Equidimensionality of universal pseudodeformation rings in characteristic p for absolute Galois groups of p-adic fields

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Contents

1 Introduction

Let K be a finite extension of $\mathbb{Q}_p$, with absolute Galois group $G_K = \mathrm{Gal}(K^{\mathrm{alg}}/K)$. Denote by $\zeta_p \in K^{\mathrm{alg}}$ a primitive p-th root of unity. Let $\mathbb{F}$ be a finite field of characteristic p, and denote by $\bar\rho\colon G_K \to \mathrm{GL}_n(\mathbb{F})$ a continuous homomorphism. Versal deformation rings $R_{\bar\rho}$ of such (and more general) $\bar\rho$ were introduced in [Maz89] by Mazur, the variant of framed deformation rings $R^{\square}_{\bar\rho}$ in [Kis09] by Kisin. These rings, often also refined by some auxiliary Hodge-theoretic conditions, have ever since Wiles' spectacular proof of Fermat's Last Theorem, [Wil95, TW95] played in important role in modularity results, to describe conditions at p of global Galois representations. More recently such rings were also important in the p-adic local Langlands program, e.g. [Col10, Kis10].

If $\bar\rho$ is absolutely irreducible, and under some further technical conditions on n and K, the rings $R_{\bar\rho}$ are well-understood. They are close to being smooth, one knows their irreducible components and their dimension, and benign crystalline points form Zariski dense subset by [Nak14, § 4] due to Nakamura, extending previous work of Chenevier. For general, not absolutely irreducible $\bar\rho$ the situation appears to be quite open. Foundational general results are lacking. If $\bar\rho$ is trivial and under some technical hypotheses on K and p, there is recent progress in [Iye19].

The present work is not directly concerned with investigating properties of the ring $R^{\square}_{\bar\rho}$. But as we shall detail later, we hope to extend the present work by a comprehensive analysis of the ring $R^{\square}_{\bar\rho}$ for any $\bar\rho$. Here we consider pseudorepresentations and their deformations, which are called pseudodeformations, as introduced by Chenevier in [Che14][1]. Pseudorepresentations are objects that aim to model the characteristic polynomial of a representation, in a functorial way in the coefficients. They were first introduced by Roby in [Rob63] and considered by Chenevier in [Che14] in the context of deformation theory. Pseudorepresentations generalize pseudocharacters, see [Tay91], that model the trace of a representation. Pseudocharacters of dimension n over a ring A are only known to be 'well-behaved' if at least $n!$ is invertible in A; see [BC09]. Since our work concerns all n and p we work with pseudorepresentations.

To any representation one can attach a pseudorepresentation. Let $\overline{D}$ denote the image of $\bar\rho$ under this assignment. Chenevier in [Che14] proved that any residual pseudorepresentation $\overline{D}$ of G_K admits a universal pseudodeformation over a universal pseudodeformation ring $R^{\mathrm{univ}}_{\overline{D}}$ and established many further foundational results; see also [WE18] for some refinements.

Let x be a point of $X^{\mathrm{univ}}_{\overline{D}} := \operatorname{Spec} R^{\mathrm{univ}}_{\overline{D}}$ with residue field $\kappa(x)$ and pseudorepresentation D_x. By [Che14, Thm. A] there exists a semisimple representation $\rho_x\colon G_K \to \mathrm{GL}_n(\kappa(x)^{\mathrm{alg}})$ unique up to conjugation whose associated pseudorepresentation is the coefficient change $D_x \otimes_{\kappa(x)} \kappa(x)^{\mathrm{alg}}$. One calls x irreducible if ρ_x is irreducible. It is shown in [Che14, Exam. 2.20] that the subset $(X^{\mathrm{univ}}_{\overline{D}})^{\mathrm{irr}}$ of irreducible points of $X^{\mathrm{univ}}_{\overline{D}}$ is open. It is a simple but important observation in [Che11] that the generic fiber of $(X^{\mathrm{univ}}_{\overline{D}})^{\mathrm{irr}}$ is contained in the regular locus $(X^{\mathrm{univ}}_{\overline{D}})^{\mathrm{reg}}$ of $X^{\mathrm{univ}}_{\overline{D}}$. A much more involved argument yields the main result of [Che11, Théorème]: The irreducible locus is open and Zariski dense in the generic fiber of $X^{\mathrm{univ}}_{\overline{D}}$ and as a consequence it is shown that the generic fiber of $X^{\mathrm{univ}}_{\overline{D}}$ is equidimensional of dimension $n^2[K:\mathbb{Q}_p]+1$.

The main result of the present work concerns the analog of [Che11] for the special fiber of $X^{\mathrm{univ}}_{\overline{D}}$. We follow the overall strategy of Chenevier but encounter new phenomena that have to be dealt with. To describe our results, we need to introduce more notation. In the following, we overline objects to denote their special fiber, e.g. $\overline{R}^{\mathrm{univ}}_{\overline{D}} := R^{\mathrm{univ}}_{\overline{D}}/(p)$ and $\overline{X}^{\mathrm{univ}}_{\overline{D}} := \operatorname{Spec} \overline{R}^{\mathrm{univ}}_{\overline{D}}$.

An elementary observation is that $\overline{X}^{\mathrm{univ}}_{\overline{D}}$ contains no regular points at all if ζ_p lies in K. This already holds for $n=1$. Our way of dealing with this problem is to study a determinant map $\det_{\overline{D}}\colon X^{\mathrm{univ}}_{\overline{D}} \to X^{\mathrm{univ}}_{\det \overline{D}}$.[2] The determinant $\det D$ of a pseudorepresentation D is the constant

[1] We use the term pseudorepresentation following [WE18]; in [Che14] the term determinant is used.

[2] This is another reason why we use the term pseuderepresentation and not the term determinant from [Che14].

coefficient of its characteristic polynomial. If D is the pseudorepresentation of a representation ρ, then $\det D$ is the pseudorepresentation attached to $\det \rho$. Eventually we show that $\det_{\overline{D}}$ is formally smooth over a dense open subset of $\overline{X}_{\overline{D}}^{\mathrm{univ}}$. This leads us to study the regular locus of $\overline{X}_{\overline{D}}^{\mathrm{univ}}$ when $\zeta_p \notin K$ and that of $\overline{X}_{\overline{D},\mathrm{red}}^{\mathrm{univ}}$ when $\zeta_p \in K$; here for a scheme X we denote by X_{red} the reduced closed subscheme underlying X. Eventually we shall show that $\overline{X}_{\overline{D}}^{\mathrm{univ}}$ is reduced when $\zeta_p \notin K$.

A next observation is that the irreducible locus $(\overline{X}_{\overline{D},\mathrm{red}}^{\mathrm{univ}})^{\mathrm{irr}}$ may contain singular points. We analyze precisely when this happens. This leads to our notion of special point. These are the points for which $\rho_x \colon G_K \to \mathrm{GL}_n(\kappa(x)^{\mathrm{alg}})$ is induced from a representation of $G_{K'}$ where $K' = K(\zeta_p)$ if $\zeta_p \notin K$ and where K' is a degree p Galois extension of K if $\zeta_p \in K$. Let $(\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{spcl}}$ denote the subset of *special* points $x \in (\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{irr}}$, set $(\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\text{n-spcl}} := (\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{irr}} \smallsetminus (\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{spcl}}$ and set $(\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{red}} := \overline{X}_{\overline{D}}^{\mathrm{univ}} \smallsetminus (\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{irr}}$.

We now state our main results. The first is on equidimensionality.

Theorem 1 (Theorem 5.4.1)**.** *The following assertions hold:*

(i) $(\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\text{n-spcl}} \subset \overline{X}_{\overline{D}}^{\mathrm{univ}}$ *is open and Zariski dense.*

(ii) If $\zeta_p \notin K$*, then* $(\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\text{n-spcl}}$ *is regular.*

(iii) If $\zeta_p \in K$*, then* $(\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\text{n-spcl}}_{\mathrm{red}}$ *is regular, and* $(\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{reg}}$ *is empty.*

(iv) $\overline{X}_{\overline{D}}^{\mathrm{univ}}$ *is equidimensional of dimension* $[K:\mathbb{Q}_p]n^2+1$*.*

We also completely characterize the singular locus if $\zeta_p \notin K$, similar to [Che11, Thm. 2.3].

Theorem 2 (Theorem 5.4.5)**.** *If* $\zeta_p \notin K$*, then the following hold:*

(i) The closure of $(\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{spcl}}$ *in* $\overline{X}_{\overline{D}}^{\mathrm{univ}}$ *lies in* $(\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{sing}}$*.*

(ii) If $n > 2$ *or* $[K:\mathbb{Q}_p] > 1$*, then* $(\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{red}} \subset (\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{sing}}$*.*

(iii) If $n = 2$*,* $K = \mathbb{Q}_p$*, and* $x \in (\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{red}}$ *is a direct sum* $D_1 \oplus D_2$ *of 1-dimensional pseudorepresentations* D_i*, then* $x \in (\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{sing}}$ *if and only if* $D_2 = D_1(m)$ *for* $m \in \{\pm 1\}$*.*

And we show the following

Theorem 3 (Theorem 5.4.7)**.** *The ring* $\overline{R}_{\overline{D},\mathrm{red}}^{\mathrm{univ}}$ *satisfies Serre's condition* (R_2)[3]*, unless* $n = 2$*,* $K = \mathbb{Q}_2$ *and* $\overline{D}$ *is trivial.*

We in fact determine the exact dimension of $(\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{red}}$ and $(\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{spcl}}$ in Lemma 5.4.2 and Corollary 5.4.3. From this, depending on n and $[K:\mathbb{Q}_p]$, one can in general establish Serre's condition (R_m) for some $m_{K,n} > 2$.

Let us explain more about Theorem 1 and its proof, also in order to give an indication of some auxiliary results and techniques of this article. The definition of the locus of special points x relies on the computation of a certain second cohomology group. Using an extension of Tate local duality to local field coefficients, see Theorem 3.4.1, this leads one to consider $H^0(G_K, \overline{\mathrm{ad}}_{\rho_x}(1))$; here ad_{ρ_x} denotes the adjoint representation of ρ_x, $\overline{\mathrm{ad}}_{\rho_x}$ the quotient of ad_{ρ_x} by its subgroup of scalars, and (1) the 1-fold Tate twist. This tangent space measures the non-smoothness of $\det_{\overline{D}}$ at such x. Using Clifford theory we analyze the non-vanishing of $H^0(G_K, \overline{\mathrm{ad}}_{\rho_x}(1))$. The auxiliary results from Clifford Theory are presented in Section 2; see also Lemma 5.1.1.

The fact that we have to deal with induction and twist by characters, i.e., by one-dimensional representations on the side of representations also necessitates that we have analogous operations

[3]see Definition A.1.3

for pseudorepresentations. Twisting is explained in Subsection 4.5. Induction of pseudorepresentations is treated in Subsection 4.6; we can only define this operation for reduced coefficients; but this turns out to be sufficient in all our applications. It is also important that the loci of $(\overline{X}^{\rm univ}_{\overline{D}})^{\rm irr}$ of special points are closed. This is proved in Lemma 5.1.3.

Let us also explain why induction of pseudorepresentations is an important tool. Suppose that $D\colon G_K \to B$ is a (pseudo-)deformation of $\overline{D}$ for B a complete noetherian domain ring with residue field $\mathbb{F}$, and that the coefficients of D generate B topologically, so that $\overline{R}^{\rm univ}_{\overline{D}} \to B$ is surjective. Suppose that for a finite extension B' we know that $D \otimes_B B'$ is induced from a pseudorepresentation $D'\colon G_{K'} \to B'$ that is a pseudodeformation of some $\overline{D}'\colon G_{K'} \to \mathbb{F}$ for some extension K' of K. Note that D' has dimension $n' = \frac{n}{[K':K]} < n$. Suppose also that by induction on n we know that $\overline{R}^{\rm univ}_{\overline{D}'}$ has the expected Krull dimension $(n')^2[K' : \mathbb{Q}_p] = \frac{1}{[K':K]} n^2 [K : \mathbb{Q}_p]$. We shall use such estimates and standard cohomological lower bounds for the dimension of local rings of $\overline{X}^{\rm univ}_{\overline{D}}$ to deduce in Theorem 5.3.1 that $(\overline{X}^{\rm univ}_{\overline{D}})^{\rm n\text{-}spcl}$ is Zariski dense in $(\overline{X}^{\rm univ}_{\overline{D}})^{\rm irr}$.

The last missing ingredient in our inductive argument to establish Theorem 1 is the proof that every neighborhood of some x in the reducible locus $(\overline{X}^{\rm univ}_{\overline{D}})^{\rm red}$ contains a point of $(\overline{X}^{\rm univ}_{\overline{D}})^{\rm irr}$. Here we follow the argument used by Chenevier [Che11, Thm. 2.1]. The key point in our setting is that étale locally $(\overline{X}^{\rm univ}_{\overline{D}})^{\rm red} \hookrightarrow \overline{X}^{\rm univ}_{\overline{D}}$ is a closed immersion. Hence if a neighborhood U of some $x \in (\overline{X}^{\rm univ}_{\overline{D}})^{\rm red}$ does not intersect $(\overline{X}^{\rm univ}_{\overline{D}})^{\rm irr}$, the local behavior at x in $\overline{X}^{\rm univ}_{\overline{D}}$ is similar to that of $\overline{X}^{\rm univ}_{\overline{D}_1} \times \overline{X}^{\rm univ}_{\overline{D}_2}$ for pseudorepresentations such that $\overline{D} = \overline{D}_1 \oplus \overline{D}_2$, after completion at x. This will ultimately yield a contradiction by comparing dimensions (of tangent spaces); see Theorem 5.2.1 and its proof. The analogous argument in [Che11] uses rigid geometry instead of the étale topology.

On the technical side, we shall often work with dimension 1 points $x \in \overline{X}^{\rm univ}_{\overline{D}}$ with D_x irreducible or at least multiplicity free. In this situation the residue field $\kappa(x)$ is a Laurent series field over a finite field. We need to rework the deformation theory at such points, which over finite field extensions over $\mathbb{Q}_p$ goes back to Kisin. These points are dense in $\overline{X}^{\rm univ}_{\overline{D}}$. We also have to clarify properties of associated representations and their deformations. This requires results from [Che14], [WE18] on pseudodeformations and generalizations thereof. Also Nekovář's treatment [Nek06] of Tate local duality with general coefficient systems, such as $\kappa(x)$, is important to us.

Let us finally explain our motivation to study $R^{\rm univ}_{\overline{D}}$. It is certainly a foundational and in light of [Che11] natural question to study the equidimensionality of $\overline{X}^{\rm univ}_{\overline{D}}$. However we also plan to use our results to investigate properties of universal framed deformation rings $R^{\square}_{\overline{\rho}}$. Following an approach similar to [Ger10, § 3.2], we see a path to deduce geometric properties of the rings $R^{\square}_{\overline{\rho}}$; inspiration from [Ger10] was also used in the recent preprint [Iye19]. This should give equidimensionality, the complete intersection property and normality of the special fiber of $R^{\square}_{\overline{\rho}}$. Building on this it seems clear how deduce a bijection between the generic components of $\operatorname{Spec} R^{\square}_{\overline{\rho}}$ and of $\operatorname{Spec} R^{\square}_{\det \overline{\rho}}$ under the map $\rho \mapsto \det \rho$. An alternative approach might be [WE13, Thm. 1.5.4.2]. It asserts that the natural map $\mathcal{R}^{\square}_{\overline{D}}/\operatorname{PGL}_n \to X^{\rm univ}_{\overline{D}}$ is a finite universal homeomorphism; here $\mathcal{R}^{\square}_{\overline{D}}$ is the space of all framed deformations with associated residual pseudorepresentation $\overline{D}$. The completion of the local rings of $\mathcal{R}^{\square}_{\overline{D}}$ at closed points are precisely the rings $R^{\square}_{\overline{\rho}}$ for those $\overline{\rho}$ that map to $\overline{D}$. Further results on $R^{\square}_{\overline{\rho}}$ might also give an approach to proving the density of crystalline points in generic fibers of universal pseudorepresentation spaces, a result that was important in the p-adic local Langlands program for $\operatorname{GL}_2(\mathbb{Q}_p)$ and that motivated [BJ15] and [Iye19].

Outline

We now give an outline of this work. In Section 2 we present those parts of Clifford theory to be used in Section 5 when defining and characterizing special points. Section 3 reviews the theory of deformations of Galois representation in the sense of Mazur with a strong emphasis on result related to deformation rings at dimension 1 points where the residue field is a local equicharacteristic field. The subsequent Section 4 is a detailed review of pseudorepresentations following largely [Che14] with some noteworthy additions that are crucial for the main results of this work: we consider the locus of reducibility in the context of pseudorepresentations, we introduce twisting and induction of pseudorepresentations (the latter under some hypotheses), and we again give a special treatment to some presumably well-known facts on equicharacteristic dimension 1 points on pseudodeformation rings.

The final Section 5 contains the proof of the main result of this work, the equidimensionality of expected dimension of the special fiber of universal pseudodeformation rings. We follow Chenevier's proof for the generic fiber [Che11], and explain how to overcome all complications that arise in the special fiber. Much of these complications are packed into our definition of special points in Subsection 5.1; see Definition 5.1.2. Non-special (irreducible) points will take the role of irreducible points in Chenevier's work; they describe that part of the irreducible locus of the special fiber of the pseudodeformation space over which the determinant map is relatively formally smooth.

Subsection 5.1 also contains some technical result on the comparison of universal pseudodeformation and universal deformation rings over local fields where the residual pseudorepresentation is a sum of two irreducible ones; see Lemma 5.1.6. In Subsection 5.2 we indicate an induction procedure to prove the main result: given a suitable induction hypothesis, we shows that the reducible locus is nowhere dense. In Subsection 5.3 we show that the non-special points are open and Zariski dense in the irreducible locus under some inductive hypotheses. By combining the previous subsections, it is then in Subsection 5.4 straightforward to proof the main theorems announced in the introduction.

Let us also note that in an appendix, we provide some results on commutative rings, on algebras over a field and on absolutely irreducible mod p representations of the absolute Galois group of a p-adic field. These results are mostly standard and they serve simply as a reference.

Some notation and conventions

- Throughout we fix a prime number p and a finite field $\mathbb{F}$ of characteristic p.
- For any field E we denote by E^{alg} an algebraic closure of E and by $G_E = \mathrm{Gal}(E^{\mathrm{alg}}/E)$ its absolute Galois group.
- We write $\mathbb{Q}_p$ for the p-adic completion of $\mathbb{Q}$ and fix an algebraic closure $\mathbb{Q}_p^{\mathrm{alg}}$ of $\mathbb{Q}_p$. All algebraic extension fields of $\mathbb{Q}_p$ will be considered as subfields of $\mathbb{Q}_p^{\mathrm{alg}}$.
- We fix a finite extension field K of $\mathbb{Q}_p$ of degree $d = [K : \mathbb{Q}_p]$ inside $\mathbb{Q}_p^{\mathrm{alg}}$.
- Throughout κ will denote a finite field of characteristic p or a local field of residue characteristic p. It will take the role of a coefficient field for deformations and pseudodeformations. If such a coefficient field is meant to be finite, we usually write $\mathbb{F}$.
- For a point x on a scheme X, we write $\mathcal{O}_{X,x}$ for the local ring at x and $\kappa(x)$ for its residue field; the latter is the second way in which the letter κ occurs; note that $\kappa(x)$ can be any field.
- By a ring, we mean a unital commutative ring. Algebras over a ring A do not need to be commutative. To make clear that an A-algebra is commutative, we will always speak of it as a commutative A-algebra.

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2 Clifford Theory

Clifford theory provides a crucial input in determining conditions that characterize the special points that we will introduce in Definition 5.1.2, building on Lemma 5.1.1. In this section, we give the representation theoretic background. We also include some results for coefficients that are not algebraically closed. The results in Subsection 2.2, and most importantly Corollary 2.2.2, are probably well-known. Those in Subsection 2.3, and in particular Lemma 2.3.1, seem more exotic nature to us. We give proofs whenever we could not locate the results in the literature.

Throughout this section, G denotes a (possibly infinite) group and H a normal subgroup of finite index. If G is a topological, we assume H to be open in G. All representations will be finite dimensional over some ring or field.

2.1 Generalities

Definition 2.1.1. *For a representation $\rho\colon H \to \mathrm{GL}_m(A)$ over a ring A and $g \in G$, we define the conjugate of ρ by g as the representation*

$$\rho^g\colon H \longrightarrow \mathrm{GL}_m(A), \qquad h \longmapsto \rho(ghg^{-1}).$$

Remark 2.1.2. Conjugation in the sense of Definition 2.1.1 defines an action of G on the set $\{[\rho^g] \,:\, g \in G\}$ of isomorphism classes $[\rho^g]$ of representations ρ^g of H. Since H acts trivially, the action factors via G/H and so, up to isomorphism, there are only finitely many conjugates of ρ.

For the remainder of this subsection, let E denote a field, of characteristic $p \geq 0$. Unless said otherwise, any representation will be of finite dimension over E.

The following lemma will be used repeatedly.

Lemma 2.1.3 (Mackey tensor product theorem for induced representations; [CR62, Cor. 44.4])**.** *Let ρ be a representation of G and ρ' of H. Then $\rho \otimes \mathrm{Ind}_H^G \rho' \cong \mathrm{Ind}_H^G((\mathrm{Res}_H^G \rho) \otimes \rho')$.*

We will also need:

Lemma 2.1.4. *For a semisimple representation ρ of H over a field E the following hold.*

(a) For any separable field extension $F \supset E$ the representation $\rho \otimes_E F$ is semisimple.

(b) One has $\mathrm{Res}_H^G \mathrm{Ind}_H^G \rho \cong \bigoplus_{g \in G/H} \rho^g$.

(c) If τ is an irreducible representation of G, then $\mathrm{Res}_H^G \tau$ is semisimple, and all irreducible summands of $\mathrm{Res}_H^G \tau$ are conjugate to one another in the sense of Definition 2.1.1.

(d) If $[G : H]$ is invertible in E, then $\mathrm{Ind}_H^G \rho$ is semisimple.

(e) If G/H acts freely on $\{[\rho^g] \,:\, g \in G\}$ and if ρ is irreducible, then $\mathrm{Ind}_H^G \rho$ is irreducible.

(f) $\mathrm{Ind}_H^G \rho$ is absolutely irreducible if and only if ρ is absolutely irreducible and G/H acts freely on $\{[\rho^g] \,:\, g \in G\}$.

(g) *Let ρ' be a second representation of H. Then $\operatorname{Ind}_H^G \rho \cong \operatorname{Ind}_H^G \rho'$ if and only if*

$$\bigoplus_{g\in G/H} \rho^g \cong \bigoplus_{g\in G/H} (\rho')^g. \tag{1}$$

(h) *If ρ is irreducible in (g), then (1) is equivalent to $\rho' \cong \rho^g$ for some $g \in G$.*

Proof. Part (a) is [CR62, Cor. 69.8] with A from there being $\operatorname{End}_{E[H]}(\rho)$. Part (b) holds by [Ser95, Prop. 22]. Part (c) follows from [CR62, Thm. 49.2]. Part (d) is immediate from [Web16, Ch. 5, Exerc. 8].

To prove Part (e), let $V' \subset \operatorname{Ind}_H^G \rho$ be an irreducible G-subrepresentation. Then by (b) the representation $\operatorname{Res}_H^G V'$ contains ρ^g for some $g \in G$. As V' is a G-representation, we deduce $\bigoplus_{g\in G/H} \rho^g \subset \operatorname{Res}_H^G V'$. Since by hypothesis the ρ^g are irreducible pairwise non-isomorphic, we find $\dim V' \geq \dim \operatorname{Ind}_H^G \rho$, and hence $\operatorname{Ind}_H^G \rho = V'$ is irreducible.

We prove Part (f). Because solving linear systems of equations have solution spaces of the same dimension over field and its algebraic closure, one has $\operatorname{Hom}_{E[H]}(\rho, \rho^g) \otimes_E E^{\mathrm{alg}} \cong \operatorname{Hom}_{E^{\mathrm{alg}}[G]}(\rho \otimes_E E^{\mathrm{alg}}.\rho^g \otimes_E E^{\mathrm{alg}})$. This allows one by base change $E \to E^{\mathrm{alg}}$ to reduces one direction of (f) to (e). For the converse assume that $\operatorname{Ind}_H^G \rho$ is absolutely irreducible. Because Ind_H^G is an exact functor, ρ must be absolutely irreducible, and hence also ρ^g for all $g \in G$. Because $\operatorname{Ind}_H^G \rho$ is absolutely irreducible, and using Frobenius reciprocity, we have

$$E \cong \operatorname{End}_{E[G]}(\operatorname{Ind}_H^G \rho) \cong \operatorname{Hom}_{E[G]}(\rho, \operatorname{Res}_H^G \operatorname{Ind}_H^G \rho) \overset{(c)}{=} \operatorname{Hom}_{E[G]}(\rho, \oplus_{g\in G/H}\rho^g).$$

Hence ρ is isomorphic to ρ^g if and only if $g \in H$, and this completes the proof of (f).

