

An explicit bound for the first sign change of the Fourier coefficients of a Siegel cusp form

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Abstract We give an explicit upper bound for the first sign change of the Fourier coefficients of an arbitrary non-zero Siegel cusp form F of even integral weight on the Siegel modular group of arbitrary genus $g \geq 2$.

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1. Introduction

Fourier coefficients of cusp forms in general are quite mysterious objects. In particular, when real this applies to the distribution of their signs. Over the past years various aspects of the latter problem have been studied by several authors.

For example, in [8] it was shown that the Fourier coefficients when real of a non-zero elliptic cusp form f on a congruence subgroup of the full modular group $\Gamma_1 := SL_2(\mathbf{Z})$ have infinitely many sign changes. The proof which is rather straightforward uses the analytic properties of the Hecke L -function and the Rankin-Selberg zeta function of f . The above result was generalized in [7] to the case of a non-zero Siegel cusp form F of even integral weight on the symplectic group $\Gamma_g := Sp_g(\mathbf{Z}) \subset GL_{2g}(\mathbf{Z})$ of arbitrary genus $g \geq 2$. In fact, in [7] it was proved that if the Fourier coefficients $a(T)$ of F are real ($T > 0$ a positive definite half-integral matrix of size g), then there exist infinitely many $T > 0$ (modulo the usual action of $GL_g(\mathbf{Z})$) such that $a(T) > 0$, and similarly such that $a(T) < 0$. The proof uses similar arguments as for $g = 1$, with e.g. the Hecke L -function of f replaced by the Koecher-Maass series of F .

A more subtle problem is to give explicit upper bounds for the first sign change in terms of the weight and the level. If f is a Hecke eigenform on the Hecke congruence group $\Gamma_0(N)$ of level N this was done e.g. in [5,6,12]. The corresponding problem for arbitrary forms (not necessarily Hecke eigenforms) f of squarefree level N was studied in [2]. In fact, in [2] the question was reduced to the case of Hecke eigenforms by writing f as a linear combination of the latter and then using Chebyshev's inequality in combination with uniform lower bounds for those Hecke eigenforms. The technical details were rather complicated.

In this paper we will give an explicit upper bound for the first sign change of the Fourier coefficients of an arbitrary non-zero Siegel cusp form F of even integral weight on Γ_g ($g \geq 2$). To the best of our knowledge, we think that this is the first general result in

this direction. The main idea will be to look at the Fourier-Jacobi expansion of F where the coefficients are Jacobi forms on the generalized Jacobi group $\Gamma_1 \propto (\mathbf{Z}^{g-1} \times \mathbf{Z}^{g-1})$. Using Taylor expansions of these coefficients we will reduce the question to the case of elliptic modular forms and then will apply the results of [2]. Though this strategy appears rather simple, the actual technical details are somewhat involved.

On the way we will also obtain a bound for the first non-vanishing Taylor coefficient function of a generalized Jacobi form $\phi(\tau, z)$ ($\tau \in \mathcal{H}$ = upper half-plane, $z \in \mathbf{C}^{g-1}$) around $z = 0$. This generalizes a basic result of [3] in the case of classical Jacobi forms (Thm. 1.2) and eventually may be of independent interest.

Although we do not have any direct and immediate application of our main result, we do think that the main steps in the proof may highlight again the theory of Jacobi forms, as an important bridge between Siegel modular forms and classical elliptic modular forms.

2. Statement of main result

We will always suppose that $g \geq 2$. For $k \in \mathbf{N}$ we let $S_k(\Gamma_g)$ be the space of Siegel cusp forms of weight k on Γ_g . Recall that this is the space of complex valued holomorphic functions $F(Z)$ on the Siegel upper half-space \mathcal{H}_g (consisting of symmetric complex matrices of size g with positive definite imaginary part) such that

$$F((AZ + B)(CZ + D)^{-1}) = \det(CZ + D)^k F(Z)$$

for all $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g$ and such that F has a Fourier expansion

$$F(Z) = \sum_{T > 0} a(T) e^{2\pi i \operatorname{tr}(TZ)} \quad (Z \in \mathcal{H}_g),$$

where T runs over all positive definite half-integral matrices of size g .

