Irreducibility of versal deformation rings in the (p, p)-case for 2-dimensional representations

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Abstract

Let G_K be the absolute Galois group of a finite extension K of $\mathbb{Q}_p$ and $\bar{\rho}\colon G_K \to \mathrm{GL}_n(\mathbb{F})$ a continuous residual representation for $\mathbb{F}$ a finite field of characteristic p. We investigate whether the versal deformation space $\mathfrak{X}(\bar{\rho})$ of $\bar{\rho}$ is irreducible. For $n = 2$ and $p > 2$ we obtain a complete answer in the affirmative based on the results of [Bö2, Bö4]. As a consequence we deduce from recent results of Colmez, Kisin and Nakamura [Col, Ki, Na1] that for $n = 2$ and $p > 2$ crystalline points are Zariski dense in the versal deformation space $\mathfrak{X}(\bar{\rho})$.

1 Introduction and statement of main results

Deformation rings of Galois representations

Let G_K be the absolute Galois group of a finite extension K of $\mathbb{Q}_p$ and let $\bar{\rho}\colon G_K \to \mathrm{GL}_n(\mathbb{F})$ be a continuous residual representation for $\mathbb{F}$ a finite field of characteristic p. Let $W(\mathbb{F})$ be the ring of Witt vectors of $\mathbb{F}$. We shall always write $\mathcal{O}$ for the ring of integers of a finite totally ramified extension of $W(\mathbb{F})[1/p]$ and denote by $\mathfrak{m}_{\mathcal{O}}$ its maximal ideal and by $\varpi_{\mathcal{O}}$ a uniformizer. To simplify notation, we shall write $\mathcal{O}_i$ for the quotient $\mathcal{O}/\varpi_{\mathcal{O}}^i\mathcal{O}$ for any integer $i \geq 1$. Denote by ad the adjoint representation of $\bar{\rho}$, i.e., the representation on $M_n(\mathbb{F})$ induced from $\bar{\rho}$ by conjugation and by ad^0 the subrepresentation on trace zero matrices.

For $\bar{\rho}$ as above, we consider the deformation functor from the category $\widehat{\mathrm{Ar}}_{\mathcal{O}}$ of complete Noetherian local $\mathcal{O}$-algebras $(R, \mathfrak{m}_R)$ to the category of sets defined by

$$D_{\bar{\rho}}(R) := \{\rho\colon G_K \to \mathrm{GL}_n(R) \mid \rho \mod \mathfrak{m}_R = \bar{\rho} \text{ and } \rho \text{ is a cont. repr.}\}/\sim$$

where $\rho \sim \rho'$ if there exists $A \in \ker\big(\mathrm{GL}_n(R) \overset{\mathrm{mod}\ \mathfrak{m}_R}{\longrightarrow} \mathrm{GL}_n(\mathbb{F})\big)$ such that $\rho' = A\rho A^{-1}$. An equivalence class $[\rho]$ of ρ under $\sim$ is called a *deformation of* $\bar{\rho}$. Since G_K satisfies the finiteness condition Φ_p from [Ma, § 1.1], by [Ma, Prop. 1] with a slight strengthening by [Ra] one deduces:

Theorem 1.1. *The functor $D_{\bar{\rho}}$ always possesses a versal hull $(R_{\bar{\rho}}, \mathfrak{m}_{\bar{\rho}})$ which is unique up to isomorphism. If in addition $\mathrm{End}_{G_K}(\bar{\rho}) = \mathbb{F}$, then $D_{\bar{\rho}}$ is representable and in particular $(R_{\bar{\rho}}, \mathfrak{m}_{\bar{\rho}})$ is unique up to unique isomorphism.*

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We denote by $\rho_{\bar\rho}\colon G_K \to \mathrm{GL}_n(R_{\bar\rho})$ a representative of the versal class.

For later use, we recall parts of the obstruction theory related to $D_{\bar\rho}$. Suppose we are given a short exact sequence

$$0 \longrightarrow J \longrightarrow R_1 \longrightarrow R_0 \longrightarrow 0,$$

where the morphism $R_1 \to R_0$ is in $\widehat{\mathrm{Ar}}_{\mathcal{O}}$, and $\mathfrak{m}_1 \cdot J = 0$ for $\mathfrak{m}_1$ the maximal ideal of R_1; such a diagram is called a *small extension of* R_0. Suppose further that we are given a deformation of $\bar\rho$ to R_0 represented by $\rho_0 : G_K \to \mathrm{GL}_n(R_0)$. Then Mazur defines a canonical obstruction class

$$\mathcal{O}(\rho_0) \in H^2(G_K, \mathrm{ad}) \otimes J$$

that vanishes if and only if ρ_0 can be lifted to a deformation $\rho_1 : G_K \to \mathrm{GL}_n(R_1)$ of $\bar\rho$, see [Ma, p. 398]. By elementary linear algebra, the obstruction class $\mathcal{O}(\rho_0)$ defines an obstruction homomorphism obs: $\mathrm{Hom}_{\mathbb{F}}(J, \mathbb{F}) \to H^2(G_K, \mathrm{ad})$, and conversely from the latter one can recover $\mathcal{O}(\rho_0)$.

The following result describes the mod $\mathfrak{m}_{\mathcal{O}}$ tangent space of $R_{\bar\rho}$ and a bound on the number of generators of an ideal in a minimal presentation of $R_{\bar\rho}$ by a power series ring over $\mathcal{O}$.

Proposition 1.2 ([Ma]). *(a) If $\mathbb{F}[\varepsilon]$ denotes the ring of dual numbers of $\mathbb{F}$ and $\overline{\mathfrak{m}}_{\bar\rho} := \mathfrak{m}_{\bar\rho}/\mathfrak{m}_{\mathcal{O}} R_{\bar\rho}$, then one has canonical isomorphisms between the two tangent spaces*

$$t_{D_{\bar\rho}} := D_{\bar\rho}(\mathbb{F}[\varepsilon]) \cong H^1(G_K, \mathrm{ad}) \cong \mathrm{Hom}_{\mathbb{F}}\left(\overline{\mathfrak{m}}_{\bar\rho}/\overline{\mathfrak{m}}_{\bar\rho}^2, \mathbb{F}\right) =: t_{R_{\bar\rho}}.$$

(b) Let $h_1 := \dim_{\mathbb{F}} H^1(G_K, \mathrm{ad})$, $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}[[x_1, \ldots, x_{h_1}]]$ and

$$0 \longrightarrow I \longrightarrow \mathcal{O}[[x_1, \ldots, x_{h_1}]] \xrightarrow{\pi} R_{\bar\rho} \longrightarrow 0$$

be a presentation of $R_{\bar\rho}$. Then the obstruction homomorphism

$$\mathrm{obs}\colon \mathrm{Hom}_{\mathbb{F}}(I/\mathfrak{m}I, \mathbb{F}) \longrightarrow H^2(G_K, \mathrm{ad}), \quad f \longmapsto (1 \otimes f)(\mathcal{O}(\rho_{\bar\rho})),$$

is injective, and thus $\dim_{\mathbb{F}} H^2(G_K, \mathrm{ad})$ *bounds the minimal number* $\dim_{\mathbb{F}} I/\mathfrak{m}I$ *of generators of I.*

If in a presentation as in (b) the number of variables is minimal, i.e., if the mod $\mathfrak{m}_{\mathcal{O}}$ tangent space of $\mathcal{O}[[x_1, \ldots, x_{h_1}]]$ is isomorphic to that of $R_{\bar\rho}$, then we call the presentation *minimal*. Now fix a character $\psi\colon G_K \to \mathcal{O}^*$ which reduces to $\det \bar\rho$ and denote by $D^{\psi}_{\bar\rho}$ the subfunctor of $D_{\bar\rho}$ of deformations whose determinant is equal to ψ (under the canonical homomorphism $\mathcal{O} \to R$).

Proposition 1.3. *If $p \not| n$, then the results of Theorem 1.1 and Proposition 1.2 hold for $D^{\psi}_{\bar\rho}$ as well, if one replaces* ad *by the adjoint representation* ad^0 *on trace zero matrices, the pair $(R_{\bar\rho}, \mathfrak{m}_{\bar\rho})$ by the versal deformation ring $(R^{\psi}_{\bar\rho}, \mathfrak{m}^{\psi}_{\bar\rho})$ and the ideal I by a* relation ideal *I^{ψ} in a minimal presentation*

$$0 \longrightarrow I^{\psi} \longrightarrow \mathcal{R} := \mathcal{O}[[x_1, \ldots, x_h]]^{1} \longrightarrow R^{\psi}_{\bar\rho} \longrightarrow 0 \qquad \text{with } h = \dim_{\mathbb{F}} H^1(G_K, \mathrm{ad}^0). \tag{1}$$

This article presents three results on the deformation rings $R_{\bar\rho}$ and $R^{\psi}_{\bar\rho}$ introduced above:

For $n = 2$, we improve the ring theoretic results from [Bö2] by showing that the rings $R^{\psi}_{\bar\rho}$ are integral domains. On the technical side, we clarify that for this result and the main results in [Bö2] the

[1] To avoid notation such as $\mathfrak{m}_{\mathcal{R}^{\psi}}$, $\mathfrak{m}_{\mathcal{R}^{\psi}}$, we use the simpler notation $\mathcal{R}$ instead of $\mathcal{R}^{\psi}$ for the frequently used ring $\mathcal{R}$

knowledge of a suitably defined *(refined) quadratic part*, see Definition 2.4, of the relation in a minimal presentation of $R^{\psi}_{\bar\rho}$ suffices.

Using the irreducibility of $R^{\psi}_{\bar\rho}$, we deduce the Zariski density of crystalline points in $\operatorname{Spec} R_{\bar\rho}$ for $n = 2$, $p > 2$ and any p-adic local field K.

For many n and K we give a cohomological description of the quadratic parts of the relations in a minimal presentation of $R^{\psi}_{\bar\rho}$ via a cup product and a Bockstein formalism in the context of Galois cohomology of p-adic fields.

We now explain these results in greater detail. From now on we assume that $p > 2$.

Ring-theoretic results on local versal deformation rings

To describe some auxiliary ring-theoretic results and some ring-theoretic properties of the versal deformation ring $R^{\psi}_{\bar\rho}$ for a fixed 2-dimensional residual representation $\bar\rho\colon G_K \to \mathrm{GL}_2(\mathbb{F})$, we fix some further notation.

For a ring R in $\widehat{\mathrm{Ar}}_{\mathcal{O}}$ and a proper ideal $\mathfrak{n}$ of R, we denote by $\mathrm{gr}_{\mathfrak{n}}(R)$ the associated graded ring $\bigoplus_{i\geq 0} \mathrm{gr}^i_{\mathfrak{n}}(R)$ with $\mathrm{gr}^i_{\mathfrak{n}}(R) = \mathfrak{n}^i/\mathfrak{n}^{i+1}$. By $\mathbf{in}_{\mathfrak{n}}\colon R \to \mathrm{gr}_{\mathfrak{n}}(R)$, we denote the map that sends $r \in R \smallsetminus \{0\}$ to its initial term in $\mathrm{gr}_{\mathfrak{n}}(R)$, i.e., if i_r is the largest integer $i \geq 0$ such that $r \in \mathfrak{n}^i$, then $\mathbf{in}_{\mathfrak{n}}(r)$ is the image of r in $\mathfrak{n}^{i_r}/\mathfrak{n}^{i_r+1}$. Further, we set $\mathbf{in}_{\mathfrak{n}}(0) = 0$ and note that $\bigcap_i \mathfrak{n}^i = \{0\}$ for R in $\widehat{\mathrm{Ar}}_{\mathcal{O}}$. If we wish to indicate i_r in the notation, we write $\mathbf{in}^{i_r}_{\mathfrak{n}}(r)$. For an ideal $I \subset R$ one denotes by $\mathbf{in}_{\mathfrak{n}}(I)$ the ideal of $\mathrm{gr}_{\mathfrak{n}}(R)$ generated by $\{\mathbf{in}_{\mathfrak{n}}(r) \mid r \in I\}$. To describe the mod $\mathfrak{m}_{\mathcal{O}}$ reduction of pairs $(R, \mathfrak{m}_R)$ in $\widehat{\mathrm{Ar}}_{\mathcal{O}}$, we define $\overline{R} := R/\mathfrak{m}_{\mathcal{O}}R$ and $\overline{\mathfrak{m}}_R := \mathfrak{m}_R/\mathfrak{m}_{\mathcal{O}}R$. Similarly, we write $\bar r$ for the image of $r \in R$ in $\overline{R}$.

The following is the key technical result to deduce ring theoretic properties of $R^{\psi}_{\bar\rho}$:

Theorem 1.4. *Suppose $\bar\rho$ is of degree 2 and $p > 2$. Fix a minimal presentation of $R^{\psi}_{\bar\rho}$ as in Proposition 1.3. Then there exists an $\mathfrak{m}_{\mathcal{R}}$-primary ideal $\mathfrak{m}_s$ of $\mathcal{R} \cong \mathcal{O}[[x_1, \ldots, x_h]]$ of the form $(\varpi^s_{\mathcal{O}}, x_1, \ldots, x_h) \supset I^{\psi}$ and generators $f_1, \ldots, f_r$ of I^{ψ} such that the following hold:*

(a) For $j = 1, \ldots, r$ we have $\mathbf{in}^2_{\mathfrak{m}_s}(f_j) \in \mathfrak{m}_s^2/\mathfrak{m}_s^3$, and the elements $\bar t_0, \bar g_1, \ldots, \bar g_r$ with $g_j := \mathbf{in}^2_{\mathfrak{m}_s}(f_j)$ form a regular sequence in $\overline{\mathrm{gr}_{\mathfrak{m}_s}(\mathcal{R})} \cong \mathbb{F}[\bar t_0, \bar t_1, \ldots, \bar t_h]$, where $t_0 = \mathbf{in}_{\mathfrak{m}_s}(\varpi^s_{\mathcal{O}})$ and $t_i = \mathbf{in}_{\mathfrak{m}_s}(x_i)$ for $i = 1, \ldots, h$.

(b) The quotient ring $\overline{\mathrm{gr}_{\mathfrak{m}_s}(\mathcal{R})}/(\bar g_1, \ldots, \bar g_r)$ is an integral domain and one has $(\bar g_1, \ldots, \bar g_r) = \overline{\mathbf{in}_{\mathfrak{m}_s}(I^{\psi})}$.

Theorem 1.4 will be proven after Corollary 3.6. A cohomological interpretation of the $\bar g_j$ is given in Theorem 1.14.

As a consequence of Theorem 1.4 and some purely ring-theoretic results summarized in Proposition 2.2, we shall obtain the following main theorem in Section 2:

Theorem 1.5. *Let the residual representation $\bar\rho$ be of degree 2 and $p > 2$. Then the following hold:*

(a) The ring $\overline{R}^{\psi}_{\bar\rho}$ is a complete intersection.

(b) The ring $R^{\psi}_{\bar\rho}$ is a complete intersection and it is flat over $\mathcal{O}$.

(c) The ring $R^{\psi}_{\bar\rho}$ is an integral domain and in particular irreducible.

In Lemma 4.1 we show that is suffices to prove Theorem 1.5 for any fixed choice of lift ψ, for instance for the Teichmüller lift of $\det \rho$.

Remark 1.6. Parts (a) and (b) of Theorem 1.5 were obtained already in [Bö2]. In fact, our present proof heavily relies on the results of [Bö2] because we shall simply quote the relations of $R^\psi_{\bar\rho}$ in a minimal presentation from there. However, the present article allows one to redo much of [Bö2] by working with the simpler ring $R^\psi_{\bar\rho}/\mathfrak{m}_s^3$, and this would avoid most of the technical difficulties occuring in [Bö2]. An example of this is given by the proof of Lemma 3.7.

Remark 1.7. It does not seem possible to show irreducibility when $n = 2$, $p = 2$ and $K = \mathbb{Q}_2$ with ideas of the present article, i.e., by using suitable initial terms in an associated graded ring of $R^\psi_{\bar\rho}$. For instance, if $\bar\rho$ is the trivial representation, then it is simple to check that the natural degrees of such initial terms are 2 and 3 and that they form a regular sequence. But the resulting associated graded ring is not an integral domain! However, when K is a proper extension of $\mathbb{Q}_2$, as shown in the Master thesis of M. Kremer, the methods of this article suffice to show that $R^\psi_{\bar\rho}$ is an integral domain for the trivial representation $\bar\rho$. For $n = 2$, $p = 2$ and $K = \mathbb{Q}_2$, see however Remark 1.13.

Irreducible components of versal deformation spaces and Zariski density of crystalline points

Denote by $\mathfrak{X}(\bar\rho)$ the *versal deformation space* of a fixed residual representation $\bar\rho : G_K \to \mathrm{GL}_n(\mathbb{F})$ that is the generic fiber over $\mathcal{O}[1/p]$ of its versal deformation ring $R_{\bar\rho}$ in the sense of Berthelot, see [deJ, § 7]. The points of $\mathfrak{X}(\bar\rho)$ are in bijection with those p-adic representations of G_K that have a mod p reduction isomorphic to $\bar\rho$. To explain the consequences of the the ring-theoretic results in Theorem 1.5 to p-adic Galois representations, we introduce the following notions due to Colmez, Kisin and Nakamura:

Definition 1.8. Let V be a potentially crystalline p-adic representation of G_K of degree n.

(i) V is called *regular* if for each embedding $\sigma\colon K \hookrightarrow \overline{\mathbb{Q}_p}$ the Hodge-Tate weights of $V \otimes_{K,\sigma} \overline{\mathbb{Q}_p}$ are pairwise distinct.

(ii) V is called *benign* if V is regular and the Frobenius eigenvalues $\alpha_1, \dots, \alpha_n$ of (the filtered φ-module corresponding to) V are pairwise distinct and satisfy $\alpha_i/\alpha_j \neq p^f$, for any i, j, with $f = [K_0 : \mathbb{Q}_p]$.

Using the following important structure result on the irreducible components of $\mathfrak{X}(\bar\rho)$, we show in Lemma 4.2 that every component of $\mathfrak{X}(\bar\rho)$ contains a regular crystalline point.

Theorem 1.9. *Suppose $p > 2$ and let $\bar\rho$ be a residual representation of G_K of degree 2. Consider the canonical map* $\mathrm{Det}\colon \mathfrak{X}(\bar\rho) \to \mathfrak{X}(\det\bar\rho)$ *induced from mapping a deformation of $\bar\rho$ to its determinant. Then* Det *induces a bijection between the irreducible components of $\mathfrak{X}(\bar\rho)$ and those of $\mathfrak{X}(\det\bar\rho)$. Moreover, for both spaces, irreducible and connected components coincide. Lastly, the connected components of $\mathfrak{X}(\det\bar\rho)$ form a principal homogeneous space over the set $\mu_{p^\infty}(K)$ of p-power roots contained in K.*

The proof follows from Theorem 1.5 and Lemma 4.1, and is thus postponed to Section 4.

Question 1.10. We wonder whether the assertions of Theorem 1.9 hold for all representations $\bar\rho\colon G_K \to \mathrm{GL}_n(\mathbb{F})$ of any degree n, and any p and any finite extension $K/\mathbb{Q}_p$? We also wonder if Theorem 1.5 holds in this generality.

The following theorem is shown by methods similar to [Ki]. It generalizes a result of Colmez and Kisin for $K = \mathbb{Q}_p$, cf. [Col, Ki], and makes crucial use of an idea of Chenevier [Na2, Thm. 2.9].

Theorem 1.11 ([Na1, Theorem 1.4])**.** *Suppose $n = 2$ and that every component of $\mathfrak{X}(\bar\rho)$ contains a regular crystalline point. Then the Zariski closure of the benign crystalline points in $\mathfrak{X}(\bar\rho)$ is non-empty and a union of irreducible components of $\mathfrak{X}(\bar\rho)$.*

We remark that the above result is also proven for arbitrary n. This is due to Chenevier [Ch2] for $K = \mathbb{Q}_p$ and to Nakamura [Na2] for arbitrary finite extensions $K/\mathbb{Q}_p$.

Using Theorems 1.5 and 1.9, we show in Section 4 that Theorem 1.11 implies:

Theorem 1.12. *Suppose $n = 2$, $p > 2$, K is a finite extension of $\mathbb{Q}_p$ and $\bar\rho\colon G_K \to \mathrm{GL}_2(\mathbb{F})$ is any residual representation. Then the benign crystalline points are Zariski dense in $\mathfrak{X}(\bar\rho)$.*

In Corollary 4.3, we prove analogs of Theorems 1.9 and 1.12 for pseudo-representations, in the sense of Chenevier [Ch3].

In the case $K = \mathbb{Q}_p$ and $n = 2$, Theorem 1.12 is an important ingredient in Colmez' proof of the p-adic local Langlands correspondence. In that case it is essentially due to Kisin, cf. [Bö4], and it is used to establish the surjectivity of Colmez' functor V, which relies on an analytic continuation argument and the knowledge of the correspondence in the crystalline case; see [Col, proof of Thm. II.3.3] or alternatively [Ki].

Remark 1.13. Suppose $p = 2$ and $K = \mathbb{Q}_2$. The assertions of Theorems 1.9 and 1.12 for the universal framed deformation space of the trivial representation $1 \oplus 1$ were proved by Colmez, Dospinescu and Paskunas [CDP, Thms. 1.1 and 1.2]. The assertion of Theorem 1.9 was proved by Chenevier in the case $n = 2$ if the residual representation is an extension of two distinct characters, and for arbitrary n if the residual representation is absolutely irreducible [Ch1, Cor. 4.2]. In these two cases the assertion of Theorem 1.12 is deduced in [CDP, Rem. 9.8].

Generation of quadratic parts of relation ideals through cohomological operations

One possible source of obstruction classes in $H^2(G_K, \mathrm{ad}^0)$ stems from the cup product in cohomology: Namely, if one composes the Lie bracket $[\cdot,\cdot]\colon \mathrm{ad}^0 \times \mathrm{ad}^0 \to \mathrm{ad}^0, (A, B) \mapsto AB - BA$, with the cup product $H^1(G_K, \mathrm{ad}^0) \times H^1(G_K, \mathrm{ad}^0) \to H^2(G_K, \mathrm{ad}^0 \otimes \mathrm{ad}^0)$, which are both alternating, one obtains a symmetric $\mathbb{F}$-bilinear pairing

$$b\colon H^1(G_K, \mathrm{ad}^0) \times H^1(G_K, \mathrm{ad}^0) \longrightarrow H^2(G_K, \mathrm{ad}^0),$$

often called the *bracket cup product*. As remarked in [Ma, §1.6], if $p \neq 2$ the pairing b *gives the quadratic relations (up to higher terms) satisfied by a minimal set of formal parameters for* $\overline{R}^{\psi}_{\bar\rho}$. We shall prove this and give a precise interpretation in Lemma 5.2.

In Section 6, we shall explain how further information on the relation ideal I^ψ may arise from cohomology, namely from a Bockstein homomorphism $\widetilde\beta_{s+1}\colon H^1(G_K, \mathrm{ad}^0) \to H^2(G_K, \mathrm{ad}^0)$. The Bockstein homomorphism can be defined whenever $\bar\rho$ admits a lift to $\mathcal{O}_s = \mathcal{O}/\varpi_{\mathcal{O}}^s\mathcal{O}$ for some s. It measures to what extent lifts from the dual number $\mathbb{F}[\varepsilon]$ can be lifted to $\mathcal{O}_s[\varepsilon]$. In Section 6 we then combine the bracket cup product with the Bockstein homomorphism, to show that these two cohomological operations (essentially) suffice to describe the refined quadratic relations in a minimal presentation of $R^{\psi}_{\bar\rho}$.

The results of Sections 5 and 6 have the following consequences. First, we complement Theorem 1.4:

Theorem 1.14. *Let the notation be as in Theorem 1.4. Then in addition to the assertions of Theorem 1.4, the following hold:*

(a) The elements $\bar{g}_j$ are the images of an $\mathbb{F}$-basis of $H^2(G_K, \mathrm{ad}^0)^\vee$ under the composite of the dual obstruction homomorphism $\mathrm{obs}^\vee \colon H^2(G_K, \mathrm{ad})^\vee \to I^\psi/\mathfrak{m}_R I^\psi$ with the canonical homomorphism $I^\psi/\mathfrak{m}_R I^\psi \to \overline{\mathfrak{m}_s^2/\mathfrak{m}_s^3}$.