We now prove Part (g). Note that by (b) the only if direction is clear. For the other direction, note first that by [CR81, Lem. 10.12] we have $\operatorname{Ind}_H^G \rho \cong \operatorname{Ind}_H^G \rho^g$ for all $g \in G$. Since induction and direct sum commute, we also have

$$\operatorname{Ind}_H^G \Big(\bigoplus_{g\in G/H} \rho^g \Big) = \bigoplus_{g\in G/H} \big(\operatorname{Ind}_H^G \rho^g\big) = \big(\operatorname{Ind}_H^G \rho\big)^{\oplus [G:H]}.$$

The same formula applies to ρ', and so our hypothesis gives $(\operatorname{Ind}_H^G \rho)^{\oplus[G:H]} \cong (\operatorname{Ind}_H^G \rho')^{\oplus[G:H]}$. The Krull-Schmidt theorem, see [CR62, Thm. 14.5], now yields $\operatorname{Ind}_H^G \rho \cong \operatorname{Ind}_H^G \rho'$. Part (h) follows from the uniqueness of composition factors and the irreducibility of the ρ^g. □

2.2 Some results when p does not divide $[G : H]$

Suppose now that $\chi\colon G \to E^\times$ is a character of finite order m, so that E contains a primitive m-th root of unity ζ and $m \cdot 1 \in E^\times$. We also set $H := \ker \chi$, and note $p \nmid m = [G : H]$. The following is a standard result of Clifford Theory, e.g. [CR62, Thm. 49.2, Cor. 50.6].

Theorem 2.2.1. *Let $\rho\colon G \to \mathrm{GL}_n(E)$ be an absolutely irreducible representation such that $\rho \cong \rho \otimes \chi$. Then the following hold:*

(a) *The order m of χ divides the degree n of ρ.*

(b) *There exists a Kummer extension $E' = E(\sqrt[m]{\lambda})$ of E for some $\lambda \in E^\times$ and an absolutely irreducible representation $\rho'\colon H \to \mathrm{GL}_{n/m}(E')$, such that*

$$\rho \otimes_E E' \cong \operatorname{Ind}_H^G \rho'.$$

(c) *The representations $(\rho')^g$, $g \in G/H$, are pairwise non-isomorphic and absolutely irreducible, and one has $\operatorname{Res}_H^G \rho \otimes_E E' \cong \bigoplus_{g\in G/H} (\rho')^g$.*

(d) *If E is local field, G is a topological group and ρ is continuous, then so is ρ'.*

(e) *If in addition to (d), G is compact, then ρ can be defined over the ring of integers $\mathcal{O}_E$ of E and ρ' can be defined over $\mathcal{O}_{E'}$.*

Proof. Lacking a precise reference, we give a proof. Let A be an invertible $n \times n$-matrix over E such that

$$A\rho(g)A^{-1} = \chi(g)\rho(g) \text{ for all } g \in G. \tag{2}$$

From (2) one deduces $A^m\rho(g)A^{-m} = \chi^m(g)\rho(g) = \rho(g)$ for all $g \in G$. Since ρ is absolutely irreducible, [CR62, (29.13)] implies that $A^m = \lambda \cdot \mathbb{1}_n$ for some $\lambda \in E$. Define $E' := E(\sqrt[m]{\lambda})$. Let $A' = \sqrt[m]{\lambda}^{-1} A$ in $\mathrm{GL}_n(E')$, so that (2) also holds for A' and also $(A')^m = \mathbb{1}_n$. Since $m\cdot 1$ is invertible in E, it follows, using the Jordan form, that A' is semisimple. Moreover A' is diagonalizable over E' since E contains a primitive m-th root of unity.

After a change of basis over E' we may write A as a block diagonal matrix with diagonal blocks $A_1, \ldots, A_m$ such that for $i = 1, \ldots, m$ A_i is a scalar matrix $\zeta^i \mathbb{1}_{n_i}$ with $0 \le n_i \le m$ and $\sum_{i=1,\ldots,m} n_i = n$. For all $g \in G$ and $i, j = 1, \ldots, m$ we decompose $\rho(g)$ correspondingly into blocks $\rho_{i,j}(g)$ so that equation (2) provides

$$\zeta^{i-j}\rho_{i,j}(g) = \chi(g)\rho_{i,j}(g). \tag{3}$$

Choose $g \in G$ such that $\chi(g) = \zeta$. Then $\rho_{i,j}(g)$ is zero unless $i - j \equiv 1 \pmod{m}$. Since $\rho(g)$ is invertible, all $\rho_{i,i+1}(g)$ and $\rho_{m,1}(g)$ must be invertible and hence square matrices and of non-zero size. We deduce that all n_i are equal, hence non-zero, and hence equal to n/m. In particular, m divides n, proving (a).

Next, for $h \in H$ and for all $i, j = 1, \ldots, m$, equation (3) becomes $\zeta^{i-j}\rho_{i,j}(h) = \rho_{i,j}(h)$ so that $\rho(h) = \bigoplus_{i=1}^m \rho_{i,i}(h)$ is a block diagonal matrix and each $\rho_{i,i}\colon H \to \mathrm{GL}_{n/m}(k)$, $h \mapsto \rho_{i,i}(h)$, is a representation of dimension n/m. In particular, the restriction satisfies

$$\mathrm{Res}_H^G \rho \otimes_E E' = \bigoplus_{i=1}^m \rho_{i,i}.$$

We choose $\rho' = \rho_{1,1}$ and consider $\mathrm{Ind}_H^G \rho'$. By [CR62, (10.8) Frobenius Reciprocity Theorem] we have

$$\mathrm{Hom}_G(\mathrm{Ind}_H^G \rho', \rho \otimes_E E') = \mathrm{Hom}_H(\rho', \mathrm{Res}_H^G \rho \otimes_E E') \neq 0.$$

Let $f\colon \mathrm{Ind}_H^G \rho' \to \rho \otimes_E E'$ be a nonzero G-homomorphism. Since ρ is irreducible, it must be surjective, and because $\dim \rho = n = m \cdot n/m = \dim \mathrm{Ind}_H^G \rho'$, its kernel must be zero, so that f is an isomorphism. Next note that Ind_H^G is an exact functor, see [CR81, § 10, Exerc. 20]. Hence ρ' is absolutely irreducible, because ρ is so. This completes the proof of (b).

Part (c) follows from Lemma 2.1.4(b) and (f). Part (d) easily follows from the continuity of $\mathrm{Res}_H^G \rho \otimes_E E' \cong \bigoplus_{g \in G/H} (\rho')^g$, using that all linear topologies on a finite dimensional vector space over E' that are compatible with the topology on E' are equivalent.

Concerning (e) we only prove the first assertion; the proof of the second the follows from (d). For this, let V be the E-vector space underlying ρ and let T be an $\mathcal{O}_E$-lattice in V. The stabilizer of T is an open subgroup of $\mathrm{GL}_n(E)$ and hence, by the continuity of ρ, the latice T is fixed by an open subgroup G' of G. Therefore G/G' is finite. Thus $T' := \bigcap_{g \in G/G'} gT$ is an $\mathcal{O}_E$-lattice in V, and this lattice is clearly G-stable. Choosing an $\mathcal{O}_E$-basis of T', that is then also an E-basis of V, assertion (e) for ρ is clear. □

Corollary 2.2.2. *Suppose that $\rho\colon G \to \mathrm{GL}_n(E)$ is a representation that is absolutely semisimple; this holds for instance if E is perfect. Then $\rho \cong \rho \otimes \chi$ holds if and only if there is a separable extension E' of E of degree less than $m^n \cdot (2n)!$ and a representation $\rho'\colon H \to \mathrm{GL}_{n/m}(E')$ such that*

$\rho\otimes_E E' \cong \operatorname{Ind}_H^G \rho'$. Furthermore, any such ρ' is absolutely semisimple, and one has $\operatorname{Res}_H^G \rho\otimes_E E' = \bigoplus_{g\in G/H}(\rho')^g$.

Proof. If $\rho \otimes_E E' \cong \operatorname{Ind}_H^G \rho'$, then Lemma 2.1.3 implies

$$(\rho \otimes_E E') \otimes \chi \cong (\operatorname{Ind}_H^G \rho') \otimes \chi \cong \operatorname{Ind}_H^G(\rho' \otimes \operatorname{Res}_H^G \chi) \cong \operatorname{Ind}_H^G \rho' \cong \rho \otimes_E E',$$

and this implies $\rho \otimes \chi \cong \rho$ by [CR62, 29.7].

Conversely, suppose that $\rho \cong \rho \otimes \chi$. After replacing E by a separable extension of degree at most $(n^2)!$, see Lemma A.2.7 and Remark A.2.8, we may assume that ρ is an absolutely completely reducible G-representation over E', i.e., $\rho = \oplus_{j\in J}\rho'_j$ for absolutely irreducible representations ρ'_j for $j \in J$. We regroup this decomposition according to orbits under iterated twisting by χ. This gives rise to a decomposition

$$\rho \cong \bigoplus_{i\in I} \Big(\bigoplus_{j=0}^{m_i-1} \rho_i \otimes \chi^j \Big)^{\oplus r_i}, \tag{4}$$

for integers $r_i > 0$, absolutely irreducible representations $\rho_i\colon G \to \mathrm{GL}_{n_i}(E')$, and divisors m_i of m, for $i \in I$, so that $\rho_i \otimes \chi^{m_i} \cong \rho_i$, and no ρ_i is isomorphic to $\rho_{i'} \otimes \chi^j$ for some $j \in \{0, \ldots, m_{i'} - 1\}$ and $i' \in I$. We have $G \supset H_i := \ker \chi^{m_i} \supset H$, $[H_i : H] = m_i$, and $\operatorname{Res}_{H_i}^G \chi$ is a character of order m_i.

By Theorem 2.2.1 we find Kummer extensions E'_i of E' of degree dividing m_i and representations $\rho''_i\colon H_i \to \mathrm{GL}_{n_i/m_i}(E'_i)$ such that $\operatorname{Ind}_{H_i}^G \rho''_i \cong \rho_i \otimes'_E E'_i$. Let $\mathbb{1}_H$ be the trivial representation of H on E'. Then

$$\begin{aligned}
\Big(\bigoplus_{j=0}^{m_i-1} \rho_i \otimes \chi^j \Big) \otimes'_E E'_i &\cong \operatorname{Ind}_{H_i}^G \rho''_i \otimes \Big(\bigoplus_{j=0}^{m_i-1} \chi^j \Big) \cong \operatorname{Ind}_{H_i}^G \Big(\rho''_i \otimes \bigoplus_{j=0}^{m_i-1} \operatorname{Res}_{H_i}^G \chi^j\Big) \\
&\cong \operatorname{Ind}_{H_i}^G \big(\rho''_i \otimes \operatorname{Ind}_H^{H_i} \mathbb{1}_H\big) \cong \operatorname{Ind}_{H_i}^G \operatorname{Ind}_H^{H_i} \big(\operatorname{Res}_H^{H_i} \rho''_i \otimes \mathbb{1}_H\big) \\
&\cong \operatorname{Ind}_H^G \big(\operatorname{Res}_H^{H_i} \rho''_i\big),
\end{aligned}$$

where the second and fourth isomorphism follows from Lemma 2.1.3. Let E'' be the composite of the E'_i and set $\rho' := \bigoplus_{i\in I}(\operatorname{Res}_H^{H_i} \rho''_i \otimes_{E'_i} E'')^{\oplus r_i}$, so that clearly $[E'' : E'] < m^n$. The first assertion of the corollary is now evident from the above and from (4). The remaining assertions follow from Lemma 2.1.4(b) and (d). □

2.3 Some results when p divides $[G : H]$

Suppose for the remainder of this subsection that $p = \operatorname{Char} E > 0$. Let $V = E^n$ and let $\rho\colon G \to \operatorname{Aut}_E(V)$ be a representation such that the canonical map $E \to \operatorname{End}_G(V)$ is an isomorphism. Except for Lemma 2.3.1, we shall also assume that E is a topological field, that ρ is continuous and that G is topologically finitely generated, and we let $\Phi(G) = \overline{G^p[G, G]}$ be the Frattini subgroup of G and we let m denote the $\mathbb{F}_p$-dimension of $G/\Phi(G)$.

Lemma 2.3.1. *Suppose that ρ is absolutely irreducible. Let $H \subset G$ be a normal subgroup of index p and set $V_H := \operatorname{Res}_H^G \rho \otimes_E E^{\mathrm{alg}}$. Then the following hold:*

- *(i) If V_H is reducible, then $V \otimes_E E^{\mathrm{alg}} \cong \operatorname{Ind}_H^G N$ for any irreducible submodule $N \subset V_H$.*
- *(ii) If V_H is irreducible, then*
  - *(1) If an $E[G]$-module N satisfies $\operatorname{Res}_H^G N \cong \operatorname{Res}_H^G V$, then $N \cong V$.*
  - *(2) $\operatorname{Ind}_H^G V_H$ is indecomposable, its socle is isomorphic to $V \otimes_E E^{\mathrm{alg}}$,*
  - *(3) $V \otimes_E E^{\mathrm{alg}}$ is not induced from any H-module,*

(4) All irreducible subquotients of $\mathrm{Ind}_H^G V_H$ *are isomorphic to* $V \otimes_E E^{\mathrm{alg}}$.

Proof. By Lemma 2.1.4(c) we have $V_H = \oplus_{g \in G/H^*} N^g$ for some irreducible H-module N over E^{alg} and some subgroup $H^* \subset G$ with $H \subset H^*$. Since $G/H \cong \mathbb{Z}/p\mathbb{Z}$, in part (i) of the present lemma, we must have $H^* = H$, and then the assertion follows from Lemma 2.1.4(e).

We now prove (ii). Let N be as in (1). By choosing the same underlying E vector space, we assume that $\mathrm{Res}_H^G N = \mathrm{Res}_H^G V$. Let $g \in G$ be a generator of G/H, and let $A, B \in \mathrm{Aut}_E(\mathrm{Res}_H^G V)$ be the automorphisms given by the action of g. Because $\mathrm{Res}_H^G V$ is absolutely irreducible, there exists a non-zero scalar $\lambda \in E$ such that $B = \lambda A$. As $g^p \in H$, we find $A^p = B^p = \lambda^p A^p$. Because $\mathrm{Char}\, E = p > 0$, we must have $\lambda = 1$, and so (1) is proved.

For (2), write $\mathrm{Ind}_H^G V_H = \bigoplus_{i \in I} N_i$ with indecomposable G-modules N_i. Let N_i' be an irreducible quotient of N_i as a G-submodule. From Lemma 2.1.4(b) and (c) and the irreducibility of V_H, we deduce $\mathrm{Res}_H^G N_i' \cong V_H$ for all i, and by (1) we find $N_i' \cong V \otimes_E E^{\mathrm{alg}}$. The following inequality implies $\#I = 1$ and the uniqueness of N_1' and thus gives (2):

$$\begin{aligned} \#I &\leq \dim_{E^{\mathrm{alg}}} \mathrm{Hom}_G(\bigoplus_i N_i, V \otimes_E E^{\mathrm{alg}}) = \dim_{E^{\mathrm{alg}}} \mathrm{Hom}_G(\mathrm{Ind}_H^G V_H, V \otimes_E E^{\mathrm{alg}}) \\ &= \dim_{E^{\mathrm{alg}}} \mathrm{Hom}_H(V_H, V_H) = 1, \end{aligned}$$

To see (3) observe that if $V \otimes_E E^{\mathrm{alg}}$ was induced, then by Lemma 2.1.4(b) then V_H had to be reducible. For (4) not that we have $\mathrm{Ind}_H^G V_H \cong (V \otimes_E E^{\mathrm{alg}}) \otimes_E \mathrm{Ind}_H^G E$. Now clearly the semisimplification of $\mathrm{Ind}_H^G E$ is the trivial module E^p, and this shows (4). ☐

In the sequel we shall write $\overline{\mathrm{End}}(V)$ for the cokernel of the natural inclusion $E \hookrightarrow \mathrm{End}_E(V)$. We shall relate the non-vanishing of the G-invariants $\overline{\mathrm{End}}_G(V)$ of this cokernel to V being induced from a subgroup of G of p-power index. We **assume** that p divides n, since otherwise the trace splits the inclusion $E \to \mathrm{End}_E(V)$ G-equivariantly, so that $\overline{\mathrm{End}}_G(V) = 0$ by our hypothesis $E = \mathrm{End}_G(V)$.

Let $\mathrm{End}_G'(V)$ be the subset of $A \in \mathrm{End}_G(V)$ such that there exists a map $\lambda_A \colon G \to E, g \mapsto \lambda_A(g)$ with

$$\forall g \in G_K : \rho(g) A \rho(g)^{-1} = A + \lambda_A(g) 1_n. \tag{5}$$

Again because $E = \mathrm{End}_G(V)$, one has the short exact sequence

$$0 \longrightarrow E = \mathrm{End}_G(V) \longrightarrow \mathrm{End}_G'(V) \longrightarrow \overline{\mathrm{End}}_G(V) \longrightarrow 0.$$

We write $\overline{A} \in \overline{\mathrm{End}}_G(V)$ for the class of $A \in \mathrm{End}_G'(V)$ under this map. Recall that $f \in E[T]$ is p-linear if $f = \sum_i a_i T^{p^i}$ and that the set $E[T]^{\text{p-lin}}$ of p-linear polynomials in $E[T]$ is a ring under addition and composition. We provide some basic properties of $\mathrm{End}_G'(V)$.

Lemma 2.3.2. *Let A be in $\mathrm{End}_G'(V)$. For $\lambda \in E^{\mathrm{alg}}$, let V_λ and V_λ' denote the eigenspace and generalized eigenspace of A for λ. Suppose from (g) on that ρ is absolutely irreducible.*

(a) The map $\lambda_\bullet$ factors via an injective homomorphism

$$\overline{\mathrm{End}}_G(V) \to \mathrm{Hom}_{\mathrm{cont}}(G, (E, +)), \overline{A} \mapsto \lambda_{\overline{A}}.$$

(b) The groups $H_A := \mathrm{Ker}\, \lambda_A$ and $H_\rho := \bigcap \{H_A \mid A \in \mathrm{End}_G'(V)\}$ contain $\Phi(G)$.

(c) The multi-set of eigenvalues of A with multiplicities is a torsor under $\Lambda_A := \lambda_A(G)$.

(d) $\Lambda_A \subset (E, +)$ is a finite-dimensional $\mathbb{F}_p$ vector space.

(e) $\mathrm{End}_G'(V)$ is a module for $E[T]^{\mathrm{lin}}$ under $(f, A) \mapsto f(A)$ and one has $\lambda_{f(A)} = f \circ \lambda_A$.

(f) If $\overline{\mathrm{End}}_G(V) \neq 0$, then there exist $A \in \mathrm{End}_G'(V)$ such that $\Lambda_A \cong (\mathbb{F}_p, +)$.

(g) The restriction $\rho|_{H_A}$ commutes with A; it preserves V_λ and V_λ' for all $\lambda \in E^{\mathrm{alg}}$.

(h) A is semisimple over E^{alg}.

(i) One has $\rho\otimes_E E^{\mathrm{alg}}\cong \operatorname{Ind}_{H_A}^{G_K} V_\lambda$ for any eigenvalue $\lambda\in E^{\mathrm{alg}}$ of A.

(j) The elements in $\operatorname{End}'_{G_K}(V)$ are simultaneously diagonalizable over E^{alg}.

(k) Suppose $\dim_{\mathbb{F}_p} E\ge m$, then there exists $A\in\operatorname{End}'_G(V)$ with $H_\rho=H_A$.

Proof. (a) The map $\lambda_\bullet$ is simply the boundary map of cohomology $H^0(G,\overline{\operatorname{End}}_E(V))\to H^1(G,E)$. Let us give a direct proof. To see that λ_A is a homomorphism, consider

$$A+\lambda_A(gh)1_n = ghAh^{-1}g^{-1} = g(A+\lambda_A(h)1_n)g^{-1} = gAg^{-1}+\lambda_A(h)1_n = A+(\lambda_A(g)+\lambda_A(h))1_n.$$

Next note that λ_A is trivial if and only if $A\in\operatorname{End}_G(V)$. Hence $\overline{A}\to\lambda_{\overline{A}}$ is defined and injective. The homomorphism property is straightforward. Note also that (b) is a direct consequence of (a).

For (c) denote by $\chi_A(T)\in E[T]$ the characteristic polynomial of A. Then (5) implies $\chi_A(T)=\chi(A+\lambda_A(g))$ for all $g\in G_K$, and this proves (c). Part (d) follows from (a) and (c).

(e) Raising (5) to the power p and using $\operatorname{Char}(E)=p$ we find

$$\forall g\in G_K:\rho(g)A^p\rho(g)^{-1}=A^p+\lambda_A(g)^p 1_n.$$

Since $\operatorname{End}'_G(V)$ is clearly an E-vector space and $\lambda_\bullet$ is E-linear, part (e) follows. To see (f), let A be in $\operatorname{End}'_G(V)\smallsetminus E$, so that $\Lambda_A\subset(E,+)$ is non-trivial and finite. Let $\Lambda\subset\Lambda_A$ be a sub $\mathbb{F}_p$-vector space of codimension 1 and let f be the p-linear polynomial $\prod_{\lambda\in\Lambda}(T-\lambda)$. Then $\Lambda_{f(A)}$ has order p by (e).

The asserted commutativity in (g) is clear from (5); the assertion on the V_λ and V'_λ is then immediate. For (h), choose an eigenvalue $\lambda\in E^{\mathrm{alg}}$ for which $\dim V_\lambda$ is minimal. By (c), we have $\dim V_\lambda\cdot\#\Lambda_A\le n$ with equality if and only if A is semisimple. Because A and $\rho|_{H_A}\otimes_E E^{\mathrm{alg}}$ commute, the action of H_A preserves V_λ. Let $V_{\overline{E}}:=V\otimes_E E^{\mathrm{alg}}$. Frobenius reciprocity gives a non-zero homomorphism in

$$\operatorname{Hom}_G(\operatorname{Ind}_{H_A}^G V_\lambda, V_{\overline{E}})\cong\operatorname{Hom}_{H_A}(V_\lambda, V_{\overline{E}}|_{H_A}).$$

Since $V_{\overline{E}}$ is an irreducible G_K-representation, it follows that

$$n\le\dim\operatorname{Ind}_{H_A}^G V_\lambda=[G:H_A]\cdot\dim V_\lambda=\#\Lambda_A\cdot\dim V_\lambda\le n.$$

Hence we must have equality and so A is semisimple. All but the last assertion of (i) follows from the proof just given.

(j) Using (5) we compute for $A,B\in\operatorname{End}'_{G_K}(E^n)$ and all $g\in G_K$ that

$$g(AB-BA)g^{-1}=(A+\lambda_A(g)1_n)(B+\lambda_B(g)1_n)-(B+\lambda_B(g)1_n)(A+\lambda_A(g)1_n)=(AB-BA).$$

Since $E=\operatorname{End}_G(V)$, we conclude that $AB-BA$ is a scalar matrix. We also know that A (and B) is semisimple. To conclude we may work over $\overline{E}$, so that we may assume that A has diagonal form. But then it is elementary to see that $AB-BA$ has entries 0 along the diagonal and hence this scalar matrix must be zero. It follows that all elements of $\operatorname{End}'_G(V)$ commute, and we conclude using (i).

(k) We need to show that for all $A,B\in\operatorname{End}'_G(V)$ there exist $\mu,\nu\in E\smallsetminus\{0\}$ such that $H_{\mu A+\nu B}=H_A\cap H_B$. Let W be the $\mathbb{F}_p$-vector space $G/(H_A\cap H_B)$ and regard λ_A and λ_B as $\mathbb{F}_p$-linear maps $W\to E$. Note that $d:=\dim_{\mathbb{F}_p}W\le m$. Let $\underline{B}:=(b_1,\ldots,b_d)$ be an $\mathbb{F}_p$ basis of W. Suppose also without loss of generality that $\dim_{\mathbb{F}_p}G_K/H_A,\dim_{\mathbb{F}_p}G_K/H_B<d$, since otherwise we are done.