We note that

$$a(T[U]) = (\det U)^k a(T)$$

for all $T > 0$ and all $U \in GL_g(\mathbf{Z})$, where we have used the standard abbreviation

$$A[B] := B^t AB$$

for complex matrices A and B of appropriate sizes.

In particular, if k is even and the $a(T)$ are real, then $a(T[U])$ is of the same sign as $a(T)$ for all unimodular U .

Theorem. *Let k be even and $F \in S_k(\Gamma_g)$, $F \neq 0$. Suppose that the Fourier coefficients $a(T)$ ($T > 0$) of F are real. Then there exist $T_1 > 0$, $T_2 > 0$ with*

$$\operatorname{tr} T_1, \operatorname{tr} T_2 \ll (k \cdot c_g)^5 \log^{26}(k \cdot c_g)$$

such that $a(T_1) > 0$, $a(T_2) < 0$. Here

$$c_g := g \cdot 2^{g-1} \cdot (4/3)^{g(g-1)/2}$$

and the constant involved in \ll is absolute and effective.

Remarks. i) There is an obvious reformulation of the Theorem for $F \in S_k(\Gamma_g)$ a Siegel cusp form with arbitrary complex Fourier coefficients, with $a(T_1)$, $a(T_2)$ replaced by $\operatorname{Re}(a(T_1))$, $\operatorname{Re}(a(T_2))$ (and $\operatorname{Im}(a(T_1))$, $\operatorname{Im}(a(T_2))$). This follows from the well-known fact that for $F \in S_k(\Gamma_g)$ the Fourier series with general coefficients $\operatorname{Re}(a(T))$ (resp. $\operatorname{Im}(a(T))$) are again in $S_k(\Gamma_g)$.

ii) Any improvement of the bound obtained in [2] for elliptic cusp forms valid for all even integral weights would lead to a corresponding improvement of the bound in the Theorem, as will be clear from the proof.

iii) One may ask more generally for the distribution of signs of the coefficients $a(T)$ where $T > 0$ runs over primitive matrices only, or (oppositely) of the "radial" coefficients $a(nT)$ ($n \geq 1$), with $T > 0$ fixed. The latter coefficients in the special case $g = 2$ are related to the corresponding Hecke eigenvalues (in case F is a Hecke eigenform), see [1]. For some results regarding sign changes of eigenvalues we refer to [9,10,13]. The method used here, however does not seem to give any new insights into the questions addressed above.

The proof of the Theorem will be given in the next section.

3. Proof

It is sufficient to show the existence of $T > 0$ with $\operatorname{tr} T$ in the given range such that $a(T) < 0$, since we can always replace F by $-F$.

Let us write

$$Z = \begin{pmatrix} \tau & z \\ z^t & \tau' \end{pmatrix} \quad (\tau \in \mathcal{H}, \tau' \in \mathcal{H}_{g-1}, z \in \mathbf{C}^{g-1}).$$

Then F has a Fourier-Jacobi expansion

$$(1) \quad F(Z) = \sum_{M>0} \phi_M(\tau, z) e^{2\pi i \operatorname{tr}(M\tau')}$$

where M runs over all positive definite half-integral matrices of size $g-1$ and the functions ϕ_M are Jacobi cusp forms of weight k and index M on the generalized Jacobi group

$$\Gamma_1 \propto (\mathbf{Z}^{g-1} \times \mathbf{Z}^{g-1}).$$

Those are complex-valued holomorphic functions $\phi(\tau, z)$ on $\mathcal{H} \times \mathbf{C}^{g-1}$ with the transformation laws

$$(2) \quad \phi\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) = (c\tau + d)^k e^{\frac{2\pi i c M[z^t]}{(c\tau + d)}} \phi(\tau, z) \quad (\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1)$$