(b) The dual of the map $H^2(G_K, \mathrm{ad}^0)^\vee \longrightarrow \overline{\mathfrak{m}_s^2/\mathfrak{m}_s^3}$ from (a) factors via $-\frac{1}{2}b \oplus -\widetilde{\beta}_{s+1}$.

We prove Theorem 1.14 at the end of Section 6 by verifying hypotheses needed to apply Theorem 6.8. In particular, this shows that cohomological information alone suffices to deduce all parts of Theorem 1.5.

Second, we observe in Example 2.3 that the bracket cup product alone need not suffice to show that $R^\psi_{\bar\rho}$ is an integral domain. Thus important ring-theoretic information is not visible by the bracket cup product but requires in addition the Bockstein homomorphism.

Third, our results show that for 2-dimensional residual representations of G_K for $p > 2$ the refined quadratic part of I^ψ in a minimal presentation of $R^\psi_{\bar\rho}$ suffices to prove Theorem 1.5. Theorem 6.8 then explains that essentially the cohomological operations suffice to deduce all ring-theoretic properties we are interested in.

The third point above is particular to the set-up we work in. For general fields K little is known about the pairing b and whether it generates a significant portion of the elements in the relation ideal I^ψ of Proposition 1.3. However for K a finite extension of $\mathbb{Q}_p$ and $p > 2$, the universal deformation $\rho_{\bar\rho} \colon G_K \to \mathrm{GL}_2(R_{\bar\rho})$ factors via a profinite group that is an extension of a finite group by the pro-p-completion $\widehat{G}_L$ of the absolute Galois group of a finite extension L of K. The group $\widehat{G}_L$ is either a free pro-p group or a Demushkin group, and topologically finitely generated. In the former case, $R_{\bar\rho}$ will be unobstructed. In the latter case $\widehat{G}_L$ is isomorphic to the pro-p completion of a group on generators $a_1, b_1, \ldots, a_g, b_g$ with a single relation $a_1^q \cdot (a_1, b_1) \cdot \ldots \cdot (a_g, b_g) = 1$, where $g = [L : \mathbb{Q}_p]$ and (x, y) denotes the commutator bracket $x^{-1}y^{-1}xy$; cf. [La] for the classification of Demushkin groups. The Demushkin case should be compared with the deformation results [GM1, GM2] by Goldmann and Millson, as already suggested in [Ma]. Goldman-Millson study the deformation theory of representations of fundamental groups of compact Kähler manifolds, and show in this context that all relations in a minimal presentation of their deformation rings are purely quadratic. A typical example is the fundamental group of a compact Riemann surface, which is a group on $2g$ generators $a_1, b_1, \ldots, a_g, b_g$ subject to a single relation $(a_1, b_1) \cdot \ldots \cdot (a_g, b_g) = 1$. The formal similarity of the relation except for the term a_1^q suggests that the deformation rings might be very similar. The term a_1^q might explain the importance of the Bockstein homomorphism when trying to detect the refined quadratic relations from cohomology.

Outline of the article

We briefly explain the organization of the article. In Section 2, we adapt some results from commutative algebra in the way we later wish to apply them. In particular, these results give a sufficient criterion for certain rings R in $\widehat{\mathrm{Ar}}_{\mathcal{O}}$ to be complete intersections and to be integral domains in terms of homogeneous initial terms of a presentation of R. The main results of Section 2 together with Theorem 1.4 imply the ring-theoretic properties stated in Theorem 1.5. In Section 3, we recall the explicit presentations of the versal deformation rings for 2-dimensional representations $\bar\rho$ from [Bö2]. In Lemma 3.7, we also give a detailed treatment of some results from [Bö2, §8], whose proofs are somewhat sketchy. At the end of Section 3, we give the proof of Theorem 1.4.

The short proof of the Zariski-density of crystalline points in local deformation spaces is the content of Section 4. We end this article with Sections 5 and 6 with (presumably well-known) results regarding the bracket cup product and the Bockstein homomorphism. These results might be relevant for tackling higher dimensional cases in future work. The proof of Theorem 1.14 ends Section 6.

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2 Results from commutative algebra

The aim of this section is to prove some results in commutative algebra in order to deduce from Theorem 1.4 the ring-theoretic results stated in Theorem 1.5. In particular, we wish to transfer ring-theoretic properties from a certain associated graded ring to the ring itself. Recall that above Theorem 1.4 we define an initial term map $\mathbf{in}_{\mathfrak{n}}$ from a ring R in $\widehat{\mathrm{Ar}}_{\mathcal{O}}$ to the associated graded ring $\mathrm{gr}_{\mathfrak{n}} R$ with respect to a proper ideal $\mathfrak{n} \subset R$, and that we write $\mathcal{O}_i$ for $\mathcal{O}/\varpi_{\mathcal{O}}^i\mathcal{O}$, $\overline{R}$ for $R/\mathfrak{m}_{\mathcal{O}}R$, $\overline{\mathfrak{m}}_R$ for $\mathfrak{m}_R/\mathfrak{m}_{\mathcal{O}}R$ and $\bar{x}$ for the image of $x \in R$ in $\overline{R}$.

Lemma 2.1. *For a ring R in $\widehat{\mathrm{Ar}}_{\mathcal{O}}$ and proper ideals $I = (f_1, \ldots, f_r), \mathfrak{n} \subset R$, the following hold:*

(a) If $\mathrm{gr}_{\mathfrak{n}} R$ is an integral domain, then so is R.

(b) If $f_1, \ldots, f_r$ is a regular sequence in R so that R/I is an integral domain, then R is an integral domain.

(c) The natural homomorphism $\mathrm{gr}_{\mathfrak{n}} R \to \mathrm{gr}_{(\mathfrak{n}+I)/I}(R/I)$ induces an isomorphism

$$(\mathrm{gr}_{\mathfrak{n}} R)/\mathbf{in}_{\mathfrak{n}}(I) \cong \mathrm{gr}_{(\mathfrak{n}+I)/I}(R/I).$$

(d) If $\mathbf{in}_{\mathfrak{n}}(f_1), \ldots, \mathbf{in}_{\mathfrak{n}}(f_r)$ is a regular sequence in $\mathrm{gr}_{\mathfrak{n}} R$, then $\mathbf{in}_{\mathfrak{n}}(I) = (\mathbf{in}_{\mathfrak{n}}(f_1), \ldots, \mathbf{in}_{\mathfrak{n}}(f_r))$.

(e) If $\mathbf{in}_{\mathfrak{n}}(f_1), \ldots, \mathbf{in}_{\mathfrak{n}}(f_r)$ is a regular sequence in $\mathrm{gr}_{\mathfrak{n}} R$, then $f_1, \ldots, f_r$ is a regular sequence in R.

Proof. Part (a) is [Ei, Cor. 5.5], and (b) follows by induction on r: For $r = 1$ we have a short exact sequence $0 \to R \xrightarrow{\cdot f_1} R \to R/(f_1) \to 0$ so that $\mathrm{gr}_{(f_1)} R \cong R/(f_1)[t]$, and R is an integral domain by (a). Parts (c), (d) and (e) are [VV, middle p. 94], [VV, Prop. 2.1] and [VV, Cor. 2.7], respectively. □

The next result is a refinement of Lemma 2.1 suited for our purposes. As a preparation we introduce the following graded ring. Denote by $\mathfrak{m}_s$ the ideal $(\varpi_{\mathcal{O}}^s, x_1, \ldots, x_h)$ of $\mathcal{R} = \mathcal{O}[[x_1, \ldots, x_h]]$[2] for some integer $s \geq 1$. Setting $t_0 := \mathbf{in}_{\mathfrak{m}_s}(\varpi_{\mathcal{O}}^s)$ and $t_i := \mathbf{in}_{\mathfrak{m}_s}(x_i)$ for $i = 1, \ldots, h$, we have $\mathrm{gr}_{\mathfrak{m}_s} \mathcal{R} = \mathcal{O}_s[t_0, \ldots, t_h]$, $\overline{\mathrm{gr}_{\mathfrak{m}_s} \mathcal{R}} = \mathbb{F}[\bar{t}_0, \bar{t}_1, \ldots, \bar{t}_h]$ and $\mathrm{gr}_{\overline{\mathfrak{m}}_{\mathcal{R}}} \overline{\mathcal{R}} \cong \mathbb{F}[\bar{t}_1, \ldots, \bar{t}_h]$, where $\bar{t}_i$ is identified with $\mathbf{in}_{\overline{\mathfrak{m}}_{\mathcal{R}}}(\bar{x}_i)$ for $i = 1, \ldots, h$.

[2] In this section, and here only, by $(\mathcal{R}, \mathfrak{m}_{\mathcal{R}})$ we denote a formally smooth ring over $\mathcal{O}$ in $\widehat{\mathrm{Ar}}_{\mathcal{O}}$ and not necessarily a ring in a presentation as in Proposition 1.3.

Proposition 2.2. *Let $\mathcal{R}, \mathfrak{m}_s$ and s be as above, and let $I \subset \mathcal{R}$ be an ideal generated by elements $f_1, \ldots, f_r \in \mathcal{R}$. Then the following hold:*

(a) If $\mathbf{in}_{\overline{\mathfrak{m}}_{\mathcal{R}}}(\bar{f}_1), \ldots, \mathbf{in}_{\overline{\mathfrak{m}}_{\mathcal{R}}}(\bar{f}_r)$ is a regular sequence in $\mathrm{gr}_{\overline{\mathfrak{m}}_{\mathcal{R}}} \overline{\mathcal{R}}$, then so is $\varpi_{\mathcal{O}}, f_1, \ldots, f_r$ in $\mathcal{R}$. In this case, $\mathbf{in}_{\overline{\mathfrak{m}}_{\mathcal{R}}}(\bar{I}) = (\mathbf{in}_{\overline{\mathfrak{m}}_{\mathcal{R}}}(\bar{f}_1), \ldots, \mathbf{in}_{\overline{\mathfrak{m}}_{\mathcal{R}}}(\bar{f}_1))$ in $\mathrm{gr}_{\overline{\mathfrak{m}}_{\mathcal{R}}} \overline{\mathcal{R}}$ and $\mathcal{R}/I$ is flat over $\mathcal{O}$.

(b) If $\mathbf{in}_{\overline{\mathfrak{m}}_{\mathcal{R}}}(\bar{f}_1), \ldots, \mathbf{in}_{\overline{\mathfrak{m}}_{\mathcal{R}}}(\bar{f}_r)$ is a regular sequence in $\mathrm{gr}_{\overline{\mathfrak{m}}_{\mathcal{R}}} \overline{\mathcal{R}}$ and $\mathrm{gr}_{\overline{\mathfrak{m}}_{\mathcal{R}}} \overline{\mathcal{R}}/\mathbf{in}_{\overline{\mathfrak{m}}_{\mathcal{R}}}(\bar{I})$ is an integral domain, then also $\mathcal{R}/I$ is an integral domain.

(c) If $\overline{\mathbf{in}_{\mathfrak{m}_s}(f_1)}, \ldots, \overline{\mathbf{in}_{\mathfrak{m}_s}(f_r)}$ is a regular sequence in $\overline{\mathrm{gr}_{\mathfrak{m}_s} \mathcal{R}}$, then so is $f_1, \ldots, f_r$ in $\mathcal{R}$. In this case, $\mathbf{in}_{\mathfrak{m}_s}(I) = (\mathbf{in}_{\mathfrak{m}_s}(f_1), \ldots, \mathbf{in}_{\mathfrak{m}_s}(f_r))$ and $\overline{\mathbf{in}_{\mathfrak{m}_s}(I)} = (\overline{\mathbf{in}_{\mathfrak{m}_s}(f_1)}, \ldots, \overline{\mathbf{in}_{\mathfrak{m}_s}(f_r)})$.

(d) If $\overline{\mathbf{in}_{\mathfrak{m}_s}(f_1)}, \ldots, \overline{\mathbf{in}_{\mathfrak{m}_s}(f_r)}, \bar{t}_0$ is a regular sequence in $\overline{\mathrm{gr}_{\mathfrak{m}_s} \mathcal{R}}$ and $\overline{\mathrm{gr}_{\mathfrak{m}_s} \mathcal{R}}/\overline{\mathbf{in}_{\mathfrak{m}_s}(I)}$ is an integral domain, then also $\mathcal{R}/I$ is an integral domain.

We postpone the proof of Proposition 2.2, and first explain some of its content.

Proof of Theorem 1.5. Theorem 1.4 together with Proposition 2.2 applied to the relation ideal I^{ψ} imply the assertions of Theorem 1.5 on $R_{\bar{\rho}}^{\psi} \cong \mathcal{R}/I^{\psi}$. ☐

The following instructive example shows the benefits of using the graded ring associated with the ideal $\mathfrak{m}_s$ in (c) and (d) instead of the one associated with $\overline{\mathfrak{m}}_{\mathcal{R}}$ in (a) and (b).

Example 2.3. Define $R := \mathcal{R}/(f)$ for $\mathcal{R} = W(\mathbb{F})[[x_1, x_2, x_3]]$, $f = qx_1 - x_2x_3$, $q = p^s$ and $s \geq 1$ an integer.[3] Then by Proposition 2.2(a) $\mathbf{in}_{\overline{\mathfrak{m}}_{\mathcal{R}}}(\bar{I}) = \big(\mathbf{in}_{\overline{\mathfrak{m}}_{\mathcal{R}}}(\bar{f})\big) = (\bar{t}_2\bar{t}_3)$ in $\mathrm{gr}_{\overline{\mathfrak{m}}_{\mathcal{R}}} \overline{\mathcal{R}} = \mathbb{F}[\bar{t}_1, \bar{t}_2, \bar{t}_3]$, and criterion (b) fails to show that R is an integral domain since $\bar{t}_2$ and $\bar{t}_3$ are nonzero zero divisors in $\mathrm{gr}_{\overline{\mathfrak{m}}_{\mathcal{R}}} \overline{\mathcal{R}}/\mathbf{in}_{\overline{\mathfrak{m}}_{\mathcal{R}}}(\bar{I})$. However, if we consider the graded ring of $\mathcal{R}$ with respect to $\mathfrak{m}_s = (q, x_1, x_2, x_3)$, then $\overline{\mathbf{in}_{\mathfrak{m}_s}(f)} = \overline{t_0t_1} - \overline{t_2t_3}$ lies in $\overline{\mathrm{gr}^2_{\mathfrak{m}_s} \mathcal{R}} \subset \overline{\mathrm{gr}_{\mathfrak{m}_s} \mathcal{R}} = \mathbb{F}[\bar{t}_0, \bar{t}_1, \bar{t}_2, \bar{t}_3]$ and R is an integral domain by Proposition 2.2(d).

In Sections 5 and 6, we show that one can use cohomological methods to compute the quadratic relations in $\mathrm{gr}^2_{\overline{\mathfrak{m}}_{\mathcal{R}}} \overline{\mathcal{R}}$ resp. $\overline{\mathrm{gr}^2_{\mathfrak{m}_s} \mathcal{R}}$ from the above example. To distinguish there between these two quadratic relations, we introduce the following notions:

Definition 2.4. Let $\mathcal{R}$, $\mathfrak{n}$, $\mathfrak{m}_s$ and s be as above, and let $f \in \mathcal{R}$.

(a) If $\bar{f} \in \overline{\mathfrak{m}}^2_{\mathcal{R}}$, then the *quadratic part of* f is $\mathbf{in}_{\overline{\mathfrak{m}}_{\mathcal{R}}}(\bar{f}^{(2)}) \in \mathrm{gr}^2_{\overline{\mathfrak{m}}_{\mathcal{R}}} \overline{\mathcal{R}}$, where $\bar{f}^{(2)}$ is the homogeneous part of $\bar{f}$ of degree 2 with respect to the grading of $\overline{\mathcal{R}}$ defined by $\overline{\mathfrak{m}}_{\mathcal{R}}$.

(b) If $f \in \mathfrak{m}_s^2$, then the *refined quadratic part of* f is $\overline{\mathbf{in}_{\mathfrak{m}_s}(f^{(2)})} \in \overline{\mathrm{gr}^2_{\mathfrak{m}_s} \mathcal{R}}$, where $f^{(2)}$ is the homogeneous part of f of degree 2 with respect to the grading of $\mathcal{R}$ defined by $\mathfrak{m}_s$.

The *(refined) quadratic part of an ideal* $I \subset \mathcal{R}$ consists of the (refined) quadratic parts of all elements in I.

[3] The relation ideal of the ring $\widetilde{R}$ from [Bö4, Thm. 5] has the shape $6d - bc$ modulo $\mathfrak{m}_1^3$, where $p = 3$. So in a qualitative sense R occurs as a versal deformation ring. At the expense of heavy notation, one could also use $\widetilde{R}$ in the example.

Proof of Proposition 2.2. It follows from the hypothesis of (a) and Lemma 2.1(e) that $(\overline{f_j})_{j=1,\ldots,r}$ is a regular sequence in $\overline{\mathcal{R}}$. Since clearly $\varpi_{\mathcal{O}}$ is a non-zero divisor of $\mathcal{R}$, the first assertion of (a) is proved. From Lemma 2.1(d) follows that $\mathbf{in}_{\overline{\mathfrak{m}}_{\mathcal{R}}}(\bar{I}) = \big(\mathbf{in}_{\overline{\mathfrak{m}}_{\mathcal{R}}}(\bar{f}_1), \ldots, \mathbf{in}_{\overline{\mathfrak{m}}_{\mathcal{R}}}(\bar{f}_r)\big)$. Finally, since $\mathcal{R}$ is local, the order of the elements in the regular sequence $\varpi_{\mathcal{O}}, f_1, \ldots, f_r$ is arbitrary. Hence from the definition of a regular sequence it follows that $\varpi_{\mathcal{O}}$ is a non-zero divisor of the $\mathcal{O}$-algebra $\mathcal{R}/(f_1, \ldots, f_r)$, which means precisely that the latter algebra is flat over $\mathcal{O}$.

To prove (b), we deduce by Lemma 2.1(c) and (a) that $\overline{\mathcal{R}/I}$ is an integral domain. By the last assertion of (a) and Lemma 2.1(b) we also have that $\mathcal{R}/I$ is an integral domain.

For the proof of (c), we define $R_i := \mathrm{gr}_{\mathfrak{m}_s} \mathcal{R}/(\mathbf{in}_{\mathfrak{m}_s}(f_1), \ldots, \mathbf{in}_{\mathfrak{m}_s}(f_i))$ for $i = 0, \ldots, r$ and g_i as the image of $\mathbf{in}_{\mathfrak{m}_s}(f_i)$ in R_i. By the remarks preceding the proposition, $\mathrm{gr}_{\mathfrak{m}_s} \mathcal{R} \cong \mathcal{O}_s[t_0, \ldots, t_h]$ and clearly this ring is flat over $\mathcal{O}_s$. By our hypothesis, $\bar{g}_i$ is a non-zero divisor of $\overline{R_{i-1}}$ for $i = 1, \ldots, r$. We claim, and prove this by induction on i, that R_i is flat over $\mathcal{O}_s$ and that g_i is a non-zero divisor of R_{i-1} for each $i = 1, \ldots, r$. If this is proved, then we have shown that $\mathbf{in}_{\mathfrak{m}_s}(f_1), \ldots, \mathbf{in}_{\mathfrak{m}_s}(f_r)$ is a regular sequence in $\mathrm{gr}_{\mathfrak{m}_s} \mathcal{R}$. Then the first assertion of (c) follows from Lemma 2.1(e). The first equality of ideals in (c) follows from Lemma 2.1(d) and the assertion just proved, the second is immediate by reduction modulo $\varpi_{\mathcal{O}}$.

To prove the claim, we consider for some $j = 2, \ldots, s$ the following diagram obtained by tensoring the short exact sequence $0 \to \mathcal{O}_{j-1} \to \mathcal{O}_j \to \mathbb{F} \to 0$ of $\mathcal{O}$-modules with the right exact sequence $R_{i-1} \to R_{i-1} \to R_i \to 0$ where the map on the left is multiplication by g_i:

$$
\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{j-1} \otimes R_{i-1} & \longrightarrow & \mathcal{O}_j \otimes R_{i-1} & \longrightarrow & \mathbb{F} \otimes R_{i-1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{id} \otimes g_i} & & \downarrow{\scriptstyle \mathrm{id} \otimes g_i} & & \downarrow{\scriptstyle \mathrm{id} \otimes g_i} & & \\
0 & \longrightarrow & \mathcal{O}_{j-1} \otimes R_{i-1} & \longrightarrow & \mathcal{O}_j \otimes R_{i-1} & \longrightarrow & \mathbb{F} \otimes R_{i-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \mathcal{O}_{j-1} \otimes R_i & \longrightarrow & \mathcal{O}_j \otimes R_i & \longrightarrow & \mathbb{F} \otimes R_i & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

We assume that the claim is proved for $i - 1$. Then the two top horizontal sequences are exact since by induction hypothesis the ring R_{i-1} is flat over $\mathcal{O}_s$. The left and middle vertical sequences are exact because the tensor product is right exact. The right vertical sequence is exact, because $\bar{g}_i$ is a non-zero divisor of $\overline{R_{i-1}}$ by hypothesis.

While i is fixed, we proceed by induction on $j = 2, \ldots, s$ to show that all rows and columns in the above diagram are in fact left exact as well: In each induction step, the left-most column is a short exact sequence by induction hypothesis. This implies the same for the middle column and it follows that all columns are short exact sequences. In this situation, the 9-lemma implies that the lower row is also a short exact sequence, and the induction step is complete. If we consider the central column for $j = s$, then this shows that g_i is a non-zero divisor of R_{i-1}. If we consider the lower row for $j = s$, we see that $\mathrm{Tor}_1^{\mathcal{O}_s}(\mathbb{F}, R_i) = 0$ and hence that R_i is flat over $\mathcal{O}_s$. This proves the claim.

Finally, we prove (d). By the proof of (c), we know that the ring $\mathrm{gr}_{\mathfrak{m}_s} \mathcal{R}/\mathbf{in}_{\mathfrak{m}_s}(I)$ is flat over $\mathcal{O}_s$

and its reduction modulo $\varpi_{\mathcal{O}}$ is $\overline{\mathrm{gr}_{\mathfrak{m}_s}\mathcal{R}}/\overline{\mathbf{in}_{\mathfrak{m}_s}(I)}$. Consider elements f, g in $\mathcal{R} \smallsetminus I$. We claim that there exist integers $a, b \in \{0, 1, \ldots, s-1\}$ such that $f' = \varpi_{\mathcal{O}}^a f$ and $g' = \varpi_{\mathcal{O}}^b g$ have non-zero image in $\overline{\mathrm{gr}_{\mathfrak{m}_s}\mathcal{R}}/\overline{\mathbf{in}_{\mathfrak{m}_s}(I)} \cong \overline{\mathrm{gr}_{(\mathfrak{m}_s+I)/I}(\mathcal{R}/I)} \cong \bigoplus_{i\geq 0}(\mathfrak{m}_s^i + I)/(\mathfrak{m}_s^{i+1} + \varpi_{\mathcal{O}}\mathfrak{m}_s^i + I)$. If the claim is shown, then this means that there exist $i, j \geq -1$ such that $f' \in (\mathfrak{m}_s^{i+1} + I) \smallsetminus (\mathfrak{m}_s^{i+2} + \varpi_{\mathcal{O}}\mathfrak{m}_s^{i+1} + I)$ and $g' \in (\mathfrak{m}_s^{j+1} + I) \smallsetminus (\mathfrak{m}_s^{j+2} + \varpi_{\mathcal{O}}\mathfrak{m}_s^{j+1} + I)$. Since by hypothesis $\overline{\mathrm{gr}_{\mathfrak{m}_s}\mathcal{R}}/\overline{\mathbf{in}_{\mathfrak{m}_s}(I)}$ is an integral domain, it follows that $f'g' \in (\mathfrak{m}_s^{i+j+2} + I) \smallsetminus (\mathfrak{m}_s^{i+j+3} + \varpi_{\mathcal{O}}\mathfrak{m}_s^{i+j+2} + I)$ and hence that the class of $f'g'$ is non-zero in $\mathcal{R}/I$. But $f'g' = \varpi_{\mathcal{O}}^{a+b} fg$, and we deduce that the class of fg is non-zero in $\mathcal{R}/I$ and thus assertion (d) follows.