For $\nu\in E^{\mathrm{alg}}$ set $C_\nu:=\lambda_A+\nu\lambda_B$. Since the common kernel of λ_A and λ_B is $0\subset W$, there exists $\nu\in E^{\mathrm{alg}}$ such that C_ν is injective, i.e., such that the vectors $(C_\nu b_i)_{i=1,\ldots,d}$ are $\mathbb{F}_p$-linearly independent in E. This means that the Moore determinant of these vectors is non-zero. In other words, the determinant of the $d \times d$-square matrix with (i,j)-entry given by $(\lambda_A(b_i)+\nu\lambda_B(b_i))^{p^{j-1}}$ is non-zero. As a function of ν, this is a polynomial of degree at most $(p^d-1)/(p-1)$. It is not identically zero because of its value at ν, and hence it can have at most $p^d - 1 < p^m - 1$ zeros. It follows that for some $\nu' \in E$ it is non-zero because $\#E \geq p^m$, and this completes the proof of (i). ☐

We will later also need the following particular result:

Corollary 2.3.3. *Suppose ρ is a non-trivial extension of an absolutely irreducible representation ρ_2 by an absolutely irreducible representation ρ_1. Suppose further that ρ_1 and ρ_2 are not isomorphic and that the ρ_i are not induced from any normal index p subgroup of G. Then $\overline{\mathrm{End}}_G(V) = 0$.*

Proof. Assume on the contrary that we can find $A \in \mathrm{End}'_G(V) \smallsetminus E$. We may assume that Λ_A has order p by Lemma 2.3.2(f), and we also may assume $E = E^{\mathrm{alg}}$, since this leaves $\dim_E \overline{\mathrm{End}}_G(V)$ unchanged. Then H_A has index p in G and we have p distinct subspaces V_λ of $V_{E^{\mathrm{alg}}}$ that are stabilized by H_A. By hypotheses and Lemma 2.3.1, the restrictions $\rho_i|_{H_A}$ are absolutely irreducible. This already implies $p = 2$. We also find that the extension of ρ_2 by ρ_1 becomes trivial when restricted to H_A, and that these restrictions must agree with the two distinct V_λ. It follows by Frobenius reciprocity that we have a non-zero map $\mathrm{Ind}_{H_A}^G(\rho_2|_{H_A}) \to \rho$ for $i = 1, 2$. By Lemma 2.3.1(ii)(4) all simple subquotients of $\mathrm{Ind}_{H_A}^G(\rho_2|_{H_A})$ are isomorphic to ρ_2. But this is absurd, since ρ is non-split and ρ_2 is not a submodule of ρ. We reach a contradiction even for $p = 2$. ☐

3 Deformations of Galois representation

This section recalls and augments the classical deformation theory [Maz89] of Mazur. Throughout we fix a profinite group G which often is G_K for K a p-adic field.

In Subsection 3.1 we fix the basic categories relevant for all deformations functors that we shall study in this work. We also recall some results on formal smoothness. Subsection 3.2 recalls Mazur's deformation theory and some extensions for residual representations $G \to \mathrm{GL}_n(\kappa)$ where κ is a finite or a local field. Subsection 3.3 studies dimension 1 points on universal deformation rings. Except perhaps for some results on equicharacteristic dimension 1 points, all results are standard. Subsection 3.4 gives a criterion for the determinant functor to be smooth. For this we recall Tate local duality for coefficient modules over local fields.

3.1 Basic categories and functors, and formal smoothness

The field κ and the ring Λ: From now on κ is either (a) a finite field of characteristic p that carries the discrete topology, or (b) κ is a local field with its natural topology and with residue characteristic p. In the latter case κ is a finite extension of $\mathbb{Q}_p$ or a finite extension of the formal Laurent series field $\mathbb{F}_p((t))$. Depending on case (a) or (b), we define a topological ring Λ. In case (b) we set $\Lambda = \kappa$. In case (a), Λ is a Noetherian complete local ring with residue field κ and equipped with the topology defined by its maximal ideal $\mathfrak{m}_\Lambda$.

The categories $\mathcal{A}r_\Lambda$ and $\widehat{\mathcal{A}r}_\Lambda$: By $\mathcal{A}r_\Lambda$ we denote the category of Artinian local Λ-algebras A with residue field isomorphic to κ and with local Λ-algebra homomorphisms as morphisms. For any local ring A we denote by $\mathfrak{m}_A$ its maximal ideal. We regard any object A of $\mathcal{A}r_\Lambda$ as a topological ring. In case (a), we give A the discrete topology. In case (b) the ring A is a κ-algebra of finite κ-dimension and we give A the unique topology that arises from any structure on A as a normed κ-vector space. This topology on A is relevant whenever we talk about continuous maps to A or to any $\mathrm{GL}_n(A)$. We further define $\widehat{\mathcal{A}r}_\Lambda$ as the category of complete Noetherian local Λ-algebras with residue field κ and with local homomorphisms as morphisms. Any object of $\widehat{\mathcal{A}r}_\Lambda$ is a limit of objects of $\mathcal{A}r_\Lambda$. We equip an object A of $\widehat{\mathcal{A}r}_\Lambda$ with the weakest topology such that all maps

$A \to A/\mathfrak{m}_A^m$, $m \geq 1$, are continuous. In case (a) this simply means that A carries the $\mathfrak{m}_A$-adic topology.

In $\mathcal{A}r_\Lambda$ the co-product of two objects A, A' is their tensor product $A \otimes_\Lambda A'$. For $A, A' \in \widehat{\mathcal{A}r}_\Lambda$, the co-product is the completed tensor product $A \widehat{\otimes}_\Lambda A' := \varprojlim_n A/\mathfrak{m}_A^n \otimes_\Lambda A'/\mathfrak{m}_{A'}^n$. Note that by [Gro64, Lem. 0_{IV}.(19.7.1.2)] the ring $A \widehat{\otimes}_\Lambda A'$ lies again in $\widehat{\mathcal{A}r}_\Lambda$. From the discussion around the Cohen structure theorem in [Sta18, § 0323] one also deduces:

Proposition 3.1.1. *Let $A \in \widehat{\mathcal{A}r}_\Lambda$ and $h := \dim_\kappa \mathfrak{m}_A/(\mathfrak{m}_\Lambda, \mathfrak{m}_A^2)$. Then there exists a surjective continuous homomorphism in $\widehat{\mathcal{A}r}_\Lambda$ from the power series ring $\Lambda[[x_1, \ldots, x_h]]$ onto A. Moreover h is minimal with this property.*

Further properties of $\mathcal{A}r_\Lambda$ and $\widehat{\mathcal{A}r}_\Lambda$ can be found in [Sta18, § 06GB] and [Sta18, § 06GV].

Functors on $\mathcal{A}r_\Lambda$ and $\widehat{\mathcal{A}r}_\Lambda$: We follow [Sch68]; see also [Sta18, Ch. 06G7]. [By $\kappa[\varepsilon] := \kappa[X]/(X^2) \in \mathcal{A}r_\Lambda$ we denote the *ring of dual numbers over κ*. Recall from [Sch68] that a *small extension in $\mathcal{A}r_\Lambda$* is a surjection $f\colon B \to A$ in $\mathcal{A}r_\Lambda$ whose kernel $\ker f$ is isomorphic to κ as a B-module, and in particular $\ker f$ is annihilated by $\mathfrak{m}_B$ and $(\ker f)^2 = 0$.

In the following we consider covariant functors F from $\mathcal{A}r_\Lambda$ or $\widehat{\mathcal{A}r}_\Lambda$ to *Sets* such that $F(\kappa)$ is a singleton.

Definition 3.1.2. *A covariant functor $F\colon \widehat{\mathcal{A}r}_\Lambda \to Sets$ is called* continuous *if the canonical map $F(A) \to \varprojlim_n F(A/\mathfrak{m}_A^i)$ is bijective for all $A \in \widehat{\mathcal{A}r}_\Lambda$.*

It is straightforward to see that there is a bijection between continuous functors $\widehat{\mathcal{A}r}_\Lambda \to Sets$ and functors $\mathcal{A}r_\Lambda \to Sets$ given by restriction. So from now on all functors on $\widehat{\mathcal{A}r}_\Lambda$ will be continuous and we use the same symbol to denote them and their restriction to $\mathcal{A}r_\Lambda$. A functor $F\colon \widehat{\mathcal{A}r}_\Lambda \to Sets$ is representable if it is isomorphic to h_B for some $B \in \widehat{\mathcal{A}r}_\Lambda$.

Definition 3.1.3 ([Sch68, Def. 2.2– 2.7])**.** *Let $F, F'\colon \widehat{\mathcal{A}r}_\Lambda \to Sets$ be functors.*

(a) The tangent space of F *is $t_F := F(\kappa[\varepsilon])$.*

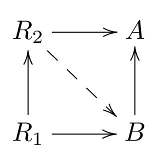
(b) A natural transformation $F' \to F$ is called smooth *if for all small extensions $B \to A$ in $\mathcal{A}r_\Lambda$, the map $F'(B) \to F'(A) \times_{F(A)} F(B)$ is surjective; cf. [Sch68, Def. 2.2].*

(c) A pair (A, ξ) consisting of an object A in $\widehat{\mathcal{A}r}_\Lambda$ and a smooth natural transformation $\xi\colon h_A \to F$ is called a hull of F *if the induced map $t_{h_B} \to t_F$ on tangent spaces is bijective; note that by Yoneda ξ corresponds to some element of $F(A)$.*

Hulls are unique up to isomorphism but in general not up to unique isomorphism. If F is representable by some $A \in \widehat{\mathcal{A}r}_\Lambda$ it clearly has a hull. Moreover one has $t_F \cong \mathfrak{m}_A/(\mathfrak{m}_A^2, \mathfrak{m}_\Lambda)$.

Formal smoothness: Recall the definition of formal smoothness:

Definition 3.1.4. *(i) A homomorphism $R_1 \to R_2$ of topological rings with R_1 and R_2 linearly topologized is called* formally smooth *if for every commutative solid diagram*



of homomorphisms of topological rings with B a discrete ring and $B \to A$ surjective with square zero kernel, a dotted arrow exists which makes the diagram commute, cf. [Sta18, Def. 07EB]

(ii) *We call a morphism* $\varphi\colon Y \to X$ *of locally Noetherian schemes* formally smooth *at* $y \in Y$, *if the induced morphism* $\widehat{\mathcal{O}}_{X,\varphi(x)} \to \widehat{\mathcal{O}}_{Y,y}$ *of topological rings is formally smooth.*

Formal smoothness is related to smoothness of natural transformations between representable functors:

Proposition 3.1.5 ([Sch68, Prop. 2.5(i)])**.** *Let* $R_1 \to R_2$ *be a morphism in* $\widehat{\mathcal{A}r}_\Lambda$ *and set* $h = \dim R_2 - \dim R_1$. *Then the following assertions are equivalent:*

(a) $R_1 \to R_2$ *is formally smooth.*

(b) *The induced map of functors* $h_{R_2} \to h_{R_2}$ *is smooth.*

(c) *There is an isomorphism* $R_1[[X_1, \ldots, x_h]] \to R_2$ *of* R_1*-algebras.*

If any of (a)–(c) holds, then h *is called the* relative dimension of R_2 over R_1.

Note that (b)⇒(c) is from [Sch68] and that (c)⇒(a)⇒(b) are straightforward. A main conclusion from Proposition 3.1.5 is that for morphisms in $\widehat{\mathcal{A}r}_\Lambda$ one can verify formal smoothness on small extensions in $\mathcal{A}r_\Lambda$.

3.2 Mazur's deformation theory and extensions

Our presentation of deformation functors follows Mazur [Maz89] and Kisin [Kis09]. Consider a continuous representation

$$\overline{\rho}\colon G \to \mathrm{GL}_n(\kappa). \tag{6}$$

We write $\mathrm{ad}_{\bar\rho}$ for $\mathrm{Mat}_{n\times n}(\kappa)$ together with the action of G induced by $\bar\rho$ composed with the conjugation action of $\mathrm{GL}_n(\kappa)$ on $\mathrm{Mat}_{n\times n}(\kappa)$. By $\mathrm{ad}^0_{\bar\rho}$ we denote the subrepresentation on trace zero matrices and by $\overline{\mathrm{ad}}_{\bar\rho}$ the quotient modulo the center κ. In the following, for a representation ρ into $\mathrm{GL}_n(A_1)$ and a ring homomorphism $A_1 \to A_2$ we write $\rho \otimes_{A_1} A_2$ for the composition of ρ with $\mathrm{GL}_n(A_1) \to \mathrm{GL}_n(A_2)$, cf. [Kis03, p. 433].

Definition 3.2.1 ([Gou01, Defs. 2.1 and 2.2])**.** *Let* A *be in* $\mathcal{A}r_\Lambda$ *with residue map* $A \to \kappa$.

(i) *A* lifting of $\overline{\rho}$ to A *is a continuous homomorphism* $\rho\colon G \to \mathrm{GL}_n(A)$ *with* $\rho \otimes_A \kappa = \overline{\rho}$.

(ii) *Let* $\Gamma_n(A)$ *denote the kernel of the canonical homomorphism* $\mathrm{GL}_n(A) \to \mathrm{GL}_n(\kappa)$.

(iii) *A* deformation of $\overline{\rho}$ to A *is a* $\Gamma_n(A)$*-conjugacy class of a liftings of* $\overline{\rho}$ *to* A.

(iv) *The* deformation functor $\mathcal{D}_{\bar\rho}$, or $\mathcal{D}_{\Lambda,\bar\rho}$ if we wish to indicate Λ, of $\overline{\rho}$ *is defined as*

$$\mathcal{D}_{\overline{\rho}}\colon \mathcal{A}r_\Lambda \longrightarrow Sets, \quad A \longmapsto \{\rho\colon G \longrightarrow \mathrm{GL}_n(A) \,:\, \rho \text{ is a deformation of } \overline{\rho}\},$$

To see when $\mathcal{D}_{\bar\rho}$ has a hull, we introduce some finiteness conditions, following [Maz89].

Definition 3.2.2. (a) *A profinite group* G *has property* Φ_p *if* $H^1(H, \mathbb{F}_p)$ *is finite for all open subgroups* H *of* G*; see [Maz89, § 1.1].*

(b) *The representation* $\overline{\rho}$ *satisfies condition* $\Phi_{\overline{\rho}}$ *if* $\dim_\kappa H^1(G, \mathrm{ad}_{\overline{\rho}})$ *is finite.*

The following well-known result is immediate from local class field theory:

Proposition 3.2.3. *The following hold:*

(i) *The profinite group* G_K *satisfies Mazur's condition* Φ_p.

(ii) *If* G *satisfies* Φ_p, *then any* $\overline{\rho}$ *satisfies* $\Phi_{\overline{\rho}}$.

Proof. Part (i) is immediate from class field theory. If κ is finite, then part (ii) is well-known; to deduce it one applies the inflation restriction sequence to $G \supset H := \operatorname{Ker} \operatorname{ad}_{\overline{\rho}}$. If κ is a local field, the assertion is proved later in Corollary 3.3.2. We invite the reader to check that there is no circular reasoning involved. □

The versal hull of $\mathcal{D}_{\bar{\rho}}$: Let $\bar{\rho}$ be as in (6). The following result is due to Mazur, with a slight extension due to Ramakrishna, if κ is finite. For κ a p-adic field, a proof was first given in [Kis03, Lem. 9.3] by Kisin. The proof for local fields of positive characteristic is analogous.

Theorem 3.2.4 ([Maz89, 1.1–1.6], [Gou01, Thm. 3.3, p. 53, Thm. 4.2])**.** *Assuming condition $\Phi_{\overline{\rho}}$, the following hold:*

(a) One has $t_{\mathcal{D}_{\bar{\rho}}} \cong H^1(G, \operatorname{ad}_{\bar{\rho}})$, and $h := \dim_\kappa H^1(G, \operatorname{ad}_{\bar{\rho}})$ is finite.

(b) The functor $\mathcal{D}_{\overline{\rho}}$ has a hull; we write $\rho^{\mathrm{ver}}_{\overline{\rho}}\colon G \longrightarrow \mathrm{GL}_n(R^{\mathrm{ver}}_{\Lambda,\overline{\rho}})$ for a representative of its versal deformation and $R^{\mathrm{ver}}_{\Lambda,\bar{\rho}} \in \widehat{\mathcal{A}r}_\Lambda$ for the versal deformation ring *of $\bar{\rho}$.*

(c) If $\operatorname{Cent}(\overline{\rho}) = \kappa$, then $\mathcal{D}_{\overline{\rho}}$ is representable; we write $\rho^{\mathrm{univ}}_{\overline{\rho}}\colon G \longrightarrow \mathrm{GL}_n(R^{\mathrm{univ}}_{\Lambda,\overline{\rho}})$ for a representative of its universal deformation and $R^{\mathrm{univ}}_{\Lambda,\overline{\rho}} \in \widehat{\mathcal{A}r}_\Lambda$ for the universal deformation ring *of $\bar{\rho}$.*

(d) If $H^2(G, \operatorname{ad}_{\bar{\rho}}) = 0$, then $\mathcal{D}_{\bar{\rho}} \to h_\Lambda, [\rho\colon G \to \mathrm{GL}_n(A)] \mapsto A$ is smooth; i.e., the map $\Lambda \to R^{\mathrm{ver}}_{\Lambda,\overline{\rho}}$ is formally smooth of relative dimension h by Proposition 3.1.5.

(e) More generally, there is always a surjection $\varphi\colon \Lambda[[x_1, \ldots, x_h]] \to R^{\mathrm{ver}}_{\Lambda,\overline{\rho}}$ such that $\operatorname{Ker}\varphi$ is generated by at most $\dim_\kappa H^2(G, \operatorname{ad}_{\bar{\rho}})$ elements.

Remark 3.2.5. The existence of $R^{\mathrm{ver}}_{\Lambda,\overline{\rho}}$ (and of $\rho^{\mathrm{ver}}_{\overline{\rho}}$) does not require condition $\Phi_{\overline{\rho}}$. However then this ring is only a profinite topological ring in general.

We shall later need to understand the change of $R^{\mathrm{ver}}_{\Lambda,\overline{\rho}}$ under maps $\Lambda \to \Lambda'$:

Lemma 3.2.6 (Cf. [Wil95, p. 457])**.** *Let $\Lambda \to \Lambda'$ be a finite injective homomorphism of complete Noetherian local rings with finite residue fields κ and κ', respectively. Let $\overline{\rho}' := \overline{\rho} \otimes_\kappa \kappa'$. Let R_Λ be a hull for $\mathcal{D}_{\Lambda,\overline{\rho}}$ and $R_{\Lambda'}$ for $\mathcal{D}_{\Lambda',\rho'}$. Then $R_{\Lambda'} \cong R_\Lambda \otimes_\Lambda \Lambda'$.*

3.3 Deformation rings at dimension 1 points

It has been first exploited by Kisin, e.g. [Kis03], that points of dimension 1 on universal deformation rings can be much easier to understand than the closed point $\operatorname{Spec} \mathbb{F}$. We recall this method and work it out further for dimension 1 points that are local fields of positive characteristic. These points will be an essential tool in our work.

Let κ be a finite field of characteristic p with the discrete topology. Let Λ be the ring of integers of a finite totally ramified extension L of $W(\kappa)[1/p]$. Let $\bar{\rho}\colon G \to \mathrm{GL}_n(\kappa)$ be a continuous representation that satisfies $\Phi_{\overline{\rho}}$ with versal deformation

$$\rho^{\mathrm{ver}}_{\overline{\rho}}\colon G \longrightarrow \mathrm{GL}_n(R^{\mathrm{ver}}_{\Lambda,\overline{\rho}}).$$

Consider a continuous homomorphism $f\colon R^{\mathrm{ver}}_{\Lambda,\overline{\rho}} \to E$ for a local field E, and suppose that the kernel of f is a prime ideal $\mathfrak{p}$ so that E is a finite extension of the fraction field of $R^{\mathrm{ver}}_{\Lambda,\overline{\rho}}/\mathfrak{p}$ under f. Let $\rho_E\colon G \to \mathrm{GL}_n(E)$ be the representation induced from $\rho^{\mathrm{ver}}_{\overline{\rho}}$ via f. Up to strict equivalence, we may and will assume $\rho_E(G) \subset \mathrm{GL}_n(\mathcal{O})$ for $\mathcal{O}$ the ring of integers of E.

Suppose first that E is of characteristic 0, in which case we follow [Kis03, § 9]. Then f factors via a map $f[1/p]\colon R^{\mathrm{ver}}_{\Lambda,\overline{\rho}}[1/p] \to E$ which is an L-algebra homomorphism, and E is a finite extension field of L. We denote by $\widehat{R}$ the completion of $R^{\mathrm{ver}}_{\Lambda,\overline{\rho}}[1/p]$ at the kernel of $f[1/p]$. Then E is the residue field of $\widehat{R}$. From the finiteness of $L \to E$ one easily deduces that in fact $\widehat{R}$ is naturally a E-algebra. Moreover we have a continuous homomorphism $\widehat{\rho}\colon G \to \mathrm{GL}_n(\widehat{R})$ induced from $\rho^{\mathrm{ver}}_{\overline{\rho}}$. Clearly $\widehat{\rho}$ is a deformation of ρ_E. Using Remark 3.2.5, this provides one with a homomorphism

$$\varphi\colon R^{\mathrm{ver}}_{E,\rho_E} \longrightarrow \widehat{R}.$$

Suppose now that E is of characteristic p. Then E is isomorphic to a Laurent series field $\kappa'((x))$ for a finite extension κ' of the finite field κ and with ring of integers $\mathcal{O} \cong \kappa'[[x]]$. Denote by $\rho_{\mathcal{O}}$ the map ρ_E with the range restricted to $\mathrm{GL}_n(\mathcal{O})$. It is a deformation of $\bar{\rho}' := \bar{\rho} \otimes_\kappa \kappa'$. Let $\Lambda' = \Lambda \otimes_{W(\kappa)} W(\kappa')$, and consider the map

$$f_E\colon R^{\mathrm{ver}}_{\Lambda',\overline{\rho}'} \otimes_{\kappa'} E \overset{\text{Lem. 3.2.6}}{\cong} R^{\mathrm{ver}}_{\Lambda,\overline{\rho}} \otimes_\kappa E \overset{f \otimes_{\mathbb{F}} \mathrm{id}_E}{\longrightarrow} E.$$

In the present case we define $\widehat{R}$ as the completion of $R^{\mathrm{ver}}_{\Lambda',\overline{\rho}'} \otimes_{\kappa'} E$ at $\ker f_E$. Clearly, $\widehat{R}$ is a E-algebra with residue field E. Note that now $\rho^{\mathrm{ver}}_{\overline{\rho}} \otimes_{R^{\mathrm{ver}}_{\Lambda,\overline{\rho}}} \widehat{R}$ defines a continuous representation

$$\hat{\rho}\colon G \longrightarrow \mathrm{GL}_n(\widehat{R})$$

which is a deformation of ρ_E. Again this yields a homomorphism

$$\varphi\colon R^{\mathrm{ver}}_{E,\rho_E} \longrightarrow \widehat{R}.$$

Theorem 3.3.1. *The map φ is formally smooth. If $R^{\mathrm{ver}}_{\Lambda,\overline{\rho}}$ is universal, it is an isomorphism.*

Proof. If $\mathrm{Char}\, E = 0$, then this is [Kis03, Prop. 9.5]. In the case $\mathrm{Char}\, E > 0$ our proof closely follows that of [Kis03, Prop. 9.5]. We consider a commutative diagram

$$\begin{array}{ccc} R^{\mathrm{ver}}_{E,\rho_E} & \longrightarrow & A \\ \downarrow & \overset{g}{\nearrow} & \downarrow \\ \widehat{R} & \longrightarrow & A/I \end{array} \tag{7}$$

with $A \in \mathcal{A}r_E$ and $I \subset A$ is a square zero ideal, with the solid arrows given, and we seek to construct a dashed arrow g so that the two triangular sub diagrams commute. If $R^{\mathrm{ver}}_{E,\rho_E}$ is universal, we also have to show that the dashed arrow is unique. Note that A and I are finite-dimensional E-vector spaces. Also, the bottom arrow induces a pair of homomorphism $R^{\mathrm{ver}}_{\Lambda',\overline{\rho}'} \to A/I$ and $E \to A/I$, where the second one is simply the E-algebra structure map.