and

$$(3) \quad \phi(\tau, z + \lambda\tau + \mu) = e^{-2\pi i(M[\lambda^t]\tau + 2\lambda z^t)} \phi(\tau, z) \quad (\forall \lambda, \mu \in \mathbf{Z}^{g-1}),$$

and having a Fourier expansion

$$\phi(\tau, z) = \sum_{n \geq 1, r \in \mathbf{Z}^{g-1}, 4n > M^{-1}[r^t]} c(n, r) e^{2\pi i(n\tau + rz^t)}$$

(we use the notation $A > B$ for symmetric real matrices A and B to indicate that $A - B > 0$)(see [17]).

Note that the (n, r) -th coefficient of $\phi_M(\tau, z)$ in (1) is equal to

$$a\left(\begin{pmatrix} n & r/2 \\ r^t/2 & M \end{pmatrix}\right)$$

(the condition $\begin{pmatrix} n & r/2 \\ r^t/2 & M \end{pmatrix} > 0$ is equivalent to $n \geq 1, 4n > M^{-1}[r^t]$ as follows from the usual Jacobi decomposition of the latter matrix).

Since $F \neq 0$ there exists

$$T_0 = \begin{pmatrix} n_0 & r_0/2 \\ r_0^t/2 & M_0 \end{pmatrix} > 0$$

such that

$$\text{tr } T_0 \ll_g k, \quad a(T_0) \neq 0$$

as is well-known. In fact, one can find such a T_0 whose trace satisfies the explicit bound

$$(4) \quad \text{tr } T_0 \leq \frac{k}{4\pi} \cdot \frac{2}{\sqrt{3}} \cdot g \cdot (4/3)^{g(g-1)/2},$$

cf. [14]. This result is a consequence of the basic theory presented in [4] and [11] and what is proven in [15,16].

Since $a(T_0) \neq 0$ the function $\phi_{M_0}(\tau, z)$ is not identically zero. We define

$$\Phi_{M_0}(\tau, z) := \prod_{\epsilon} \phi_{M_0}(\tau, \epsilon z)$$

where $\epsilon = (\epsilon_1, \dots, \epsilon_{g-1})$ runs over all elements of $\{-1, 1\}^{g-1}$ and $\epsilon z := (\epsilon_1 z_1, \dots, \epsilon_{g-1} z_{g-1})$.

We will simply write $\Phi(\tau, z)$ instead of $\Phi_{M_0}(\tau, z)$.

Note that

$$\phi_{M_0}(\tau, \epsilon z) = \phi_{M_0[D_\epsilon]}(\tau, z)$$

where D_ϵ is the diagonal matrix with entries on the diagonal in the given order $\epsilon_1, \dots, \epsilon_{g-1}$. This follows immediately if one acts on F with the unimodular matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & D_\epsilon \end{pmatrix}.$$

Therefore $\phi_{M_0}(\tau, \epsilon z)$ is a non-zero Jacobi cusp form of weight k and index $M_0[D_\epsilon]$ (which of course can also be checked by direct inspection).

Hence we conclude that $\Phi(\tau, z)$ is a non-zero Jacobi cusp form of weight $2^{g-1}k$ and index

$$(5) \quad \mathcal{M}_0 := \sum_{\epsilon} M_0[D_\epsilon].$$

By construction this function is *even* w.r.t. each of the variables z_ν , $\nu \in \{1, \dots, g-1\}$. We observe that

$$\text{tr } \mathcal{M}_0 = 2^{g-1} \cdot \text{tr } M_0.$$

We now develop $\Phi(\tau, z)$ in a Taylor series around $z = 0$, i.e. write

$$\Phi(\tau, z) = \sum_{\nu_1, \dots, \nu_{g-1} \geq 0} \chi_{\nu_1, \dots, \nu_{g-1}}(\tau) z_1^{\nu_1} \dots z_{g-1}^{\nu_{g-1}}.$$