It suffices to prove the claim for f; the proof for g is analogous. Choose $i \geq 0$ such that $f \in (\mathfrak{m}_s^i + I) \smallsetminus (\mathfrak{m}_s^{i+1} + I)$. This is equivalent to $\mathbf{in}_{\mathfrak{m}_s}(f)$ lying in the i-th graded piece of $\mathrm{gr}_{\mathfrak{m}_s}\mathcal{R}/\mathbf{in}_{\mathfrak{m}_s}(I)$. If the image of $\mathbf{in}_{\mathfrak{m}_s}(f)$ is non-zero in $\overline{\mathrm{gr}_{\mathfrak{m}_s}\mathcal{R}}/\overline{\mathbf{in}_{\mathfrak{m}_s}(I)}$, we choose $a = 0$ and are done. Else we have $f \in \varpi_{\mathcal{O}}\mathfrak{m}_s^i + \mathfrak{m}_s^{i+1} + I$, and since $\mathrm{gr}_{\mathfrak{m}_s}\mathcal{R}/\mathbf{in}_{\mathfrak{m}_s}(I)$ is annihilated by $\varpi_{\mathcal{O}}^s$, we can find $a \in \{1, \ldots, s-1\}$ such that $\varpi_{\mathcal{O}}^{a-1} f \notin (\mathfrak{m}_s^{i+1} + I)$ but $\varpi_{\mathcal{O}}^a f \in (\mathfrak{m}_s^{i+1} + I)$. To prove the claim, it remains to show that $\varpi_{\mathcal{O}}^a f$ does not lie in $\varpi_{\mathcal{O}}\mathfrak{m}_s^{i+1} + \mathfrak{m}_s^{i+2} + I$.

Because $\mathrm{gr}_{\mathfrak{m}_s}\mathcal{R}/\mathbf{in}_{\mathfrak{m}_s}(I)$ is flat over $\mathcal{O}_s$ and the integer $a \geq 1$ is minimal such that $\varpi_{\mathcal{O}}^a \mathbf{in}_{\mathfrak{m}_s}(f) = 0$, there exists $f_0 \in \mathcal{R}$ such that $\varpi_{\mathcal{O}}^{s-a}\mathbf{in}_{\mathfrak{m}_s}(f_0) = \mathbf{in}_{\mathfrak{m}_s}(f)$ in $\mathrm{gr}_{\mathfrak{m}_s}\mathcal{R}/\mathbf{in}_{\mathfrak{m}_s}(I)$. In terms of ideals this means $\varpi_{\mathcal{O}}^{s-a} f_0 - f \in \mathfrak{m}_s^{i+1} + I$ and $f_0 \notin \varpi_{\mathcal{O}}\mathfrak{m}_s^i + \mathfrak{m}_s^{i+1} + I$. Since by hypothesis $\bar{t}_0 = \overline{\mathbf{in}_{\mathfrak{m}_s}(\varpi_{\mathcal{O}}^s)}$ is a non-zero divisor of $\overline{\mathrm{gr}_{\mathfrak{m}_s}\mathcal{R}}/\overline{\mathbf{in}_{\mathfrak{m}_s}(I)}$, we have $\varpi_{\mathcal{O}}^s f_0 \notin \varpi_{\mathcal{O}}\mathfrak{m}_s^{i+1} + \mathfrak{m}_s^{i+2} + I$. On the other hand, we find

$$\varpi_{\mathcal{O}}^a \cdot (\varpi_{\mathcal{O}}^{s-a} f_0 - f) = \varpi_{\mathcal{O}}^s f_0 - \varpi_{\mathcal{O}}^a f \in \varpi_{\mathcal{O}}^a \mathfrak{m}_s^{i+1} + \varpi_{\mathcal{O}}^a I \subset \varpi_{\mathcal{O}}\mathfrak{m}_s^{i+1} + \mathfrak{m}_s^{i+2} + I.$$

We deduce that $f' = \varpi_{\mathcal{O}}^a f$ does not lie in $\varpi_{\mathcal{O}}\mathfrak{m}_s^{i+1} + \mathfrak{m}_s^{i+2} + I$, as had to be shown. ☐

We end this section with a simple result on regular sequences, flatness and integral domains:

Lemma 2.5. *Suppose I is an ideal of $\mathcal{R} \cong \mathcal{O}[[x_1, \ldots, x_h]]$ such that I is minimally generated by $m := \dim_{\mathbb{F}} I/\mathfrak{m}_{\mathcal{R}} I$ elements. Suppose $g_1, \ldots, g_l$ are elements of $\mathcal{R}$ and let $J = I + \mathcal{R}g_1 + \ldots + \mathcal{R}g_l$.*

(a) If $\mathcal{R}/J$ is a complete intersection ring of Krull dimension $h + 1 - l - m$, then $\mathcal{R}/I$ is a complete intersection ring and I is generated by a regular $\mathcal{R}$-sequence.

(b) If (a) holds and if $\mathcal{R}/J$ is flat over $\mathcal{O}$, then $\mathcal{R}/I$ is flat over $\mathcal{O}$.

(c) If (a) holds and if $\mathcal{R}/J$ is an integral domain, then $\mathcal{R}/I$ is an integral domain.

Proof. By induction, it suffices to prove the lemma for $l = 1$. Let $f_1, \ldots, f_m$ denote a minimal set of generators of I. The hypothesis of (a) implies that $\mathcal{R}/(f_1, \ldots, f_m, g_1)$ is a complete intersection ring of dimension $h + 1 - m - 1 = h - m$. It follows that $f_1, \ldots, f_m, g_1$ must be a regular sequence, and now (a) is immediate. To see (b), observe that its hypothesis implies that $\mathcal{R}/(J + \varpi_{\mathcal{O}})$ is a complete intersection ring of Krull dimension $h - l - m$. It follows from (a) that $f_1, \ldots, f_m, g_1, \varpi_{\mathcal{O}}$ is a regular sequence. Part (b) is now clear. For (c) note that since $g_1 \pmod I$ is a non-zero divisor in $\mathcal{R}/I$ by the proof of (a), we may now apply Lemma 2.1(b) to complete (c). ☐

3 Explicit presentations of the versal deformation rings

In order to prove Theorem 1.4 using the explicit minimal presentations of versal deformation rings computed in [Bö2], we note that we can work over the ring of Witt vectors $W(\mathbb{F})$ by [CDT, A.1].

First we need to introduce some notation: Denote by H the image of a fixed residual representation $\bar\rho : G_K \to \mathrm{GL}_2(\mathbb{F})$ of degree two, and by U a p-Sylow subgroup of H. Since G_K is prosolvable, the group H is solvable. Either $\#H$ is of order prime to p, or U is a normal subgroup of H. By the lemma of Schur-Zassenhaus, we can find a subgroup G of H of order prime to p such that $U \rtimes G = H$. By $\overline{H}, \overline{G}$ we denote the images of H, G in $\mathrm{PGL}_2(\mathbb{F})$. Note that U is isomorphic to its image in $\mathrm{PGL}_2(\mathbb{F})$ because its order is prime to the order of $\mathbb{F}^*$ and hence we may identify U with its image. The following can now be deduced from Dickson's classification of finite subgroups of $\mathrm{PGL}_2(\mathbb{F})$, see [Hu, II.7]. The group $\overline{G}$ is either cyclic or dihedral and if U is non-trivial, $\overline{G}$ must be cyclic (we assume $p > 2$). We also introduce finite extensions $L \supset F \supset K$ in a fixed algebraic closure of K by the conditions $G_L = \ker(\bar\rho) \subset G_F = \bar\rho^{-1}(U) \subset G_K$.

For a character $\xi : G_K \to \mathbb{F}^*$ we denote by $\mathbb{F}^\xi$ the one-dimensional vector space $\mathbb{F}$ together with the action via ξ. We let $\mathrm{triv} : G_K \to \mathbb{F}^*$ be the trivial character and $\varepsilon : G_K \to \mathbb{F}^*$ be the mod p cyclotomic character. Observe that $\mathrm{ad} = \mathrm{End}(\bar\rho) \cong \bar\rho \otimes_{\mathbb{F}} \bar\rho^\vee \cong \mathbb{F}^{\mathrm{triv}} \oplus \mathrm{ad}^0$ since $p > 2$ and thus $\mathrm{ad}^0 \cong \mathrm{Hom}_{\mathbb{F}}\left(\mathrm{ad}^0, \mathbb{F}\right)$. Using local Tate duality, one obtains that $H^2(G_K, \mathrm{ad}^0) \cong \left((\mathrm{ad}^0)^U \otimes \mathbb{F}^\varepsilon\right)^G$.

In the remainder of this section, we distinguish the following five cases.

(A) $\overline{G} \neq \{1\}$ is cyclic and U is trivial. Then $\bar\rho \sim \begin{pmatrix} 1 & 0 \\ 0 & \xi \end{pmatrix} \otimes \eta$ for some characters $\xi, \eta : G_K \to \mathbb{F}^*$. Moreover,

$$\mathrm{ad} \cong (\mathbb{F}^{\mathrm{triv}})^2 \oplus \mathbb{F}^\xi \oplus \mathbb{F}^{\xi^{-1}} \quad \text{and} \quad (\mathrm{ad}^0)^U \otimes \mathbb{F}^\varepsilon \cong \mathbb{F}^\varepsilon \oplus \mathbb{F}^{\xi\varepsilon} \oplus \mathbb{F}^{\xi^{-1}\varepsilon}.$$

(B) $\overline{G} \neq \{1\}$ is cyclic and U is nontrivial. Then $\bar\rho \sim \begin{pmatrix} 1 & \star \\ 0 & \xi \end{pmatrix} \otimes \eta$ for some characters $\xi, \eta : G_K \to \mathbb{F}^*$; here $\star$ denotes a non-trivial extension, i.e., a non-trivial class in $H^1(G_K, \mathbb{F}^\xi)$. Moreover,

$$(\mathrm{ad})^U \cong \mathbb{F}^{\mathrm{triv}} \oplus \mathbb{F}^{\xi^{-1}} \quad \text{and} \quad (\mathrm{ad}^0)^U \otimes \mathbb{F}^\varepsilon \cong \mathbb{F}^{\xi^{-1}\varepsilon}.$$

(C) $\overline{G}$ is dihedral. Then $\overline{H} = \overline{G}$, and $\overline{U}$ is trivial. By [Mu, Prop. 2.1.1], there exists a character ξ' of a normal cyclic subgroup C_n of $\overline{G}$ of index 2 such that $\bar\rho \sim \mathrm{Ind}_{C_n}^{\overline{G}}(\xi')$. Then we have

$$\mathrm{ad} \cong \mathbb{F}^{\mathrm{triv}} \oplus \mathbb{F}^\varphi \oplus \mathrm{Ind}_{C_n}^{\overline{G}} \mathbb{F}^\xi \quad \text{and} \quad \mathrm{ad}^0 \otimes \mathbb{F}^\varepsilon \cong \mathbb{F}^{\varphi\varepsilon} \oplus \mathrm{Ind}_{C_n}^{\overline{G}} \mathbb{F}^\xi \otimes \mathbb{F}^\varepsilon,$$

where $\varphi \colon \overline{G}/C_n \to \mathbb{F}^*$ is the unique non-trivial character of order two and $\xi : C_n \to \mathbb{F}^*$ is the character $g \mapsto \xi'(g)^{1-\#k_K}$ for k_K the residue field of K.

(D) $\overline{G}$ and $\overline{U}$ are trivial. Then $\overline{H}$ is trivial, and H is in the scalars of $\mathrm{GL}_2(\mathbb{F})$. Moreover,

$$\mathrm{ad} \cong (\mathbb{F}^{\mathrm{triv}})^4 \quad \text{and} \quad \mathrm{ad}^0 \otimes \mathbb{F}^\varepsilon \cong (\mathbb{F}^\varepsilon)^3.$$

(E) $\overline{G}$ is trivial and $\overline{U}$ is nontrivial. Then $\bar\rho \sim \begin{pmatrix} 1 & \star \\ 0 & 1 \end{pmatrix} \otimes \eta$ for some character $\eta : G_K \to \mathbb{F}^*$, where $\star$ denotes a non-trivial extension. Moreover,

$$(\mathrm{ad})^U \cong (\mathbb{F}^{\mathrm{triv}})^2 \quad \text{and} \quad (\mathrm{ad}^0)^U \otimes \mathbb{F}^\varepsilon \cong \mathbb{F}^\varepsilon.$$

Remark 3.1. We would like to correct a mistake in [Bö2, Lem. 6.1] when U is nontrivial. As the character ψ defined at the beginning of [Bö2, §5] corresponds to the character ξ^{-1} in the notation used here, in [Bö2, Lem. 6.1] the line $((\mathrm{ad}_{\bar\rho})^U \otimes \mu_p(L))^G \cong (k^\chi \oplus k^{\psi^{-1}\chi})^G$ should be replaced by $((\mathrm{ad}_{\bar\rho})^U \otimes \mu_p(L))^G \cong (k^\chi \oplus k^{\psi\chi})^G$. Further, in case (ix) the condition in case (ix) should read $\chi = \psi^{-1}$ and not $\chi = \psi$ as written.

We know from [Bö2, Theorem 2.6] that the versal deformation ring is isomorphic to the quotient $W(\mathbb{F})[[x_1,\ldots,x_h]]/I^\psi$, where I^ψ is generated by exactly $h_2 := \dim_{\mathbb{F}} H^2(G_K, \mathrm{ad}^0)$ relations.

Lemma 3.2 (Cf. [Bö2, Lem. 6.1])**.** *If $\mu_{p^\infty}(F) = \{1\}$, then $h_2 = 0$. Else, the dimensions h_2 and h take the following values in the cases (A)-(E) introduced above.*

(A) *(i) If $\varepsilon = \mathrm{triv}$, then $h_2 = 1$ and $h = 3[K:\mathbb{Q}_p] + 2$;*
(ii) If $\varepsilon = \xi$ and the order of ξ is two, then $h_2 = 2$ and $h = 3[K:\mathbb{Q}_p] + 3$;
(iii) If $\varepsilon = \xi$ or $\varepsilon = \xi^{-1}$ and $\xi \neq \xi^{-1}$, then $h_2 = 1$ and $h = 3[K:\mathbb{Q}_p] + 2$;
(iv) In all other cases $h_2 = 0$ and $h = 3[K:\mathbb{Q}_p] + 1$.

(B) *(i) If $\varepsilon = \xi^{-1}$ and the order of ξ is two, then $h_2 = 1$ and $h = 3[K:\mathbb{Q}_p] + 1$;*
(ii) If $\varepsilon = \xi^{-1}$ and $\xi \neq \xi^{-1}$, then $h_2 = 1$ and $h = 3[K:\mathbb{Q}_p] + 1$;[4]
(iii) In all other cases $h_2 = 0$ and $h = 3[K:\mathbb{Q}_p]$.

(C) *(i) If $\varepsilon = \varphi$, then $h_2 = 1$ and $h = 3[K:\mathbb{Q}_p] + 1$;*
(ii) In all other cases $h_2 = 0$ and $h = 3[K:\mathbb{Q}_p]$.

(D) *(i) If $\varepsilon = \mathrm{triv}$, then $h_2 = 3$ and $h = 3[K:\mathbb{Q}_p] + 6$;*
(ii) In all other cases $h_2 = 0$ and $h = 3[K:\mathbb{Q}_p] + 3$.

(E) *(i) If $\varepsilon = \mathrm{triv}$, then $h_2 = 1$ and $h = 3[K:\mathbb{Q}_p] + 2$;*
(ii) In all other cases $h_2 = 0$ and $h = 3[K:\mathbb{Q}_p] + 1$.

Proof. If F contains no p-power roots of unity, then the maximal pro-p quotient $G_F(p)$ of G_F is a free pro-p group and $h_2 = 0$ by [La, §1.4]. Otherwise we use the above decompositions of $\mathrm{ad} \cong \mathbb{F}^{\mathrm{triv}} \oplus \mathrm{ad}^0$ and $(\mathrm{ad}^0)^U \otimes \mathbb{F}^\varepsilon$ in the cases (A)–(E), and obtain the values of h_2 and $h_0 := \dim_{\mathbb{F}} H^0(G_K, \mathrm{ad}^0)$. Recall next that the Euler-Poincaré characteristic of ad^0 is $3[K:\mathbb{Q}_p] = -h_0 + h - h_2$ from which one computes h. □

For the following explicit descriptions of minimal presentations of $R^\psi_{\bar\rho}$, we recall the functor E_Π from [Bö2, Proposition 2.3]. It is always representable and its universal ring is a versal hull for $D^\psi_{\bar\rho}$. To describe E_Π we need to fix some notation. Since U is a p-group in $\mathrm{GL}_2(\mathbb{F})$ we shall assume that U lies in the set of unipotent upper triangular matrices $U_2(\mathbb{F})$. If U is non-trivial, let $\{g_n\}_n$ be a minimal set of topological generators of the maximal pro-p quotient $G_F(p)$ of G_F so that the $\bar\rho(g_n) = \left(\begin{smallmatrix} 1 & \bar u_n \\ 0 & 1 \end{smallmatrix}\right)$ for $\bar u_n \in \mathbb{F}$ generate U as a G-module. If U is non-trivial, there is a smallest index i_0 for which $\bar u_{i_0}$ is a unit. Then by conjugation by an element of the form $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $\lambda \in \mathbb{F}^*$, which clearly lifts to $\mathrm{GL}_2(W(\mathbb{F}))$, we will assume from now on that $\bar u_{i_0} = 1$. For any ring R in $\widehat{\mathrm{Ar}}_{W(\mathbb{F})}$ we denote by $\widetilde{\Gamma}_2(R)$ the inverse image of $U_2(\mathbb{F})$ under the reduction homomorphism $\mathrm{SL}_2(R) \to \mathrm{SL}_2(\mathbb{F})$. We set $\bar\alpha := \bar\rho|_{G_F(p)}$. If $\bar G = \{1\}$, then we define the functor $E_\Pi \colon \widehat{\mathrm{Ar}}_{W(\mathbb{F})} \to \mathbf{Sets}$ by sending $(R, \mathfrak{m}_R)$ to the set

$$\left\{ \alpha \in \mathrm{Hom}_G(G_F(p), \tilde\Gamma_2(R)) \,\middle|\, \alpha(g_{i_0}) = \begin{pmatrix} 1 & 1 \\ * & * \end{pmatrix} \text{ and } \alpha \equiv \bar\alpha \pmod{\mathfrak{m}_R} \right\}$$

[4]The reason for not combining (i) and (ii) in case (B) into a single case is that the cases of Lemma 3.2 are used throughout this section, and in later parts the distinction is necessary.

if U is non-trivial, and else to $\mathrm{Hom}_G(G_F(p), \tilde{\Gamma}_2(R))$. Observe that $E_\Pi(\mathbb{F}) = \{\bar{\alpha}\}$. As noted above, E_Π is representable and its universal ring, we write $R_{\bar{\alpha}}$, is isomorphic to $R^{\psi}_{\bar{\rho}}$. The gain is that it is rather elementary to write down explicitly $R_{\bar{\alpha}}$.

Lastly, we define q as the number of p-power roots of unity contained in F and g_q as the polynomial

$$g_q(x) := \sum_{k=0}^{(q-1)/2} \frac{q}{(2k+1)!} \prod_{j=0}^{k-1} (q^2 - (2j+1)^2) x^k.$$

Note that the polynomial g_q lies in fact in $\mathbb{Z}[x]$.

Remark 3.3. We take this opportunity to correct another mistake from [Bö2, Rem. 5.5 (i)]: In the formulas for $a_{n,k}$ and $b_{n,k}$, the expressions $(2k)!$ and $(2k+1)!$, respectively, should be in the denominator.

Theorem 3.4 (Cf. [Bö2, Thm. 6.2 and Rem. 6.3 (iv)])**.** *Suppose $\mu_{p^\infty}(F) \neq \{1\}$ and set $m := [K : \mathbb{Q}_p]$. There exist a minimal presentation $0 \to I^\psi \to \mathcal{R} \to R^{\psi}_{\bar{\rho}} \to 0$ of $R^{\psi}_{\bar{\rho}}$, where $\mathcal{R}$ and I^ψ are as follows in the respective cases of the previous lemma.*

(A) *(i)* $\mathcal{R} = W(\mathbb{F})[[\{b_i, c_i\}_{i=1}^m, \{d_j\}_{j=0}^{m+1}]]$ *and* $I^\psi = \Big(\sum_{i=1}^m c_i b_{m-i+1} - ((1+d_0)^q - 1)(1+d_0)^{-\frac{q}{2}}\Big)$;

(ii) $\mathcal{R} = W(\mathbb{F})[[\{b_i, c_i, d_i\}_{i=0}^m]]$ *and* $I^\psi = \Big(\sum_{i=0}^m b_i d_{m-i} - b_0 g_q(b_0 c_0), -\sum_{i=0}^m c_i d_{m-i} - c_0 g_q(b_0 c_0)\Big)$;

(iii) *If* $\varepsilon = \psi$, *then* $\mathcal{R} = W(\mathbb{F})[[\{b_i, d_i\}_{i=0}^m, \{c_j\}_{j=1}^m]]$ *and* $I^\psi = \Big(\sum_{i=0}^m d_i b_{m-i} - q b_0\Big)$;
If $\varepsilon = \psi^{-1}$, *then* $\mathcal{R} = W(\mathbb{F})[[\{b_i\}_{i=1}^m, \{c_j, d_j\}_{j=0}^m]]$ *and* $I^\psi = \Big(\sum_{i=0}^m d_i c_{m-i} - q c_0\Big)$;

(iv) $\mathcal{R} = W(\mathbb{F})[[\{b_i, c_i\}_{i=1}^m, \{d_j\}_{j=0}^m]]$ *and* $I^\psi = (0)$.

(B) *(i)* $\mathcal{R} = W(\mathbb{F})[[\{b_i, c_i, d_i\}_{i=0}^m]]/(b_{i_0}, d_{m-i_0})$ *and* $I^\psi = \Big(-\sum_{i=0, i\neq i_0}^m c_i d_{m-i} - \delta_{i_0}(c_0 + 2c_{i_0} b_0) \cdot g_q(b_0(c_0 + c_{i_0} b_0)) - 2(1-\delta_{i_0}) c_0 g_q(c_0)\Big)$, *where* $\delta_{i_0} \in \{0, 1\}$ *is 0 if* $i_0 = 0$ *and else 1;*

(ii) $\mathcal{R} = W(\mathbb{F})[[\{b_i\}_{i=1}^m, \{c_j, d_j\}_{j=0}^m]]/(b_{i_0})$ *and* $I^\psi = \Big(\sum_{i=0}^m c_i d_{m-i} - q c_0\Big)$;

(iii) $\mathcal{R} = W(\mathbb{F})[[\{b_i, c_i, d_i\}_{i=1}^m]]$ *and* $I^\psi = (0)$.

(C) *(i)* $\mathcal{R} = W(\mathbb{F})[[\{b_i\}_{i=1}^{2m}, \{d_j\}_{j=0}^m]]$ *and* $I^\psi = \Big(\sum_{i=1}^m b_i b_{2m-i+1} - \big((1+d_0)^{\frac{q}{2}} - (1+d_0)^{-\frac{q}{2}}\big)\Big)$;

(ii) $\mathcal{R} = W(\mathbb{F})[[\{b_i\}_{i=1}^{2m}, \{d_j\}_{j=1}^m]]$ *and* $I^\psi = (0)$.

Remark 3.5. We point out that in front of the sum $\sum c_i d_{m-i}$ in the second generator of the relation ideals in [Bö2, Theorem 6.2(ii) and Prop. 7.3(ii)] a minus sign is missing. This originates from a sign mistake in [Bö2, Lem. 5.6(B)]: There the matrix $\begin{pmatrix} 0 & b_i \\ c_i & 0 \end{pmatrix}$ should read instead $\begin{pmatrix} 0 & b_i \\ -c_i & 0 \end{pmatrix}$.