By possibly conjugating $\hat{\rho}$ by some matrix in $\Gamma_n(\widehat{R})$ we can assume that $\rho^{\mathrm{ver}}_{\rho_E} \otimes_{R^{\mathrm{ver}}_{E,\rho_E}} \widehat{R} = \hat{\rho}$. Following the proof in [Kis03, Prop. 9.5], one shows that there exists an $\mathcal{O}$-subalgebra A° of A such that

(a) A° is free over $\mathcal{O}$ of rank equal to $\dim_\kappa A$ and $A^\circ \otimes_{\mathcal{O}} \kappa = A$,

(b) the image of A° under $A \to E$ is $\mathcal{O}$, and so $A^\circ \in \widehat{\mathcal{A}r}_{\kappa'}$

(c) the image of $\rho^{\mathrm{ver}}_{\rho_E} \otimes_{R^{\mathrm{ver}}_{E,\rho_E}} A$ lies in $\mathrm{GL}_n(A^\circ)$,

(d) the homomorphism $R^{\mathrm{ver}}_{\Lambda',\overline{\rho}'} \to A/I$ factors via A°/I° where $I^\circ = I \cap A^\circ$.

Write ρ_{A° for $\rho^{\mathrm{ver}}_{\rho_E} \otimes_{R^{\mathrm{ver}}_{E,\rho_E}} A$ considered with its image in $\mathrm{GL}_n(A^\circ)$. Then ρ_{A° reduces to $\rho^{\mathrm{ver}}_{\overline{\rho}'} \otimes_{R^{\mathrm{ver}}_{\Lambda',\overline{\rho}'}} A^\circ/I^\circ$ modulo I°, and thus by the versality of $R^{\mathrm{ver}}_{\Lambda',\overline{\rho}'}$ there is a homomorphism $g^\circ\colon R^{\mathrm{ver}}_{\Lambda',\overline{\rho}'} \to A^\circ$ such that $\rho^{\mathrm{ver}}_{\overline{\rho}'} \otimes_{R^{\mathrm{ver}}_{\Lambda',\overline{\rho}'}} A^\circ$ is strictly equivalent to ρ_{A°. Let $g\colon \widehat{R} \to A$ be the the homomorphism obtained from $g^\circ \otimes \mathrm{id}$ under completion. It is now not difficult to see that both triangles in (7) commute with this choice of g.

It remains to show the uniqueness of g if $R^{\mathrm{ver}}_{\Lambda',\overline{\rho}'}$ is universal. The argument in [Kis03, Prop. 9.5] shows that there is in fact a directed system A_n°, $n \in \mathbb{N}_{\geq 1}$, satisfying (a) – (d) such that $\bigcup_n A_n^\circ = A$. Now if one has g_1, g_2 completing the diagram (7) to two commutative diagrams, there have to be homomorphisms $g_1^\circ, g_2^\circ \colon R^{\mathrm{ver}}_{\Lambda',\overline{\rho}'} \to A_n^\circ$ for n sufficiently large that give rise to g_1 and g_2, respectively. The corresponding deformations $G \to \mathrm{GL}_n(A_n^\circ)$ of $\overline{\rho}'$ do agree over A and then they will agree for n sufficiently large. Hence they represent the same strict equivalence class. Because $R^{\mathrm{ver}}_{\Lambda',\overline{\rho}'}$ is universal, they define the same ring maps $g_1^\circ = g_2^\circ$ and hence $g_1 = g_2$. $\square$

We would like to point out one consequence of Theorem 3.3.1.

Corollary 3.3.2. *Suppose E is a local field and $\rho\colon G \to \mathrm{GL}_n(E)$ is a continuous homomorphism. Let κ be the (finite) residue field of E and let $\bar{\rho}$ be the semisimple reduction of ρ to κ. Then*

$$\dim_E H^1(G, \mathrm{ad}_\rho) \leq \dim_\kappa H^1(G, \mathrm{ad}_{\bar{\rho}}).$$

Proof. Essentially the corollary will simply follow from the fact that the rank of a coherent sheaf cannot decrease under specialization: Let $\mathcal{O}$ be the valuation ring of E. By possibly passing to a finite extension of E, we may assume that E^n contains a $\rho(G)$-stable $\mathcal{O}$-lattice whose reduction is $\bar{\rho}$. Let $R := R^{\mathrm{ver}}_{\mathcal{O},\bar{\rho}}$. We may assume that R is noetherian, i.e., that $\dim_\kappa H^1(G, \mathrm{ad}_{\bar{\rho}})$ is finite, since else there is nothing to show. Let also $\widehat{R}$ and $R' := R^{\mathrm{ver}}_{E,\rho}$ be as in Theorem 3.3.1.

Denote by $\widehat{\Omega}_{R/\mathcal{O}}$ the module of continuous Kähler differentials. Since R is a power series ring over $\mathcal{O}$, as an R-module $\widehat{\Omega}_{R/\mathcal{O}}$ is finitely generated. By Nakayama's Lemma we have

$$\dim_E \widehat{\Omega}_{R/\mathcal{O}} \otimes_R E \leq \dim_\kappa \widehat{\Omega}_{R/\mathcal{O}} \otimes_R \kappa.$$

Now $\widehat{\Omega}_{R/\mathcal{O}} \otimes_R E \cong \widehat{\Omega}_{\widehat{R}/E} \otimes_{\widehat{R}} E$. Since $\widehat{R}$ is formally smooth over R', we also have

$$\dim_E \widehat{\Omega}_{R'/E} \otimes_{R'} E \leq \dim_E \widehat{\Omega}_{R/\mathcal{O}} \otimes_R E.$$

By [Maz97, § 17, § 21], the dual of $\widehat{\Omega}_{R/\mathcal{O}} \otimes_R \kappa$ is the mod $\mathfrak{m}_{\mathcal{O}}$-tangent space of R at $\mathfrak{m}_R$ and the dual of $\widehat{\Omega}_{R'/E} \otimes_{R'} E$ is the tangent space of R' at $\mathfrak{m}_{R'}$, and the latter can be identified with $H^1(G, \mathrm{ad}_{\bar{\rho}})$ and $H^1(G, \mathrm{ad}_\rho)$, respectively. This proves the corollary. $\square$

The following result helps to derive consequences on $\operatorname{Spec} R^{\mathrm{ver}}_{\Lambda,\bar{\rho}}$. For later applications we will focus on special fibers.

Lemma 3.3.3. *Let R be in $\widehat{\mathcal{A}r}_\kappa$, let $\mathfrak{p} \in \operatorname{Spec} R$ be a 1-dimensional point of characteristic p, i.e., $\dim R/\mathfrak{p} = 1$. Let $\kappa(\mathfrak{p}) = \operatorname{Quot}(R/\mathfrak{p})$, consider the homomorphism*

$$\varphi\colon R \otimes_\kappa \kappa(\mathfrak{p}) \to \kappa(\mathfrak{p}), \quad r \otimes \alpha \mapsto (r \mod \mathfrak{p}) \cdot \alpha,$$

set $\mathfrak{q} := \ker \varphi$ and denote by $\widehat{R}$ the completion of $R \otimes_\kappa \kappa(\mathfrak{p})$ at the maximal ideal $\mathfrak{q}$ and by $\widehat{R}_\mathfrak{p}$ the completion of $R_\mathfrak{p}$ at $R_\mathfrak{p}\mathfrak{p}$. Then the following hold:

- *(a) One has an isomorphism $\widehat{R}_\mathfrak{p}[\![T]\!] \cong \widehat{R}$.*
- *(b) If $\widehat{R}$ is formally smooth over $\kappa(\mathfrak{p})$ of dimension d, then $R_\mathfrak{p}$ is regular of dimension $d-1$.*

Proof. Consider $R \to R_\mathfrak{p} \to \widehat{R}_\mathfrak{p}$. Tensoring with $\kappa(\mathfrak{p})$ over κ, it yields a diagram

$$\begin{array}{ccccc} R \otimes_\kappa \kappa(\mathfrak{p}) & \longrightarrow & R_\mathfrak{p} \otimes_\kappa \kappa(\mathfrak{p}) & \longrightarrow & \widehat{R}_\mathfrak{p} \otimes_\kappa \kappa(\mathfrak{p}) = \left(\varprojlim R_\mathfrak{p}/R_\mathfrak{p}\mathfrak{p}^n\right) \otimes_\kappa \kappa(\mathfrak{p}) \\ \downarrow \iota & \swarrow \iota' & & \swarrow \iota'' & \\ \widehat{R} = \varprojlim (R \otimes_\kappa \kappa(\mathfrak{p}))/\mathfrak{q}^n, & & & & \end{array}$$

where ι is completion and where initially the dashed arrows ι' and ι'' do not exist. For the existence of ι', we use the universal property of localization. Thus we need to show that $R \smallsetminus \mathfrak{p} \otimes 1$ is mapped under ι to the units in $\widehat{R}$. The ring $\widehat{R}$ is local with residue map induced from φ, and therefore we need to show that $\varphi \circ \iota(R \smallsetminus \mathfrak{p} \otimes 1)$ lies in $\kappa(\mathfrak{p})^\times$, but this is clear from the definitions and the inclusion $R/\mathfrak{p} \hookrightarrow \kappa(\mathfrak{p})$. Regarding ι'', we first note that $\mathfrak{p} \otimes_\kappa \kappa(\mathfrak{p})$ maps to $\mathfrak{q}$ under ι and hence $\mathfrak{p}^n \otimes_\kappa \kappa(\mathfrak{p})$ to $\mathfrak{q}^n$. Hence the existence of ι' gives a compatible system of homomorphisms $R_\mathfrak{p}/R_\mathfrak{p}\mathfrak{p}^n \to (R \otimes_\kappa \kappa(\mathfrak{p}))/\mathfrak{q}^n$ und this provides the construction of ι''.

Let π denote the reduction map $\pi\colon \widehat{R} \to \kappa(\mathfrak{p})$, set $\varphi' = \pi \circ \iota'$ and $\varphi'' = \pi \circ \iota''$, and define $\mathfrak{q}' = \ker \varphi'$ and $\mathfrak{q}'' = \ker \varphi''$. Then the arguments just given provide a commutative diagram with canonical isomorphisms in the bottom row

$$\begin{array}{ccccc} R \otimes_\kappa \kappa(\mathfrak{p}) & \longrightarrow & R_\mathfrak{p} \otimes_\kappa \kappa(\mathfrak{p}) & \longrightarrow & \widehat{R}_\mathfrak{p} \otimes_\kappa \kappa(\mathfrak{p}) = \big(\varprojlim R_\mathfrak{p}/R_\mathfrak{p}\mathfrak{p}^n\big) \otimes_\kappa \kappa(\mathfrak{p}) \\ \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \iota'} & & \downarrow{\scriptstyle \iota''} \\ \widehat{R} = \varprojlim (R \otimes_\kappa \kappa(\mathfrak{p}))/\mathfrak{q}^n & \xrightarrow{\simeq} & \widehat{R}' := \varprojlim (R_\mathfrak{p} \otimes_\kappa \kappa(\mathfrak{p}))/\mathfrak{q}'^n & \xrightarrow{\simeq} & \widehat{R}'' = \varprojlim (\widehat{R}_\mathfrak{p} \otimes_\kappa \kappa(\mathfrak{p}))/\mathfrak{q}''^n, \end{array}$$

where by slight abuse of notation we denote the middle and right vertical maps again ι' and ι''. Note that by the Cohen structure theorem in equal characteristic the ring $\widehat{R}_\mathfrak{p}$ contains $\kappa(\mathfrak{p})$ as a subfield. Focussing on the right must arrow and using that $R_\mathfrak{p}$ is regular if and only if $\widehat{R}_\mathfrak{p}$ is so, it will suffice to prove the following assertion:

Let $\mathcal{R}$ be a complete Noetherian local $\kappa(\mathfrak{p})$-algebra with residue field $\kappa(\mathfrak{p})$ and residue homomorphism $\pi\colon \mathcal{R} \to \kappa(\mathfrak{p})$, let $\psi\colon \mathcal{R} \otimes_\kappa \kappa(\mathfrak{p}) \to \kappa(\mathfrak{p})$ be the homomorphism $r \otimes x \mapsto \pi(r) \cdot x$, let $\mathfrak{Q} = \ker \psi$ and let $\widehat{\mathcal{R}}$ be the completion of $\mathcal{R} \otimes_\kappa \kappa(\mathfrak{p})$ at $\mathfrak{Q}$. Then $\widehat{\mathcal{R}} \cong \mathcal{R}[\![t]\!]$.

To prove the assertion, note first that if $\mathcal{S}_1$ and $\mathcal{S}_2$ are $\kappa(\mathfrak{p})$-algebras with maximal ideals $\mathfrak{P}_1$ and $\mathfrak{P}_2$ such that $\kappa(\mathfrak{p})$ is in both cases the residue field, then the completion of $\mathcal{S} := \mathcal{S}_1 \otimes_{\kappa(\mathfrak{p})} \mathcal{S}_2$ at the maximal ideal $\mathfrak{m} := \mathfrak{P}_1 \otimes_{\kappa(\mathfrak{p})} \mathcal{S}_2 + \mathcal{S}_1 \otimes_{\kappa(\mathfrak{p})} \mathfrak{P}_2$ is isomorphic to

$$\varprojlim \mathcal{S}_1/\mathfrak{P}_1^n \widehat{\otimes}_{\kappa(\mathfrak{p})} \varprojlim \mathcal{S}_2/\mathfrak{P}_2^n.$$

If furthermore $\mathcal{S}_1$ is complete with respect to $\mathfrak{P}_1$ and if $\varprojlim \mathcal{S}_2/\mathfrak{P}_2^n \cong \kappa(\mathfrak{p})[\![T]\!]$, then the completion of $\mathcal{S}$ at $\mathfrak{m}$ is $\mathcal{S}_1[\![T]\!]$. We apply this to $\mathcal{S}_1 = \mathcal{R}$, $\mathcal{S}_2 = \kappa(\mathfrak{p}) \otimes_\kappa \kappa(\mathfrak{p})$, $\mathfrak{P}_2 = \ker(\kappa(\mathfrak{p}) \otimes_\kappa \kappa(\mathfrak{p}) \to \kappa(\mathfrak{p}), x \otimes y \mapsto xy)$. Then by the following lemma we have $\varprojlim \mathcal{S}_2/\mathfrak{P}_2^n \cong \kappa(\mathfrak{p})[\![T]\!]$, and we deduce $\widehat{\mathcal{R}} \cong \mathcal{R}[\![T]\!]$. ☐

Lemma 3.3.4. *Let κ' be a finite extension of κ and let L be the Laurent series field over κ'. Let $\mathfrak{q}$ be the kernel of the multiplication map $L \otimes_\kappa L \to L$. Then there is an isomorphism*

$$L \widehat{\otimes} L := \varprojlim_n (L \otimes_\kappa L)/\mathfrak{q}^n \xrightarrow{\simeq} L[\![T]\!].$$

Proof. We think this result ought to be known. But in lack of a reference, we give a proof. We first explain why one can assume $\kappa = \kappa'$.

For this observe that $L \cong \kappa'(\!(s)\!) \cong \kappa(\!(s)\!) \otimes_\kappa \kappa'$. Hence $L \otimes_\kappa L \to L$ can be written as the map

$$\kappa(\!(s)\!) \otimes_\kappa \kappa(\!(s)\!) \otimes_\kappa (\kappa' \otimes_\kappa \kappa') \to \kappa'(\!(s)\!), \quad f \otimes g \otimes \alpha \otimes \beta \mapsto fg\alpha\beta.$$

Since κ' is a finite field, the ring $T = \kappa' \otimes_\kappa \kappa'$ is a finite product of fields isomorphic to κ', i.e., T contains $[\kappa' : \kappa]$ elementary idempotents, and one easily checks that all but one of them map to zero under the multiplication map $T \to \kappa'$. Hence all but one of these idempotents lie in $\mathfrak{q}$ and therefore they also lie in all powers of $\mathfrak{q}$. Thus under completion these components will vanish. Hence from now on, we shall assume $\kappa' = \kappa$.

The first observation we make is that $\mathfrak{q}/\mathfrak{q}^2$ is isomorphic to the module of differentials $\Omega_{L/\kappa}$, by one of the definitions of the latter. Now the single element s is a p-basis of L over κ, i.e., L is a vector space over $L^p\kappa = L^p$ in the basis $1, s, \ldots, s^p$. It follows from [Eis95, Thm. 16.14.b] that $\Omega_{L/\kappa}$ is a vector space of dimension 1 over L. Consequently, we have $\dim_L \mathfrak{q}^n/\mathfrak{q}^{n+1} \leq 1$ for all $n \geq 0$, and by smoothness of $L[\![T']\!]$ there is a surjective ring homomorphism $\psi\colon L[\![T']\!] \to L\widehat{\otimes}L$.

We will now construct explicit surjective homomorphisms

$$\varphi_n\colon \kappa(\!(s)\!) \otimes_\kappa \kappa(\!(s)\!) \to \kappa(\!(s)\!)[T]/(T^n)$$

and verify that $\mathfrak{q}$ lies in the kernel of φ_n. The idea will be that T should be the image of $s\otimes 1 - 1\otimes s$ and that morally $\mathfrak{q}^n$ is generated by $(s\otimes 1 - 1\otimes s)^n$. However we think that in fact the $\mathfrak{q}^n$ are infinitely generated. So we provide an explicit construction. For a formal Laurent series $f = \sum_{i\gg -\infty} a_i s^i$ and $j \in \mathbb{N}_0$ we define the hyperderivatives

$$D^j f := \sum_{i\gg -\infty} a_i \binom{i}{j} s^{i-j}.$$

The operators D^j are continuous in the s-adic topology. We observe that

$$D^j(fg) = \sum_{k=0}^{j} D^k f D^{j-k} g. \tag{8}$$

By continuity it reduces to verifying this for f and g being powers of s, and this comes down to the Vandermonde convolution for binomials $\sum_{k=0}^{j} \binom{i_1}{k}\binom{i_2}{k} = \binom{i_1+i_2}{k}$. We now define the map $\varphi_n\colon L\widehat{\otimes}L \to L[\![T]\!]/(T^n)$ by

$$f \otimes g \longmapsto \sum_{j=0}^{n-1} (-1)^j T^j \cdot f \cdot D^j g.$$

The map is well-defined, and hence additive, since the D^j are κ-linear. It is also clear that it is L-linear with L acting from the left. Using (8) and $T^l = 0$ for $l \geq n$, one verifies that the map is a ring homomorphism. To see that φ_n is surjective, one computes the images of elements of the form $f \otimes s^i$ for $i = 0, \ldots, n-1$. This results in an L-linear homomorphism $\oplus_{i=0}^{n-1} L \otimes s^i \to \oplus_{i=0}^{n-1} L \cdot T^i$ of which the obvious matrix representative is upper triangular with ± 1 on the diagonal.

It is also rather straightforward to see that $\mathfrak{q}^n$ lies in the kernel of φ_n: the ideal $\mathfrak{q}$ is generated as an L-vector space by the expressions $g \otimes 1 - 1 \otimes g$, $g \in L$. Therefore $\mathfrak{q}^n$ is the L-linear span of expressions $\prod_{k=1}^{n} (g_k \otimes 1 - 1 \otimes g_k)$. Their image under φ_n is

$$\prod_{k=1}^{n} \Big(g_k - \sum_{j=0}^{n-1} (-1)^j T^j D^j g_k \Big) = \prod_{k=1}^{n} \Big(-T \sum_{j=1}^{n-1} (-1)^j T^{j-1} D^j g_k \Big),$$

and the right hand side is a multiple of T^n and hence 0 in $L[\![T]\!]/(T^n)$.

Now the composition $\varphi_n \circ \psi\colon L[\![T']\!] \to L\widehat{\otimes}L \to L[\![T]\!]/(T^n)$ is a surjective L-algebra homomorphism for all $n \geq 0$ with the first and second arrows being surjective. In the limit we therefore obtain an isomorphism $L[\![T']\!] \to L\widehat{\otimes}L \to L[\![T]\!]$ as asserted. □

Remark 3.3.5. We shall obtain an analog of Theorem 3.3.1 for pseudodeformation in Corollary 4.8.7.

3.4 Relative formal smoothness of the determinant functor

We first recall a generalization of Tate local duality from [Nek06].

Tate local duality and generalizations: Recall that for a profinite group G and a discrete G-module M, one defines the continuous cohomology $H^i(G,M)$ as $\varinjlim_{U\in\mathfrak{U}} H^i(G/U,M^U)$, where $\mathfrak{U}$ is the set of all normal open subgroups of G; the form a basis of open neighborhoods near the identity of G. This applies for instance if M is a κ-vector space with a continuous G-action and if κ is finite. Suppose however that κ is a local field and that M is a finite dimensional κ-vector space that carries the natural topology induced from κ and a continuous κ-linear G-action. Let $\mathcal{O}$ be the valuation ring of κ with maximal ideal $\mathfrak{m}_{\mathcal{O}}$. Because G is compact a standard argument shows that M contains a G-stable $\mathcal{O}$-lattice L. In this case one defines continuous cohomology via

$$H^i(G,M) := \varprojlim_n H^i(G, L/\mathfrak{m}_{\mathcal{O}}^n L)\otimes_L \kappa,$$

and one shows that this definition is independent of any choices; cf. [Nek06, § 3]. Note that one also has $H^1(G,M)=Z^1(G,M)/B^1(G,M)$, where $Z^1(G,M)$ denotes the continuous 1-cocycles $c\colon G\to M$ (with $c(gh)=gc(h)+c(g)$ for all $g,h\in G$) and $B^1(G,M)$ the continuous 1-coboundaries – in fact all 1-coboundaries are continuous by the continuity of the action of G on M; for n-coboundaries with $n\geq 2$, this is no longer true. Note also that similar descriptions hold for all $H^i(G,M)$ and in particular for $i=2$ (and for $i=0$ when $H^0(G,M)=M^G$).

The next result is a generalization of Tate local duality from finite field to local field coefficients. In the form needed it is due to Nekovář. Let V be a finite-dimensional κ-vector space with the topology induced from κ, and suppose that V carries a continuous κ-linear action by G_K. Write $V^\vee$ for the dual $\mathrm{Hom}_\kappa(V,\kappa)$ of V, and $V(1)$ for the twist of V by the cyclotomic character. Set $h^j(K,V) := \dim_\kappa H^j_{\mathrm{cont}}(G_K,V)$.

Theorem 3.4.1 (Tate and Nekovář)**.** *The following assertions hold:*

(a) One has $h^j(K,V)<\infty$ for $j\in\mathbb{Z}$ and $h^j(K,V)=0$ for $j\notin\{0,1,2\}$;

(b) For $j\in\{0,1,2\}$ one has natural isomorphisms

$$H^{2-j}_{\mathrm{cont}}(G_K,V^\vee(1)) \xrightarrow{\simeq} H^j_{\mathrm{cont}}(G_K,V)^\vee;$$

(c) One has the Euler characteristic formula

$$\sum_{j\geq 0}(-1)^j h^j(K,V) = -[K:\mathbb{Q}_p]\cdot \dim_E V.$$

Proof. If κ is finite, the above statement is just the usual Tate local duality. If κ is local, let $\mathcal{O}$ be its valuation ring. Because G_K is compact one can find an $\mathcal{O}$-lattice T in V that is stable under G_K. Let $j\geq 0$. Then [Nek06, Thm. 5.2.6] asserts that each $H^j_{\mathrm{cont}}(G_K,V)$ is a finitely generated $\mathcal{O}$-module and moreover it gives a spectral sequence

$$\mathrm{Ext}^i_{\mathcal{O}}(H^{2-j}_{\mathrm{cont}}(G_K,T^\vee(1)),\mathcal{O}) \Longrightarrow H^{i+j}_{\mathrm{cont}}(G_K,T).$$

Because $\mathcal{O}$ is regular and of dimension 1, the groups $\mathrm{Ext}^1_{\mathcal{O}}(\cdot,\mathcal{O})$ are finitely generated $\mathcal{O}$-torsion modules. After tensoring the results just quoted with κ over $\mathcal{O}$ part (b) and (a) are clear. Part (c) follows from [Nek06, Thm. 4.6.9 and 5.2.11] applied to T, again after tensoring with κ over $\mathcal{O}$. □

The determinant map: The determinant of representations induces a natural transformation

$$\det\colon \mathcal{D}_{\bar\rho} \to \mathcal{D}_{\deg\bar\rho} \tag{9}$$

that maps the class of $\rho\colon G \to \mathrm{GL}_n(A)$ to the class of $\det\circ\rho$. The induced map on adjoint representations is the trace map in the short exact sequence

$$0 \longrightarrow \mathrm{ad}^0_{\bar\rho} \longrightarrow \mathrm{ad}_{\bar\rho} \xrightarrow{\mathrm{tr}} \mathrm{ad}_{\det\bar\rho} \cong \kappa \longrightarrow 0. \tag{10}$$

Using that $\mathrm{ad}_{\bar\rho}$ is self-dual it is easy to see that the sequence dual to (10) is

$$0 \longrightarrow \kappa \xrightarrow{\mathrm{diag}} \mathrm{ad}_{\bar\rho} \longrightarrow \overline{\mathrm{ad}}_{\bar\rho} \longrightarrow 0. \tag{11}$$

We have the following explicit result on det for $G = G_K$.