Let us choose $\alpha_{g-1}, \alpha_{g-2}, \dots, \alpha_2, \alpha_1$ in a “minimal” way such that

$$\chi_{\alpha_1, \alpha_2, \dots, \alpha_{g-2}, \alpha_{g-1}}(\tau)$$

is not the zero function. Here “minimal” means that for all τ we have

$$\begin{aligned} \chi_{\nu_1, \dots, \nu_{g-1}}(\tau) &= 0 \quad (0 \leq \forall \nu_{g-1} < \alpha_{g-1}, \forall \nu_{g-2}, \dots, \nu_1 \geq 0), \\ \chi_{\nu_1, \dots, \nu_{g-2}, \alpha_{g-1}}(\tau) &= 0 \quad (0 \leq \forall \nu_{g-2} < \alpha_{g-2}, \forall \nu_{g-3}, \dots, \nu_1 \geq 0), \dots, \\ \chi_{\nu_1, \alpha_2, \dots, \alpha_{g-1}}(\tau) &= 0 \quad (0 \leq \forall \nu_1 < \alpha_1). \end{aligned}$$

In the following, we will denote the diagonal elements of M_0 by

$$m_{11}^0, m_{22}^0, \dots, m_{g-1, g-1}^0.$$

Lemma. *The function*

$$(6) \quad \sum_{\nu_1 \geq 0} \chi_{\nu_1, \alpha_2, \dots, \alpha_{g-1}}(\tau) z_1^{\nu_1}$$

is a (classical) non-zero Jacobi cusp form of weight $2^{g-1}k + \alpha_2 + \dots + \alpha_{g-1}$ and index $2^{g-1}m_{11}^0$.

Proof. Up to a non-zero universal factor the function in (6) is equal to

$$\left(\partial_{z_2}^{\alpha_2} \dots \partial_{z_{g-1}}^{\alpha_{g-1}} \Phi(\tau, z_1, \dots, z_{g-1}) \right)_{|(z_2, \dots, z_{g-1})=(0, \dots, 0)}.$$

We differentiate equation (2) successively w.r.t. z_2, \dots, z_{g-1} up to the orders $\alpha_2, \dots, \alpha_{g-1}$, respectively and use Leibniz rule together with the “minimality” of $\alpha_{g-1}, \dots, \alpha_2$. Then we see that indeed (6) behaves like a Jacobi form of weight $2^{g-1}k + \alpha_2 + \dots + \alpha_{g-1}$ and index $2^{g-1}m_{11}^0$ w.r.t. to the action of Γ_1 .

Likewise in (3) we take $\lambda = (\lambda_1, 0, \dots, 0)$ and $\mu = (\mu_1, 0, \dots, 0)$ and differentiate successively to see the correct behavior of (6) under the action of \mathbf{Z}^2 .

The conditions $n \geq 1, 4n > \mathcal{M}_0^{-1}[r^t]$ in the Fourier expansion of Φ are equivalent to

$$\begin{pmatrix} n & r/2 \\ r^t/2 & \mathcal{M}_0 \end{pmatrix} > 0.$$

Hence taking into account (5) (which implies that the $(1, 1)$ -entry of \mathcal{M}_0 is $2^{g-1}m_{11}^0$), it follows that

$$\begin{pmatrix} n & r_1/2 \\ r_1/2 & 2^{g-1}m_{11}^0 \end{pmatrix} > 0.$$

Therefore $4n \cdot 2^{g-1}m_{11}^0 > r_1^2$ and so the Fourier expansion of (6) is indeed as required.

Finally we note that the function in (6) is not identically zero since $\chi_{\alpha_1, \alpha_2, \dots, \alpha_{g-1}}(\tau)$ is not the zero function by hypothesis. This proves the Lemma.