Proof. The relation ideal of the versal hull of the deformation functor without fixing the determinant is listed in the respective cases in [Bö2, Theorem 6.2]. We remark that the relation $(1+a_0)^q - 1$ from there is omitted due to our condition on the determinant, and we used a change of variables according to [Bö2, Remark 6.3(iv)] to simplify the expressions for the relations and variables. In order to obtain the right number of indeterminates of the power series ring $\mathcal{R}$, we follow the steps described in the proof of [Bö2, Theorem 2.6].

Since by assumption F contains a p-power root of unity, $G_F(p)$ is a Demuškin group, and its Frattini quotient $\bar{G}_F(p)$ is isomorphic to $\mathbb{F}^{\mathrm{triv}} \oplus \mathbb{F}^{\varepsilon} \oplus \mathbb{F}_p[G]^m$ as a G-module. By the Burnside basis theorem, there are closed subgroups P_n of $G_F(p)$ such that the Frattini quotients $\bar{P}_n$ of P_n are irreducible and $\overline{G}_F(p) = \oplus_n \bar{P}_n$. Since the tangent space $t_E := E_\Pi(\mathbb{F}[t]/(t^2))$ of E_Π is isomorphic to the tangent space t_D and $\mathrm{ad}^0 \cong \tilde{\Gamma}_2(\mathbb{F}[t]/(t^2))$ as a G-module, we have $h = \dim_{\mathbb{F}} t_D = \dim_{\mathbb{F}} t_E \leq \dim_{\mathbb{F}} \mathrm{Hom}_G(\overline{G}_F(p), \mathrm{ad}^0)$. We can compute the right hand side in terms of those G-submodules $\bar{P}_n$ of $\overline{G}_F(p)$ that occur in decompositions of both $\overline{G}_F(p)$ and ad^0 into irreducible G-modules, because the G-submodules that do not occur in a decomposition of ad^0 have trivial image (prime-to-adjoint principle). As remarked in [Bö2, §6], the multiplicities of the G-submodules occuring in a decomposition of ad^0 are $(\overline{G}_F(p), \mathrm{Ind}_{\bar{C}_n}^{\bar{G}} \mathbb{F}^{\psi})_G = 2m$ if $\bar{G}$ is dihedral, $(\overline{G}_F(p), \mathbb{F}^{\tau})_G = m$ for any non-trivial character $\tau \neq \varepsilon$, and $(\overline{G}_F(p), \mathbb{F}^{\mathrm{triv}})_G = (\overline{G}_F(p), \mathbb{F}^{\varepsilon})_G = m + 1 + \delta_K$, where $(X, Y)_G := \dim_{\mathbb{F}}(\mathrm{Hom}_G(X, Y))$ for G-modules and δ_K is 1 if ε acts trivially and 0 otherwise. By [Bö2, Lem. 5.3], we can choose $x_n \in P_n$ such that Gx_n topologically generates P_n, and whose image under a homomorphism $\alpha : P_n \to \Gamma_n(R)$ is either the identity if $\bar{P}_n$ does not occur in a decompositions of ad^0 or a matrix of the type

$$S(b,c) := \begin{pmatrix} \sqrt{1+bc} & b \\ c & \sqrt{1+bc} \end{pmatrix} \qquad \text{or} \qquad D(d) := \begin{pmatrix} \sqrt{1+d} & 0 \\ 0 & \sqrt{1+d}^{-1} \end{pmatrix}$$

for any ring R in $\widehat{\mathrm{Ar}}_{W(\mathbb{F})}$ and $b, c, d \in \mathfrak{m}_R$. If U is non-trivial, we shall take for the g_n in the definition of E_Π the generators x_n. If $\bar{\rho}(x_0) \neq \mathrm{id}$, then we take $g_1 := x_0$, else we shall assume that $g_1 := x_1$ by a suitable permutation of the indices n. In cases (A)–(C), we will consider the power series ring $\mathcal{R}$ over $W(\mathbb{F})$ in the variables b, c, d occurring in the images $S(b,c)$ and $D(d)$ of all generators. Then we will obtain the universal object $(R_{\bar{\alpha}}, \alpha_{\bar{\alpha}})$ representing E_Π, where $R_{\bar{\alpha}}$ is the quotient ring of $\mathcal{R}$ modulo the respective relations in terms of the variables b, c, d from [Bö2, Lemma 5.6 and Theorem 6.2] .

We begin with explicitly describing $\mathcal{R}$ and the relation ideal I^{ψ} in case (A). Then we have that $\mathrm{ad}^0 \cong \mathbb{F}^{\mathrm{triv}} \oplus \mathbb{F}^{\xi} \oplus \mathbb{F}^{\xi^{-1}}$ and $h = \dim_{\mathbb{F}} t_E = (\overline{G}_F(p), \mathbb{F}^{\mathrm{triv}})_G + (\overline{G}_F(p), \mathbb{F}^{\xi})_G + (\overline{G}_F(p), \mathbb{F}^{\xi^{-1}})_G$. The following table displays the respective multiplicities of the subrepresentations $\mathbb{F}^{\mathrm{triv}}$, $\mathbb{F}^{\xi}$ and $\mathbb{F}^{\xi^{-1}}$ in $\overline{G}_F(p)$:

		$k_1 = (\overline{G}_F(p), \mathbb{F}^{\mathrm{triv}})_G$	$k_2 = (\overline{G}_F(p), \mathbb{F}^{\xi})_G$	$k_3 = (\overline{G}_F(p), \mathbb{F}^{\xi^{-1}})_G$
(i)	$\varepsilon = \mathrm{triv}$	m+2	m	m
(ii)	$\varepsilon = \xi$, $\varepsilon = \varepsilon^{-1}$	m+1	m+1	m+1
(iii)	$\varepsilon = \xi$, $\varepsilon \neq \varepsilon^{-1}$	m+1	m+1	m
	$\varepsilon = \xi^{-1}$, $\varepsilon \neq \varepsilon^{-1}$	m+1	m	m+1
(iv)	$\varepsilon \notin \{\mathrm{triv}, \xi, \xi^{-1}\}$	m+1	m	m

By [Bö2, Lem. 5.3(ii)-(iv)], there exist $b_n, c_{n'}, d_{n''} \in \mathfrak{m}_R$ (with $n = (m+1-k_2), \ldots, m$, $n' = (m+1-k_3), \ldots, m$ and $n'' = 0, \ldots, k_1 - 1$) such that a generator x_n of a subgroup P_n gets mapped to either

$$S(b_n, 0), \quad S(0, c_{n'}), \quad S(b_n, c_n), \quad D(d_{n''}) \quad \text{or} \quad D(0)$$

under a G-equivariant homomorphism $P_n \to \mathrm{GL}_2(R)$. Finally, in [Bö2, Lem. 5.6(A)-(D),(F)] the image of the Demuškin relation involving these matrices is completely described. The thereby obtained equations define the respective relation ideal I^{ψ} (as in [Bö2, Theorem 6.2(i)-(iv)]).

In case (B), we have that $\mathrm{ad}^0 \cong \mathbb{F}^{\mathrm{triv}} \oplus \mathbb{F}^{\xi} \oplus \mathbb{F}^{\xi^{-1}}$ and $h = \dim_{\mathbb{F}} t_D = \dim_{\mathbb{F}} t_E < h' := (\overline{G}_F(p), \mathbb{F}^{\mathrm{triv}})_G + (\overline{G}_F(p), \mathbb{F}^{\xi})_G + (\overline{G}_F(p), \mathbb{F}^{\xi^{-1}})_G$ due to the further conditions that $\alpha \in E_\Pi(\mathbb{F}[t]/t^2)$ has to satisfy if U is non-trivial. As in case (A), there exist $b_n, c_{n'}, d_{n''} \in \mathfrak{m}_R$ (with $n = (m+1-k_2), \ldots, m$, $n' =$

$(m+1-k_3),\dots,m$ and $n''=0,\dots,k_1-1$) such that a generator x_n of a subgroup P_n gets mapped to either

$$S(\bar u_n+b_n,0),\quad S(0,c_{n'}),\quad S(\bar u_n+b_n,c_n),\quad D(d_{n''})\quad\text{or}\quad D(0)$$

under a G-equivariant homomorphism $P_n\to\mathrm{GL}_2(R)$. Due to the condition on the image of x_{i_0}, the variable b_{i_0} occuring in the image of x_{i_0} must vanish. In [Bö2, Lem. 5.6(A)-(D),(F)] the image of the Demuškin relation involving these matrices is completely described. By [Bö2, Theorem 6.2(ii)-(iii)], this gives rise to the following generators of I^ψ:

$$\sum_{i=0}^m(\bar u_i+b_i)d_{m-i}-(\bar u_0+b_0)g_q((\bar u_0+b_0)c_0)\quad\text{and}\quad -\sum_{i=0}^m c_id_{m-i}-c_0g_q((\bar u_0+b_0)c_0)\quad\text{in case (i)}$$

and in case (ii) to $\sum_{i=0,i\neq i_0}^m c_id_{m-i}-qc_0$. In (i), we use the first relation $d_{m-i_0}=(\bar u_0+b_0)g_q((\bar u_0+b_0)c_0)-\sum_{i=0,i\neq i_0}^m(\bar u_i+b_i)d_{m-i}$ to also elimate d_{m-i_0}. Then the second equation reads

$$-\sum_{i=0}^m c_id_{m-i}-c_0g_q((\bar u_0+b_0)c_0)=-\sum_{i=0,i\neq i_0}^m\big(c_i-c_{i_0}(\bar u_i+b_i)\big)d_{m-i}-(c_0+c_{i_0}(\bar u_0+b_0))g_q((\bar u_0+b_0)c_0).$$

We perform a linear change of coordinates by replacing $c_i+c_{i_0}(\bar u_i+b_i)$ by c_i for $i\neq i_0$. Note that $\bar u_0=0$ if $i_0>0$ so that we obtain the respective generators of I^ψ displayed in case (B).

In case (C), we have that $\mathrm{ad}^0\cong\mathbb F^\varphi\oplus\mathrm{Ind}_{C_n}^{\overline G}\mathbb F^\xi$ and $h=\dim_{\mathbb F}t_E=(\overline G_F(p),\mathbb F^\varphi)_G+2(\overline G_F(p),\mathrm{Ind}_{C_n}^{\overline G}\mathbb F^\xi)_G$. This means that the multiplicities of the subrepresentations $\mathrm{Ind}_{C_n}^{\overline G}\mathbb F^\xi$ in $\overline G_F(p)$ are $2m$, and the ones of the subrepresentations $\mathbb F^\varphi$ are $m+1$ if $\varepsilon=\varphi$ and m if $\varepsilon\neq\varphi$. By [Bö2, Lem. 5.3(ii),(v)-(vii)], there exist $b_n,d_{n'}\in\mathfrak m_R$ (with $n=1,\dots,2m$, $n'=0,\dots,m$ in (i) and $n'=1,\dots,m$ in (ii)) such that a generator x_n of a subgroup P_n gets mapped to either

$$S(b_n,b_n),\quad S(b_n,-b_n),\quad D(d_{n'})\quad\text{or}\quad D(0)$$

under a G-equivariant homomorphism $P_n\to\mathrm{GL}_2(R)$. Finally, in [Bö2, Lem. 5.6(E)-(F)] the image of the Demuškin relation involving these matrices is completely described. The thereby obtained equations define the respective relation ideal I^ψ (as in [Bö2, Theorem 6.2(v)-(vii)]). □

We define $\mathfrak n$ to be the ideal in $\mathcal R$ generated by all the variables $b_i,c_{i'},d_{i''}$ occuring in the respective definitions of $\mathcal R$ in the previous theorem. Further, define the ideal $\mathfrak m_s\subset\mathcal R$ as $\mathfrak m_s:=q\mathcal R+\mathfrak n$. In cases (A)–(C) it is now a simple matter to read off from the previous theorem the initial terms for the graded rings naturally associated to $\mathcal R$. Checking that these initial terms form a regular sequence will imply most parts of Theorem 1.4 and, when combined with Proposition 2.2, the assertions of Theorem 1.5 in cases (A)–(C).

Corollary 3.6. *In the cases (A)–(C) of the previous lemma, denote the two generators of I^ψ in case (A)(ii) by f_1 and f_2, and in the other cases the generator of I^ψ by f_1.*

- *(a) Let* **in** *be the initial term map $\overline{\mathcal R}\to\mathrm{gr}_{\overline{\mathfrak n}}\overline{\mathcal R}$. Then the following are the initial terms of the generators of I^ψ in $\overline{\mathbf{in}(I^\psi)}\subset\overline{\mathrm{gr}_{\mathfrak n}\mathcal R}$ in the cases (A)–(C) of Lemma 3.2, where we only list those cases in which h_2 is non-zero.*
  - *(A) (i)* $\mathbf{in}(\overline{f_1})=\sum_{i=1}^m\bar c_i\bar b_{m-i+1}$;
    (ii) $\mathbf{in}(\overline{f_1})=-\sum_{i=0}^m\bar b_i\bar d_{m-i}$ *and* $\mathbf{in}(\overline{f_2})=\sum_{i=0}^m\bar c_i\bar d_{m-i}$;

(iii) *If* $\varepsilon = \psi$, *then* $\mathbf{in}(\overline{f_1}) = \sum_{i=0}^{m} \bar{d}_i \bar{b}_{m-i}$*;*
If $\varepsilon = \psi^{-1}$, *then* $\mathbf{in}(\overline{f_1}) = \sum_{i=0}^{m} \bar{d}_i \bar{c}_{m-i}$*;*

(B) *(i)* $\mathbf{in}(\overline{f_1}) = -\sum_{i=0, i\neq i_0}^{m} \bar{c}_i \bar{d}_{m-i} - \overline{1-\delta_{i_0}} \cdot \overline{\tfrac{q}{3}} \cdot \bar{c}_0^2$*;*[5]

(ii) $\mathbf{in}(\overline{f_1}) = \sum_{i=0}^{m} \bar{c}_i \bar{d}_{m-i}$*;*

(C) *(i)* $\mathbf{in}(\overline{f_1}) = \sum_{i=1}^{m} \bar{b}_i \bar{b}_{2m-i+1}$.

(b) *Let* $\mathbf{in}$ *be the initial term map* $\mathcal{R} \to \mathrm{gr}_{\mathfrak{m}_s} \mathcal{R}$ *and set* $t_0 := \mathbf{in}(q)$. *Then the following are the initial terms of the generators of* I^ψ *in* $\overline{\mathbf{in}(I^\psi)} \subset \overline{\mathrm{gr}_{\mathfrak{m}_s} \mathcal{R}}$ *in the cases (A)–(C) of Lemma 3.2, where we only list those cases in which* h_2 *is non-zero.*

(A) *(i)* $\overline{\mathbf{in}(f_1)} = \sum_{i=1}^{m} \bar{c}_i \bar{b}_{m-i+1} - \bar{t}_0 \bar{d}_0$*;*

(ii) $\overline{\mathbf{in}(f_1)} = \sum_{i=0}^{m} \bar{b}_i \bar{d}_{m-i} - \bar{t}_0 \bar{b}_0$ *and* $\overline{\mathbf{in}(f_2)} = -\sum_{i=0}^{m} \bar{c}_i \bar{d}_{m-i} - \bar{t}_0 \bar{c}_0$*;*

(iii) *If* $\varepsilon = \psi$, *then* $\overline{\mathbf{in}(f_1)} = \sum_{i=0}^{m} \bar{d}_i \bar{b}_{m-i} - \bar{t}_0 \bar{b}_0$*;*
If $\varepsilon = \psi^{-1}$, *then* $\overline{\mathbf{in}(f_1)} = \sum_{i=0}^{m} \bar{d}_i \bar{c}_{m-i} - \bar{t}_0 \bar{c}_0$*;*

(B) *(i)* $\overline{\mathbf{in}(f_1)} = -\sum_{i=0, i\neq i_0}^{m} \bar{c}_i \bar{d}_{m-i} - \overline{2-\delta_{i_0}} \cdot \bar{t}_0 \bar{c}_0 - \overline{1-\delta_{i_0}} \cdot \overline{\tfrac{q}{3}} \cdot \bar{c}_0^2$*;*[5]

(ii) $\overline{\mathbf{in}(f_1)} = \sum_{i=0}^{m} \overline{c_i} \bar{d}_{m-i} - \bar{t}_0 \bar{c}_0$*;*

(C) *(i)* $\overline{\mathbf{in}(f_1)} = \sum_{i=1}^{m} \bar{b}_i \bar{b}_{2m-i+1} - \bar{t}_0 \bar{d}_0$.

Proof of Theorem 1.4. We first give the proof in cases (A)–(C): In all cases of (A)–(C) with $h_2 \neq 0$, Theorem 1.4(a) holds since the initial terms given in Corollary 3.6(b) together with $\bar{t}_0$ form regular sequences in $\overline{\mathrm{gr}_{\mathfrak{m}_s} \mathcal{R}}$ with $t_0 := \mathbf{in}(q)$. Moreover, by Proposition 2.2(c) the initial terms from Corollary 3.6(b) generate $\overline{\mathbf{in}(I^\psi)}$ in the respective cases and one checks that $\overline{\mathrm{gr}_{\mathfrak{m}_s} \mathcal{R}}/\overline{\mathbf{in}(I^\psi)}$ is an integral domain. Thus Theorem 1.4(b) follows from Proposition 2.2(d) in cases (A)–(C).

Theorem 1.4 in the remaining cases (D) and (E) is a direct consequence of the following lemma. □

Lemma 3.7. *In the cases (D) and (E) let* q *denote the minimum of* p *and the number of* p*-power roots of unity in* K. *Then there exists a minimal presentation*

$$0 \longrightarrow I^\psi = (r_1, \ldots, r_m) \longrightarrow \mathcal{R} \cong W(\mathbb{F})[[x_1, \ldots, x_h]] \longrightarrow R_{\bar{\rho}}^{\psi} \longrightarrow 0$$

such that, letting $\mathfrak{m}_s = (q, x_1, \ldots, x_h)$, *the following hold:*

(a) $\mathfrak{m}_s^2 \supset I^\psi$ *and* $\overline{\mathbf{in}(q)}, \overline{\mathbf{in}(r_1)}, \ldots, \overline{\mathbf{in}(r_m)} \in \overline{\mathrm{gr}^2_{\mathfrak{m}_s} \mathcal{R}}$ *is a regular sequence in* $\overline{\mathrm{gr}_{\mathfrak{m}_s} \mathcal{R}}$*;*

(b) $\overline{\mathrm{gr}_{\mathfrak{m}_s} \mathcal{R}}/\big(\overline{\mathbf{in}(r_1)}, \ldots, \overline{\mathbf{in}(r_m)}\big)$ *is an integral domain and* $\overline{\mathbf{in}(I^\psi)} = \big(\overline{\mathbf{in}(r_1)}, \ldots, \overline{\mathbf{in}(r_m)}\big)$*;*

(c) $\mathbf{in}(\bar{r}_1), \ldots, \mathbf{in}(\bar{r}_m) \in \mathrm{gr}^2_{\overline{\mathfrak{m}}_{\mathcal{R}}} \overline{\mathcal{R}}$ *form a regular sequence in* $\mathrm{gr}_{\overline{\mathfrak{m}}_{\mathcal{R}}} \overline{\mathcal{R}}$*;*

(d) $\mathrm{gr}_{\overline{\mathfrak{m}}_{\mathcal{R}}} \overline{\mathcal{R}}/\big(\mathbf{in}(\bar{r}_1), \ldots, \mathbf{in}(\bar{r}_1)\big)$ *is an integral domain and* $\mathbf{in}(\overline{I^\psi}) = \big(\mathbf{in}(\bar{r}_1), \ldots, \mathbf{in}(\bar{r}_1)\big)$*;*

(e) $m = \dim_{\mathbb{F}} H^2(G_K, \mathrm{ad}^0)$ *and* $\dim_{\mathrm{Krull}} R_{\bar{\rho}}^{\psi} = h + 1 - m$.

[5] Note that the term involving $\bar{c}_0^2$ vanishes unless $q = 3$.

Proof. The proof proceeds along the lines of the proof of [Bö2, Theorem 2.6], but it is simpler in our case as we shall only determine the initial parts of the g_i and r_j, and since there is no action of a finite group of order prime to p. We recall that $\bar\rho \sim \left(\begin{smallmatrix} 1 & \star \\ 0 & 1 \end{smallmatrix}\right) \otimes \eta$ for some character $\eta : G_K \to \mathbb{F}^*$, where $\star$ denotes an extension. As a preliminary reduction, we may twist $\bar\rho$ by η^{-1} so that the image of $\bar\rho$ is a p-group. Twisting all deformations by the Teichmüller lift of η^{-1} provides an isomorphism to the deformation functor of the twist of $\bar\rho$. In particular both functors are represented by isomorphic versal rings. Since now $\det\bar\rho$ is trivial, we shall also assume that its fixed lift ψ is the trivial character, since again, changing ψ has no effect on the versal deformation ring up to isomorphism. After this reduction, the first case to consider is that when K does not contain a non-trivial p-power root of unity. Then by Lemma 3.2 we have $h_2 = 0$. Hence $R^\psi_{\bar\rho}$ is unobstructed and thus formally smooth, and assertions (a)–(e) are obvious.

Suppose from now on that K contains a primitive p-th root of unity ζ_p. Then the maximal pro-p-quotient $G_K(p)$ of G_K is known to be a Demushkin group of rank $2g = [K : \mathbb{Q}_p] + 2$, cf. [La, §5]. By the classification of Demushkin groups with $q > 2$ [La, Theorem 7], the pro-p group $G_K(p)$ is isomorphic to the pro-p completion Π of the discrete group

$$\langle x_1, \ldots, x_{2g} \mid r\rangle$$

for the Demushkin relation $r = x_1^q(x_1, x_2)(x_3, x_4)\ldots(x_{2g-1}, x_{2g})$ – recall that $(x, y) = x^{-1}y^{-1}xy$. In the following we fix an isomorphism $G_K(p) \cong \Pi$.[6] Note also that $2g \geq 4$, because K has to contain $\mathbb{Q}_p(\zeta_p)$ and $[\mathbb{Q}_p(\zeta_p) : \mathbb{Q}_p] = p - 1 \geq 2$. If $\mathrm{im}(\bar\rho)$ is non-trivial, the functor $E_\Pi\colon \widehat{\mathrm{Ar}}_{W(\mathbb{F})} \to \mathbf{Sets}$ is given by

$$(R, \mathfrak{m}_R) \longmapsto \left\{\alpha \in \mathrm{Hom}(\Pi, \tilde\Gamma_2(R)) \,\middle|\, \alpha(x_{i_0}) = \begin{pmatrix} 1 & 1 \\ * & * \end{pmatrix},\ \bar\alpha(x_i) \equiv \begin{pmatrix} 1 & \bar u_i \\ 0 & 1 \end{pmatrix} (\mathrm{mod}\ \mathfrak{m}_R) \text{ for all } i\right\},$$

and else by $(R, \mathfrak{m}_R) \mapsto \mathrm{Hom}(\Pi, \tilde\Gamma_2(R))$. As the elements $\{g_n\}$ from the bottom of page 12 we take $x_1, \ldots, x_{2g}$. As noted there, E_Π is always representable and its universal ring $R_{\bar\alpha}$ is isomorphic to $R^\psi_{\bar\rho}$. In order to find an explicit presentation of $R_{\bar\alpha}$, we define $\mathcal{S} := W(\mathbb{F})[[b_i, c_i, d_i : i = 1, \ldots, 2g]]$. For each $1 \leq i \leq 2g$ let

$$M_i := \mathbb{1}_2 + \begin{pmatrix} a_i & b_i + u_i \\ c_i & d_i \end{pmatrix} \qquad \text{with } \mathbb{1}_2 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

where we choose a lift $u_i \in W(\mathbb{F})$ of $\bar u_i \in \mathbb{F}$, subject to the requirement $u_i = 0$ whenever $\bar u_i = 0$, and where $a_i \in \mathcal{S}$ is chosen so that $\det M_i = 1$, i.e., $a_i = ((b_i + u_i)c_i - d_i)\sum_{n\geq 0} d_i^n$. Observe that in case (D) all $u_i = 0$. We define polynomials r_k in $\mathcal{S}$ by

$$\mathbb{1}_2 + \begin{pmatrix} r_1 & r_2 \\ r_3 & r_4 \end{pmatrix} := M_1^q[M_1, M_2]\ldots[M_{2g-1}, M_{2g}],$$

where $[M_i, M_{i'}]$ is the commutator bracket $M_i^{-1}M_{i'}^{-1}M_iM_{i'}$. Note that $(1 + r_1)(1 + r_4) - r_2r_3 = 1$ and that, as we shall explain in a moment, $(r_1, \ldots, r_4) \subset \mathfrak{m}_s = (q, b_i, c_i, d_i : i = 1, \ldots, 2g)$. It is now straightforward to see that the ring

$$R_{\bar\alpha} := \begin{cases} W(\mathbb{F})[[b_i, c_i, d_i : i = 1, \ldots, 2g]]/(r_1, r_2, r_3) & \text{in case (D)} \\ W(\mathbb{F})[[b_i, c_i, d_i : i = 1, \ldots, 2g]]/(r_1, r_2, r_3, b_{i_0}, d_{i_0} - c_{i_0}) & \text{in case (E)} \end{cases}$$

[6] By slight abuse of notation we shall therefore regard the topological generators x_i of Π as elements of G_K.

together with the homomorphism $\alpha_{\bar\alpha} : \Pi \to \mathrm{SL}_2(R_{\bar\alpha})$ defined by mapping x_i to M_i – the latter regarded as a matrix over $R_{\bar\alpha}$ – is a universal object for E_Π. Note that $\alpha_{\bar\alpha}$ is well-defined precisely because we imposed the condition that all r_k vanish. In case (E) we may and shall assume that $i_0 \le 4$ by permuting the **indices** of the x_i in pairs $(2i'-1, 2i')$ for $i' \in \{2, \dots, g\}$.