Lemma 3.4.2. *Suppose that $H^0(G_K, \overline{\mathrm{ad}}_{\bar\rho}(1)) = 0$. Then $\det\colon \mathcal{D}_{\bar\rho} \to \mathcal{D}_{\det\bar\rho}$ is smooth of relative dimension $d(n^2-1)$. This holds in particular, if $p\nmid n$, $\kappa \cong \mathrm{End}_{\kappa[G_K]}(\bar\rho)$ and $\zeta_p \in K$.*

Proof. Let $A \to B$ be a small extension in $\mathcal{A}r_\Lambda$. Let I be its kernel so that $I^2 \subset \mathfrak{m}_A I = 0$. For the relative smoothness, we need to show the surjectivity of

$$\mathcal{D}_{\bar\rho}(A) \longrightarrow \mathcal{D}_{\bar\rho}(B) \times_{\mathcal{D}_{\det\bar\rho}(B)} \mathcal{D}_{\det\bar\rho}(A).$$

So suppose we are given deformations $\rho_B \in \mathcal{D}_{\bar\rho}(B)$ and $\tau_A \in \mathcal{D}_{\det\bar\rho}(A)$ with $\det\rho_B = \tau_A \otimes_A B \in \mathcal{D}_{\det\bar\rho}(B)$. We need to find a deformation $\rho_A \in \mathcal{D}_{\bar\rho}(A)$ such that $\rho_A \otimes_A B = \rho_B$ and $\det\rho_A = \tau_A$.

Recall that there is a canonical obstruction class $\mathcal{O}(\rho_B) \in H^2(G_K, \mathrm{ad}_\rho) \otimes_\kappa I$, which vanishes if and only if there exists a deformation of $\bar\rho$ to A that lifts ρ_B. Because of the existence of the deformation τ_A that maps to $\det\rho_B$, the obstruction class $\mathcal{O}(\det\rho_B) \in H^2(G_K, \mathrm{ad}_{\det\bar\rho}) \otimes_\kappa I$ vanishes. By Theorem 3.4.1 the long exact sequence of Galois cohomology arising from (10) gives the left exact sequence

$$H^2(G_K, \mathrm{ad}^0_{\bar\rho}) \longrightarrow H^2(G_K, \mathrm{ad}_{\bar\rho}) \xrightarrow{H^2(\mathrm{tr})} H^2(G_K, \kappa) \longrightarrow 0$$

By Theorem 3.4.1 the sequence is dual to the right exact sequence

$$0 \longrightarrow H^0(G_K, \kappa(1)) \xrightarrow{H^0(\mathrm{diag}(1))} H^0(G_K, \mathrm{ad}_{\bar\rho}(1)) \longrightarrow H^0(G_K, \overline{\mathrm{ad}}_{\bar\rho}(1)),$$

that arises from (11). By our hypothesis the map $H^0(\mathrm{diag}(1))$ is an isomorphism, and so by duality the same holds for $H^2(\mathrm{tr})$. By a short explicit computation one sees that $\mathcal{O}(\rho_B)$ maps to $\mathcal{O}(\det\rho_B) = 0$ under $H^2(\mathrm{tr}) \otimes_\kappa \mathrm{id}_I$, and this implies the vanishing of $\mathcal{O}(\rho_B)$.

We have now proved that there exists $\rho'_A \in \mathcal{D}_{\bar\rho}(A)$ mapping to $\rho_B \in \mathcal{D}_{\bar\rho}(B)$. However, this lift need not satisfy $\det\rho'_A = \tau_A$. At this point we note that our hypothesis in fact implies that $H^2(G_K, \mathrm{ad}^0_{\bar\rho}) = 0$, so that

$$H^1(G_K, \mathrm{ad}_{\bar\rho}) \longrightarrow H^1(G_K, \mathrm{ad}_{\det\bar\rho}) = H^1(G_K, \kappa) \tag{12}$$

is surjective. Now $\det\rho'_A$ and τ_A are deformations of τ_B and the space of all such deformations is a principal homogeneous space under $H^1(G_K, \kappa)$, i.e., the tangent space of the deformation problem, by [Sch68, Rem. 2.15], and likewise the deformations of ρ_B form a principal homogeneous space under $H^1(G_K, \mathrm{ad}_{\bar\rho})$. Since (12) is surjective we can thus alter ρ'_A by a class in $H^1(G_K, \mathrm{ad}_{\bar\rho})$ into some other deformation ρ_A of ρ_B that also satisfies $\det\rho_A = \tau_A$. This completes the proof of the formal smoothness. Note also that if $p\nmid n$, then the above two sequences are exact on both sides and hence $H^0(G_K, \overline{\mathrm{ad}}_{\bar\rho}(1)) = 0$.

By Proposition 3.1.5 it follows that the natural map $R^{\mathrm{univ}}_{\det\bar\rho} \to R^{\mathrm{univ}}_{\bar\rho}$ is formally smooth of relative dimension $h = h^1(K, \mathrm{ad}_{\bar\rho}) - h^1(K, \mathrm{ad}_{\det\bar\rho})$. It remains to identify h with the number in the lemma. Since (12) is surjective, by the long exact sequence for $H^*(G_K, \cdot)$ applied to (10), we deduce

$$h = h^1(K, \mathrm{ad}^0_{\bar\rho}) - h^0(K, \mathrm{ad}_{\det\bar\rho}) + h^0(K, \mathrm{ad}_{\bar\rho}) - h^0(K, \mathrm{ad}^0_{\bar\rho}) = h^1(K, \mathrm{ad}^0_{\bar\rho}) - 1 + 1 - h^0(K, \mathrm{ad}^0_{\bar\rho}).$$

Since $h^2(K, \mathrm{ad}^0_{\bar\rho}) = 0$ by hypothesis and the duality statement of Theorem 3.4.1, the Euler characteristic formula of Theorem 3.4.1 implies $h = d(n^2-1)$.

The last assertion is straightforward. If $p \nmid n$, then the sequence (10) splits; the second assumption now yields $H^0(G_K, \mathrm{ad}^0_{\bar\rho}) = 0$. If now κ has characteristic zero, then $0 = H^0(G_K, \kappa(1)) \cong H^0(G_K, \mathrm{ad}_{\bar\rho}(1))$ and we are done. If on the other hand κ has characteristic p and $\zeta_p \in K$, then $\mathrm{ad}^0_{\bar\rho} = \mathrm{ad}^0_{\bar\rho}(1)$ and we are done, as well. □

Let $\bar\rho_1 \colon G_K \to \mathrm{GL}_1(\kappa)$ be a continuous character and denote by $\bar\rho_0$ the trivial such character. If κ is finite, denote by $\rho_1 \colon G_K \to \mathrm{GL}_1(\Lambda)$ the Teichmüller lift of $\bar\rho_1$, if κ is a local field, set $\rho_1 := \bar\rho_1$. There is a natural isomorphism $\mathcal{D}_{\bar\rho_0} \to \mathcal{D}_{\bar\rho_1}$ mapping a deformation $\rho \colon G_K \to \mathrm{GL}_1(A)$ to $\rho \otimes \rho_1$. As already observed in [Maz89, § 1.4], $\mathcal{D}_{\bar\rho_0}$ is representable by the completed group ring $\Lambda[[G_K^{\mathrm{ab},p}]]$, where $G_K^{\mathrm{ab},p}$ is the completion of the abelianization G_K^{ab} of G_K along normal open subgroups of p-power index; the universal homomorphism

$$G_K \to (\Lambda[[G_K^{\mathrm{ab},p}]])^\times$$

factors via $G_K^{\mathrm{ab},p}$ and sends $g \in G_K^{\mathrm{ab},p}$ to itself as a unit element in $\Lambda[[G_K^{\mathrm{ab},p}]]$. The reciprocity homomorphisms of local class field theory, yields an isomorphism

$$\mathrm{rec}^p \colon K^{\times,p} \to G_K^{\mathrm{ab},p},$$

where $K^{\times,p}$ is the pro-p completion of the multiplicative group $K^\times$. The torsion subgroup of $K^{\times,p}$ is naturally identified with the group $\mu_{p^\infty}(K)$ of p-power roots of unity in K. Combining this with det from (9), we the following chain of natural ring homomorphisms in $\widehat{\mathcal{A}r}_\Lambda$

$$\Lambda[\mu_{p^\infty}(K)] \longrightarrow \Lambda[[K^{\times,p}]] \xrightarrow{\simeq} R^{\mathrm{ver}}_{\Lambda,\det\bar\rho} \longrightarrow R^{\mathrm{ver}}_{\Lambda,\bar\rho}. \tag{13}$$

Corollary 3.4.3 (Cf. [Nak14, § 4].)**.** *Let κ be finite or a local field of characteristic p and suppose $\Lambda = \kappa$. Suppose that $H^0(G_K, \overline{\mathrm{ad}}_{\bar\rho}(1)) = 0$. Then the following hold:*

(a) Both morphisms in (13) are formally smooth.

(b) Both morphism in the following induced diagram are formally smooth:

$$\Lambda = \Lambda[\mu_{p^\infty}(K)]_{\mathrm{red}} \longrightarrow \Lambda[[K^{\times,p}]]_{\mathrm{red}} \xrightarrow{\simeq} (R^{\mathrm{ver}}_{\Lambda,\det\bar\rho})_{\mathrm{red}} \longrightarrow (R^{\mathrm{ver}}_{\Lambda,\bar\rho})_{\mathrm{red}}.$$

The relative dimensions in both cases are $d+1$ and $d(n^2-1)$, respectively

Proof. Let $q := \mathrm{ord}\,\mu_{p^\infty}(K)$. By Lemma 3.4.2 the natural map $R^{\mathrm{univ}}_{\det\rho} \longrightarrow R^{\mathrm{univ}}_{\rho}$ is formally smooth of relative dimension h. From the theory of local fields one has $K^{\times,p} \cong \mathbb{Z}_p^{[K:\mathbb{Q}_p]+1} \times \mu_{p^\infty}(K)$, where $\mu_{p^\infty}(K)$ is a finite cyclic group of p-power order q. By our hypothesis on the characteristic of κ, the right hand morphism in (13) can be identified with

$$\kappa[x]/(x^q) \to \kappa[[x_1, \ldots, x_d, x]]/(x^q),$$

and parts (a) and (b) for it are now obvious. To see the second part of (b), note that the kernel of the reduction map $\kappa[x]/(x^q) \to \kappa$ is nilpotent. Hence the kernel of the induced map $\varphi \colon R^{\mathrm{ver}}_{\Lambda,\bar\rho} \to R^{\mathrm{ver}}_{\Lambda,\bar\rho} \otimes_{\kappa[x]/(x^q)} \kappa$ is nilpotent as well, and the map $R^{\mathrm{ver}}_{\Lambda,\bar\rho} \to (R^{\mathrm{ver}}_{\Lambda,\bar\rho})_{\mathrm{red}}$ factors via φ. At the same time, formal smoothness is preserved under base change. Hence $\kappa \to R^{\mathrm{ver}}_{\Lambda,\bar\rho} \otimes_{\kappa[x]/(x^q)} \kappa$ is formally smooth, and therefore $R^{\mathrm{ver}}_{\Lambda,\bar\rho} \otimes_{\kappa[x]/(x^q)} \kappa$ is regular and in particular a domain. We deduce that $R^{\mathrm{ver}}_{\Lambda,\bar\rho} \otimes_{\kappa[x]/(x^q)} \kappa \to (R^{\mathrm{ver}}_{\Lambda,\bar\rho})_{\mathrm{red}}$ is an isomorphism, and this completes (b). □

We end this subsection with a computation of $H^0(G_K, \mathrm{ad}_{\overline{\rho}} \otimes \chi)$.

Lemma 3.4.4. *Let E be a finite or local field with its natural topology. Denote by $\rho_i \colon G_K \to \mathrm{GL}_{n_i}(E)$ continuous Galois representations for $i = 1, 2$, and let $\rho = \begin{pmatrix} \rho_1 & c \\ 0 & \rho_2 \end{pmatrix}$ be an extension of ρ_1 by ρ_2. Let $\chi \colon G_K \to E^\times$ be a continuous character and write 1 for the trivial character. Suppose that*

(a) $\mathrm{Hom}_{G_K}(\rho_1, \rho_2 \otimes \chi) = 0$ *and* $\mathrm{Hom}_{G_K}(\rho_2, \rho_1 \otimes \chi) = 0$.

(b) For $i = 1, 2$, we have $\mathrm{End}_{G_K}(\rho_i) \cong E$ if $\chi = 1$ and $\mathrm{Hom}_{G_K}(\rho_i, \rho_i \otimes \chi) = 0$ if $\chi \neq 1$.

(c) If $\chi = 1$, then the class $c \in \mathrm{Ext}^1_{G_K}(\rho_2, \rho_1)$ is nontrivial,

Then $\mathrm{End}_{G_K}(\rho) \cong E$ if $\chi = 1$ and $\mathrm{Hom}_{G_K}(\rho, \rho \otimes \chi) = 0$ if $\chi \neq 1$.

Proof. To determine $\mathrm{Hom}_{G_K}(\rho, \rho \otimes \chi)$, we consider $A_{ij} \in \mathrm{Mat}_{n_i, n_j}(E)$, for $1 \leq i, j \leq 2$ such that

$$\begin{aligned} 0 &\stackrel{!}{=} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} \rho_1 & c \\ 0 & \rho_2 \end{pmatrix} - \begin{pmatrix} \rho_1 \otimes \chi & c \\ 0 & \rho_2 \otimes \chi \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \\ &= \begin{pmatrix} A_{11}\rho_1 & A_{11}c + A_{12}\rho_2 \\ A_{21}\rho_1 & A_{21}c + A_{22}\rho_2 \end{pmatrix} - \begin{pmatrix} \rho_1 \otimes \chi \cdot A_{11} + c \otimes \chi \cdot A_{21} & \rho_1 \otimes \chi \cdot A_{12} + c \otimes \chi \cdot A_{22} \\ \rho_2 \otimes \chi \cdot A_{21} & \rho_2 \otimes \chi \cdot A_{22} \end{pmatrix}. \end{aligned}$$

From hypothesis (a) and considering the (2,1)-entry we deduce $A_{21} = 0$. From hypothesis (b) and considering the (1,1)- and (2,2)-entries, we deduce, depending on χ the following: If $\chi = 1$, then A_{ii} are scalar for $i = 1, 2$, say equal to $\lambda_i \mathbb{1}_{n_i}$ for some $\lambda_i \in E$, respectively; if $\chi \neq 1$, then both $A_{ii} = 0$. Considering the $(1, 2)$-entry, we obtain the relation

$$A_{11}c - c \otimes \chi \cdot A_{22} = \rho_1 \otimes \chi \cdot A_{12} - A_{12}\rho_2$$

If $\chi \neq 1$, the left hand side is zero, and from (a) we deduce $A_{12} = 0$, so that the proof in this case is complete. If $\chi = 1$, then we have $(\lambda_1 - \lambda_2)c = \rho_1 A_{12} - A_{12}\rho_2$. Now $g \mapsto \rho_1(g)A_{12} - A_{12}\rho_2(g)$ is a 1-coboundary with values in $\mathrm{Hom}_{G_K}(\rho_2, \rho_1)$, and so if $\lambda_1 \neq \lambda_2$, the last condition implies that c is the trivial class in $\mathrm{Ext}^1_{G_K}(\rho_2, \rho_1)$ which is excluded by (c). This shows $\lambda_1 = \lambda_2$, and $A_{12} \in \mathrm{Hom}_{G_K}(\rho_2, \rho_1)$, and hence $A_{12} = 0$, again by (a). This completes the proof. ☐

4 Pseudorepresentations and their deformations

In this section we recall main definitions and results on polynomial laws and pseudorepresentations. We assume that the reader is familiar with [Che11, Che14, WE18]. Nevertheless, we will give many reminders. Each subsection gives a short survey over its contents. In Proposition 4.3.9 we prove an analog of the locus of reducibility of [BC09, Prop. 1.5.1] in the context of pseudorepresentations. In Subsections 4.5 and 4.6 we introduce twisting and induction as operations on pseudorepresentations. In Proposition 4.7.4 we sketch the existence of a universal ring for continuous pseudodeformation where the residue field is a local field and in Proposition 4.7.6 we consider such rings under change of coefficients. These are adaptions of well-known results. Subsection 4.8 presents in detail many results on dimension 1 points in universal pseudodeformation spaces.

Throughout this section, A will be a commutative unital ring with $0 \neq 1$. If A is local, we write $\mathfrak{m}_A$ for its maximal ideal and $\kappa(A)$ for its residue field. We write $\mathcal{A}lg_A$ for the category of A-algebras and $\mathcal{CA}lg_A$ for the full subcategory of commutative A-algebras. By R, S we always denote objects of $\mathcal{A}lg_A$ and by B an object of $\mathcal{CA}lg_A$. For an A-algebra R we denote by R^o the A-algebra with the multiplication of R reversed. By G we denote a group and by $B[G]$ the group algebra over B for any $B \in \mathcal{CA}lg_A$. The letters m, n (also with indices) will denote non-negative integers. If $\rho \colon G \to \mathrm{GL}_n(B)$ is a representation, then by $\rho^{\mathrm{lin}} \colon B[G] \to M_n(B)$ we denote its linearization given by $\sum_i b_i g_i \mapsto \sum_i b_i \rho(g_i)$.

4.1 Pseudorepresentations

In this subsection, we introduce pseudorepresentations, Azumaya algebras and Cayley-Hamilton A-algebras. Of particular importance is Proposition 4.1.10, which says that a pseudorepresentation is determined by its characteristic polynomial coefficients.

For an A-module M we define the functor $\underline{M}\colon \mathcal{CA}lg_A \to Sets,\ B \mapsto M \otimes_A B$.

Definition 4.1.1 ([Che14, § 1.1])**.** *Let M and N be A-modules.*

(i) An A-polynomial law $P\colon M \to N$ *is a natural transformation $\underline{M} \to \underline{N}$. I.e., P is a family of maps $P_B\colon M \otimes_A B \to N \otimes_A B$ for all $B \in \mathrm{Ob}(\mathcal{CA}lg_A)$ that induce commutative diagrams for every morphism in $\mathcal{CA}lg_A$.*

(ii) An A-polynomial law $P\colon M \to N$ is called homogeneous of degree n *if*

$$P_B(bx) = b^n P_B(x) \quad \text{for all } B \in \mathrm{Ob}(\mathcal{CA}lg_A),\ b \in B \text{ and } x \in M \otimes_A B.$$

We let $\mathcal{P}^n_A(M,N)$ denote the set of all such.

Let S, S' be objects in $\mathcal{A}lg_A$, so that in particular they are A-modules.

(iii) An A-polynomial law $P\colon S \to S'$ is called multiplicative *if*

$$P_B(1) = 1 \quad \text{and} \quad P_B(xy) = P_B(x)P_B(y) \quad \text{for all } B \in \mathrm{Ob}(\mathcal{CA}lg_A) \text{ and } x, y \in S \otimes_A B.$$

(iv) We write $\mathcal{M}^n_A(S,S')$ for the set of multiplicative A-polynomial laws $P\colon S \to S'$ that are homogeneous of degree n.

(v) A pseudorepresentation on S of dimension n *is an A-polynomial law $D : S \to A$ that is multiplicative and homogeneous of degree n. We let $\mathcal{P}s\mathcal{R}^n_S(A)$ be the set of all such.*

(vi) If $S = A[G]$ in (v), we call D an A-valued pseudorepresentation of G of dimension n, *and we write $D\colon G \to A$ for $D\colon A[G] \to A$, as well as $\mathcal{P}s\mathcal{R}^n_G(A)$ for $\mathcal{P}s\mathcal{R}^n_{A[G]}(A)$.*

Remark 4.1.2 ([Che14, after Exmp. 1.2]). A homogeneous polynomial law P of degree n is uniquely determined by $P_{A[T_1,\dots,T_m]}\colon M[T_1,\dots,T_m] \to N[T_1,\dots,T_m]$ for all $m \geq 0$.

Facts 4.1.3. *The following facts are easy to verify.*

(i) The only multiplicative polynomial law of degree zero is the constant map with value 1.
(ii) Multiplicative polynomial laws that are homogeneous of degree 1 are A-algebra homomorphisms, and vice versa.
(iii) The composition of polynomial laws is a polynomial law; if they are homogeneous, the composition is homogeneous and its degree is product of the individual degrees.
(iv) The composition of multiplicative polynomial laws is multiplicative.
(v) If $D\colon S \to A$ is an A-valued pseudorepresentation, then for any $B \in \mathcal{CA}lg_A$, the base change $D \otimes_A B\colon S \otimes_A B \to B$ is a B-valued pseudorepresentation.

Definition 4.1.4 (Pseudorepresentation of a representation)**.** *Let $\rho\colon G \to \mathrm{GL}_n(A)$ be a representation. The pseudorepresentation D_ρ attached to ρ is the composition of the determinant* $\det\colon M_n(B) \to B$ *with the morphism $(\rho \otimes_A B)^{\mathrm{lin}}$.*

Definition 4.1.5 (Determinant of a pseudorepresentation)**.** *Let D be a pseudorepresentation of G over A. Then the restriction $\varphi := D|_G\colon G \to A^\times$ is a group homomorphism. We define* $\det(D) := D_\varphi$ *with D_φ from Definition 4.1.4 as* the determinant of D.

Reminder 4.1.6. From [Mil80, § IV.1 – IV.2] we recall the notion of Azumaya algebra and some of its properties. Let first A be a local ring with residue field κ. An algebra $C \in \mathcal{A}lg_A$ is called an *Azumaya* A*-algebra* if C free of finite rank as an A-module and if in $\mathcal{A}lg_A$ the map

$$C \otimes_A C^\circ \longrightarrow \mathrm{End}_A(C), \quad c \otimes c' \longmapsto (x \mapsto cxc'),$$

is an isomorphism. Then there exists a finite étale homomorphism $A \to B$ such that $C \otimes_A B \cong M_m(B)$ for some m. One calls m the *degree of* C; it satisfies $\mathrm{rank}_A C = m^2$. Moreover C carries a *reduced norm map* $\det_C \colon C \to A$ characterized by the property that $\det_C \otimes_A B$ is the determinant on $M_m(B)$. Its extension to $C[t]$ defines a reduced characteristic polynomial $\chi_c := \det_{C[t]}(t-c) \in A[t]$, monic of degree m, for any $c \in C$. Lastly, $C \otimes \kappa$ is a central simple algebra over κ.

Let now X be a scheme. An $\mathcal{O}_X$-algebra $\mathcal{C}$ is called an *Azumaya algebra over* X if $\mathcal{C}$ is coherent as an $\mathcal{O}_X$-module and if for all closed points $x \in X$, the stalk $\mathcal{C}_x$ is an Azumaya algebra over $\mathcal{O}_{X,x}$. There is a Zariski cover $\{U_i\}$ of X and for each i a finite étale surjective cover $U_i' \to U_i$ such that one has an isomorphism $\mathcal{C} \otimes_{\mathcal{O}_X} \mathcal{O}_{U_i'} \xrightarrow{\simeq} \mathrm{Mat}_{m_i}(\mathcal{O}_{U_i'})$ for suitable $m_i \in \mathbb{N}_{\geq 1}$. In particular, there is a locally constant function $\underline{m} \colon X \to \mathbb{N}_{\geq 1}$ such that $\mathrm{rank}_{\mathcal{O}_X} \mathcal{C} = \underline{m}^2$. Also, the reduced norm exists as a map $\det_{\mathcal{C}} \colon \mathcal{C} \to \mathcal{O}_X$. For $X = \mathrm{Spec}\, A$ affine, one calls $C = \mathcal{C}(X)$ an Azumaya A-algebra.

Example 4.1.7. *Let C be an Azumaya A-algebra of degree n with reduced norm $\det_C \colon C \to A$.*

(i) The family of maps $(\det_C \otimes_A B \colon C \otimes_A B \to B)_{B \in \mathcal{CA}lg_A}$ defines a pseudorepresentation, also called $\det_C$, of dimension n; see [Che14, § 1.5].

(ii) If $D \colon C \to A$ is any pseudorepresentation of dimension n', then by [Che14, Lem. 2.15], we have $n | n'$ and $D = \det_C^{n'/n}$.

An important notion for pseudorepresentation is that of characteristic polynomial.

Lemma 4.1.8 ([Che14, § 1.10])**.** *Let $D \in \mathcal{P}s\mathcal{R}^n_S(A)$. Define $\chi_{D,B}(\cdot, t) \colon S \otimes_A B \longrightarrow B[t]$ by $s \mapsto D_{B[t]}(t-s)$ for all $B \in \mathrm{Ob}(\mathcal{CA}lg_A)$ and $s \in S \otimes_A B$. Then the following hold:*

(i) $\chi_D(\cdot, t) \colon S \to A[t]$ is a multiplicative homogeneous polynomial law of degree n.