We now observe that $\chi_{\alpha_1, \dots, \alpha_{g-1}}(\tau)$ is a non-zero cusp form of weight

$$(7) \quad k_1 := 2^{g-1}k + \alpha_1 + \alpha_2 + \dots + \alpha_{g-1}$$

on Γ_1 , by the “minimality” of α_1 and a similar argument as above (compare [3, p. 31]). By Thm. 1.2, p. 10 in [3] it follows that

$$\alpha_1 \ll 2^{g-1}m_{11}^0.$$

We now proceed inductively regarding the other variables z_2, z_3, \dots . Thus we successively choose $\beta_{g-1}, \beta_{g-2}, \dots, \beta_3, \beta_1, \beta_2 \geq 0$ in a “minimal” way such that $\chi_{\beta_1, \beta_2, \dots, \beta_{g-1}}(\tau)$ is not the zero function, then successively $\gamma_{g-1}, \gamma_{g-2}, \dots, \gamma_4, \gamma_2, \gamma_1, \gamma_3 \geq 0$ in a “minimal” way such that $\chi_{\gamma_1, \gamma_2, \gamma_3, \dots, \gamma_{g-1}}(\tau)$ is not the zero function and so on.

In the same way as above, using “minimality” one then shows that

$$\sum_{\nu_2 \geq 0} \chi_{\beta_1, \nu_2, \beta_3, \dots, \beta_{g-1}}(\tau) z_2^{\nu_2}$$

is a non-zero Jacobi cusp form of weight $2^{g-1}k + \beta_1 + \beta_3 + \dots + \beta_{g-1}$ and index $2^{g-1}m_{22}^0$, the function

$$\sum_{\nu_3 \geq 0} \chi_{\gamma_1, \gamma_2, \nu_3, \gamma_4, \dots, \gamma_{g-1}}(\tau) z_3^{\nu_3}$$

is a non-zero Jacobi cusp form of weight $2^{g-1}k + \gamma_1 + \gamma_2 + \gamma_4 + \dots + \gamma_{g-1}$ and index $2^{g-1}m_{33}^0$, and so on.

Likewise as above, the functions $\chi_{\beta_1, \beta_2, \dots, \beta_{g-1}}(\tau)$ resp. $\chi_{\gamma_1, \gamma_2, \dots, \gamma_{g-1}}(\tau)$ and so on are non-zero cusp forms on Γ_1 of weights $2^{g-1}k + \beta_1 + \dots + \beta_{g-1}$ resp. $2^{g-1}k + \gamma_1 + \dots + \gamma_{g-1}$ and so on, and $\beta_2 \ll 2^{g-1}m_{22}^0$, $\gamma_3 \ll 2^{g-1}m_{33}^0, \dots$

Since both the α 's and the β 's are “minimal”, we conclude immediately from these conditions that

$$\beta_{g-1} = \alpha_{g-1}, \beta_{g-2} = \alpha_{g-2}, \dots, \beta_3 = \alpha_3,$$

in a successive way. It then follows that

$$\alpha_2 \leq \beta_2,$$

for otherwise we had

$$0 = \chi_{\beta_1, \beta_2, \alpha_3, \dots, \alpha_{g-1}}(\tau) = \chi_{\beta_1, \beta_2, \beta_3, \dots, \beta_{g-1}}(\tau),$$

a contradiction.