For $k = 2, 3$ and $j = 0, 1$ we define

$$G^{k,k-j}(\mathcal{S}) := \left\{ \mathbb{1}_2 + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathcal{S}) : c \in \mathfrak{m}_s^k, a, b, d \in \mathfrak{m}_s^{k-j} \right\}.$$

We set $\delta_{q=3} = 1$ if $q = 3$ and $\delta_{q=3} = 0$ if $q \neq 3$. One can easily check the following facts, where starting from (2) we let $j = 0$ in case (D) and $j = 1$ in case (E):

(1) the sets $G^{k,k-j}(\mathcal{S})$ defined above are subgroups of $\mathrm{SL}_2(\mathcal{S})$, and moreover $G^{2,2-j}(\mathcal{S})$ is a normal subgroup of $G^{3,3-j}(\mathcal{S})$ for $j \in \{0, 1\}$;

(2) the matrices M_1^q and $[M_{2i-1}, M_{2i}]$, for $i = 1, \dots, g$, lie in $G^{2,2-j}(\mathcal{S})$;

(3) in case (D), computing modulo $G^{3,3}(\mathcal{S})$, for $i = 1, \dots, g$ one has $M_1 \equiv \mathbb{1}_2 + \begin{pmatrix} -qd_1 & qb_1 \\ qc_1 & qd_1 \end{pmatrix}$ and

$$[M_{2i-1}, M_{2i}] \equiv \mathbb{1}_2 + \begin{pmatrix} b_{2i-1}c_{2i} - b_{2i}c_{2i-1} & 2b_{2i-1}d_{2i} - 2b_{2i}d_{2i-1} \\ -2c_{2i-1}d_{2i} + 2c_{2i}d_{2i-1} & -b_{2i-1}c_{2i} + b_{2i}c_{2i-1} \end{pmatrix};$$

(4) in case (E), computing modulo $G^{3,2}(\mathcal{S})$, for $i = 1, \dots, g$ one has

$$M_1^q \equiv \begin{cases} \mathbb{1}_2 + \begin{pmatrix} 0 & 1 \\ c_1 & 0 \end{pmatrix} \cdot (q + \delta_{q=3} c_1), & \text{if } i_0 = 1, \\ \mathbb{1}_2 + \begin{pmatrix} 0 & 0 \\ qc_1 & 0 \end{pmatrix}, & \text{if } i_0 > 1, \end{cases}$$

$$[M_{2i-1}, M_{2i}] \equiv$$
$$\mathbb{1}_2 + \begin{pmatrix} u_{2i-1}c_{2i} - u_{2i}c_{2i-1} & u_{2i-1}^2 c_{2i} - u_{2i}^2 c_{2i-1} + 2u_{2i-1}d_{2i} - 2u_{2i}d_{2i-1} \\ u_{2i}c_{2i-1}^2 - 2(u_{2i-1} - u_{2i})c_{2i-1}c_{2i} - u_{2i-1}c_{2i}^2 - 2c_{2i-1}d_{2i} + 2c_{2i}d_{2i-1} & -u_{2i-1}c_{2i} + u_{2i}c_{2i-1} \end{pmatrix};$$

(5) for $M, M' \in G^{2,2-j}(\mathcal{S})$ one has $MM' \equiv M + M' - \mathbb{1}_2 \pmod{G^{3,3-j}(\mathcal{S})}$.

Using these facts we can explicitly compute the initial terms of the relations r_i since for $j = 0, 1$:

$$\mathbb{1}_2 + \begin{pmatrix} r_1 & r_2 \\ r_3 & r_4 \end{pmatrix} = M_1^q \cdot \prod_{i=1}^{g} [M_{2i-1}, M_{2i}] \equiv M_1^q + \sum_{i=1}^{g} \left([M_{2i-1}, M_{2i}] - \mathbb{1}_2 \right) \mod G^{3,3-j}(\mathcal{S}).$$

In case (D) we have $r_1, r_2, r_3 \in \mathfrak{m}_s^2$ from (3) and (5), and in case (E) we deduce $r_1, r_2 \in \mathfrak{m}_s$ and $r_3 \in \mathfrak{m}_s^2$ from (4) and (5). Below we make the initial terms of the r_k more explicit. To then analyze properties of $R_{\bar\alpha}$, we shall need the following results from commutative algebra, which are simple exercises:

(α) if R is a ring and $a_1, a_2, a_3 \in R$, then using total degrees $w = xy - a_1x - a_2y + a_3$ is a non-zero divisor in the polynomial ring $R[x, y]$ over R; if moreover R is an integral domain and $a_3 \neq a_1a_2$, then $R[x, y]/(w)$ is an integral domain, as can be seen by performing a linear coordinate change with x and y, and then passing to $\mathrm{Frac}(R)[x, y]/(w)$.

(β) if R is an $\mathbb{N}$-graded Noetherian ring and if $f_1,\dots,f_\omega\in R$ are homogeneous of positive degree, then they form a regular sequence if they do so in any order (see [Mat, Remark after Thm. 16.3]);

(γ) if R and $f_1,\dots,f_\omega$ are as in (β), if the f_i form a regular sequence and if $R[\frac{1}{f_{\omega'+1}\cdot\ldots\cdot f_\omega}]/(f_1,\dots,f_{\omega'})$ is an integral domain for any $1\le\omega'\le\omega$, then $R/(f_1,\dots,f_{\omega'})$ is an integral domain, as well.

We first show assertions (a)–(e) in case (D). Here we take $\mathcal{R}=\mathcal{S}$. Because $\mathfrak{m}_s^2$ contains (r_1,r_2,r_3), the presentation $0\to(r_1,r_2,r_3)\to\mathcal{R}\to R_\alpha^\psi\to 0$ is minimal. We shall consider the canonical reduction map $\pi\colon\mathcal{R}\to\mathcal{R}'=\mathcal{R}/(b_i,c_i,d_i,i=5,\dots,2g)$, and we let $\mathfrak{m}'_s=\pi(\mathfrak{m}_s)$ and $r'_k=\pi(r_k)$ for $k=1,2,3$. The ring $\mathcal{R}'$ is a power series ring over $W(\mathbb{F})$ in 12 variables. Thus $\mathrm{gr}_{\overline{\mathfrak{m}}_{\mathcal{R}'}}\overline{\mathcal{R}}'$ and $\overline{\mathrm{gr}_{\mathfrak{m}'_s}\mathcal{R}'}/(\bar t_0)$, for $t_0:=\mathbf{in}(q)$, are polynomials rings over $\mathbb{F}$ in 12 variables. The elements $\mathbf{in}^2(\bar r'_k)\equiv\overline{\mathbf{in}^2(r'_k)}\pmod{\bar t_0}$ are homogeneous elements of degree 2 for $k=1,2,3$, which by (3) are given by the expressions

$$\bar b_1\bar c_2-\bar b_2\bar c_1+\bar b_3\bar c_4-\bar b_4\bar c_3,\quad \bar b_1\bar d_2-\bar b_2\bar d_1+\bar b_3\bar d_4-\bar b_4\bar d_3\quad\text{and}\quad \bar c_1\bar d_2-\bar c_2\bar d_1+\bar c_3\bar d_4-\bar c_4\bar d_3.$$

Using (α) and (β), one easily deduces that $\bar c_1,\bar d_2,\bar b_3,\bar b_4$ together with the three displayed relations above form a regular sequence in any order in $R=\mathbb{F}[\bar b_k,\bar c_k,\bar d_k:k=1,\dots,4]$. To complete the argument, we wish to apply (γ). If we invert $\bar b_4$ in R, then forming the quotient of R by the first two relations is equivalent to eliminating $\bar c_3,\bar d_3$ in R. This will change $\overline{\mathbf{in}^2(r'_3)}\pmod{\bar t_0}$ to

$$w:=c_1d_2-\frac{b_2}{b_4}d_4c_1-\frac{b_1}{b_4}c_4d_2-c_2d_1+\frac{b_2}{b4}c_4d_1+\frac{b_1}{b_4}c_2d_4\in R':=\mathbb{F}[b_1,b_2,b_3,b_4,c_2,c_4,d_1,d_4,\frac{1}{b_4}][c_1,d_2].$$

Since $(b_2d_4)(b_1c_4)\ne b_4^2(-c_2d_1+\frac{b_2}{b4}c_4d_1+\frac{b_1}{b_4}c_2d_4)$ in the polynomial ring $\mathbb{F}[b_1,b_2,b_3,b_4,c_2,c_4,d_1,d_4]$, the ring $R'/(w)$ is an integral domain by (α). Therefore by (γ) the ring $\overline{\mathrm{gr}_{\mathfrak{m}'_s}\mathcal{R}'}/(\bar t_0,\overline{\mathbf{in}(r'_1)},\overline{\mathbf{in}(r'_2)},\overline{\mathbf{in}(r'_3)})$ is an integral domain, as well. This implies that $\bar t_0,\overline{\mathbf{in}(r_1)},\overline{\mathbf{in}(r_2)},\overline{\mathbf{in}(r_3)},\bar b_5,\bar c_5,\bar d_5,\dots,\bar b_{2g},\bar c_{2g},\bar d_{2g}$ is a regular sequence in $\overline{\mathrm{gr}_{\mathfrak{m}_s}\mathcal{R}}$ and that the corresponding quotient ring is an integral domain. Invoking Lemma 2.1(b) for the domain property, this completes the proof of (a) and (b) in case (D). The proof of (c) and (d) is analogous since the elements $\mathbf{in}(\bar r'_k)$ and $\overline{\mathbf{in}(r'_k)}\pmod{\bar t_0}$ are formally given by the same expressions for $k=1,2,3$. Part(e) follows from Lemma 3.2.

We now turn to case (E). Recall that here we have $u_{i_0}=1$ by definition of E_Π. Let $i_1\ne i_0$ denote the index in $\{1,2,3,4\}$ such that $\{i_0,i_1\}$ is either $\{1,2\}$ or $\{3,4\}$. Using (4) above, one finds that the coefficients of c_{i_1} in r_1 and of d_{i_1} in r_2 are in $\{\pm1,\pm2\}\subset W(\mathbb{F})^*$. In particular $\mathbf{in}^1(\bar r_k)$, $k=1,2$, and $\overline{\mathbf{in}^1(r'_k)}\pmod{\bar t_0}$, $k=1,2$, are $\mathbb{F}$-linearly independent elements in $\overline{\mathfrak{m}}_{\mathcal{R}'}/\overline{\mathfrak{m}}^2_{\mathcal{R}'}$ and $\overline{\mathfrak{m}'_s/(\mathfrak{m}'_s)^2}\pmod{\bar t_0}$, respectively. We define $\mathcal{R}=\mathcal{S}/(r_1,r_2,,b_{i_0},d_{i_0}-c_{i_0})$. Using r_1 and r_2 as replacement rules to eliminate the variables c_{i_1} and d_{i_1}, we find that the homomorphism

$$W(\mathbb{F})[[c_{i_0},b_{i_1},b_k,c_k,d_k:k\in\{1,\dots,2g\}\smallsetminus\{i_0,i_1\}]]\to\mathcal{R},$$

which sends each formal variable to the same named variable in $\mathcal{R}$, is an isomorphism. By $\widetilde r_3$ we denote the image of r_3 in $\mathcal{R}$. It is clear from (2) that $\widetilde r_3$ lies in $\mathfrak{m}_s^2$, where now $\mathfrak{m}_s$ is the image of $(q,b_i,c_i,d_i,i=1,\dots,2g)$ in $\mathcal{R}$. In particular, $0\to(\widetilde r_3)\to\mathcal{R}\to R_\alpha^\psi\to0$ is a minimal presentation.

As in the analysis of (D), we consider the reduction map $\pi\colon\mathcal{R}\to\mathcal{R}'=\mathcal{R}/(b_i,c_i,d_i,i=5,\dots,2g)$, we define $\mathfrak{m}'_s=\pi(\mathfrak{m}_s)$ and $r'_3=\pi(\widetilde r_3)$. The ring $\mathcal{R}'$ is now a power series ring over $W(\mathbb{F})$ in 8 variables. A short computation shows

$$w:=\mathbf{in}^2(\bar r'_3)\equiv\overline{\mathbf{in}^2(r'_3)}\pmod{\bar t_0}\equiv\begin{cases}2\bar d_3\bar c_4-2\bar d_4\bar c_3+\text{ other terms}, & \text{if } i_0\in\{1,2\},\\ 2\bar d_1\bar c_2-2\bar d_2\bar c_1, & \text{if } i_0\in\{3,4\}.\end{cases}$$

From $w \neq 0$ we deduce (a) and (c). The proof of (e) follows from Lemma 3.2. Arguing as for (D), to prove (b) and (d) it suffices to show that w is a non-zero divisor in

$$\mathrm{gr}_{\overline{\mathfrak{m}}_{\mathcal{R}'}} \overline{\mathcal{R}}' \cong \overline{\mathrm{gr}_{\mathfrak{m}'_s} \mathcal{R}'}/(\bar{t}_0) \cong \mathbb{F}[[c_{i_0}, b_{i_1}, b_k, c_k, d_k : k \in \{1,2,3,4\} \smallsetminus \{i_0, i_1\}]].$$

We need to show that w is irreducible, i.e., not a product of two linear terms. For this one may consider w as a bilinear from. If w was reducible, the representing Gram matrix would have rank at most 2. However, the displayed coefficients of w imply that this rank is at least 4. □

Remark 3.8. (a) In Section 2, we showed Theorem 1.5 by combining Theorem 1.4 with Proposition 2.2. Alternatively, in cases (D) and (E) Theorem 1.5 follows easily from Lemma 3.7(c),(d) combined with Proposition 2.2(a),(b).

(b) In cases (D) and (E), Theorem 1.4 can also be deduced from [Bö2, §8]. However, we felt that the arguments there are somewhat sketchy. To make them more precise, we would have needed to introduce much notation. Since the above proof follows nicely from the ideas of Section 2, we chose this path.

4 Crystalline points in components of versal deformation spaces

Let $\mathfrak{X}(\bar{\rho})$ be the versal deformation space of a fixed residual representation $\bar{\rho} : G_K \to \mathrm{GL}_n(\mathbb{F})$. The Zariski density of benign crystalline points in $\mathfrak{X}(\bar{\rho})$ for $n = 2$ is an important consequence of the integrality results of the previous sections. The purpose of this section is to prove Theorem 1.9 on irreducible components of $\mathfrak{X}(\bar{\rho})$, and Theorem 1.12 on the Zariski density of crystalline points by showing that any component of $\mathfrak{X}(\bar{\rho})$ contains a crystalline point.

We fix a character $\psi\colon G_K \to \mathcal{O}^*$ that reduces to $\det \bar{\rho}$. As is well-known, e.g. [Bö1, Prop. 2.1] for results of this type, one has the following result:

Lemma 4.1. *Suppose p does not divide n and $\psi'\colon G_K \to \mathcal{O}^*$ is a second lift of $\det \bar{\rho}$. Then*

(a) $D_{\bar{\rho}} \to D^{\psi}_{\bar{\rho}} \times D_{\det \bar{\rho}}, [\rho] \mapsto ([\rho \otimes (\psi \det \rho^{-1})^{1/n}], \det \rho)$ is an isomorphism of functors with inverse $([\rho'], \varphi') \mapsto [\rho' \otimes (\varphi' \psi^{-1})^{1/n}]$. In particular one has a natural isomorphism $R_{\bar{\rho}} \cong R^{\psi}_{\bar{\rho}} \hat{\otimes}_{\mathcal{O}} R_{\det \bar{\rho}}$.

(b) $D^{\psi}_{\bar{\rho}} \to D^{\psi'}_{\bar{\rho}}, [\rho] \mapsto [\rho \otimes \sqrt{\psi^{-1}\psi'}]$ is an isomorphism of functors so that $R^{\psi}_{\bar{\rho}}$ and $R^{\psi'}_{\bar{\rho}}$ are isomorphic.

Lemma 4.1 shows that is suffices to prove Theorem 1.5 for any fixed choice of lift ψ, for instance for the Teichmüller lift of $\det \rho$. Furthermore, together with Theorem 1.5, it implies Theorem 1.9:

Proof of Theorem 1.9. By Theorem 1.5 and part (a) of the previous lemma, the map $\mathrm{Det}\colon \mathfrak{X}(\bar{\rho}) \to \mathfrak{X}(\det \bar{\rho})$ of Theorem 1.9 induces a bijection of irreducible components. Moreover the irreducible components of both spaces will be connected components if this holds for $\mathfrak{X}(\det \bar{\rho})$. To prove this and the remaining assertion of Theorem 1.9, it will suffice to describe $R_{\det \bar{\rho}}$ explicitly. This however has been carried out in [Ma, § 1.4]: Denote by Π the abelianized pro-p completion of G_K, which by class field theory is isomorphic to $(\mathbb{Z}_p, +) \times (1 + \mathfrak{m}_K, \cdot)$. Then $R_{\bar{\eta}} \cong \mathcal{O}[[\Pi]] \cong \mathcal{O}[[T_0, \ldots, T_{[K:\mathbb{Q}_p]}]][X]/((1+X)^q - 1)$ for any character $\bar{\eta}\colon G_K \to \mathbb{F}^*$, where $q = \#\mu_{p^\infty}(K)$. The remaining assertions are now immediate. □

Proof of Theorem 1.12. By [Mu, Thm. 0.0.4], we may choose a crystalline p-adic Galois representation $\rho_0\colon G_K \to \mathrm{GL}_2(\overline{\mathbb{Q}_p})$ which is a lift of $\bar\rho$, i.e., so that $[\rho_0] \in \mathfrak{X}(\bar\rho)$. By the construction in [Mu], we can assume ρ_0 to be regular. We want to show that any component of $\mathfrak{X}(\bar\rho)$ contains a regular crystalline point so that the hypothesis of Theorem 1.11 holds. Denote by ψ the determinant of ρ_0, so that ψ is crystalline, and by $\mathfrak{X}(\bar\rho)^\psi$ the rigid analytic space that is the generic fiber of $R_{\bar\rho}^\psi$ in the sense of Berthelot. By Lemma 4.1, we have the isomorphism

$$\mathfrak{X}(\bar\rho)^\psi \times \mathfrak{X}(\det\bar\rho) \xrightarrow{\cong} \mathfrak{X}(\bar\rho),\ \ ([\rho'],\varphi') \longmapsto [\rho' \otimes (\varphi'\psi^{-1})^{1/2}].$$

By the following lemma, we have a crystalline point φ_i' in any component i of $\mathfrak{X}(\det\bar\rho)$. Now the components form a torsor over $\mu_{p^\infty}(K)$, which is a finite cyclic group of p-power order. Because 2 is prime to p, the characters $(\varphi_i')^2$ still exhaust all components of $\mathfrak{X}(\det\bar\rho)$, and the same holds for the translates $\psi(\varphi_i')^2$. Now under the above map we have $([\rho_0],\psi(\varphi_i')^2) \mapsto [\rho_0 \otimes \varphi_i']$, and by Theorem 1.9 we see that the latter representations give a regular crystalline lift in any component of $\mathfrak{X}(\bar\rho)$. Applying Theorem 1.11 completes the proof of Theorem 1.12. □

Lemma 4.2. *Any component of $\mathfrak{X}(\det\bar\rho)$ contains a crystalline point.*

Proof. By twisting by ψ^{-1} it will suffice to prove the lemma for the trivial character 1 in place of $\det\bar\rho$. The crystalline points in $\mathfrak{X}(1)$ correspond to characters $G_K \to \overline{\mathbb{Q}_p}^*$ with trivial reduction 1. We shall use the classification of one-dimensional crystalline representations to describe the crystalline points. Let $\mathrm{r}_K\colon \hat{\mathbb{Z}} \times \mathcal{O}_K^* \xrightarrow{\sim} G_K^{\mathrm{ab}}$ be the local Artin map. Consider the induced projection $\mathrm{pr}_2\colon G_K^{\mathrm{ab}} \to \mathcal{O}_K^*$, and let $\mathcal{P}_K$ be the set of embeddings $K \hookrightarrow \overline{\mathbb{Q}_p}$. Then for any $\tau_0 \in \mathcal{P}_K$ one defines a character χ_{τ_0} as the composite

$$\chi_{\tau_0} : G_K \longrightarrow G_K^{\mathrm{ab}} \xrightarrow{\mathrm{pr}_2} \mathcal{O}_K^* \xrightarrow{\tau_0} \overline{\mathbb{Q}_p}^*.$$

One has the following assertions, cf. [Con, App. B]:

(a) The character χ_{τ_0} is crystalline with labelled Hodge-Tate weights $(a_\tau)_{\tau\in\mathcal{P}_K}$ where $a_{\tau_0} = 1$ and $a_\tau = 0$ for $\tau \in \mathcal{P}_K \smallsetminus \{\tau_0\}$.[7]

(b) Any crystalline character of G_K is of the form $\nu\prod_{\tau\in\mathcal{P}_K}\chi_\tau^{\ell_\tau}$ for integers ℓ_τ and an unramified character ν. The tuple $(\ell_\tau)_{\tau\in\mathcal{P}_K}$ is its labelled Hodge-Tate weight.

As discussed in the proof of Theorem 1.9, $R_1 \cong \mathcal{O}[[\Pi]] \cong \mathcal{O}[[T_0,\dots,T_{[K:\mathbb{Q}_p]}]][X]/((1+X)^q-1)$ so that $\mathfrak{X}(1)$ has $q = \#\mu_{p^\infty}(K)$ connected components. In order to find a crystalline point in any component of $\mathfrak{X}(1)$, we introduce a labelling of its connected components by $\mu_{p^\infty}(K)$: Any point in $\mathfrak{X}(1)$ corresponds to a character $G_K \to \overline{\mathbb{Q}_p}^*$ with trivial mod p reduction, which factors via the abelianized pro-p completion Π of G_K, i.e., it induces a character $\eta : \Pi \to \overline{\mathbb{Q}_p}^*$. Via the isomorphism $\mathrm{r}_{K,p}\colon \mathbb{Z}_p \times (1+\mathfrak{m}_{\mathcal{O}_K}) \xrightarrow{\sim} \Pi$ induced from r_K by pro-p completion, the torsion subgroup $\mu_{p^\infty}(K)$ of $(1+\mathfrak{m}_{\mathcal{O}_K})$ is isomorphic to the torsion subgroup of Π so that we can define the label of η to be $\eta\circ\mathrm{r}_{K,p}|_{\mu_{p^\infty}(K)}(\zeta) \in \mu_{p^\infty}(K)$ for a chosen generator ζ of $\mu_{p^\infty}(K)$. Equivalently, one can say that the component of $\mathfrak{X}(1)$ that contains η is determined by the restriction $\eta\circ\mathrm{r}_{K,p}|_{\mu_{p^\infty}(K)}$.

