. Denote by $[K]$, the $G(k)$-conjugacy class of K. If π is a representation of $G(k)$, we denote by π^K the K-fixed points of the space realizing π. By the notation $\pi^{[K]}\neq 0$, we mean that π has a non-zero vector fixed by some (therefore any) conjugate of K.

The conjugacy classes of hyperspecial subgroups of $G(k)$ form a single orbit under $T_{\mathrm{ad}}(k)$. The author, in his Ph.D. thesis [Mis13, Theorem 2.2.1] (also [Mis12, Thm. 1]) constructs a map $T_{\mathrm{ad}}(k)/p(T(k)) \twoheadrightarrow \widehat{\mathcal{S}_\varphi}$. For the action of $T_{\mathrm{ad}}(k)$ on $\widehat{\mathcal{S}_\varphi}$ given by this map, he shows that $\pi_{t\cdot\rho}^{t\cdot[K]} \neq 0 \iff \pi_\rho^{[K]} \neq 0$ for all $t \in T_{\mathrm{ad}}(t)$, $\rho \in \widehat{\mathcal{S}_\varphi}$, where K is a hyperspecial subgroup of $G(k)$. Using this result, we have

Theorem 12. *Let Π_φ be an unramified L-packet associated to a Langlands parameter φ. Then $\pi_\rho \in \Pi_\varphi$ is ψ-generic iff $\pi_{t\cdot\rho}$ is $t\cdot\psi$ generic for all $t \in T_{\mathrm{ad}}(k)$.*

Proof. Given $\pi_\rho \in \Pi_\varphi$, let K be a hyperspecial subgroup such that $\pi_\rho^{[K]} \neq 0$. We can write K as the stabilizer $G(k)_x$ of some hyperspecial point x in the Bruhat-Tits building of $G(k)$. Without loss of generality we can assume x to lie in the apartment associated to T. We have that $(\mathrm{Ind}_{U(k)}^{G(k)}\psi)^{G(k)_x} \neq 0$ iff there exists $g \in G$ such that $\psi|_{g^{-1}G(k)_x g\cap U(k)} \equiv 1$. Without loss of generality, we can assume that $g = 1$. Let $t \in T_{\mathrm{ad}}(k)$.

$$\begin{aligned}
\mathrm{Hom}_{G(k)}(\pi_\rho, (\mathrm{Ind}_{U(k)}^{G(k)}\psi)) \neq 0 \quad &\text{iff} \quad (\mathrm{Ind}_{U(k)}^{G(k)}\psi)^{G(k)_x} \neq 0\\
&\text{iff} \quad \psi|_{G(k)_x\cap U(k)} \equiv 1\\
&\text{iff} \quad t\cdot\psi|_{G(k)_{t\cdot x}\cap U(k)} \equiv 1\\
&\text{iff} \quad (\mathrm{Ind}_{U(k)}^{G(k)} t\cdot\psi)^{t\cdot[G(k)_x]} \neq 0\\
&\text{iff} \quad \mathrm{Hom}_{G(k)}(\pi_{t\cdot\rho}, (\mathrm{Ind}_{U(k)}^{G(k)} t\cdot\psi)) \neq 0
\end{aligned}$$

□

Remark 13. Note that we do not assume φ to be tempered. However, if the associated L-packet is not generic, then the above statement could be vacuous.

Remark 14. Theorem 12 is a very special case of Conjecture $6'$. In [Kal13, Thm. 3.3], Kaletha proves Conjecture $6'$ for tempered representations in the case when G is a quasi-split real K-group or a quasi-split p-adic classical group (in the sense of Arthur).

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Current address: The Ruprecht-Karls-Universität Heidelberg, Im Neuenheimer Feld 288 Heidelberg 69120

E-mail address: manish.mishra@gmail.com