We therefore conclude that

$$\alpha_2 \ll 2^{g-1}m_{22}^0.$$

Proceeding in the same way with the α 's and the γ 's, we infer that

$$\alpha_3 \ll 2^{g-1}m_{33}^0.$$

Working on inductively, we finally conclude that

$$\alpha_1 \ll 2^{g-1}m_{11}^0, \dots, \alpha_{g-1} \ll 2^{g-1}m_{g-1, g-1}^0,$$

hence

$$(8) \quad \alpha_1 + \dots + \alpha_{g-1} \ll 2^{g-1}tr M_0.$$

Remark. We note that the arguments used above more generally show that if $\phi(\tau, z)$ ($\tau \in \mathcal{H}, z \in \mathbf{C}^{g-1}$) is a generalized Jacobi form of weight k and index $M > 0$ on $\Gamma_1 \propto (\mathbf{Z}^{g-1} \times \mathbf{Z}^{g-1})$ and we let $\chi_{\nu_1, \dots, \nu_{g-1}}(\tau)$ ($\nu_1, \dots, \nu_{g-1} \geq 0$) its Taylor coefficients around $z = 0$, then there exists $(\alpha_1, \dots, \alpha_{g-1}) \in \mathbf{N}_0^{g-1}$ such that $\alpha_1 + \dots + \alpha_{g-1} \ll tr M$ and $\chi_{\alpha_1, \dots, \alpha_{g-1}}(\tau)$ is not the zero function. This result generalizes Thm. 1.1 in [3] in the classical case $\Gamma_1 \propto \mathbf{Z}^2$ and may be of independent interest.

With the definition (7) it now follows from (8) that

$$(9) \quad k_1 \ll 2^{g-1}(k + tr M_0).$$

Let us write $a(n)$ ($n \geq 1$) for the Fourier coefficients of $\chi_{\alpha_1, \dots, \alpha_{g-1}}(\tau)$. Then by [2] (in the case of level 1) there exists $\tilde{n} \geq 1$ such that

$$\tilde{n} \ll k_1^5 \log^{26} k_1, \quad a(\tilde{n}) < 0.$$

By (9) it follows that

$$(10) \quad \tilde{n} \ll (2^{g-1}(k + tr M_0))^5 \log^{26}(2^{g-1}(k + tr M_0)).$$

Observe that $\chi_{\alpha_1, \dots, \alpha_{g-1}}(\tau)$ up to a non-zero universal scalar equals

$$\left(\partial_{z_1}^{\alpha_1} \dots \partial_{z_{g-1}}^{\alpha_{g-1}} \Phi(\tau, z) \right)_{|z=0}.$$

Therefore if the Fourier coefficients of $\Phi(\tau, z)$ are denoted by $C(n, r)$, it follows that up to a non-zero scalar $a(\tilde{n})$ is equal to

$$\sum_{r \in \mathbf{Z}^{g-1}, 4\tilde{n} > \mathcal{M}_0^{-1}[r^t]} C(\tilde{n}, r) r_1^{\alpha_1} \dots r_{g-1}^{\alpha_{g-1}}.$$

Since $\Phi(\tau, z)$ is an even function w.r.t. each of the variables z_1, \dots, z_{g-1} , the $\alpha_1, \dots, \alpha_{g-1}$ are all even integers. Hence it follows that there exists $\tilde{r} \in \mathbf{Z}^{g-1}$ such that $C(\tilde{n}, \tilde{r}) < 0$. However, $C(\tilde{n}, \tilde{r})$ is a finite sum of products of Fourier coefficients

$$a\left(\begin{pmatrix} n_\epsilon & * \\ * & M_0[D_\epsilon] \end{pmatrix}\right)$$

where ϵ runs over $\{-1, 1\}^{g-1}$ as before, and with

$$\sum_{\epsilon} n_\epsilon = \tilde{n}.$$

Hence at least one of the coefficients

$$a\left(\begin{pmatrix} n_\epsilon & * \\ * & M_0[D_\epsilon] \end{pmatrix}\right)$$

must be negative.

Since $n_\epsilon \leq \tilde{n}$ and $tr M_0 \leq tr T_0$ we infer from (10)

$$tr \begin{pmatrix} n_\epsilon & * \\ * & M_0[D_\epsilon] \end{pmatrix} = n_\epsilon + tr M_0$$

$$\begin{aligned} &\ll (2^{g-1}(k + tr M_0))^5 \log^{26}(2^{g-1}(k + tr M_0)) + tr T_0 \\ &\ll (2^{g-1}(k + tr T_0))^5 \log^{26}(2^{g-1}(k + tr T_0)). \end{aligned}$$

Inserting from (4) we then obtain our assertion.

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