(ii) There exist unique A-polynomial laws $\Lambda_{D,i} \colon S \to A$ of degree i, $i = 0, \ldots, n$, such that

$$\chi_D(\cdot, t) = \sum_{i=0}^{n} (-1)^i \Lambda_{D,i}(s) t^{n-i}.$$

(iii) $\Lambda_{D,0} = 1$ and $\Lambda_{D,n} = D$.

(iv) The maps $s \mapsto \sum_{i=0}^{n} (-1)^i \Lambda_{D,i}(s) s^{n-i}$ for all $B \in \mathrm{Ob}(\mathcal{CA}lg_A)$ and $s \in S \otimes_A B$ define a multiplicative A-polynomial law $\chi_D \colon S \to S$ that is homogeneous of degree n.

Definition 4.1.9. *[Che14, § 1.10] Let S, D, $\chi_D(\cdot, t)$ and $\Lambda_{D,i}$ be as in Lemma 4.1.8.*

(i) The polynomial law $\chi_D(\cdot, t)$ is called the characteristic polynomial of D.

(ii) The polynomial law $\Lambda_{D,i}$ is called the i^{th} characteristic polynomial coefficient of D.

(iii) The A-linear map $\tau_D := \Lambda_{D,1}$ is called the trace associated with D.

An important tool to extract properties of multiplicative homogeneous polynomial laws is Amitsur's formula; see [Che14, Formula (1.5)]. It expresses values of such laws in terms of characteristic polynomial coefficients, from which one deduces the following result:

Proposition 4.1.10 ([Che14, Cor. 1.14],[WE13, 1.1.9.15])**.** *Let $D \in \mathcal{P}s\mathcal{R}^n_G(A)$.*

(i) The characteristic polynomial coefficients $(\Lambda_{D,i} \colon G \to A)_{i=1,\ldots,n}$ characterize D.

(ii) *Let $C \subset A$ be the subring generated by $\{\Lambda_{D,i}(g) : g \in G, i = 1, \ldots, n\}$. Then D factors through a unique C-valued pseudorepresentation D_C of G of dimension n.*

An natural operation on pseudorepresentations is the formation of direct sums.[4]

Definition 4.1.11 ([WE13, § 1.1.11]). *Let S, S_1, S_2 be $\mathcal{A}lg_A$ and B in $\mathcal{CA}lg_A$.*

(i) *The* direct sum *of multiplicative homogeneous A-polynomial laws $P_i \colon S_i \to B$ of degree n_i, $i = 1, 2$, is the multiplicative homogeneous A-polynomial law of degree $n_1 + n_2$ given by*

$$P_1 \oplus P_2 \colon S_1 \times S_2 \to B, \quad (x_1, x_2) \mapsto P_1(x_1)P_2(x_2).$$

(ii) *The* direct sum *of psuedorepresentations $D_i \colon S \to A$ of dimension n_i, $i = 1, 2$, is the pseudorepresentation of dimension $n_1 + n_2$ given by $D_1 \oplus D_2 \colon S \to A, x \mapsto D_1(x)D_2(x)$.*

Remark 4.1.12. Note that $\det^{n'/n}$ from Example 4.1.7 could now also be written as $\det^{\oplus(n'/n)}$.

Lemma 4.1.13 ([Che14, Lem. 2.2]). *Let S_1, S_2 be in $\mathcal{A}lg_A$. Let $B \neq 0$ be in $\mathcal{CA}lg_A$ such that* $\operatorname{Spec} B$ *is connected. Let $P \colon S_1 \times S_2 \longrightarrow B$ be a multiplicative A-polynomial law that is homogeneous of degree n. Then there exist for $i = 1, 2$ unique $n_i \geq 0$ with $n_1 + n_2 = n$ and multiplicative homogeneous A-polynomial laws $P_i \colon S_i \to B$ of degree n_i such that $P = P_1 \oplus P_2$.*

Lemma 4.1.14 ([WE13, Lem. 1.1.11.7]). *For $i = 1, 2$ let $\rho_i \colon G \to \mathrm{GL}_{n_i}(A)$ be a representation, and set $\rho := \rho_1 \oplus \rho_2$. Then $D_\rho = D_{\rho_1} \oplus D_{\rho_2}$ for the associated pseudorepresentations from Definition 4.1.4.*

To any $D \in \mathcal{P}s\mathcal{R}^n_S(A)$, one can naturally assign its kernel $\mathrm{Ker}(D)$.

Definition 4.1.15 ([Che14, 1.17]). *Let $P \colon M \to N$ be a polynomial law for A-modules M, N.*

(i) *The* kernel $\ker(P)$ of P *is the A-submodule of M defined as*

$$\{x \in M \; : \; P(x \otimes b + m) = P(m) \quad \text{for all } B \in \mathrm{Ob}(\mathcal{CA}lg_A), \; b \in B \text{ and } m \in M \otimes_A B\}$$

(ii) *If* $\ker(P) = 0$*, then P is called* faithful.

Proposition 4.1.16 ([Che14, 1.19–1.21]). *For $D \in \mathcal{P}s\mathcal{R}^n_S(A)$ the following hold.*

(i) $\ker D$ *is a two-sided ideal of S; there exists $\widetilde{D} \in \mathcal{P}s\mathcal{R}^n_{S/\ker D}(A)$ such that $D = \widetilde{D} \circ \pi$ for π the projection $S \to S/\ker D$, and* $\ker D$ *is maximal with this property.*

(ii) *If C is an Azumaya A-algebra then its reduced norm $\det_C$ is faithful.*

Over fields, the following is a fundamental result on faithful pseudorepresentations.

Theorem 4.1.17 ([Che14, Thm. 2.16]). *Let k be a field such that k is perfect, or k has characteristic $p > 0$ and $[k : k^p] < \infty$. Let $D \colon S \to k$ be a pseudorepresentation of dimension n. Then $S/\ker D$ is of finite k-dimension and semisimple as a ring.*

Choose a k-algebra isomorphism $S/\ker D \xrightarrow{\simeq} \prod_{i=1}^s S_i$ where each S_i is a simple k-algebra. Let n_i be the degree of S_i over its center k_i, let $f_i := [k_i \cap k^{\mathrm{sep}} : k]$ and let q_i be the smallest p-power such that $k_i^{q_i} \subset k^{\mathrm{sep}}$; note that that all $q_i = 1$ if k is perfect. Then under the above isomorphism D coincides with the product determinant

$$D = \prod_{i=1}^{s} \det_{S_i}^{m_i}$$

for some uniquely determined integers $m_i \geq 1$, and one has $n = \sum_i m_i n_i q_i f_i$.

[4] We use the term direct sum in analogy with the case of representations; Chenevier uses the term product.

We record the following consequence that will be used in the proof of Proposition 4.4.7.

Corollary 4.1.18. *Let $\overline{D}\colon G \to \mathbb{F}$ be an n-dimensional pseudorepresentation. Let $\mathbb{F}'$ be the extension of $\mathbb{F}$ of degree $n!$. Then $\overline{D} \otimes_{\mathbb{F}} \mathbb{F}'$ is a direct sum of split irreducible representations.*

Proof. Over finite fields the Brauer group is zero. Thus by Theorem 4.1.17 we have an isomorphism $\mathbb{F}[G]/\ker D \xrightarrow{\simeq} \prod_{i=1}^{s} M_{d_i}(\mathbb{F}_i)$ for integers $d_i \geq 1$ and finite field $\mathbb{F}_i$ over $\mathbb{F}$ such that $n = \sum_i d_i f_i m_i$ for $f_i = [\mathbb{F}_i : \mathbb{F}]$. Over perfect fields semisimple rings are absolutely semisimple, see Definition A.2.1 and Remark A.2.2, and thus $\mathbb{F}'[G]/\ker(D \otimes_{\mathbb{F}} \mathbb{F}') \xrightarrow{\simeq} \prod_{i=1}^{s} \prod_{j=1}^{f_i} M_{d_i}(\mathbb{F}')$. We conclude using Lemma 4.1.13, Example 4.1.7(ii) and Remark 4.1.12. ☐

Over algebraically closed field, the following consequence of Theorem 4.1.17 is important.

Theorem 4.1.19 ([Che14, Thm. 2.12])**.** *Suppose that k is an algebraically closed field and S is a k-algebra. If $D\colon S \to k$ is an n-dimensional pseudorepresentation, then there is a semisimple representation $\rho_D\colon S \to \mathrm{Mat}_n(k)$ unique up to isomorphism with associated pseudorepresentation D, and one has $\ker \rho_D^{\mathrm{lin}} = \ker D$.*

Definition 4.1.20. *Let k be a field and let $D \in \mathcal{P}s\mathcal{R}^n_G(k)$.*

(i) We call $\rho_{D\otimes_k k^{\mathrm{alg}}}$ from Theorem 4.1.19 the semisimple representation associated to $D \otimes_k k^{\mathrm{alg}}$.

(ii) We call D

  (1) irreducible *if $\rho_{D\otimes_k k^{\mathrm{alg}}}$ is irreducible, and reducible otherwise.*

  (2) multiplicity free *if $\rho_{D\otimes_k k^{\mathrm{alg}}}$ is a direct sum of pairwise non-isomorphic irreducible k^{alg}-linear representations of $S \otimes_k k^{\mathrm{alg}}$.*

  (3) split *if $D = D_\rho$ for some representation $\rho\colon S \to \mathrm{Mat}_n(k)$.*

Next we recall the concept of the Cayley-Hamilton property for pseudorepresentations.

Definition 4.1.21 ([Che14, 1.17])**.** *Let S be an A-algebra and let D be in $\mathcal{P}s\mathcal{R}^n_S(A)$.*

(i) The Cayley-Hamilton ideal $\mathrm{CH}(D)$ of D *is the 2-sided ideal of S generated by the coefficients of the polynomials*

$$\chi_{D,A[t_1,\ldots,t_m]}(s) \in S[t_1,\ldots,t_m]$$

where m ranges over all positive integers and s over all elements of $S[t_1,\ldots,t_m]$.[5]

(ii) One calls D Cayley-Hamilton *if $\mathrm{CH}(D) = 0$, or, equivalently, if χ_D is identically zero.*

Proposition 4.1.22 ([Che14, 1.20f.],[WE13, 1.1.8.6])**.** *For $D \in \mathcal{P}s\mathcal{R}^n_S(A)$ the following hold.*

(i) $\ker(D) \supset \mathrm{CH}(D)$*, and hence D factors via some $\widetilde{D} \in \mathcal{P}s\mathcal{R}^n_{S/\mathrm{CH}(D)}(A)$.*

(ii) If D is Cayley-Hamilton and $S' \subset S$ is any A-subalgebra, then $D|_{S'}$ is Cayley-Hamilton.

(iii) For any morphism $S \to S'$ in $\mathcal{A}lg_A$ one has $S'/\mathrm{CH}(D \otimes_S S') \cong (S/\mathrm{CH}(D)) \otimes_S S'$.

Definition 4.1.23 ([Che14, 1.17])**.** *Let S be an A-algebra and let D be in $\mathcal{P}s\mathcal{R}^n_S(A)$.*

(i) One calls $S^{\mathrm{CH}}_D := S/\mathrm{CH}(D)$ the Cayley-Hamilton quotient of S with respect to D.

(ii) One calls the induced A-algebra homomorphism $\rho^{\mathrm{CH}}_D\colon S \to S/\mathrm{CH}(D)$ D the Cayley-Hamilton representation attached to D.

[5] It suffices to let the s range over all elements of the form $\sum_{j=1}^{m} s_j t^j$ with $s_j \in S$.

Any pseudorepresentation $D \in \mathcal{P}s\mathcal{R}^n_S(A)$ possesses a factorization

$$S \xrightarrow{\rho^{\mathrm{CH}}_D} S^{\mathrm{CH}}_D \xrightarrow{\widetilde{D}} A \tag{14}$$

with $\widetilde{D}$ from Proposition 4.1.22(i). In the special case $S = A[G]$ the factorization is a composition of a group homomorphism $G \to (S^{\mathrm{CH}}_D)^\times$ with $\widetilde{D}$, i.e., $D = \widetilde{D} \circ \rho^{\mathrm{CH}}_D \colon G \to A$. Because of the following result and the good behavior of $\mathrm{CH}(\cdot)$ under base change, one might think of ρ^{CH}_D as a natural substitute of a representation ρ with $D = D_\rho$ also in cases when such a representation does not exist. First we need one more piece of notation.

Definition 4.1.24. *Let $B \in \mathcal{CA}lg_A$, $D \in \mathcal{P}s\mathcal{R}^n_G(B)$ and $X := \operatorname{Spec} B$. Let $x \in X$ with residue homomorphism $\pi_x \colon B \to \kappa(x)$, and let $\overline{x}$ be a geometric point of X above x.*

(i) We call $D_x := \pi_x \circ D$ the pseudorepresentation of D at x *and set $D_{\bar{x}} := D_x \otimes_{\kappa(x)} \kappa(x)^{\mathrm{alg}}$.*

(ii) We call $\rho_{\bar{x}} := \rho_{D_{\bar{x}}} \colon G \to \mathrm{GL}_n(\kappa(x)^{\mathrm{alg}})$ the (semisimple) representation at $\bar{x}$.[6]

(iii) We say that x has a property if $D_{\overline{x}}$ satisfies this property.

If B is a universal ring for some space of pseudorepresentations and $x \in \operatorname{Spec} B$, then writing D_x it will be implicitly understood that D refers to the corresponding universal pseudorepresentation.

The following is a significant generalization of Theorem 4.1.17 to families.

Proposition 4.1.25 (Cf. [Che14, Cor. 2.23])**.** *Let $D \in \mathcal{P}s\mathcal{R}^n_G(A)$ be such that D_x is irreducible for all $x \in \operatorname{Spec} A$. Then $C := A[G]^{\mathrm{CH}}_D$ is an A-Azumaya algebra of degree n, and $D = \det_C \circ \rho^{\mathrm{CH}}_D$ for $\rho^{\mathrm{CH}}_D \colon A[G] \to C$ the Cayley-Hamilton representation restricted to G.*

Remark 4.1.26. We shall use the notation ρ_D for $D \in \mathcal{P}s\mathcal{R}^n_G(A)$ in two situations: Either A is an algebraically closed field and then it is the semisimple representation ρ_D from Theorem 4.1.19. Or A is arbitrary and D_x is irreducible for all $x \in \operatorname{Spec} A$, and then it is an abbreviation for ρ^{CH}_D. Because of Proposition 4.1.25 this assignment is well-defined.

4.2 Universal rings of pseudorepresentations

Here we recall the existence of a universal pseudodeformation ring and that irreducible points form an open subscheme. Moreover we introduce morphisms related to the addition of pseudorepresentations.

Proposition 4.2.1 ([Che14, Prop. 1.6, Ex. 1.7])**.** *The functor $\mathcal{P}s\mathcal{R}^n_S(\,\cdot\,) \colon \mathcal{CA}lg_A \to Sets$ is representable for any S in $\mathcal{A}lg_A$ by some ring $R^{\mathrm{univ}}_{S,n} \in \mathcal{CA}lg_A$. Moreover for any $B \in \mathcal{CA}lg_A$, the natural map $B \otimes_A R^{\mathrm{univ}}_{S,n} \to R^{\mathrm{univ}}_{B \otimes_A S,n}$ is an isomorphism.*

The above means that there is a natural isomorphism $\mathrm{Hom}_{\mathcal{CA}lg}(R^{\mathrm{univ}}_{S,n}, \,\cdot\,) \to \mathcal{P}s\mathcal{R}^n_S(\,\cdot\,)$. Let the pseudorepresentation corresponding to $\mathrm{id}_{R^{\mathrm{univ}}_{S,n}}$ be

$$D^{\mathrm{univ}}_{S,n} \colon S \otimes_A R^{\mathrm{univ}}_{S,n} \longrightarrow R^{\mathrm{univ}}_{S,n}.$$

Definition 4.2.2. *The commutative A-algebra $R^{\mathrm{univ}}_{S,n}$ and the A-scheme $X^{\mathrm{univ}}_{S,n} := \operatorname{Spec} R^{\mathrm{univ}}_{S,n}$ are called the* n-dimensional universal pseudorepresentation ring *and* space*, respectively, and $D^{\mathrm{univ}}_{S,n}$ is called the* n-dimensional universal pseudorepresentation.

For $S = \mathbb{Z}[G]$, we abbreviate $R^{\mathrm{univ}}_{G,n} := R^{\mathrm{univ}}_{S,n}$, $D^{\mathrm{univ}}_{G,n} := D^{\mathrm{univ}}_{S,n}$ and $X^{\mathrm{univ}}_{G,n} := X^{\mathrm{univ}}_{S,n}$.

[6] We sometimes ignore the subtlety of geometric points and simply write ρ_x.

Remark 4.2.3. (a) In [Che14] the ring $R^{\text{univ}}_{S,n}$ is denoted by $\Gamma^n_A(S)^{\text{ab}}$; in our notation A is implicit in the structural map of S as an A-algebra.

(b) For A-schemes X there is an obvious notion of $\mathcal{O}(X)$-*valued pseudorepresentation* $S \to \mathcal{O}(X)$ *of dimension* n. The space $X^{\text{univ}}_{S,n}$ represents the resulting functor of pseudorepresentations on the category of A-schemes.

Example 4.2.4. *Recall the determinant* $\det(D) \in \mathcal{P}s\mathcal{R}^1_G(A)$ *of any* $D \in \mathcal{P}s\mathcal{R}^n_G(A)$ *from Definition 4.1.5. If we apply this to* $D^{\text{univ}}_{A[G],n}$*, we obtain*

$$\det(D^{\text{univ}}_{A[G],n}) \in \mathcal{P}s\mathcal{R}^1_{S\otimes_A R^{\text{univ}}_{A[G],n}}(R^{\text{univ}}_{A[G],n}).$$

The last assertion in Proposition 4.2.1 and the universality of $R^{\text{univ}}_{A[G],1}$ *now yields a homomorphism in* $\mathcal{CA}lg_A$

$$\det\colon R^{\text{univ}}_{A[G],1} \to R^{\text{univ}}_{A[G],n}.$$

and an induced morphism of schemes $\det\colon X^{\text{univ}}_{A[G],n} \to X^{\text{univ}}_{A[G],1}$*, which we both denote by* $\det$.

Lemma 4.2.5 ([Rob63, Thm. III.4])**.** *(i) The canonical map* $R^{\text{univ}}_{S_1\times S_2,n} \to \bigoplus_{i=0}^n R^{\text{univ}}_{S_1,i} \otimes R^{\text{univ}}_{S_2,n-i}$*, induced from the universal property of these rings, is an isomorphism in* $\mathcal{CA}lg_A$*.*

(ii) Let $B \neq 0$ *be in* $\mathcal{CA}lg_A$ *such that* $\operatorname{Spec} B$ *is connected. Then any* A*-algebra homomorphism* $R^{\text{univ}}_{S_1\times S_2,n} \to B$ *corresponding to* P *factors via some summand* $R^{\text{univ}}_{S_1,i} \otimes R^{\text{univ}}_{S_2,n-i}$ *in* (i).

Corollary 4.2.6 ([WE13, Lem. 1.1.11.7])**.** *Suppose* $n_1 + n_2 = n$ *for* $n_i \geq 0$*. Then the map*

$$\iota_{n_1,n_2}\colon X^{\text{univ}}_{S,n_1} \times_A X^{\text{univ}}_{S,n_2} \longrightarrow X^{\text{univ}}_{S,n}, (D_1, D_2) \mapsto D_1 \oplus D_2$$

is a morphism of affine A*-schemes that corresponds to the ring homomorphism*

$$R^{\text{univ}}_{S,n} \xrightarrow{\Delta} R^{\text{univ}}_{S\times S,n} \xrightarrow{4.2.5(\text{i})} R^{\text{univ}}_{S,n_1} \otimes R^{\text{univ}}_{S,n_2}$$

where Δ *is induced from the diagonal map* $S \to S \times S$ *and the universality of the rings.*

4.3 Generalized matrix algebras

Generalized matrix algebras are important in the study of Cayley-Hamilton pseudorepresentations over Henselian local rings, and were introduced for that purpose in [BC09, § 1.3] in the context of pseudocharacters. This subsection recalls some basic result. In Proposition 4.3.9 we shall generalize [BC09, Prop. 1.5.1], in Proposition 4.3.9 on the ideal of total reducibility to pseudorepresentations.

Definition 4.3.1 (Cf. [BC09, Def. 1.3.1])**.** *A* generalized matrix algebra (or simply GMA) over A (of type $(n_1, \ldots, n_r)$) *is an* A*-algebra* S *together with*

(a) a set of orthogonal idempotents $e_1, \ldots, e_r \in S$ *with* $\sum_{i=1}^r e_i = 1_S$*, and*

(b) a set of A*-algebra isomorphisms* $\psi_i : e_i S e_i \xrightarrow{\sim} \operatorname{Mat}_{n_i}(A)$ *for* $i = 1, \ldots, r$

such that the associated trace map $\tau : S \to A, x \mapsto \sum_{i=1} \operatorname{tr}(\psi_i(e_i x e_i))$ *is central, i.e., it satisfies* $\tau(xy) = \tau(yx)$ *for all* $x, y \in S$*. We write* $\mathcal{E} := \{e_i, \psi_i\}_{i=1,\ldots,r}$ *the* data of idempotents *of* S*. If we wish to emphasize the entire structure of a GMA we write* $(S, \mathcal{E})$ *instead of* S*. The dimension of* S *will be* $\sum_i n_i$*.*

Notation 4.3.2. *Let* S *be an GMA over* A *of type* $(n_1, \ldots, n_r)$*. For* $1 \leq i \leq r$ *and* $1 \leq k, l \leq n_i$ *we denote by* $E_i^{k,l}$ *the unique element in* $e_i S e_i$ *that maps under* ψ_i *to the matrix in* $\operatorname{Mat}_{n_i}(A)$ *that has 1 in the* (k,l)*-entry and everywhere else 0. For later use, we also introduce elements* $E^j := E^{i'}_{i+1}$ *for* $j = 1, \ldots, n$*, where* $i, i' \geq 1$ *are unique such that* $j = n_1 + \ldots + n_i + i'$ *with* $1 \leq i' \leq n_{i+1}$*. We write* $\mathcal{A}^j$ *for* $E^j S E^j$ *and* φ^j *for the isomorphism* $\mathcal{A}^j \to A$ *induced from* τ*.*

The following result explains why GMA are generalizations of matrix algebras.

Lemma 4.3.3 (Structure of a GMA [BC09, p. 21ff.]). *(i) Attach to a GMA $(S, \mathcal{E})$ over A of type $(n_1, \dots, n_r)$ the following data:*

(1) A-modules $\mathcal{A}_{i,j} := E_i^{1,1} S E_j^{1,1}$ for $1 \le i, j \le r$,

(2) isomorphisms $\mathcal{A}_{i,i} \cong A$ under τ for $i = 1, \dots, r$,

(3) A-linear maps $\varphi_{i,j,k} : \mathcal{A}_{i,j} \otimes_A \mathcal{A}_{j,k} \to \mathcal{A}_{i,k}$ induced from the product in S.

Then they satisfy the following conditions:

(UNIT) *For $1 \le i, j \le r$ we have $\mathcal{A}_{i,i} = A$ and both $\varphi_{i,i,j}$ and $\varphi_{i,j,j}$ agree with the A-module structure on $A_{i,j}$.*

(ASSO) *For $1 \le i, j, k, l \le r$ and $x \otimes y \otimes z \in \mathcal{A}_{i,j} \otimes_A \mathcal{A}_{j,k} \otimes_A \mathcal{A}_{k,l}$ we have*

$$\varphi_{i,k,l}\big(\varphi_{i,j,k}(x \otimes y) \otimes z\big) = \varphi_{i,j,l}\big(x \otimes \varphi_{j,k,l}(y \otimes z)\big) \quad \textit{in } \mathcal{A}_{i,l}.$$

(COMM) *For $1 \le i, j \le r$, $x \in \mathcal{A}_{i,j}$ and $y \in \mathcal{A}_{j,i}$ we have $\varphi_{i,j,i}(x \otimes y) = \varphi_{j,i,j}(y \otimes x)$.*

Then the structures in (1)–(3) induce an A-algebra structure on

$$\begin{pmatrix} \mathrm{Mat}_{n_1}(\mathcal{A}_{1,1}) & \cdots & \mathrm{Mat}_{n_1,n_r}(\mathcal{A}_{1,r}) \\ \vdots & \ddots & \vdots \\ \mathrm{Mat}_{n_r,n_1}(\mathcal{A}_{r,1}) & \cdots & \mathrm{Mat}_{n_r}(\mathcal{A}_{r,r}) \end{pmatrix} \tag{15}$$

and the latter is isomorphic to S.