Now we use the above labelling of components to find a crystalline character in each component. Recall that $f = [K_0 : \mathbb{Q}_p]$, and denote by $\tau_0 \in \mathcal{P}_K$ our usually chosen embedding $K \hookrightarrow \overline{\mathbb{Q}_p}$. By (b) above, for any $\ell \in \mathbb{Z}$ the character $\chi_{\tau_0}^{\ell(q^f-1)}\colon G_K \to \overline{\mathbb{Q}_p}^*$ is crystalline. Because of the factor q^f-1 in the exponent,

[7] For the definition of labelled Hodge-Tate weights, see [DS, Def. 3.2].

its image is a pro-p group, and it is straightforward to see that for the induced character $\eta\colon \Pi \to \overline{\mathbb{Q}_p}$ we have $\eta \circ \mathrm{r}_{K,p}|_{1+\mathfrak{m}_{\mathcal{O}_K}} = \tau_0^{\ell(q^f-1)}|_{1+\mathfrak{m}_{\mathcal{O}_K}}$. Hence $\eta \circ \mathrm{r}_{K,p}|_{\mu_{p^\infty}(K)}$ is equal to the homomorphism

$$\mu_{p^\infty}(K) \longrightarrow \mu_{p^\infty}(K), \quad \alpha \longmapsto \alpha^{\ell(q^f-1)} = \alpha^{-\ell}.$$

By choosing ℓ suitably, it is clear that η can be made to lie in any connected component of $\mathfrak{X}(1)$. ☐

For the following result, we assume that the reader is familiar with the theory of determinants as introduced in [Ch3]. Following [WE] we shall call them pseudo-representations. Let R be in $\mathrm{Ar}_{\mathcal{O}}$. To any representation $\rho\colon G_K \longrightarrow \mathrm{GL}_n(R)$ one can attach a pseudo-representation of degree n, i.e., a multiplicative R-polynomial law $\tau = \tau_\rho\colon R[G_K] \to R$ homogeneous of degree n. To describe the latter, denote for any R-module M by $\underline{M}$ the functor from R-algebras A to sets that assigns to A the set $M \otimes_R A$. Then τ is the natural transformation $\underline{R[G_K]} \to \underline{R}$ that on any R-algebra A is given by $\tau_A\colon A[G_K] \to A$, $\sum r_i g_i \mapsto \det\left(\sum r_i \rho(g_i)\right)$. In particular, any residual representation $\bar\rho\colon G_K \longrightarrow \mathrm{GL}_n(\mathbb{F})$ has an associated pseudo-representation $\bar\tau$. By [Ch3], if τ arises from a representation ρ over R, then the characteristic polynomial $\chi_\rho(g)$ of ρ is equal to $\chi_\tau(g,T) := \tau_{R[T]}(T-g) \in R[T]$ for any $g \in G_K$. The determinant of τ is defined as the representation $\det\tau := \tau_R = (-1)^n\chi_\tau(\_\_,0)\colon G_K \to \mathrm{GL}_1(R)$.

In [Ch3, § 3.1], Chenevier defines a deformation functor D_τ for a residual pseudo-representations $\bar\tau\colon \mathbb{F}[G_K] \to \mathbb{F}$. By [Ch3, Prop. 3.3 and Ex. 3.7], the functor $D_{\bar\tau}$ is representable by a ring $R_{\bar\tau}$ in $\mathrm{Ar}_{\mathcal{O}}$. By $\mathfrak{X}^{\mathrm{ps}}(\bar\tau)$ we denote the generic fiber of $\mathrm{Spf}\, R_{\bar\tau}$ in the sense of Berthelot, see [deJ, § 7]. If $\bar\tau$ is associated to $\bar\rho$ then there are natural functors

$$\mathfrak{X}(\bar\rho) \xrightarrow{\pi_1} \mathfrak{X}^{\mathrm{ps}}(\bar\tau) \xrightarrow{\pi_2} \mathfrak{X}(\det\bar\rho), \tag{2}$$

where π_1 is defined by mapping a deformation to the associated pseudo-representation, and π_2 by mapping a pseudo-representation to its determinant. Note that the composite is defined by the usual determinant of representations.

Corollary 4.3. *Suppose $\bar\rho$ is a semisimple 2-dimensional residual representation of G_K and $p > 2$.*

(a) The morphisms of connected components

$$\pi_0(\mathfrak{X}(\bar\rho)) \xrightarrow{\pi_0(\pi_1)} \pi_0(\mathfrak{X}^{\mathrm{ps}}(\bar\tau)) \xrightarrow{\pi_0(\pi_2)} \pi_0(\mathfrak{X}(\det\bar\rho)),$$

induced from (2) are bijective.

(b) The benign crystalline points are Zariski dense in $\mathfrak{X}^{\mathrm{ps}}(\bar\rho)$.

Proof. To prove (a) observe that by Theorem 1.9, the composite $\pi_0(\pi_2)\circ\pi_0(\pi_1)$ is a bijection. Moreover the map $\pi_0(\pi_1)$ is surjective: For this it suffices to show that any pseudo-representation τ over $\overline{\mathbb{Q}_p}$, i.e. any closed point in $\mathfrak{X}^{\mathrm{ps}}(\bar\tau)$, arises from a representation ρ, i.e. a closed point in $\mathfrak{X}(\bar\rho)$. By [Ch3, Thm. 2.12], it is known that τ is the pseudo-representation for a semisimple representation $G_K \to \mathrm{GL}_2(\overline{\mathbb{Q}_p})$. The latter can be realised over a finite extension E of $\mathbb{Q}_p$ and then, in turn by a representation $\rho'\colon G_K \to \mathrm{GL}_2(\mathcal{O}_E)$ for $\mathcal{O}_E$ the valuation ring of E. Moreover, by possibly enlarging E and choosing a suitable lattice, one can also assume that the reduction $\bar\rho'$ of ρ' modulo $\mathfrak{m}_{\mathcal{O}_E}$ is semisimple. Now on the one hand, we have $\chi_{\bar\tau} = \chi_{\bar\rho}$. On the other hand $\pi_1(\rho') = \tau$ yields $\chi_\tau = \chi_{\rho'}$, and reducing mod $\mathfrak{m}_{\mathcal{O}_E}$ we deduce $\chi_{\bar\rho'} = \chi_{\bar\tau} = \chi_{\bar\rho}$. By the semisimplicty of $\bar\rho$ and $\bar\rho'$, the theorem of Brauer-Nesbitt now implies $\bar\rho \cong \bar\rho'$. But then ρ' represents an element of $\mathfrak{X}(\bar\rho)$ that maps to τ, completing the proof of (a).

To prove (b), observe that, by what we just proved, the map π_1 is surjective on (closed) points. Moreover for rigid spaces all Zariski closed subsets are the Zariski closures of their closed points. But then the image under π_1 of a Zariski dense subset is Zariski dense. It follows from Theorem 1.12 that the set of benign crystalline points in $\mathfrak{X}^{\mathrm{ps}}(\bar{\tau})$, which is the image of the set of benign crystalline points in $\mathfrak{X}(\bar{\rho})$, is Zariski dense in $\mathfrak{X}^{\mathrm{ps}}(\bar{\tau})$. □

5 The cup product and quadratic obstructions

In the remainder of the article, we consider a residual representation $\bar{\rho}\colon G_K \longrightarrow \mathrm{GL}_n(\mathbb{F})$ for $n \in \mathbb{N}$ arbitrary. Let $0 \to I^\psi \to \mathcal{R} \xrightarrow{\pi} R^\psi_{\bar{\rho}} \to 0$ be a minimal presentation of $R^\psi_{\bar{\rho}}$ as in (1) of Proposition 1.3. In this section, we show that the bracket cup product $b\colon \mathrm{Sym}^2(H^1(G_K, \mathrm{ad}^0)) \to H^2(G_K, \mathrm{ad}^0)$ determines the quadratic part of the relation ideal $\bar{I}^\psi$ in the sense of Definition 2.4.

As recalled in Proposition 1.2 and 1.3, Mazur attaches to any small extension $0 \to J \to R_1 \to R_0 \to 0$ in $\widehat{\mathrm{Ar}}_{\mathcal{O}}$ and deformation $\rho_0\colon G_K \to \mathrm{GL}_n(R_0)$ with determinant ψ an obstruction class $\mathcal{O}(\rho_0) \in H^2(G_K, \mathrm{ad}^0) \otimes J$ for lifting ρ_0 to a deformation to R_1. First one chooses a continuous set-theoretic lift $\rho_1\colon G_K \to \mathrm{GL}_n(R_1)$ of ρ_0 which still satisfies $\det \circ \rho_1 = \psi$.[8] Then $\mathcal{O}(\rho_0) \in H^2(G_K, \mathrm{ad}^0 \otimes J)$ is given by the 2-cocycle

$$(g, h) \longmapsto \rho_1(gh)\rho_1(h)^{-1}\rho_1(g)^{-1} - 1. \tag{3}$$

Similarly, $\mathcal{O}(\rho_0)$ can be described by the obstruction homomorphism $\mathrm{obs}\colon \mathrm{Hom}_{\mathbb{F}}(J, \mathbb{F}) \to H^2(G_K, \mathrm{ad}^0)$. The latter is defined as follows: For any $f \in \mathrm{Hom}_{\mathbb{F}}(J, \mathbb{F})$, form the pushout on the left of the given small extension and denote the result by $0 \to \mathbb{F} \to R_f \to R_0 \to 0$. If $\rho_f\colon G_K \to \mathrm{GL}_n(R_f)$ is a continuous set-theoretic lift of ρ_0 satisfying $\det \circ \rho_f = \psi$, then we set $\mathrm{obs}(f) := \big(\mathcal{O}(\rho_0), f\big) := (\mathrm{id} \otimes f)\big(\mathcal{O}(\rho_0)\big) \in H^2(G_K, \mathrm{ad}^0)$, i.e., $\mathrm{obs}(f)$ is given by the 2-cocycle $(g, h) \longmapsto \rho_f(gh)\rho_f(h)^{-1}\rho_f(g)^{-1} - 1$.

The following lemma shows that the obstruction class is independent of a chosen small extension. Its simple proof is left as an exercise.

Lemma 5.1. *Consider a morphism of small extensions*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J & \longrightarrow & R_1 & \longrightarrow & R_0 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_0} & & \\ 0 & \longrightarrow & J' & \longrightarrow & R'_1 & \longrightarrow & R'_0 & \longrightarrow & 0, \end{array}$$

i.e., a commuting diagram with both rows a small extension and the right hand square in $\widehat{\mathrm{Ar}}_{\mathcal{O}}$. Let $\mathcal{O}(\rho_0) \in H^2(G_K, \mathrm{ad}^0 \otimes J)$ be the obstruction of a deformation $\rho_0 : G_K \to \mathrm{GL}(R_0)$ of $\bar{\rho}$. Then

$$(\mathrm{id} \otimes \pi)\big(\mathcal{O}(\rho_0)\big) = \mathcal{O}(\pi_0 \circ \rho_0) \in H^2(G_K, \mathrm{ad}^0 \otimes J') \cong H^2(G_K, \mathrm{ad}^0) \otimes J'.$$

Recall that $^-$ means that we pass to rings mod $\mathfrak{m}_{\mathcal{O}}$, and minimality of the presentation of $R^\psi_{\bar{\rho}}$ implies that π induces an isomorphism $\overline{\mathfrak{m}}_{\mathcal{R}}/\overline{\mathfrak{m}}^2_{\mathcal{R}} \cong \overline{\mathfrak{m}}^\psi_{\bar{\rho}}/(\overline{\mathfrak{m}}^\psi_{\bar{\rho}})^2$. In particular, $\bar{I}^\psi \subset \overline{\mathfrak{m}}^2_{\mathcal{R}}$. In this section, we consider the filtration $\{\overline{\mathfrak{m}}^i_{\mathcal{R}}\}_{i \geq 0}$ on $\overline{\mathcal{R}}$, and let **in** denote the initial term map $\overline{\mathcal{R}} \to \mathrm{gr}_{\overline{\mathfrak{m}}_{\mathcal{R}}} \overline{\mathcal{R}}$. The following basic result relates the bracket cup product and the quadratic part of I^ψ:

[8] Such a map always exists: For instance choose a continuous set-theoretic splitting $R_0 \to R_1$ of the given homomorphism $R_1 \to R_0$. Observe that since the R_i are local, it induces a continuous set-theoretic splitting of $\mathrm{GL}_n(R_1) \to \mathrm{GL}_n(R_0)$. Finally, fix the determinant similar to Lemma 4.1.

Lemma 5.2. *We assume $p > 2$. Then the following diagram is commutative:*

$$\begin{array}{ccccc} \mathrm{Hom}_{\mathbb{F}}\left(H^2(G_K,\mathrm{ad}^0),\mathbb{F}\right) & \xrightarrow{\ \mathrm{obs}^\vee\ } & \bar{I}^\psi/\overline{\mathfrak{m}}_{\mathcal{R}}\bar{I}^\psi & \twoheadrightarrow & \left(\bar{I}^\psi+\overline{\mathfrak{m}}^3_{\mathcal{R}}\right)/\overline{\mathfrak{m}}^3_{\mathcal{R}} \\ \Big\downarrow{\scriptstyle -\frac{1}{2}b^\vee} & & & & \Big\downarrow \\ \mathrm{Sym}^2\left(\mathrm{Hom}_{\mathbb{F}}\left(H^1(G_K,\mathrm{ad}^0),\mathbb{F}\right)\right) & \xrightarrow{\ \sim\ } & \mathrm{Sym}^2\left(\overline{\mathfrak{m}}_{\mathcal{R}}/\overline{\mathfrak{m}}^2_{\mathcal{R}}\right) & \xrightarrow{\ \sim\ } & \overline{\mathfrak{m}}^2_{\mathcal{R}}/\overline{\mathfrak{m}}^3_{\mathcal{R}}, \end{array}$$

where $b^\vee$ is induced by the dual of the bracket cup product, and $\mathrm{obs}^\vee$ *is dual to the obstruction homomorphism. In particular, the quadratic part $\mathbf{in}^2(\bar{I}^\psi)$ of $\bar{I}^\psi$ in $\overline{\mathfrak{m}}^2_{\mathcal{R}}/\overline{\mathfrak{m}}^3_{\mathcal{R}}$ agrees with the image of $b^\vee$.*

Proof. Let $\bar{J} := (\bar{I}^\psi+\overline{\mathfrak{m}}^3_{\mathcal{R}})/\overline{\mathfrak{m}}^3_{\mathcal{R}}$. We prove that the following diagram is commutative:

$$\begin{array}{ccccc} \mathrm{Sym}^2\left(H^1(G_K,\mathrm{ad}^0)\right) & \xrightarrow{\ \sim\ } & \mathrm{Sym}^2\left(\mathrm{Hom}_{\mathbb{F}}\left(\overline{\mathfrak{m}}_{\mathcal{R}}/\overline{\mathfrak{m}}^2_{\mathcal{R}},\mathbb{F}\right)\right) & \xrightarrow{\ \sim\ } & \mathrm{Hom}_{\mathbb{F}}\left(\overline{\mathfrak{m}}^2_{\mathcal{R}}/\overline{\mathfrak{m}}^3_{\mathcal{R}},\mathbb{F}\right) \\ \Big\downarrow{\scriptstyle -\frac{1}{2}b} & & & & \Big\downarrow \\ H^2(G_K,\mathrm{ad}^0) & \xleftarrow[\ \mathrm{obs}\ ]{} & \mathrm{Hom}_{\mathbb{F}}\left(\bar{I}^\psi/\overline{\mathfrak{m}}_{\mathcal{R}}\bar{I}^\psi,\mathbb{F}\right) & \hookleftarrow & \mathrm{Hom}_{\mathbb{F}}\left(\bar{J},\mathbb{F}\right). \end{array}$$

The first isomorphism in the upper row is the canonical isomorphism from Proposition 1.2(a). We shall show that the image of any $c_1 \in H^1(G_K,\mathrm{ad}^0)$ in $H^2(G_K,\mathrm{ad}^0)$ is independent of whether we apply $-\frac{1}{2}b$ or the clockwise composite morphism that passes via obs. Since both maps are $\mathbb{F}$-linear and elements of the form c_1^2 generate $\mathrm{Sym}^2\left(H^1(G_K,\mathrm{ad}^0)\right)$ as an $\mathbb{F}$-vector space, this will prove commutativity. Before we embark on the lengthy computation of the composite morphism, we observe that the bracket cup product of c_1 with itself is represented by the explicit 2-cocycle $(g,h)\mapsto[c_1(g),\mathrm{Ad}\,\bar{\rho}(g)c_1(h)]$, see [Wa, §2] – we write $\mathrm{Ad}\,\bar{\rho}$ for the adjoint action of G_K on ad^0 to have clear notation.

We now compute the clockwise composite morphism that passes via obs. First we extend c_1 to a basis $\{c_1,\dots,c_h\}$ of $H^1(G_K,\mathrm{ad}^0)$. Via the isomorphisms $H^1(G_K,\mathrm{ad}^0)\cong\mathrm{Hom}_{\mathbb{F}}\left(\overline{\mathfrak{m}}_{\mathcal{R}}/\overline{\mathfrak{m}}^2_{\mathcal{R}},\mathbb{F}\right)$, we obtain a basis of $\mathrm{Hom}_{\mathbb{F}}\left(\overline{\mathfrak{m}}_{\mathcal{R}}/\overline{\mathfrak{m}}^2_{\mathcal{R}},\mathbb{F}\right)$, which by slight abuse of notation, we also denote $\{c_1,\dots,c_h\}$. For the corresponding dual basis of $\overline{\mathfrak{m}}_{\mathcal{R}}/\overline{\mathfrak{m}}^2_{\mathcal{R}}$ we write $\{\bar{x}_1,\dots,\bar{x}_h\}$ so that $c_i(\bar{x}_j)$ is the Kronecker symbol δ_{ij}. We lift the latter elements to a system of parameters $\{x_1,\dots,x_h\}$ of $\overline{\mathfrak{m}}_{\mathcal{R}}$; this defines an isomorphism $\overline{\mathcal{R}}\cong\mathbb{F}[[x_1,\dots,x_h]]$. With this notation, the image of c_1^2 in $\mathrm{Hom}_{\mathbb{F}}\left(\overline{\mathfrak{m}}^2_{\mathcal{R}}/\overline{\mathfrak{m}}^3_{\mathcal{R}},\mathbb{F}\right)$ is characterized by $c_1^2(\bar{x}_i\bar{x}_j)=0$ if one of i,j is at least 2 and $c_1^2(\bar{x}_1^2)=1$. The image of c_1^2 in $\mathrm{Hom}_{\mathbb{F}}\left(\bar{J},\mathbb{F}\right)$ is the restriction $c_1^2|_{\bar{J}}$ to the subspace $\bar{J}\subset\overline{\mathfrak{m}}^2_{\mathcal{R}}/\overline{\mathfrak{m}}^3_{\mathcal{R}}$. Finally, the composition of the canonical homomorphism $\bar{I}^\psi/\overline{\mathfrak{m}}_{\mathcal{R}}\bar{I}^\psi\to\bar{J}$ and $c_1^2|_{\bar{J}}$ defines an element f in $\mathrm{Hom}_{\mathbb{F}}\left(\bar{I}^\psi/\overline{\mathfrak{m}}_{\mathcal{R}}\bar{I}^\psi,\mathbb{F}\right)$. To evaluate $\mathrm{obs}(f)=(\mathcal{O}(\rho_{\bar{\rho}}),f)$, we consider the following diagram which displays three morphisms of small extensions:

$$\begin{array}{ccccccccc} 0 & \to & \bar{I}^\psi/\overline{\mathfrak{m}}_{\mathcal{R}}\bar{I}^\psi & \to & \bar{\mathcal{R}}/\overline{\mathfrak{m}}_{\mathcal{R}}\bar{I}^\psi & \to & \bar{\mathcal{R}}/\bar{I}^\psi & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \bar{J} & \to & \bar{\mathcal{R}}/\overline{\mathfrak{m}}^3_{\mathcal{R}} & \to & \bar{\mathcal{R}}/(\bar{I}^\psi+\overline{\mathfrak{m}}^3_{\mathcal{R}}) & \to & 0 \\ & & \big\downarrow\!\!\!\!\!\!\hookrightarrow & & \| & & \downarrow & & \\ 0 & \to & \overline{\mathfrak{m}}^2_{\mathcal{R}}/\overline{\mathfrak{m}}^3_{\mathcal{R}} & \to & \bar{\mathcal{R}}/\overline{\mathfrak{m}}^3_{\mathcal{R}} & \to & \bar{\mathcal{R}}/\overline{\mathfrak{m}}^2_{\mathcal{R}} & \to & 0 \\ & & \downarrow{\scriptstyle c_1^2} & & \downarrow & & \| & & \\ 0 & \to & \mathbb{F} & \to & \bar{\mathcal{R}}/\ker(c_1^2) & \to & \bar{\mathcal{R}}/\overline{\mathfrak{m}}^2_{\mathcal{R}} & \to & 0, \end{array}$$

where the last row is obtained by pushout along c_1^2 and where we denote by $\ker(c_1^2)$ the ideal of $\overline{\mathcal{R}}$ that is the preimage under $\overline{\mathcal{R}} \to \overline{\mathcal{R}}/\overline{\mathfrak{m}}_{\mathcal{R}}^3$ of the kernel of $c_1^2\colon \overline{\mathfrak{m}}_{\mathcal{R}}^2/\overline{\mathfrak{m}}_{\mathcal{R}}^3 \to \mathbb{F}$. Note that since $\overline{\mathcal{R}}/\overline{\mathfrak{m}}_{\mathcal{R}}^2 \cong \overline{R}_{\bar\rho}^{\psi}/(\overline{\mathfrak{m}}_{\bar\rho}^{\psi})^2$, the right column is the morphism defining the deformation $\rho_{\bar\rho}^{\psi} \pmod{(\overline{\mathfrak{m}}_{\bar\rho}^{\psi})^2}$ to $\overline{\mathcal{R}}/\overline{\mathfrak{m}}_{\mathcal{R}}^2$.