(ii) Conversely, suppose we are given a family $(\mathcal{A}_{i,j})_{1 \le i,j \le r}$ of A-modules together with A-linear maps $\varphi_{i,j,k} : \mathcal{A}_{i,j} \otimes_A \mathcal{A}_{j,k} \to \mathcal{A}_{i,k}$ for $1 \le i, j, k \le r$ satisfying the above conditions (UNIT), (ASSO) and (COMM). Then there is a unique structure of a GMA of type $(n_1, \dots, n_r)$ on the A-module $S := \oplus_{i,j=1}^r \mathrm{Mat}_{n_i,n_j}(\mathcal{A}_{i,j})$.

Next we provide some technical lemmas:

Lemma 4.3.4. *Let S be a GMA over A of type $(n_1, \dots, n_r)$ over A, and $B \in \mathrm{Ob}(\mathcal{CA}lg_A)$. Then $S \otimes_A B$ is a GMA over B of type $(n_1, \dots, n_r)$.*

The proof of Lemma 4.3.4 is straightforward and left as an exercise.

Proposition 4.3.5 ([WE18, Prop. 2.23]). *Given a GMA $(S, \mathcal{E})$ over A of dimension n, there exists a natural n-dimensional Cayley-Hamilton pseudorepresentation $\det_{(S,\mathcal{E})} : S \to A$, called the* determinant of the GMA $(S, \mathcal{E})$, *and given, for any B in $\mathcal{CA}lg_A$, by the formula*

$$\det_{(S.\mathcal{E})}(x) = \sum_{\sigma \in \mathfrak{S}_d} \mathrm{sgn}(\sigma) \prod_{\text{cycles } \gamma \text{ of } \sigma} \varphi^{l_0}\Big(\prod_{l \in \gamma} E^l x E^{\sigma(l)}\Big)$$

for any $x \in R \otimes_A B$. Here the product is first over the cycles γ in the cycle decomposition of σ and then over the elements l of the cycle γ taken in the order that they appear in the cycle, where l_0 is the initial element of γ chose. We also have $\tau = \Lambda^1_{D_\mathcal{E}}$.

The next results are auxiliary for Proposition 4.3.9 on the locus of reducibility of a GMA.

Lemma 4.3.6 ([Che14, Lem. 1.12(i)]). *One has $D(1 + ss') = D(1 + s's)$ for all $s, s' \in S$.*

Lemma 4.3.7. *Let $(S, \mathcal{E})$ be a GMA and let $D : S \to A$ a pseudorepresentation. Then for any $x \in \mathrm{Mat}_{n_i \times n_j}(\mathcal{A}_{i,j})$ for some $1 \le i, j \le r$ with $i \ne j$, we have $D(1 + e_i x e_j) = 1$.*

Proof. By Lemma 4.3.6 we have $D(1+e_ixe_j) = D(1+e_je_ix) = D(1) = 1$. □

Lemma 4.3.8 ([Che14, Lem. 2.4])**.** *Let S be an A-algebra, $e \in S$ be an idempotent, and $D\colon S \to A$ be a pseudorepresentation of dimension n. Suppose that $\operatorname{Spec}(A)$ is connected.*

(i) The polynomial law $D_e\colon eSe \to A$, $s \mapsto D(s+1-e)$, is a pseudorepresentation of dimension $r(e) \le n$. One has $r(1-e)+r(e) = n$.

(ii) The restriction of D to the A-subalgebra $eSe \oplus (1-e)S(1-e)$ is the sum $D_e \oplus D_{1-e}$. It is a pseudorepresentation of dimension n.

(iii) If D is faithful or Cayley-Hamilton, then D_e is faithful or Cayley-Hamilton, respectively.

(iv) Suppose that D is Cayley-Hamilton. Then $e = 1$ if and only if $D(e) = 1$, and $e = 0$ if and only if $r(e) = 0$. If $e_1, \dots, e_s$ is a family of nonzero orthogonal idempotents of S, then $s \le n$ and $\sum_{i=1}^s r(e_i) \le n$. Further, $\sum_{i=1}^s r(e_i) = n$ if and only if $\sum_{i=1}^s e_i = 1$.

The next result is the adaption of [BC09, Prop. 1.5.1] to pseudorepresentations.

Proposition 4.3.9. *Let $(S, \mathcal{E})$ be a GMA over A and let $\mathcal{A}_{i,j}$ and $\varphi_{i,j,k}$ be as in Lemma 4.3.3. Define $I = \sum_{i \ne j} \mathcal{A}_{i,j}\mathcal{A}_{j,i}$ as the ideal of total reducibility in A.*

(i) (1) If $I = 0$, then the map $\pi\colon S \to \sum_i e_iSe_i \subset S, x \mapsto \sum_i e_ixe_i$ is a ring homomorphism.

(2) Denoting by D_i the map $e_iSe_i \xrightarrow{\psi_i} \operatorname{Mat}_{n_i}(A) \xrightarrow{\det} A$ for $i = 1, \dots, r$, one has

$$\det_{(S,\mathcal{E})} = \oplus_{i=1}^r D_i \circ \pi \mod I.$$

(ii) Suppose that there exist m_i-dimensional pseudorepresentations $D'_i\colon S \to A$ with $m_i > 0$ for $i \in \{1, \dots, r\}$ such that one has $\det_{(S,\mathcal{E})} = \oplus_{i=1}^r D'_i$. Then $I = 0$ and for a unique permutation $\sigma \in \mathfrak{S}_r$ we have $D'_{\sigma(i)} = D_i \circ \pi$ with D_i and π from (i).

Proof. Part (1) of (i) is a straightforward matrix calculation using $\mathcal{A}_{i,j}\mathcal{A}_{j,i} = 0$ for all $i \ne j$ from $\{1, \dots, r\}$. To see part (2) of (i) note that by our definitions we have the explicit formula

$$D_i \mod I\colon e_iSe_i \longrightarrow A/I, \quad x \longmapsto \sum_{\sigma_i \in \mathfrak{S}_{n_i}} \operatorname{sgn}(\sigma_i) \prod_{j=1}^{n_i} E_{i,j}xE_{i,\sigma_i(j)} \mod I,$$

and using distributivity for $x \in S$

$$\begin{aligned}\prod_{i=1}^r (D_i \circ \pi)(x) \mod I &= \prod_{i=1}^r \sum_{\sigma_i \in \mathfrak{S}_{n_i}} \operatorname{sgn}(\sigma_i) \prod_{j=1}^{n_i} E_{i,j}xE_{i,\sigma_i(j)} \mod I \\ &= \sum_{\sigma_1 \in \mathfrak{S}_{n_1}} \dots \sum_{\sigma_r \in \mathfrak{S}_{n_r}} \prod_{i=1}^r \operatorname{sgn}(\sigma_i) \prod_{j=1}^{n_i} E_{i,j}xE_{i,\sigma_i(j)} \mod I.\end{aligned}$$

Now in the sum $\operatorname{sgn}(\sigma)\prod_{i=1}^r \prod_{j=1}^{n_i} f^{\mathrm{univ}}_{i,\mathbf{i}(\sigma(i,j))}(E_{i,j}xE_{\sigma(i,j)})$, modulo I only those summands are nonzero for which $\sigma \in \mathfrak{S}_n$ satisfies $\mathbf{i}(\sigma(i,j)) = i$. Therefore, in a nonzero summand we can write $\sigma = (\sigma_1, \dots, \sigma_r) \in \mathfrak{S}_{n_1} \times \dots \times \mathfrak{S}_{n_r}$ and

$$\begin{aligned}\det_{(S,\mathcal{E})}(x) &= \sum_{(\sigma_1,\dots,\sigma_r) \in \mathfrak{S}_{n_1} \times \dots \times \mathfrak{S}_{n_r}} \prod_{i=1}^r \operatorname{sgn}(\sigma_i) \prod_{j=1}^{n_i} f_{i,i}(E_{i,j}xE_{i,\sigma_i(j)}) \mod I \\ &= \sum_{\sigma_1 \in \mathfrak{S}_{n_1}} \dots \sum_{\sigma_r \in \mathfrak{S}_{n_r}} \prod_{i=1}^r \operatorname{sgn}(\sigma_i) \prod_{j=1}^{n_i} f_{i,i}(E_{i,j}xE_{i,\sigma_i(j)}) \mod I.\end{aligned}$$

This completes the proof of (i).

We now prove (ii). In a first step, we show the **claim** that there is a unique permutation $\sigma \in \mathfrak{S}_r$ such that $D_i = (D'_{\sigma(i)})_{e_i}$ and $(D'_{i'})_{e_i} = 1$ for $i' \neq \sigma(i)$. For this, we restrict $\oplus_{i'=1}^r D'_{i'}$ to $e_i S e_i$, so that

$$D_i = (\det_{(S,\mathcal{E})})_{e_i} = \oplus_{i'}(D'_{i'})_{e_i}.$$

By Lemma 4.3.8 the $(D'_{i'})_{e_i}$ are pseudorepresentations, and one has $m_{i'} \geq m_{i',i} := \dim(D'_{i'})_{e_i}$. Now under addition in the sense of Corollary 4.2.6 dimensions are added, and it follows that

$$n_i = \sum\nolimits_{i'=1}^{r} m_{i',i}.$$

Since $e_i S e_i = \mathrm{Mat}_{n_i}(A)$ it follows from Example 4.1.7(ii) that each $m_{i',i}$ is divisible by n_i. Hence there is a map $\sigma\colon \{1,\ldots,r\} \to \{1,\ldots,r\}$ such that $m_{\sigma(i),i} = n_i$ and $m_{i',i} = 0$ for $i' \neq \sigma(i)$, and moreover $D_i = (D'_{\sigma(i)})_{e_i}$. The uniqueness of σ is clear from the construction. It remains to show that σ is bijective. It will suffice to show that σ is surjective.

For this, let $S'_{i'} := \oplus_{i\in\sigma^{-1}(i')} e_i S e_i$, so that $S = \oplus_{i'} S'_{i'}$. The restriction of $D'_{i''}$ to $S'_{i'}$ is zero if $i'' \neq i'$, and the restriction of $D'_{i'}$ to $S'_{i'}$ is a pseudorepresentation with

$$m_{i'} \overset{4.3.8}{\geq} \dim D'_{i'}|_{S'_{i'}} = \dim \oplus_{i''=1}^{r} D'_{i''}|_{S'_{i'}} = \dim \det_{(S,\mathcal{E})}|_{S'_{i'}} = \sum_{i\in\sigma^{-1}(i')} n_i.$$

Summing over all i' in the image of σ implies $\sum_{i'\in\sigma(\{1,\ldots,r\})} m_{i'} \geq n$. However, all $m_{i'}$ are strictly positive and $\sum_{i'=1}^{r} m_{i'} = n$, and this implies that σ is surjective, and hence the claim is proved.

For simplicity of notation we assume from here on, without loss of generality, that $\sigma = \mathrm{id}$. We now show that $I = 0$. For this, it suffices to show that $\mathcal{A}_{i,j}\mathcal{A}_{j,i} = 0$ for all $i \neq j$. By restricting to the subalgebra $S' = e_i S e_i + e_j S e_j + e_i S e_j + e_j S e_i$ with $\mathcal{E}' = (e_i, \psi_i, e_j, \psi_j,)$, i.e., by considering $D_{e_i+e_j}$, and using $\det_{(S,\mathcal{E})}|_{S'} = \det_{(S',\mathcal{E}')}$, we may assume $r = 2$ for the proof of $I = 0$.

Let b be in $\mathcal{A}_{1,2}$ and c in $\mathcal{A}_{2,1}$, and write x for $e_1 E_1^{1,1} b E_2^{1,1} e_2$ and y for $e_2 E_2^{1,1} c E_1^{1,1} e_1$ with $E_i^{k,l}$ from Notation 4.3.2. Using the description of GMA's from Lemma 4.3.3 one easily verifies that

$$1 + xy = 1 + E_1^{1,1} bc \in e_1 S e_1 + (1 - e_1),\ 1 + yx = 1 + E_2^{1,1} bc \in (1 - e_2) + e_2 S e_2.$$

Note moreover that by Lemma 4.3.6 we have $D(1+xy) = D(1+yx)$ for every pseudorepresentation $D\colon S \to A$. If we apply this to D'_i and our earlier observations on $(D'_i)_{e_{i'}}$, we find that

$$D'_i(1 + xy) = D'_i(1 + yx) = 1$$

for $i = 1, 2$ and hence from hypothesis (2) that $\det_{(S,\mathcal{E})}(1 + E_1^{1,1} bc) = 1$. From the formula for $\det_{(S,\mathcal{E})}$ on $e_1 S e_1 + e_2 S e_2 \cong \mathrm{Mat}_{n_1}(A) \times \mathrm{Mat}_{n_2}(A)$, we deduce that

$$\det_{(S,\mathcal{E})}(1 + E_1^{1,1} bc) = 1 + bc,$$

and hence that $bc = 0$, as was to be shown.

For the second assertion, observe that by Lemma 4.3.7 we have $D'_i(1 + e_i x e_j) = 1$ for any $i \neq j$ and $x \in \mathrm{Mat}_{n_i,n_j}(\mathcal{A}_{i,j})$. It follows that $D'_i(1 + u) = 1$ for any u in the kernel of π. And now the second assertion follows from knowing the restriction of D'_i to $\sum_i e_i S e_i$ given in the first claim of the proof of (ii). □

The following result of Chenevier shows one relation between GMA's and pseudorepresentations.

Theorem 4.3.10 ([Che14, Thm. 2.22])**.** *Assume that A is a henselian local ring with maximal ideal m_A and residue field $\kappa(A)$. Let S be an A-algebra and suppose that $D \in \mathcal{P}s\mathcal{R}_S^n(A)$ is Cayley-Hamilton. Denote by $\overline{D} = D \otimes_A \kappa(A)\colon S/m_A S \longrightarrow \kappa(A)$ the residual pseudorepresentation of D. Suppose that $\overline{D}$ is split. Then the following hold:*

(i) If $\overline{D}$ is irreducible, then $D = \det \circ \rho$ for some A-algebra isomorphism $\rho\colon S \xrightarrow{\sim} \mathrm{Mat}_n(A)$.

(ii) If $\overline{D}$ is multiplicity free, then S is a generalized matrix algebra $(S, \mathcal{E})$ and $D = \det_{(S,\mathcal{E})}$.

4.4 Continuous Pseudorepresentations

In our application, mainly continuous pseudorepresentations (of a profinite group G) will play a role. In this subsection we will recall this concept and some of its properties. We denote throughout this subsection be G a profinite group. Let us refer to [Gro60, Ch. 0 § 7, Ch. 1 § 10] for a more thorough introduction to topological rings and formal schemes.

We introduce in Definition 4.4.2 a category of admissible κ-algebras that is perhaps not standard. In Proposition 4.4.7 we prove a finiteness statement for continuous pseudorepresentations of G_K with K p-adic and values in a finite field of characteristic p.

Definition 4.4.1 (Cf. [Che14, § 2.30])**.** *Let A be a commutative topological ring. Then $D \in \mathcal{P}s\mathcal{R}^n_G(A)$ is called* continuous *if and only if the characteristic polynomial functions (restricted to G) $\Lambda_{D,i}\colon G \to A$ are continuous for $i = 1, \dots, n$.*

We shall study continuity only for two types of commutative rings A that we now describe. Consider a directed set J with minimal element 0 and an inverse system A_λ, $\lambda \in J$, of topological commutative rings with continuous transition maps and such that $A_\lambda \to A_0$ is surjective with nilpotent kernel for any $\lambda \in J$. Then the inverse limit

$$\lim_{\lambda \in J} A_\lambda \tag{16}$$

is a topological ring with respect to the weakest topology for which the projections to all A_λ are continuous.

Definition 4.4.2. *Let κ be a local or a finite field with its natural topology.*

(i) We say that a commutative topological ring A is κ-admissible *if there is an inverse system $(A_\lambda)_{\lambda \in J}$ as above (16) and an isomorphism of topological rings $A \cong \lim_{\lambda \in J} A_\lambda$ such that each A_λ is a finite-dimensional continuous κ-algebra with the natural topology of a finite dimensional κ-vector space.*

(ii) We denote by $\mathcal{A}\mathrm{dm}_\kappa$ the category whose objects are κ-admissible commutative topological rings and whose morphisms are continuous κ-algebra homomorphisms.

Note that $\widehat{\mathcal{A}r}_\kappa$ is a full subcategory of $\mathcal{A}\mathrm{dm}_\kappa$; but objects in $\mathcal{A}\mathrm{dm}_\kappa$ are in general only semilocal, and with residue field of finite κ-dimension.

Definition 4.4.3 ([Che14, § 3.9])**.** *Let $W(\mathbb{F})$ be the topological ring of Witt vectors over $\mathbb{F}$.*

(i) A commutative topological ring A is admissible *if there is an inverse system $(A_\lambda)_{\lambda \in J}$ as above (16) and an isomorphism of topological rings $A \cong \lim_{\lambda \in J} A_\lambda$ such that each A_λ carries the discrete topology.*

(ii) We denote by $\mathcal{A}\mathrm{dm}_{W(\mathbb{F})}$ the category whose objects are admissible commutative topological rings A together with a continuous homomorphism $W(\mathbb{F}) \to A$ and whose morphisms are continuous $W(\mathbb{F})$-algebra homomorphisms.

Remark 4.4.4. Suppose A is admissible or κ-admissible, and suppose that $A = \lim_{\lambda \in J} A_\lambda$ for an inverse system $(A_\lambda)_{\lambda \in J}$ as in the above definitions. Then one can form the completed group ring as the inverse limit

$$A[[G]] := \lim_{\lambda, H} A_\lambda[G/H],$$

where H ranges over all open normal subgroups of G; it contains $A[G]$ and is in fact the completion of $A[G]$ with respect to the topology of $A[G]$ inherited from $A[[G]]$.

Using Amitsur's formula, one can verify that the above definition of continuity is equivalent to the condition that for every commutative continuous A-algebra B, with $B \in \mathcal{A}\mathrm{dm}$ or $\mathcal{A}\mathrm{dm}_\kappa$, respectively, the map $D_B\colon B[G] \to B$ is continuous; see [WE13, Def. 3.1.0.10]. This allows one also to extend D_B to a (continuous) pseudorepresentation $B[[G]] \to B$.

The following is the basic result on continuity if A is discrete.

Lemma 4.4.5 ([Che14, Lem. 2.33]). *Let A be a discrete and let $D\colon A[G] \to A$ be a pseudorepresentation. Then D is continuous if and only if $\ker(D)$ is contained in the kernel of the canonical map $A[G] \to A[G/H]$ for some normal open subgroup $H \subset G$. In this case, the natural representation $G \longrightarrow (B[G]/\ker(D))^\times$ factors through G/H.*

We record the following consequence:

Corollary 4.4.6 ([Che14, Exem. 2.34]). *Let k be a discrete field and let $D \in \mathcal{P}s\mathcal{R}^n_G(k^{\mathrm{alg}})$ be continuous. Then the representation $\rho_D\colon G_K \to \mathrm{GL}_n(k^{\mathrm{alg}})$ associated by Theorem 4.1.19 is continuous, its image is finite and it is defined over a finite extension of k.*

Proof. We provide a proof, as none is given in [Che14, Exem. 2.34]: Because D is continuous, Lemma 4.4.5 shows that $\ker D$ contains the kernel of $k[G] \to k[G/H]$ for some open subgroup H of G. By Theorem 4.1.19, the kernels of ρ_D^{lin} and of D are the same, and hence ρ_D is continuous since it is trivial on the open subgroup H. Since G/H is finite, this also shows that $\rho_D(G) \subset \mathrm{GL}_n(k^{\mathrm{alg}})$ is finite. It follows that the entries of the matrices in the image of ρ_D lie in a finite extension of k, and this proves the last assertion. ☐

When combined with earlier results, we deduce the following finiteness statement:

Proposition 4.4.7. *Let $\mathbb{F}$ be a finite field of characteristic p and let $n \geq 1$ be an integer. Then there exist only finitely many continuous pseudorepresentations $\overline{D}\colon G_K \to \mathbb{F}$ of dimension n.*

Denote by $\mathbb{F}' \supset \mathbb{F}$ the unique field extension of degree $n!$. Then for any $\overline{D}$ as above $\overline{D} \otimes_{\mathbb{F}} \mathbb{F}'$ is a direct sum of split irreducible pseudorepresentations $\overline{D}_i\colon G_K \to \mathbb{F}'$.

Proof. The second part is immediate from Corollary 4.1.18. Hence it suffices to prove the first part for split irreducible $\overline{D}$. It moreover suffices to assume that $\mathbb{F}$ contains the unique extension of the residue field of K of degree $n!$. The result follows from Lemma A.3.1. ☐

The next result shows the existence of a minimal ring of definition for any continuous pseudorepresentation, and it gives an important result on their structure.

Lemma 4.4.8 ([Che14, Lem. 3.10]). *Let A be in $\mathcal{A}\mathrm{dm}_{W(\mathbb{F})}$, let $D\colon G \to A$ be a continuous pseudorepresentation, and let $C \subset A$ be the closure of the $W(\mathbb{F})$-algebra generated by the characteristic polynomial coefficients $\Lambda_{D,i}(g)$ for $g \in G$ and $i \geq 1$.*

(i) The ring C is an admissible profinite subring of A. In particular, $C = \varprojlim_i C_i$ is a finite product of local $W(\mathbb{F})$-algebras with finite residue fields.

(ii) If further $\iota\colon A \longrightarrow A'$ is a continuous $W(\mathbb{F})$-algebra homomorphism, $D'\colon G \to A'$ is the induced pseudorepresentation and $C' \subset A'$ is the closure $C' \subset A'$ of the $W(\mathbb{F})$-algebra generated by the characteristic polynomial coefficients $\Lambda_{D',i}(g)$ for $g \in G$ and $i \geq 1$, then ι induces a surjection $C \to C'$ in $\mathcal{A}\mathrm{dm}_{W(\mathbb{F})}$.

We use the Lemma 4.4.8 to make the following useful definitions.

Definition 4.4.9 ([Che14, Def. 3.11]). *For a finite field $\mathbb{F}$ one defines*

$$|G(n)| := \{z \in \mathrm{Spec}(R^{\mathrm{univ}}_{W(\mathbb{F})[G],n}) \,:\, z \text{ is closed and } \kappa(z) \text{ is finite}\}.$$

Definition 4.4.10 ([Che14, Def. 3.12]). *Let A be in $\mathcal{A}\mathrm{dm}_{W(\mathbb{F})}$, let $D \in \mathcal{P}s\mathcal{R}^n_G(A)$ be continuous, let $C \subset A$ be the ring from Lemma 4.4.8 and let $D_C\colon G \to C$ be the pseudorepresentation from Proposition 4.1.10.*

(i) We call C the ring of definition of D over $W(\mathbb{F})$.

(ii) If if C is local, so that $\kappa(C)$ is finite, one calls D residually constant.

(iii) One calls D residually equal *to D_z for some $z \in |G(n)|$, if C is local and $D_z \cong D_C \otimes_C \kappa(C)$.*

4.5 Twisting of pseudorepresentations

In this subsection, we introduce a twisting operation for pseudorepresentations that is the analog of the twist of a representation by a character, and we state some of its basic properties. Our approach require us to recall a number of results on the universal pseudorepresentation that go back to Roby. Our main construction is only carried out for pseudorepresentation of a topological group G. Our exposition of background material follows [WE13, 1.1].

Definition 4.5.1. *Let M be an A-module. The commutative A-algebra $\Gamma_A(M)$ is the quotient algebra of the polynomial algebra generated by the symbols $m^{[i]}$, $m \in M$, $i \in \mathbb{N}$, subject to the relations*

(i) $m^{[0]} = 1$ *for all* $m \in M$,

(ii) $(am)^{[i]} = a^i m^{[i]}$ *for* $a \in A$, $m \in M$, $i \in \mathbb{N}$,

(iii) $m^{[i]}m^{[j]} = \binom{i+j}{i} m^{[i+j]}$ *for* $m \in M$, $i, j \in \mathbb{N}$ *and*

(iv) $(m+n)^{[i]} = \sum_{j=0}^{i} m^{[j]} n^{[i-j]}$ *for* $m, n \in M$ *and* $i \in \mathbb{N}$,

The ring $\Gamma_A(M)$ is a graded A-algebra $\Gamma_A(M) = \bigoplus_{i \geq 0} \Gamma_A^i(M)$ with its i-th graded piece $\Gamma_A^i(M)$ being the A-module generated by the element $m^{[i]}$, $m \in M$. The A-algebra $\Gamma_A(M)$ is called the divided power algebra of M. The construction $M \mapsto \Gamma_A(M)$ defines a functor from A-modules to graded A-algebras. If $\varphi\colon M \to N$ is an A-module homomorphism, the induced map $\Gamma_A(\varphi)\colon \Gamma_A(M) \to \Gamma_A(N)$ is characterized by $m^{[i]} \mapsto (\varphi(m))^{[i]}$, $m \in M$, $i \in \mathbb{N}$. One has compatibility with base change, i.e., natural isomorphisms $\Gamma_A^d(M) \otimes_A B \cong \Gamma_B^d(M \otimes_A B)$.