By the Lemma 5.1, we can use the last row to compute $\mathrm{obs}(f)$. For this, we need a suitable set-theoretic lift of $\rho_{\bar\rho}^{\psi} \pmod{(\overline{\mathfrak{m}}_{\bar\rho}^{\psi})^2}$ to $\bar{\mathcal{R}}/\ker(c_1^2)$. We begin with a cohomological description of $\rho_{\bar\rho}^{\psi} \pmod{(\overline{\mathfrak{m}}_{\bar\rho}^{\psi})^2}$: using vector space duality, the canonical isomorphism $H^1(G_K, \mathrm{ad}^0) \cong \mathrm{Hom}_{\mathbb{F}}\left(\overline{\mathfrak{m}}_{\mathcal{R}}/\overline{\mathfrak{m}}_{\mathcal{R}}^2, \mathbb{F}\right)$ can be described equivalently by the 1-cocycle $\sum_{i=1}^h c_i \otimes \bar{x}_i$ in $Z^1(G_K, \mathrm{ad}^0 \otimes \overline{\mathfrak{m}}_{\mathcal{R}}/\overline{\mathfrak{m}}_{\mathcal{R}}^2)$. Therefore, $\rho_{\bar\rho}^{\psi} \pmod{(\overline{\mathfrak{m}}_{\bar\rho}^{\psi})^2}$ is given by the formula

$$g \longmapsto \Big(1 + \sum_{i=1}^h c_i(g) \otimes \bar{x}_i\Big)\bar\rho(g).$$

We want to obtain a formula for a set-theoretic lift to $\bar{\mathcal{R}}/\ker(c_1^2)$. It will be convenient to use the exponential map $\exp_2(x) = 1 + x + \frac{1}{2}x^2$ to level 2, which is well-defined as the rings $(R, \mathfrak{m}_R)$ in $\widehat{\mathrm{Ar}}_{\mathcal{O}}$ have characteristics different from 2. Moreover, $\exp_2$ can be applied to matrices $A \in M_n(\mathfrak{m}_R)$. If in addition $\mathfrak{m}_R^3 = 0$, then one can also verify that $\det(\exp_2(A)) = \exp_2(\mathrm{Tr}(A))$. In particular, $\exp_2(A)$ has determinant equal to 1 if A is of trace zero. Now we take as our set-theoretic lift

$$\rho_0'\colon G_K \longrightarrow \mathrm{GL}_n(\bar{\mathcal{R}}/\ker(c_1^2)), \quad g \longmapsto \exp_2\Big(\sum_{i=1}^h c_i(g) \otimes x_i\Big)\bar\rho(g) \pmod{\ker(c_1^2)}.$$

By the remark above on $\exp_2$, we have $\det(\rho_0'(g)) = \det(\bar\rho(g)) = \psi(g) \pmod{\mathfrak{m}_{\mathcal{O}}}$ for all $g \in G_K$. In $\mathcal{R}/\ker(c_1^2)$, we have $x_i x_j = 0$ whenever $i > 1$ or $j > 1$. Hence, the expressions $\exp_2(c_i(g) \otimes x_i)$ commute for all i and we have $\exp_2\left(\sum_{i=1}^h c_i(g) \otimes x_i\right) = \prod_{i=1}^h \exp_2(c_i(g)x_i)$. Using these properties, the class $\mathrm{obs}(f)$ is represented by the 2-cocycle

$$(g,h) \longmapsto \rho_0'(gh)\rho_0'(h)^{-1}\rho_0'(g)^{-1} - 1 = \rho_1'(gh)\rho_1'(h)^{-1}\rho_1'(g)^{-1} - 1,$$

where ρ_1' is the lift $G_K \to \mathrm{GL}_n(\bar{\mathcal{R}}/\ker(c_1^2))$, $g \mapsto \exp_2\left(c_1(g) \otimes x_1\right)\bar\rho(g) \pmod{\ker(c_1^2)}$, of $\bar\rho$. At this point, it is a simple if lengthy computation to verify that the right hand side of the previous expression is the 2-cocycle $(g,h) \mapsto -\frac{1}{2}[c_1(g), \mathrm{Ad}\,\bar\rho(g)c_1(h)] \otimes x_1^2$. Now x_1^2 is our chosen $\mathbb{F}$-basis of the lower left term in the above diagram and via c_1^2 it is mapped to 1. Hence, $\mathrm{obs}(f)$ agrees with the expression for $-\frac{1}{2}b(c_1, c_1)$ given above. □

Remark 5.3. The use of the exponential map in the above proof seems standard, e.g. [Gol, 1.3].

Corollary 5.4. *Suppose $\bar\rho$ is of degree 2 and $p > 2$. Then the homomorphism*

$$b\colon \mathrm{Sym}^2 H^1(G_K, \mathrm{ad}^0) \longrightarrow H^2(G_K, \mathrm{ad}^0)$$

induced from the bracket cup product is surjective.

Proof. Consider a minimal presentation $0 \to I^{\psi} \to \mathcal{R} \to R_{\bar\rho} \to 0$ of $R_{\bar\rho}$. By Lemma 5.2, it suffices to show that the images of the quadratic parts of generators of I^{ψ} span a subspace of dimension equal to $\dim_{\mathbb{F}} H^2(G_K, \mathrm{ad}^0)$. This follows from Corollary 3.6(a) in case (A)–(C), Lemma 3.7(c)–(e) in case (D)–(E) and Lemma 3.2 by direct inspection in the respective cases of Section 3. □

6 Further quadratic obstructions from the Bockstein homomorphism

Let $\bar{\rho}\colon G_K \to \mathrm{GL}_n(\mathbb{F})$ be a residual representation and $0 \to I^\psi \to \mathcal{R} \xrightarrow{\pi} R^\psi_{\bar{\rho}} \to 0$ be a fixed minimal presentation as in Proposition 1.3. In the previous section we gave a description of the contribution of the bracket cup product $b\colon \mathrm{Sym}^2(H^1(G_K, \mathrm{ad}^0)) \to H^2(G_K, \mathrm{ad}^0)$ to the relation ideal I^ψ. By Lemma 5.2, knowing b is equivalent to knowing the quadratic part of $\bar{I}^\psi$. In Example 2.3 we saw that knowing the refined quadratic part may have stronger ring-theoretic implications than knowing the quadratic part only. The theme of this section is the Bockstein homomorphism and its additional contribution to the relation ideal I^ψ. The upshot is a cohomological description of the refined quadratic part of I^ψ in cohomological terms in Lemma 6.6 and Theorem 6.8.

We suppose that there is a representation $\rho_{s+1}\colon G_K \to \mathrm{GL}_n(\mathcal{O}_{s+1})$ lifting $\bar{\rho}$ for some integer s.[9] Observe that ρ_{s+1} defines a homomorphism $\alpha_{s+1}\colon R^\psi_{\bar{\rho}}/(\varpi_{\mathcal{O}}^{s+1}) \to \mathcal{O}_{s+1}$. For the following discussion it will be convenient to choose a regular sequence of parameters of $\mathcal{R}$ that is compatible with α_{s+1} in the following sense: Since the morphism $\alpha_{s+1} \circ (\pi \pmod{\varpi_{\mathcal{O}}^{s+1}})\colon \mathcal{R}/(\varpi_{\mathcal{O}}^{s+1}) \to \mathcal{O}_{s+1}$ is a surjective homomorphism of formally smooth $\mathcal{O}_{s+1}$-algebras, it possesses an $\mathcal{O}_{s+1}$-splitting. Thus we may choose $x_1, \ldots, x_h$ of $\mathcal{R}$ with $h = \dim_{\mathbb{F}} H^1(G_K, \mathrm{ad}^0)$ such that $\mathcal{R}/\varpi_{\mathcal{O}}^{s+1} \cong \mathcal{O}_{s+1}[[x_1, \ldots, x_h]]$ and such that under this identification the homomorphism $\alpha_{s+1} \circ (\pi \pmod{\varpi_{\mathcal{O}}^{s+1}})$ sends all x_i to zero. For $1 \le i \le s$, let $\alpha_i := \alpha_{s+1} \pmod{\varpi_{\mathcal{O}}^i}\colon R^\psi_{\bar{\rho}}/(\varpi_{\mathcal{O}}^i) \to \mathcal{O}_i$ and $\rho_i := \rho_{s+1} \pmod{\varpi_{\mathcal{O}}^i}\colon G_K \to \mathrm{GL}_n(\mathcal{O}_i)$. Further, for $1 \le i \le s+1$, the adjoint representation $\mathrm{Ad}\,\rho_i : G_K \to \mathrm{ad}^0_i$ of G_K on trace zero matrices $\mathrm{ad}^0_i := \mathrm{Mat}^0_n(\mathcal{O}_i)$ is given by conjugation with ρ_i so that $\mathrm{ad}^0_1 = \mathrm{ad}^0$. Then for all $2 \le i \le s+1$ there is a short exact sequence of G_K-modules

$$0 \longrightarrow \mathrm{ad}^0_{i-1} \xrightarrow{\cdot \varpi_{\mathcal{O}}} \mathrm{ad}^0_i \xrightarrow{\mathrm{pr}_i} \mathrm{ad}^0 \longrightarrow 0.$$

Definition 6.1. For $2 \le i \le s+1$, the *i-th Bockstein operator* or the *p^i-Bockstein homomorphism* is the connecting homomorphism β_i in the induced long exact cohomology sequence

$$\begin{array}{l} \ldots \longrightarrow H^1(G_K, \mathrm{ad}^0_{i-1}) \xrightarrow{\cdot \varpi_{\mathcal{O}}} H^1(G_K, \mathrm{ad}^0_i) \xrightarrow{\mathrm{pr}_i^*} H^1(G_K, \mathrm{ad}^0) \xrightarrow{\beta_i} \\ \qquad \longrightarrow H^2(G_K, \mathrm{ad}^0_{i-1}) \xrightarrow{\cdot \varpi_{\mathcal{O}}} H^2(G_K, \mathrm{ad}^0_i) \xrightarrow{\mathrm{pr}_i^*} H^2(G_K, \mathrm{ad}^0) \longrightarrow \ldots \end{array} \tag{4}$$

Now we give an explicit description of β_i that will be useful later.

Lemma 6.2. *Let $2 \le i \le s+1$, let $c \in Z^1(G_K, \mathrm{ad}^0)$, and let $\widetilde{\ }$ denote a set-theoretic splitting of $\mathrm{ad}^0_i \to \mathrm{ad}^0$. The i-th Bockstein operator is given explicitly by*

$$\beta_i([c]) = \Big((g,h) \longmapsto \varpi_{\mathcal{O}}^{-1} \cdot \big(\mathrm{Ad}\,\rho_i(g)\widetilde{c}(h) - \widetilde{c}(gh) + \widetilde{c}(g) \big) \Big) \pmod{B^2(G_K, \mathrm{ad}^0_{i-1})}. \tag{5}$$

Proof. The connecting homomorphism β_i is defined by applying the snake lemma to the following

[9] The delicate matter of the correct choice of $q = p^s$ is discussed in Lemma 6.9 and Remark 6.10.

commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
 & & \bar{C}^1(G_K, \mathrm{ad}^0_{i-1}) & \xrightarrow{\cdot \varpi_{\mathcal{O}}} & \bar{C}^1(G_K, \mathrm{ad}^0_i) & \xrightarrow{\mathrm{pr}_i^*} & \bar{C}^1(G_K, \mathrm{ad}^0) & \longrightarrow & 0 \\
 & & \downarrow \partial_{i-1} & & \downarrow \partial_i & & \downarrow \partial_1 & & \\
0 & \longrightarrow & Z^2(G_K, \mathrm{ad}^0_{i-1}) & \xrightarrow[\cdot \varpi_{\mathcal{O}}]{} & Z^2(G_K, \mathrm{ad}^0_i) & \xrightarrow[\mathrm{pr}_i^*]{} & Z^2(G_K, \mathrm{ad}^0), & &
\end{array}$$

where we let $\bar{C}^1(G_K, \mathrm{ad}^0_j) := C^1(G_K, \mathrm{ad}^0_j)/B^1(G_K, \mathrm{ad}^0_j)$ and ∂_j is induced by the coboundary map

$$C^1(G_K, \mathrm{ad}^0_j) \longrightarrow C^2(G_K, \mathrm{ad}^0_j), \quad b \longmapsto \big((g,h) \mapsto \big(\operatorname{Ad}\rho_j(g) b(h) - b(gh) + b(g)\big)\big).$$

for any $1 \le j \le s+1$. We lift the given 1-cocycle $c \in Z^1(G_K, \mathrm{ad}^0)$ to the 1-cochain $b_0 := \big(g \mapsto \widetilde{c}(g)\big) : G_K \to \mathrm{ad}^0_i$, and denote the image of c and b_0 in $\bar{C}^1(G_K, \mathrm{ad}^0_i)$ by $\bar{c}$ and $\bar{b}_0$, respectively. Since by assumption $\partial_1(\bar{c})$ vanishes and the right hands side of the diagram is commutative, we conclude that $\partial_i(\bar{b}_0) \in \ker(\mathrm{pr}_i^*)$. Using the exactness of the lower row, we may define $\beta_i([c]) := \varpi_{\mathcal{O}}^{-1} \cdot \partial_i(\bar{b}_0)$ (mod $B^2(G_K, \mathrm{ad}^0_{i-1})$) so that the desired formula (5) follows from the definition of ∂_i. □

The meaning of the Bockstein operator for obstructions is given by the following straightforward result.

Lemma 6.3. *Let $i \in \{2, \dots, s+1\}$ and consider a deformation $\bar{\rho}_c = (1 + c\varepsilon) \cdot \bar{\rho} : G_K \to \mathrm{GL}_n(\mathbb{F}[\varepsilon])$ of $\bar{\rho}$ for some $c \in Z^1(G_K, \mathrm{ad}^0)$. Then ρ_i has a deformation to $\mathcal{O}_i[\varepsilon]$ that lifts $\bar{\rho}_c$ if and only if $\beta_i([c]) = 0$.*

Proof. As in the mod $\varpi_{\mathcal{O}}$ case we can write any deformation to $\mathcal{O}_i[\varepsilon]$ of ρ_i as

$$\rho_{i,c_i} = (1 + c_i\varepsilon) \cdot \rho_i : G_K \to \mathrm{GL}_n(\mathcal{O}_i[\varepsilon])$$

for some $c_i \in Z^1(G_K, \mathrm{ad}^0_i)$. Using the functorial homomorphism $\mathrm{pr}_i^* : C^1(G_K, \mathrm{ad}^0_i) \to C^1(G_K, \mathrm{ad}^0)$, we find that the image of ρ_{i,c_i} under reduction mod $\varpi_{\mathcal{O}}$ is given by

$$(1 + \mathrm{pr}_i^*(c_i)\varepsilon) \cdot \bar{\rho} : G_K \to \mathrm{GL}_n(\mathbb{F}[\varepsilon]).$$

Hence, such a deformation $\rho_{i,c_i} \colon G_K \to \mathrm{GL}_n(\mathcal{O}_i[\varepsilon])$ of ρ_i that lifts $\bar{\rho}_c$ exists if and only if $\mathrm{pr}_i^*(c_i) = c$. The long exact sequence of group cohomology (4) implies that the latter holds if and only if $[c]$ lies in the kernel of β_i. □

Corollary 6.4. *Let i be in $\{2, \dots, s+1\}$ and consider the presentation*

$$0 \longrightarrow I_i \longrightarrow \mathcal{R}_i := \mathcal{O}_i[x_1, \dots, x_h]/(x_1, \dots, x_h)^2 \xrightarrow{\pi_i} R_i := R^{\psi}_{\bar{\rho}}/\pi((x_1, \dots, x_h)^2 + \varpi^i_{\mathcal{O}}\mathcal{R}) \longrightarrow 0 \qquad (6)$$

induced from (1) in Proposition 1.3. Then $\beta_i = 0$ if and only if $I_i = 0$, i.e., if and only if π_i is an isomorphism. In particular, if $\beta_s = 0$, then $\beta_j = 0$ for all $j = 2, \dots, s$.

Proof. Suppose that I_i is non-zero and let $f \neq 0$ be an element of I_i. By multiplying f by a suitable power of $\varpi_{\mathcal{O}}$, we may assume that f lies in $\varpi_{\mathcal{O}}^{i-1}\mathcal{R}_i$, i.e., that f is of the form $\varpi_{\mathcal{O}}^{i-1}(\sum_{j=1}^h \lambda_j x_j)$ for suitable $\lambda_j \in \mathcal{O}_i$ such that at least one λ_j lies in $\mathcal{O}_i^*$. Let $\bar{\alpha}_\varepsilon \colon R_i \to \mathbb{F}[\varepsilon]$ be an $\mathcal{O}$-algebra homomorphism such that $\bar{\alpha}_\varepsilon\big(\sum_{j=1}^h \lambda_j x_j\big)$ is non-zero. Since $\beta_i = 0$, there exists an $\mathcal{O}$-algebra homomorphism

$$\alpha_{i,\varepsilon} \colon R_i \to \mathcal{O}_i[\varepsilon]$$

such that $\alpha_{i,\varepsilon} \equiv \bar{\alpha}_\varepsilon \pmod{\varpi_{\mathcal{O}}}\colon R_i \to \mathbb{F}[\varepsilon]$. We deduce

$$0 \overset{\pi_i(I_i)=0}{=} (\alpha_{i,\varepsilon} \circ \pi_i)\Big(\varpi_{\mathcal{O}}^{i-1}\big(\sum_{j=1}^{h} \lambda_j x_j\big)\Big) \overset{\mathcal{O}\text{-hom.}}{=} \varpi_{\mathcal{O}}^{i-1}(\alpha_{i,\varepsilon} \circ \pi_i)\Big(\sum_{j=1}^{h} \lambda_j x_j\Big) \in \mathcal{O}_i[\varepsilon],$$

and it follows that $(\alpha_{i,\varepsilon} \circ \pi_i)\big(\sum_{j=1}^h \lambda_j x_j\big)$ lies in $\varpi_{\mathcal{O}}\mathcal{O}_i[\varepsilon]$, or, in other words, that $\bar{\alpha}_\varepsilon\big(\sum_{j=1}^h \lambda_j x_j\big) = 0$. This is a contradiction. □

Lemma 6.5. *Suppose that $\beta_s = 0$, so that also $\beta_2 = \ldots = \beta_{s-1} = 0$. Then the following hold:*

(a) For $i = 2, \ldots, s$, the short exact sequence

$$0 \longrightarrow \mathrm{ad}^0 \xrightarrow{\cdot\varpi_{\mathcal{O}}^{i-1}} \mathrm{ad}^0_i \xrightarrow{\gamma_i} \mathrm{ad}^0_{i-1} \longrightarrow 0.$$

yields a short exact sequence $0 \longrightarrow H^2(G_K, \mathrm{ad}^0) \xrightarrow{\cdot\varpi_{\mathcal{O}}^{i-1}} H^2(G_K, \mathrm{ad}^0_i) \xrightarrow{\gamma_i^*} H^2(G_K, \mathrm{ad}^0_{i-1}) \longrightarrow 0$.

(b) The Bockstein homomorphism $\beta_{s+1}\colon H^1(G_K, \mathrm{ad}^0) \to H^2(G_K, \mathrm{ad}^0_s)$ induces a homomorphism

$$\widetilde{\beta}_{s+1}\colon H^1(G_K, \mathrm{ad}^0) \longrightarrow H^2(G_K, \mathrm{ad}^0)$$

with $\beta_{s+1} = \varpi_{\mathcal{O}}^{s-1}\widetilde{\beta}_{s+1}$ and the following property: A deformation $\rho_c = (1 + c\varepsilon)\bar{\rho} : G_K \to \mathrm{GL}_n(\mathbb{F}[\varepsilon])$ of $\bar{\rho}$ given by $c \in Z^1(G_K, \mathrm{ad}^0)$ lifts to a deformation of ρ_{s+1} to $\mathcal{O}_{s+1}[\varepsilon]$ if and only if $\widetilde{\beta}_{s+1}([c]) = 0$.

(c) If $\widetilde{c} \in Z^1(G_K, \mathrm{ad}^0_{s+1})$ denotes a set-theoretic lift of $c \in Z^1(G_K, \mathrm{ad}^0)$, then one has the explicit formula

$$\widetilde{\beta}_{s+1}([c]) = \Big((g, h) \longmapsto \varpi_{\mathcal{O}}^{-s}\big(\operatorname{Ad}\rho_{s+1}(g)\widetilde{c}(h) - \widetilde{c}(gh) + \widetilde{c}(g)\big)\Big) \pmod{B^2(G_K, \mathrm{ad}^0)}.$$

Proof. For (a), recall that one has $\operatorname{scd} G_K = 2$ for the strict cohomological dimension of K. Thus from $\beta_s = 0$ and from (4) we obtain the short exact sequence

$$0 \longrightarrow H^2(G_K, \mathrm{ad}^0_{s-1}) \xrightarrow{\cdot\varpi_{\mathcal{O}}} H^2(G_K, \mathrm{ad}^0_s) \xrightarrow{\mathrm{pr}_i^*} H^2(G_K, \mathrm{ad}^0) \longrightarrow 0.$$

The groups $H^2(G_K, \mathrm{ad}^0_i)$ are finite, and we deduce

$$\# H^2(G_K, \mathrm{ad}^0_s) = \# H^2(G_K, \mathrm{ad}^0_{s-1}) \cdot \# H^2(G_K, \mathrm{ad}^0) \tag{7}$$

The sequence in (a) of second cohomology groups is part of a long exact cohomology sequence. Its right exactness thus follows from $\operatorname{scd} G_K = 2$, and then its left exactness is immediate from (7).

For (b) and (c) we consider the commutative diagram

$$\begin{array}{ccccccc} & & & H^1(G_K,\mathrm{ad}^0) & & & \\ & \swarrow^{\widetilde{\beta}_{s+1}} & & \downarrow{\scriptstyle \beta_{s+1}} & \searrow^{\beta_s} & & \\ 0 \longrightarrow & H^2(G_K,\mathrm{ad}^0) & \xrightarrow{\cdot\varpi_{\mathcal{O}}^{s-1}} & H^2(G_K,\mathrm{ad}^0_s) & \xrightarrow{\gamma_s^*} & H^2(G_K,\mathrm{ad}^0_{s-1}) & \longrightarrow 0 \end{array}$$

with exact second row. Because $\beta_s = 0$, the dashed arrow $\widetilde{\beta}_{s+1}$ exists, and this proves (b). Finally the formula for $\widetilde{\beta}_{s+1}$ in (c) follows from multiplying the formula (5) for β_{s+1} by $\varpi_{\mathcal{O}}^{-(s-1)}$. □

The next result gives the meaning of the Bockstein operator for the relation ideal I^ψ.

Lemma 6.6. *For $i = 1, \dots, s+1$, let $\mathfrak{m}_i$ be the kernel of the composition morphism $\mathcal{R} \xrightarrow{\pi} R^\psi_{\bar\rho} \twoheadrightarrow R^\psi_{\bar\rho}/(\varpi^i_{\mathcal{O}}) \xrightarrow{\alpha_i} \mathcal{O}_i$, i.e., $\mathfrak{m}_i = (\varpi^i_{\mathcal{O}}, x_1, \dots, x_h)$. Let I_{s+1} be the relation ideal in (6) and denote by $I^\psi \to I_{s+1}$ the canonical homomorphism. Suppose $\beta_s = 0$. Then one has the following commutative diagram:*

$$\begin{array}{ccccc} H^2(G_K, \mathrm{ad}^0)^\vee & \xrightarrow{\mathrm{obs}^\vee} & I^\psi/\mathfrak{m}_{\mathcal{R}} I^\psi & \longrightarrow & I_{s+1} \\ \downarrow{\scriptstyle -\widetilde{\beta}^\vee_{s+1}} & & & & \downarrow \\ H^1(G_K, \mathrm{ad}^0)^\vee & \xrightarrow{\sim} & \overline{\mathfrak{m}}_{\mathcal{R}}/\overline{\mathfrak{m}}^2_{\mathcal{R}} & \xrightarrow[\varpi^s_{\mathcal{O}}\cdot]{\sim} & V, \end{array}$$

where $V := \big(\mathfrak{m}_s^2 + \varpi_{\mathcal{O}}^{s+1}\mathcal{R}\big)/\big(\mathfrak{m}_{s+1}^2 + \varpi_{\mathcal{O}}^{s+1}\mathcal{R}\big)$ is an $\mathbb{F}$-vector space with basis $\{\varpi^s_{\mathcal{O}} x_j\}_{j=1,\dots,h}$.

Proof. As in the proof of Lemma 5.2, we prove commutativity of the dual diagram

$$\begin{array}{ccccc} H^2(G_K, \mathrm{ad}^0) & \xleftarrow{\mathrm{obs}} & \mathrm{Hom}_{\mathbb{F}}\left(I^\psi/\mathfrak{m}_{\mathcal{R}} I^\psi, \mathbb{F}\right) & \longleftarrow & \mathrm{Hom}_{\mathbb{F}}\left(I_{s+1}, \mathbb{F}\right) \\ \uparrow{\scriptstyle -\widetilde{\beta}_{s+1}} & & & & \uparrow \\ H^1(G_K, \mathrm{ad}^0) & \xleftarrow{\sim} & \mathrm{Hom}_{\mathbb{F}}\left(\overline{\mathfrak{m}}_{\mathcal{R}}/\overline{\mathfrak{m}}^2_{\mathcal{R}}, \mathbb{F}\right) & \xleftarrow[\varpi^{-s}_{\mathcal{O}}\cdot]{\sim} & \mathrm{Hom}_{\mathbb{F}}\left(V, \mathbb{F}\right). \end{array}$$

We start by computing $\mathrm{obs}(\bar f) \in H^2(G_K, \mathrm{ad}^0)$, where $\bar f$ is the image in $\mathrm{Hom}_{\mathbb{F}}\left(I^\psi/\mathfrak{m}_{\mathcal{R}} I^\psi, \mathbb{F}\right)$ of a homomorphism $f \in \mathrm{Hom}_{\mathbb{F}}(V, \mathbb{F})$. For this, we use f to construct certain deformations of $\bar\rho$ and corresponding 1-cocycles that at the end of the proof also determine the image of f in $H^2(G_K, \mathrm{ad}^0)$ under the other composite morphism passing through $H^1(G_K, \mathrm{ad}^0)$.

In order to compute $\mathrm{obs}(\bar f)$ with the help of Lemma 5.1, let $\widetilde{f}_{s+1} \colon V = \oplus_{j=1}^h \mathbb{F}\varpi^s_{\mathcal{O}} x_j \to \mathcal{O}_{s+1}$ be a set-theoretic lift of f, and define $f_{s+1} \colon \mathcal{R}_{s+1} \to \mathcal{O}_{s+1}[\varepsilon]$ by mapping x_i to $\widetilde{f}_{s+1}(\varpi^s_{\mathcal{O}} x_i) \cdot \varepsilon$. Then we consider the quotient $R_{s+} := \mathcal{R}_{s+1}/\big(\mathfrak{m}_s^2 + \varpi_{\mathcal{O}}^{s+1}\mathcal{R}\big)$ of $\mathcal{R}_{s+1}$. Note that R_{s+} is the ring fiber product

$$\begin{array}{ccc} R_{s+} & \longrightarrow & \mathcal{R}_s \\ \downarrow & \square & \downarrow \\ \mathcal{O}_{s+1} & \longrightarrow & \mathcal{O}_s. \end{array}$$

The deformation ρ_{s+1} defines a homomorphism $R^\psi_{\bar\rho} \to \mathcal{O}_{s+1}$, and since $\beta_s = 0$ there is a surjection $R^\psi_{\bar\rho} \twoheadrightarrow \mathcal{R}_s$ by the previous lemma. By universality of the fiber product R_{s+}, there exists a homomorphism $g : R^\psi_{\bar\rho} \to R_{s+}$ that corresponds to a deformation $\rho_{s+} \colon G_K \to \mathrm{GL}_n(R_{s+})$ of $\bar\rho$. Moreover, the homomorphism $\big(f_{s+1} \pmod{\varpi^s_{\mathcal{O}}\varepsilon\mathcal{O}_{s+1}}\big) \circ g \colon R^\psi_{\bar\rho} \to R_{s+} \to \mathcal{O}_{s+1}[\varepsilon]/(\varpi^s_{\mathcal{O}}\varepsilon\mathcal{O}_{s+1})$ defines a deformation $\bar\rho_{s+} \colon G_K \longrightarrow \mathrm{GL}_n(\mathcal{O}_{s+1}[\varepsilon]/(\varpi^s_{\mathcal{O}}\varepsilon\mathcal{O}_{s+1}))$. Finally, we form the pushout R_f of $V \hookrightarrow \mathcal{R}_{s+1}$ and f so that there is a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I^{\psi}/\mathfrak{m}_{\mathcal{R}} I^{\psi} & \longrightarrow & \mathcal{R}/\mathfrak{m}_{\mathcal{R}} I^{\psi} & \longrightarrow & R^{\psi}_{\bar\rho} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \quad \searrow g & & \\
0 & \longrightarrow & I_{s+1} & \longrightarrow & \mathcal{R}_{s+1} & \longrightarrow & R_{s+1} & \longrightarrow & 0 \\
 & & \hookdownarrow & & \| & & \downarrow & & \\
0 & \longrightarrow & V & \longrightarrow & \mathcal{R}_{s+1} & \longrightarrow & R_{s+} & \longrightarrow & 0 \\
 & & \downarrow f & & \downarrow \quad \searrow f_{s+1} & & \| & & \\
0 & \longrightarrow & \mathbb{F} & \longrightarrow & R_f & \longrightarrow & R_{s+} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \varpi_{\mathcal{O}}^s \varepsilon \mathcal{O}_{s+1} & \longrightarrow & \mathcal{O}_{s+1}[\varepsilon] & \longrightarrow & \mathcal{O}_{s+1}[\varepsilon]/\varpi_{\mathcal{O}}^s \varepsilon \mathcal{O}_{s+1} & \longrightarrow & 0
\end{array}$$

whose rows are small extensions in $\widehat{\mathrm{Ar}}_{\mathcal{O}}$. Using Lemma 5.1, we obtain

$$\mathrm{obs}(\bar f) \otimes \varpi_{\mathcal{O}}^s \varepsilon \mathcal{O}_{s+1} = \big(\mathcal{O}(\rho_{s+}), f\big) \otimes \varpi_{\mathcal{O}}^s \varepsilon \mathcal{O}_{s+1} = \mathcal{O}(\bar\rho_{s+}) \in H^2\big(G_K, \mathrm{ad}^0 \otimes_{\mathbb{F}} \varpi_{\mathcal{O}}^s \varepsilon \mathcal{O}_{s+1}\big). \tag{8}$$

Now we follow the steps explained above Lemma 5.1: Namely, we first define a suitable set-theoretic lift $G_K \longrightarrow \mathrm{GL}_n(\mathcal{O}_{s+1}[\varepsilon])$ of $\bar\rho_{s+}$ and then compute the obstruction class (8) by applying formula (3). Composing the surjection $R^{\psi}_{\bar\rho} \to R_s$ and $f_{s+1} \pmod{\varpi_{\mathcal{O}}^s}\colon R_s \to \mathcal{O}_s[\varepsilon]$ determines a deformation $\rho_{s,\varepsilon} = (1+\varepsilon c_s)\rho_s\colon G_K \longrightarrow \mathrm{GL}_n(\mathcal{O}_s[\varepsilon])$ for some 1-cocylcle $c_s \in H^1(G_K, \mathrm{ad}_s)$. Let $\widetilde{c_s} \in Z^1(G_K, \mathrm{ad}^0_{s+1})$ be a set-theoretic lift of c_s that by construction defines a set-theoretic lift

$$\widetilde{\rho}_{s,\varepsilon} := (1+\varepsilon\widetilde{c_s})\rho_{s+1}\colon G_K \longrightarrow \mathrm{GL}_n(\mathcal{O}_{s+1}[\varepsilon])$$

of $\bar\rho_{s+}$. Using formula (3), we calculate a representative in $Z^2\big(G_K, \mathrm{ad}^0 \otimes (\varpi_{\mathcal{O}}^s\varepsilon)\big)$ for (8) by evaluating

$$\begin{array}{rcl}
(h,k) & \longmapsto & \widetilde{\rho}_{s,\varepsilon}(hk)\widetilde{\rho}_{s,\varepsilon}(k)^{-1}\widetilde{\rho}_{s,\varepsilon}(h)^{-1} - 1 \\
 & \overset{(6)}{=} & \big(1+\varepsilon\widetilde{c_s}(hk)\big)\rho_{s+1}(hk)\rho_{s+1}(k)^{-1}\big(1-\varepsilon\widetilde{c_s}(k)\big)\rho_{s+1}(h)^{-1}\big(1-\varepsilon\widetilde{c_s}(h)\big) - 1 \\
 & = & \varpi_{\mathcal{O}}^{-s}\big(\widetilde{c_s}(hk) - \mathrm{Ad}_{\rho_{s+1}(h)}\,\widetilde{c_s}(k) - \widetilde{c_s}(h)\big)\cdot \varpi_{\mathcal{O}}^s\varepsilon.
\end{array}$$

Hence, the class $\mathrm{obs}(\bar f) \in H^2(G_K, \mathrm{ad}^0)$ is obtained from dividing by $\varepsilon\varpi_{\mathcal{O}}^s$.

It remains to compute the image of the homomorphism $f \in \mathrm{Hom}_{\mathbb{F}}(V, \mathbb{F})$ under the composite morphism passing through $H^1(G_K, \mathrm{ad}^0)$. First note that the map $f_{s+1} \pmod{\varpi_{\mathcal{O}}}\colon \mathbb{F}[x_1,\ldots,x_h]/(x_1,\ldots,x_h)^2 \to \mathbb{F}[\varepsilon]$ induces a homomorphism $f_1 \in \mathrm{Hom}_{\mathbb{F}}\big(\overline{\mathfrak{m}}_{\mathcal{R}}/\overline{\mathfrak{m}}^2_{\mathcal{R}}, \mathbb{F}\big)$, which under multiplication by $\varpi_{\mathcal{O}}^s$ is mapped to $f \in \mathrm{Hom}_{\mathbb{F}}(V, \mathbb{F})$. We want to compute $\widetilde{\beta}_{s+1}([c])$, where $c \in Z^1(G_K, \mathrm{ad}^0)$ is a representative of the image of f_1 under the canonical isomorphism $\mathrm{Hom}_{\mathbb{F}}\big(\overline{\mathfrak{m}}_{\mathcal{R}}/\overline{\mathfrak{m}}^2_{\mathcal{R}}, \mathbb{F}\big) \xrightarrow{\sim} H^1(G_K, \mathrm{ad}^0)$. Since by construction $\rho_{s,\varepsilon} = (1+\varepsilon c_s)\rho_s\colon G_K \to \mathrm{GL}_n(\mathcal{O}_s[\varepsilon])$ lifts $(1+\varepsilon c)\bar\rho\colon G_K \to \mathrm{GL}_n(\mathbb{F}[\varepsilon])$, it is clear that $\widetilde{c_s} \in Z^1(G_K, \mathrm{ad}^0_{s+1})$ is a set-theoretic lift of c. By Lemma 6.5(c), it thus provides us with the representative

$$(h,k) \longmapsto \varpi_{\mathcal{O}}^{-s}\left(\mathrm{Ad}_{\rho_{s+1}(h)}\,\widetilde{c_s}(k) - \widetilde{c_s}(hk) + \widetilde{c_s}(h)\right) \quad \in Z^2(G_K, \mathrm{ad}^0)$$

for $\widetilde{\beta}_{s+1}([c])$. This shows that $\widetilde{\beta}_{s+1}([c]) = -\mathrm{obs}(\bar f)$, proving the lemma. □

If $\bar\rho$ has a lift to $\mathcal{O}_{2s}$, then there is a natural refinement of the above with regards to the filtration of $\mathcal{R}$ given by $\mathfrak{m}_s$. Denoting by **in** the initial term map with respect to this filtration, one has isomorphisms

$$\overline{\mathfrak{m}_s^2/(\mathfrak{m}_s^3 + \mathfrak{m}_{2s}^2)} \cong \mathbb{F}\cdot\mathbf{in}(\varpi_{\mathcal{O}}^s)^2 \oplus \bigoplus_{i=1}^{h} \mathbb{F}\cdot\mathbf{in}(\varpi_{\mathcal{O}}^s)\cdot\mathbf{in}(x_i) \cong \mathbb{F}\cdot\mathbf{in}(\varpi_{\mathcal{O}}^s)^2 \oplus V$$

and
$$\ker\left(\overline{\mathfrak{m}_s^2/\mathfrak{m}_s^3} \to \overline{\mathfrak{m}_s^2/(\mathfrak{m}_s^3+\mathfrak{m}_{2s}^2)}\right) \cong \bigoplus_{1\le i\le j\le n} \mathbb{F}\cdot \mathbf{in}(x_i)\cdot \mathbf{in}(x_j) \cong \overline{\mathfrak{m}}_{\mathcal{R}}^2/\overline{\mathfrak{m}}_{\mathcal{R}}^3.$$

In other words, we have a natural 2-step filtration of $\overline{\mathrm{gr}^2_{\mathfrak{m}_s}\mathcal{R}}$ whose first subquotient is isomorphic to $\overline{\mathfrak{m}}_{\mathcal{R}}^2/\overline{\mathfrak{m}}_{\mathcal{R}}^3$ and whose second subquotient is isomorphic to $\mathbb{F}\cdot\mathbf{in}(\varpi_{\mathcal{O}}^s)^2\oplus V$ with V as above. A variant of the above lemma is the following whose proof we leave to the reader:

Lemma 6.7. *Let I_{s+1} be the relation ideal in (6) and let $I^\psi\to I_{s+1}$ be the canonical homomorphism. Suppose that ρ_{s+1} possesses a lift to $\mathcal{O}_{2s}$ and that $\beta_s=0$. Then one has the following commutative diagram:*

$$\begin{array}{ccccc}
H^2(G_K,\mathrm{ad}^0)^\vee & \longrightarrow & I^\psi/\mathfrak{m}_{\mathcal{R}}I^\psi & \longrightarrow & I_{s+1}\\
\downarrow{\scriptstyle -\widetilde{\beta}_{s+1}^\vee} & & & & \downarrow\\
H^1(G_K,\mathrm{ad}^0)^\vee & \xrightarrow{\sim} & \overline{\mathfrak{m}}_{\mathcal{R}}/\overline{\mathfrak{m}}_{\mathcal{R}}^2 & \xhookrightarrow[\varpi_{\mathcal{O}}^s\cdot]{} & \mathbb{F}\cdot\mathbf{in}(\varpi_{\mathcal{O}}^s)^2\oplus V.
\end{array}$$

The use of the above 2-step filtration of $\overline{\mathrm{gr}^2_{\mathfrak{m}_s}\mathcal{R}}$ allows one to apply our results on the Bockstein operator on one piece and that of the bracket cup product on the other. This gives precise information on the *refined quadratic parts* in $\overline{\mathrm{gr}^2_{\mathfrak{m}_s}\mathcal{R}}$ which arise from $H^2(G_K,\mathrm{ad}^0)$ – with the possible exception of the quotient $\mathbf{in}(\varpi_{\mathcal{O}}^s)^2\cdot\mathbb{F}$. Namely, we have the following result:

Theorem 6.8. *Suppose $\bar\rho$ has a lift to $\mathcal{O}_{s+1}$ and that $\beta_s=0$. Then I_ψ is contained in $\mathfrak{m}_s^2+\varpi_{\mathcal{O}}^{s+1}\mathcal{R}$ and the following diagram is commutative, where all homomorphisms are the natural ones, as given either in Lemma 5.2 or Lemma 6.6:*

$$\begin{array}{ccccc}
H^2(G_K,\mathrm{ad}^0)^\vee & \longrightarrow & I^\psi/\mathfrak{m}_{\mathcal{R}}I^\psi & \longrightarrow & \overline{I^\psi/\big(I^\psi\cap(\mathfrak{m}_s^3+\varpi_{\mathcal{O}}^{s+1}\mathcal{R})\big)}\\
\downarrow{\scriptstyle -\widetilde{\beta}_{s+1}^\vee\oplus-\frac{1}{2}b^\vee} & & & & \downarrow\\
H^1(G_K,\mathrm{ad}^0)^\vee\oplus\mathrm{Sym}^2H^1(G_K,\mathrm{ad}^0)^\vee & \xrightarrow{\sim} & \overline{\mathfrak{m}}_{\mathcal{R}}/\overline{\mathfrak{m}}_{\mathcal{R}}^2\oplus\overline{\mathfrak{m}}_{\mathcal{R}}^2/\overline{\mathfrak{m}}_{\mathcal{R}}^3 & \xhookrightarrow[(\varpi_{\mathcal{O}}^s\cdot__)\oplus\mathrm{incl.}]{} & \overline{\mathrm{gr}^2_{\mathfrak{m}_s}\mathcal{R}}/(\mathbb{F}\cdot\mathbf{in}(\varpi_{\mathcal{O}}^s)^2).
\end{array}$$

If in addition $\bar\rho$ has a lift to $\mathcal{O}_{2s}$, then the above diagram still commutes if one removes the symbols '$+\varpi_{\mathcal{O}}^{s+1}\mathcal{R}$' in the top right and '$\mathbb{F}\cdot\mathbf{in}(\varpi_{\mathcal{O}}^s)^2$' in the lower right corner.

We now discuss various issues about the Bockstein homomorphism that were left open so far, for instance the existence of lifts ρ_{s+1} and the choice of s.

Lemma 6.9. *Let $p>2$ and $\bar\rho\colon G_K\to\mathrm{GL}_n(\mathbb{F})$ be a representation. Let s be an integer.*[10] *We fix a minimal presentation of $R_{\bar\rho}^\psi$ as in Proposition 1.3 and an isomorphism $\mathcal{R}\cong\mathcal{O}[[x_1,\ldots,x_h]]$, and set $\mathfrak{m}_s:=(\varpi_{\mathcal{O}}^s,x_1,\ldots,x_h)$. Then the following hold:*

(a) If $n=2$, then $\bar\rho$ has a lift to $W(\mathbb{F})$.

(b) For general n, if $\bar\rho(G_K)$ is a p-group and if $p^s=\#\mu_{p^\infty}(K)>1$, then $\bar\rho$ has a lift to the ring $W(\mathbb{F})/p^{s+1}W(\mathbb{F})$.

(c) If the relation ideal I^ψ lies in $\mathfrak{m}_s^2$, then $\bar\rho$ has a lift to $\mathcal{O}_{2s}$.

[10]In different items, s may take different values.

(d) *If $\bar\rho$ has a lift to $\mathcal{O}_{2s}$ and if $\beta_s = 0$, then any choice $y_i \in x_i + \varpi_{\mathcal{O}}^s \mathcal{R}$, $i = 1, \dots, h$, induces a change of coordinates isomorphism $\mathcal{R} \cong \mathcal{O}[[x_1, \dots, x_h]] \cong \mathcal{O}[[y_1, \dots, y_h]]$ such that $\mathfrak{m}_s$ is independent of whether we use the x_i or the y_i to define it.*

(e) *If $R^\psi_{\bar\rho}$ is flat over $\mathcal{O}$, then there exists a finite totally ramified extension of $\mathcal{O}[1/p]$ with ring of integers $\mathcal{O}'$ and a homomorphism $R^\psi_{\bar\rho} \to \mathcal{O}'$ in $\widehat{\mathrm{Ar}}_{\mathcal{O}}$, i.e., $\bar\rho$ has a lift to characteristic zero, and in particular lifts to $\mathcal{O}'/(\varpi'_{\mathcal{O}})^s$ for every integer s.*

Regarding (e) note that A. Muller [Mu] has constructed crystalline lifts of a large class of mod p Galois representations $\bar\rho$ for any n. Whether such a lift always exists is still an open question.

Proof. Part (a) can be obtained from a simple adaption of [Kha, Theorem 2]: Khare's proof using Kummer theory works for all field of characteristic zero. Part (b) is [Bö3, Prop. 2.1]. Part (c) is rather trivial: the hypothesis implies that $R^\psi_{\bar\rho} \cong \mathcal{R}/I^\psi$ surjects onto $\mathcal{R}/(p^{2s}, x_1, \dots, x_h) \cong \mathcal{O}_{2s}$. Part (d) is also obvious. For (e) observe that by flatness the ring $R^\psi_{\bar\rho}[1/p]$ is non-zero. Hence, its generic fiber $\mathfrak{X}(\rho)^\psi$ is a non-empty rigid analytic space over $\mathcal{O}[1/p]$. Thus it has points over some finite extension of $\mathcal{O}[1/p]$. These points are the desired lifts. □

Remark 6.10. The definition of the Bockstein operators β_i depends on a choice of a *base point*, i.e., a lift ρ_{s+1} of $\bar\rho$ to $\mathcal{O}_{s+1}$. We do not know in general in what sense the vanishing of β_s and the non-vanishing of β_{s+1} could be independent of such a lift. A change of base point as described in Lemma 6.9(d), clearly does not change the integer s for which $\beta_s = 0$ and $\beta_{s+1} \neq 0$, assuming the existence of ρ_{s+1}. We also do not know, what an optimal choice of s, independently of a choice of the lift ρ_{s+1} means, although Lemma 6.9 provides some reasonable guesses. If one does have an explicit choice of ρ_{s+1}, and a situation where one can then determine its infinitesimal deformations, then one can determine whether $\beta_s = 0$ and $\beta_{s+1} \neq 0$. Such an approach is sketched in the proof of Proposition 6.11.

Before giving the proof of Theorem 1.14, we discuss the existence of such a base point in cases (D) and (E) of Section 3. For the remainder, suppose that $q = \#\mu_{p^\infty}(K) > 1$ and set $s := \log_p q$. Suppose also that the image of $\bar\rho$ is a p-group and that the fixed lift ψ of $\det\bar\rho$ is the trivial character – both can be assumed without loss of generality by twisting; cf. the proof of Lemma 3.7.

Proposition 6.11. *In cases (D) and (E) of Section 3 there exists a deformation ρ_∞ in $D^\psi_{\bar\rho}(W(\mathbb{F}))$ such that $\beta_s = 0$ and $\beta_{s+1} \neq 0$.*

Proof. We ask the reader to have the notation and concepts used in the proof of Lemma 3.7 at hand. We define

$$M_i := \begin{pmatrix} 1 & u_i \\ 0 & 1 \end{pmatrix} \text{ for } i = 1, 3, \dots, 2g \quad \text{ and } \quad M_2 := \begin{pmatrix} \sqrt{1-q}^{-1} & u_2 \\ 0 & \sqrt{1-q} \end{pmatrix},$$

where the u_i are all zero in case (D). Then it is easy to verify that the M_i satisfy the Demushkin relation $M_1^q[M_1, M_2] \dots [M_{2g-1}, M_{2g}] = 1$. Hence, the map $\Pi \to \mathrm{GL}_2(W(\mathbb{F}))$ defined by mapping x_i to M_i yields the desired lift ρ_∞.

We use this base point to determine the Bockstein relations, and thus to determine the correct value of s such that $\beta_s = 0$ and $\beta_{s+1} \neq 0$, by computing explicitly infinitesimal deformations of ρ_∞. Namely, we define $N_i := M_i(1 + \varepsilon A_i) \in \mathrm{GL}_2(W(\mathbb{F})[\varepsilon])$ for matrices $A_i = \begin{pmatrix} a_i & b_i \\ c_i & -a_i \end{pmatrix}$. Computing the Demushkin relation $N_1^q[N_1, N_2]\dots[N_{2g-1}, N_{2g}] = 1$, we obtain a linear relation whose coefficients lie in $qW(\mathbb{F})$ but not in $pqW(\mathbb{F})$. The assertion follows. □

Remark 6.12. We note that the base point lift chosen in the proof of the previous proposition is obtained as a specialization of the variables in the proof of Lemma 3.7 within $qW(\mathbb{F})$. Hence, by Lemma 6.9(d), the trivial specialization that sends all variables to zero gives a lift to $W(\mathbb{F})/q^2W(\mathbb{F})$ (in fact to $W(\mathbb{F})$) so that $\beta_s = 0$ and $\beta_{s+1} \neq 0$.

Proof of Theorem 1.14. By Theorem 3.4, we have in cases (A)–(C) of Section 3 that a lift $\rho_{2s}\colon G_K \to \mathrm{GL}_2(W(\mathbb{F})/p^{2s}W(\mathbb{F}))$ exists for $s = \log_p q$ if we specialize all variables to zero. Then all the specialized relations will vanish modulo q^2. Moreover for this choice, we have $\beta_s = 0$ and $\beta_{s+1} \neq 0$ because the linear terms of the relations vanish modulo q but not modulo pq. By Corollary 3.6(b), the images of the quadratic parts of generators of I^ψ span a subspace of dimension equal to $h_2 = \dim_{\mathbb{F}} H^2(G_K, \mathrm{ad}^0)$. Thus Theorem 1.14 follows from Theorem 6.8.

It remains to consider cases (D) and (E). We take the specialization from Remark 6.12 as our lift to $W(\mathbb{F})/q^2W(\mathbb{F})$ so that $\beta_s = 0$ and $\beta_{s+1} \neq 0$. By Lemma 3.7(c),(e), there exists a presentation

$$0 \longrightarrow (r_1, \dots, r_m) \longrightarrow \mathcal{R} \longrightarrow R_{\bar{\rho}}^{\psi} \longrightarrow 0$$

such that $\mathbf{in}(\bar{r}_1), \dots, \mathbf{in}(\bar{r}_m) \in \mathrm{gr}^2_{\overline{\mathfrak{m}}_{\mathcal{R}}} \overline{\mathcal{R}}$ form a regular sequence in $\mathrm{gr}_{\overline{\mathfrak{m}}_{\mathcal{R}}} \overline{\mathcal{R}}$ and $m = \dim_{\mathbb{F}} H^2(G_K, \mathrm{ad}^0)$. We complete the proof of Theorem 1.14 by a further appeal to Theorem 6.8. □

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