Definition 4.5.2. *The universal degree d homogenous polynomial law $L_M^d \in \mathcal{P}_A(M, \Gamma_A^d(M))$ is defined by the maps*

$$L_{M,B}^d\colon M \otimes_A B \longrightarrow \Gamma_A^d(M) \otimes_A B \cong \Gamma_B(M \otimes_A B), m \otimes b \mapsto (bm)^{[i]}, \quad m \in M, b \in B.$$

The universality of L^d is expressed by the following result:

Theorem 4.5.3 ([Rob63, Thme. IV.1])**.** *Let M, N be two A-modules and let d be in $\mathbb{N}$. There is a canonical isomorphism*

$$\mathrm{Hom}_A(\Gamma_A^d(M), N) \xrightarrow{\simeq} \mathcal{P}_A^d(M, N), f \longmapsto f \circ L_M^d.$$

To describe the map in the converse direction, let $P \in \mathcal{P}_A^d(M, N)$. Define $I_d := \{\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d \mid \sum \alpha_i = d\}$. Given $\alpha = (\alpha_1, \ldots, \alpha_d) \in I_d$ set $T^\alpha = \prod_{j=1}^d T_j^{\alpha_j}$ for indeterminates $(T_1, \ldots, T_d)$, and set $m^{[\alpha]} = \prod_{j=1}^d m_i^{[\alpha_i]} \in \Gamma_A^d(M)$ for $m = (m_1, \ldots, m_d) \in M^d$. Define now for all $\alpha \in I_d$ simultaneously maps $P^{[\alpha]}\colon M^d \to N$ by

$$P_{A[T_1,\ldots,T_d]}(T_1 m_1 + \ldots + T_d m_d) = \sum_{\alpha \in I_d} P^{[\alpha]}(m) T^\alpha$$

for $m = (m_1, \ldots, m_d) \in M^d$. In the proof of Theorem 4.5.3 by Roby, it is shown that given any $P \in \mathcal{P}_A^d(M, N)$, there exists an A-module homomorphism $f\colon \Gamma_A^d(M) \to N$ such that

$$f\big(m^{[\alpha]}\big) = P^{[\alpha]}(m), \quad \forall \alpha \in I_d \text{ and } m \in M^d, \tag{17}$$

and that $f \circ L_M^d = P$.

If M is a free A-module, the A-module $\Gamma_A^d(M)$ has the following explicit description.

Theorem 4.5.4 ([Rob63, Thme. IV.2]). *Suppose that M is a free A-module with basis $(e_i)_{i\in I}$. Then for $d \in \mathbb{N}$, the A-module $\Gamma^d_A(M)$ is free with basis*

$$\Big\{e_{i_1}^{[k_1]} \cdot \ldots \cdot e_{i_j}^{[k_h]} \mid h \in \mathbb{N}, (i_1, \ldots, i_h) \in I^h, (k_1, \ldots, k_h) \in \mathbb{N}_{\geq 1}, \sum_{j=1}^{h} k_h = d\Big\}.$$

If M is an A-algebra R, then [Rob80] defines an A-algebra structure on each $\Gamma^d_A(R)$, different from that on $\Gamma_A(R)$, by defining a multiplication $\Gamma^d_A(R) \otimes_A \Gamma^d_A(R) \to \Gamma^d_A(R)$. This multiplication is defined as a composition of two maps. The first map exists for any A-module M, the second is built from the ring structure of R. Let first M be an arbitrary A-module. Then the map $\beta_M : M \oplus M \to M \otimes_A M, (m, m') \mapsto m \otimes m'$ is a homogeneous polynomial law of degree 2, and thus $L^d_{M\otimes M} \circ \beta_M$ lies in $\mathcal{P}^{2d}_A(M \oplus M, M \otimes M)$. By Theorem 4.5.3 we have $L^d_{M\otimes M} \circ \beta_M = \eta_M \circ L^{2d}_{M\oplus M}$ for a unique A-linear map

$$\eta_M : \Gamma^{2d}_A(M \oplus M) \to \Gamma^d_A(M \otimes_A M).$$

[Rob63, Thme. III.4] gives an isomorphism $\bigoplus_{i=0}^{e} \Gamma^i_A(M) \otimes \Gamma^{e-i}_A(M) \to \Gamma^e_A$ for any $e \in \mathbb{N}$. It is further shown in [Rob80, p. 869] that the maps $\Gamma^i_A(M) \otimes \Gamma^{2d-i}_A(M) \to \Gamma^d_A(M \otimes M)$ induced from η_M are zero for $i \neq d$, and that the induced map $\widetilde{\eta}_M : \Gamma^d_A(M) \otimes \Gamma^d_A(M) \to \Gamma^d_A(M \otimes M)$ is given by the explicit formula

$$\widetilde{\eta}_M\big(m^{[\alpha]} \otimes n^{[\beta]}\big) = \sum_{\gamma \in M^{\alpha,\beta}_{d\times d}(\mathbb{N})} \prod_{(i,j)\in\{1,\ldots,d\}^2} (m_i \otimes n_j)^{[\gamma_{ij}]}, \tag{18}$$

for $m, n \in M^d$, $\alpha, \beta \in I_d$, and where $M^{\alpha,\beta}_{d\times d}(\mathbb{N})$ denotes the set of all matrices $\gamma = (\gamma_{ij})$ in $M_{d\times d}(\mathbb{N})$ whose rows sum to β and whose columns sum to α. Let now $M = R$ be an A algebra. Then the multiplication map $\mu_R : R \otimes_A R \to R$ is A-linear, and thus it induces a graded map $\Gamma_A(\mu_R)$ whose d-th graded piece is a homomorphism $\Gamma^d_A(\mu_R) : \Gamma^d_A(R \otimes R) \to \Gamma^d_A(R)$. Roby defines

$$\mu^d_R := \Gamma^d_A(\mu_R) \circ \widetilde{\eta}_R : \Gamma^d_A(R) \otimes_A \Gamma^d_A(R) \to \Gamma^d_A(R)$$

It is shown in [Rob80, p. 870] that if R is unital, associative or commutative, respectively, then the same property holds for $\Gamma^d_A(R)$ with the multiplication μ^d_R, for any $d \in \mathbb{N}$. It turns out that L^d_R is multiplicative with respect to this multiplication on $\Gamma^d_A(R)$. The key result is the following:

Theorem 4.5.5 ([Rob80, Thme.]). *For A-algebras S, S', the following map is a bijection*

$$\mathrm{Hom}_{A\text{-}\mathcal{A}lg}(\Gamma^d_A(S), S') \to \mathcal{M}^d_A(S, S'), f \mapsto f \circ L^d_R.$$

Suppose now that $R = A[G]$ for a group G. Note that the elements of G form an A-basis of $A[G]$, and hence an A-basis of $\Gamma^n_A(A[G])$ is described in Theorem 4.5.4. Let $D : G \to A$ be a pseudorepresentation of dimension d. From Theorems 4.5.3 and 4.5.5, und using (17), we deduce:

Proposition 4.5.6. *There exists a unique homomorphism $f_D : \Gamma^d_A(A[G]) \to A$ such that*

$$f_D\big(g^{[\alpha]}\big) = D^{[\alpha]}(g), \quad \forall \alpha \in I_d \text{ and } g \in G^d.$$

It is multiplicative for the product on $\Gamma^d_A(A[G])$ given by $\mu^d_{A[G]}$.

Let now $\chi : G \to A^\times$ be a group homomorphism. Define for $\alpha = (\alpha_1, \ldots, \alpha_d)$ and $g = (g_1, \ldots, g_d) \in G^d$ the notation $\chi(g^{[\alpha]})$ to by $\chi(g^{[\alpha]}) := \prod_{i=1}^{d} \chi(g_i)^{\alpha_i}$. Because $\{g^{[\alpha]} \mid \alpha \in I_d, g \in G^d\}$ is a basis of $\Gamma^d_A(A[G])$ we have a unique A-linear map $f_{D,\chi} : \Gamma^d_A(A[G]) \to A$ such that

$$f_{D,\chi}\big(g^{[\alpha]}\big) = D^{[\alpha]}(g) \cdot \chi(g^{[\alpha]}), \quad \forall \alpha \in I_d \text{ and } g \in G^d.$$

Proposition 4.5.7. *Suppose that $D \in \mathcal{P}s\mathcal{R}^n_G(A)$ and that $\chi\colon G \to A^\times$ is a group homomorphism. Then the following hold:*

(i) The map $f_{D,\chi}$ defined above is multiplicative.

Define the d-dimensional pseudorepresentation $D \otimes \chi$ to be $f_{D,\chi} \circ L^d_{A[G]}$.

(ii) The characteristic polynomial coefficients of $D \otimes \chi$ satisfy the identities

$$\Lambda_{D\otimes\chi,i}(g) = \Lambda_{D,i}(g)\cdot\chi(g)^i \quad \text{for all } i \text{ and all } g \in G.$$

(iii) If D and χ are continuous, then so is $D \otimes \chi$.

Proof. To see Part (i) we need to show that $f_{D,\chi}(g\cdot h) = f_{D,\chi}(g)f_{D,\chi}(h)$ for $g = (g_1,\ldots,g_d), h = (h_1,\ldots,h_d) \in G^d$ and for $\cdot$ the multiplication given by $\mu^d_{A[G]}$. Using (18) we compute

$$\begin{aligned} g^{[\alpha]}\cdot h^{[\beta]} &= \mu^d_{A[G]}\Big(\widetilde{\eta}_M\big(g^{[\alpha]}\otimes h^{[\beta]}\big)\Big) = \mu^d_{A[G]}\Big(\sum_{\gamma\in M^{\alpha,\beta}_{d\times d}(\mathbb{N})}\prod_{(i,j)\in\{1,\ldots,d\}^2}(g_i\otimes h_j)^{[\gamma_{ij}]}\Big) \\ &= \sum_{\gamma\in M^{\alpha,\beta}_{d\times d}(\mathbb{N})}\prod_{(i,j)\in\{1,\ldots,d\}^2}(g_ih_j)^{[\gamma_{ij}]}. \end{aligned}$$

Observe that $\sum_{i,j}\gamma_{ij} = d$ for $\gamma \in M^{\alpha,\beta}_{d\times d}(\mathbb{N})$, and that index pairs (i,j) with $\gamma_{ij} = 0$ can be ignored. We write $\underline{\gamma}$ for the flattening of γ truncated to length d, i.e., we first regard γ as a d^2-tuple in one index and then omit the highest d^2-d indices where $\gamma_{ij} = 0$. Using in $(*)$ the definition of $M^{\alpha,\beta}_{d\times d}(\mathbb{N})$, we find

$$\begin{aligned} f_{D,\chi}(g^{[\alpha]}\cdot h^{[\beta]}) &= \sum_{\gamma\in M^{\alpha,\beta}_{d\times d}(\mathbb{N})} f_{D,\chi}\Big(\prod_{(i,j)\in\{1,\ldots,d\}^2}(g_ih_j)^{[\gamma_{ij}]}\Big) \\ &= \sum_{\gamma\in M^{\alpha,\beta}_{d\times d}(\mathbb{N})} D^{[\underline{\gamma}]}((g_ih_j)_{(i,j)\in\underline{\gamma}})\Big(\prod_{(i,j)\in\{1,\ldots,d\}^2}\chi(g_ih_j)^{\gamma_{ij}}\Big) \\ &\overset{(*)}{=} \sum_{\gamma\in M^{\alpha,\beta}_{d\times d}(\mathbb{N})} D^{[\underline{\gamma}]}((g_ih_j)_{(i,j)\in\underline{\gamma}})\chi(g^{[\alpha]})\chi(h^{[\beta]}) \\ &= \chi(g^{[\alpha]})\chi(h^{[\beta]})f_D(g^{[\alpha]}\cdot h^{[\beta]}) \\ &\overset{f_D \text{ multipl.}}{=} \chi(g^{[\alpha]})\chi(h^{[\beta]})f_D(g^{[\alpha]})f_D(h^{[\beta]}) = f_{D,\chi}(g^{[\alpha]})f_{D,\chi}(h^{[\beta]}). \end{aligned}$$

Concerning (ii), note that

$$D_{A[T']}(1-Tg') = D_{A[T_1,\ldots,T_d]}\Big(\sum_{i=1}^d T_ig_i\Big)|_{g=(e,\ldots,e,g'),T=(1,0,\ldots,0,T')},$$

so that

$$\begin{aligned} \Lambda_{D\otimes\chi,i}(g') &= (-1)^i(D\otimes\chi)^{[d-i,0,\ldots,0,i]}(e,\ldots,e,g') \\ &= (-1)^iD^{[d-i,0,\ldots,0,i]}(e,\ldots,e,g')\chi((e,\ldots,e,g')^{[d-i,0,\ldots,0,i]}) \\ &= (-1)^iD^{[d-i,0,\ldots,0,i]}(e,\ldots,e,g')\chi(g)^i = \Lambda_{D,i}(g')\chi(g)^i. \end{aligned}$$

Part (iii) follows from Definition 4.4.1 and Part (ii). □

Definition 4.5.8 (Twist of pseudorepresentations)**.** *We call the multiplicative polynomial law $D \otimes \chi \in \mathcal{P}s\mathcal{R}^n_G(A)$ from Proposition 4.5.7 the* twist of D by χ.

Remark 4.5.9. (a) If $D = D_\rho$ for a representation ρ of G, then $D \otimes \chi = D_{\rho\otimes\chi}$. This can for instance be deduced from Proposition 4.5.7(ii) and the theorem of Brauer-Nesbitt.

(b) It should be interesting to define the tensor product of two pseudorepresentations of any dimensions n, n'.

Lemma 4.5.10. *Let D, D' be in $\mathcal{P}s\mathcal{R}^n_G(A)$ and let $\chi\colon G \to A^\times$ be a group homomorphism. Then $D' = D \otimes \chi$ if and only if $\Lambda_{D',i}(g) = \Lambda_{D,i}(g) \cdot \chi(g)^i$ for all i and all $g \in G$.*

Proof. Proposition 4.5.7(ii) shows that the condition given is necessary. That it is also sufficient follows from Proposition 4.1.10(i), which says that a pseudorepresentation is determined by its characteristic polynomial coefficients. □

Corollary 4.5.11. *Let D be in $\mathcal{P}s\mathcal{R}^n_G(A)$ and let $\chi\colon G \to A^\times$ be a character of finite order. Suppose that $\chi(g) - 1$ lies in $A^\times$ whenever $g \in G \smallsetminus \ker\chi$. Then the following hold:*

(i) $D = D \otimes \chi$ if and only if

$$\forall g \in G, \forall i = 0, \dots, n : \Lambda_{D,i}(g) = 0 \text{ or } \operatorname{ord}\chi(g) \text{ divides } i.$$

(ii) Let I be the ideal of A generated by the set

$$\{\Lambda_{D,i}(g) \ : \ (g, i) \in G \times \{1, \dots, n\} \text{ such that } \operatorname{ord}\chi(g) \nmid i\}.$$

Then the locus of $\operatorname{Spec} A$ on which $D = D \otimes \chi$ is the closed subscheme $\operatorname{Spec} A/I$.

Proof. To see Part (i), note that by Lemma 4.5.10 we have $D = D \otimes \chi$ if and only if

$$\Lambda_{D,i}(g) = \Lambda_{D,i}(g) \cdot \chi^i(g) \quad \text{for all } i \text{ and all } g \in G.$$

Since $1 - \chi^i(g)$ is a unit in $A^\times$ whenever $\operatorname{ord}\chi(g) \nmid i$, and is zero otherwise, the latter is clearly equivalent to the condition given in the corollary.

Part (ii) follows from Part (i) since the latter implies that for any ideal J of A one has

$$(D \otimes_A A/J) \otimes \chi = D \otimes_A A/J \quad \Longleftrightarrow \quad I \subset J.$$

□

4.6 Induction for pseudorepresentations

In this subsection we introduce the operation of inducing a pseudorepresentation from a finite index subgroup (under some condition). First, the characteristic polynomial of an induced representation with values in an Azumaya algebra is described in Lemma 4.6.6. As the characteristic polynomial coefficients determine a pseudorepresentation by Proposition 4.1.10, Lemma 4.6.6 allows us to define an induced pseudorepresentation in Theorem 4.6.10 under Assumption 4.6.7.

We fix a group G and a subgroup $H \subset G$ of finite index m.

Lemma 4.6.1. *Let C be an Azumaya A-algebra. Consider a representation $\rho\colon H \longrightarrow C^\times$. There exists a representation $\rho^*\colon G \to \operatorname{Mat}_m(C)^\times$ such that for any étale extension $A \to A'$ that splits C, there is an isomorphism $\rho^* \otimes_A A' \cong \operatorname{Ind}_H^G(\rho \otimes_A A')$ of G-representations over A.*

The linearization $(\rho^*)^{\mathrm{lin}}\colon A[G] \to \operatorname{Mat}_m(C)$ of ρ^* takes values in the Azumaya algebra $\operatorname{Mat}_m(C)$, and by Example 4.1.7 therefore D_{ρ^*} is a pseudorepresentation with values in A.

Proof. To prove the lemma, we adapt the description of the induced matrix representation from [CR81, pp. 227-230] to the setting of Azumaya-algebras. Let $g_1, \dots, g_m$ be a set of representatives of left cosets of G/H such that $G = \bigsqcup_{i=1}^m g_i H$. For $g \in G$ we define for each $j \in \{1, \dots, m\}$ an $i = i_j$ in $\{1, \dots, m\}$ by the condition

$$gg_j \in g_i H.$$

The assignment $j \to i_j$ is a permutation of $\{1, \dots, m\}$. We extend ρ from H to G by defining

$$\widetilde{\rho}\colon G \longrightarrow C, \quad g \longmapsto \begin{cases} \rho(g) & \text{if } g \in H, \\ 0 & \text{if } g \in G \smallsetminus H. \end{cases}$$

Consider the map

$$\rho^*\colon G \longrightarrow \mathrm{Mat}_m(C), \quad g \longmapsto \begin{pmatrix} \widetilde{\rho}(g_1^{-1} g g_1) & \cdots & \widetilde{\rho}(g_1^{-1} g g_m) \\ \vdots & \ddots & \vdots \\ \widetilde{\rho}(g_m^{-1} g g_1) & \cdots & \widetilde{\rho}(g_m^{-1} g g_m) \end{pmatrix}.$$

Then for all $g \in G$ the image $\rho^*(g)$ is a monomial matrix over the skew field C since for $1 \le i, j \le m$ the only non-zero entry in the i^{th} row and j^{th} column of $\rho^*(g)$ is $\rho(g_{i_j}^{-1} g g_j) \in C^\times$. In particular, this shows that $\rho^*(g)$ lies in $\mathrm{GL}_m(C)$.

We claim that ρ^* has the properties asserted in the lemma. Let $A \to A'$ be finite étale so that $C \otimes_A A' = \mathrm{Mat}_r(A')$ for a suitable $r \in \mathbb{N}_{\ge 1}$. Then $\rho^* \otimes_A A'$ is the matrix representation of the induced representation of

$$\rho \otimes_A A'\colon H \longrightarrow \mathrm{GL}_r(A')$$

simply by our construction following [CR81]. This implies the multiplicativity of the map ρ^*, i.e., that it is a homomorphism. Moreover, it shows that $\rho^* \otimes_A A'$ is the usual induced representation of $\rho \otimes_A A'$. □

Remark 4.6.2. It can be shown that $\rho \mapsto \rho^*$ in Lemma 4.6.1 is uniquely characterized as the right adjoint of the restriction homomorphism from G-representations to H-representations on Azumaya algebras.

Definition 4.6.3. *We call* ρ^* *in Lemma 4.6.1 the* representation induced from ρ under $H \subset G$ *and denote it by* $\mathrm{Ind}_H^G \rho$.

Example 4.6.4. Before we go on and since we have just seen an explicit form of ρ^*, for later use we consider the following example: Let G be a group and let $H \subset G$ be a normal subgroup of index p. Let C be an Azumaya algebra of characteristic p and let $\rho\colon H \to C^\times$ be a representation. Fix $g_0 \in G \smallsetminus H$ and set $g_i := g_0^i$ for $i = 1, \dots, p$, so that $G = \bigsqcup_{i=1}^p g_i H$. Define the induced representation $\rho^*\colon G \to \mathrm{Mat}_p(C)^\times$ as in the above proof. Let also $A \in \mathrm{Mat}_p(C)$ be the diagonal matrix with diagonal $(i \cdot 1_C)_{i=0,\dots,p-1}$. Define the group isomorphism $\lambda\colon G/H \to \mathbb{F}_p, g_0^i H \mapsto i \pmod p$. *Then one has for all* $g \in G$ *the relation*

$$\rho^*(g) A \rho^*(g^{-1}) - A = -\lambda(g) 1_{\mathrm{Mat}_p(C)}.$$

Let $g \in G$. We shall verify $\rho^*(g)A - A\rho^*(g) = \lambda(g)\rho^*(g)$. Observe that $\widetilde{\rho}(g_i^{-1} g g_j) = \widetilde{\rho}(g_0^{-i} g g_0^j) = 0$ unless $gH = g_0^{i-j}$, i.e., unless $\lambda(g) = i - j$. In the following, we write a lower subscript $_{i,j}$ to indicate the (i,j)-entry of a matrix in $\mathrm{Mat}_p(C)$. Then

$$\begin{aligned} (\rho^*(g)A - A\rho^*(g))_{i,j} &= \rho^*(g)_{i,j} \cdot j - i \cdot \rho^*(g)_{i,j} \;=\; (j - i) \cdot \widetilde{\rho}(g_0^{-i} g g_0^j) \\ &\overset{\text{observ.}}{=} -\lambda(g) \cdot \rho^*(g)_{i,j}, \end{aligned}$$

and this completes the proof of our assertion.

Below we want to have a rather explicit description of the characteristic polynomial of $\operatorname{Ind}_H^G \rho$. This is prepared in the following lemmas. We could not locate these presumably well-known results in the literature, so we indicate some proofs. We also need to fix some notation: Let C be an Azumaya A-algebra of degree n. Recall from Reminder 4.1.6 that elements $c \in C$ have a reduced characteristic polynomial χ_c; we define its coefficients $\Lambda_{c,i}$ by $\chi_c(t) = \sum_{i=0}^n (-1)^i \Lambda_{c,i}(c) t^{n-i}$. Recall also that if C is an Azumaya A-algebra, then so is $\operatorname{Mat}_m(C)$. We write χ_c^m for the characteristic polynomial (of degree nm) of $c \in \operatorname{Mat}_m(C)$.

Lemma 4.6.5. *Let $c = (c_{i,j})$ be in $\operatorname{Mat}_m(C)$. Suppose that there is a permutation $\sigma \in \mathfrak{S}_m$ such that $c_{i,j} = 0$ for $i \neq \sigma(j)$ and such that $c_{\sigma(j),j}$ lies in $C^\times$ for all j. Then χ_c^m has the following description:*

Write σ in its cycle decomposition $\sigma = \sigma_1 \cdot \ldots \cdot \sigma_v$, where the σ_l are disjoint cycles of length m_l such that $\sum_{l=1}^v m_l = m$ and let j_l be in the support of σ_l such that $\sigma_l = (j_l, \sigma(j_l), \ldots, \sigma^{m_l-1}(j_l))$. Then

$$\chi_c^m(t) = \prod_{l=1}^{v} \chi_{c(l)}(t^{m_l}) \qquad \textit{with } c(l) := c_{j_l,\sigma^{m_l-1}(j_l)} c_{\sigma^{m_l-1}(j_l),\sigma^{m_l-2}(j_l)} \cdot \ldots \cdot c_{\sigma(j_l),j_l}$$

Proof. Let $s_i = m_1 + \ldots + m_{l-1}$ for $l = 1, \ldots, v$, with $m_0 = 0$, and let $\tau \in \mathfrak{S}_m$ be the permutation whose inverse is given by

$$\begin{pmatrix} s_1+1 & s_1+2 & \cdots & s_1+m_1 \\ j_1 & \sigma(j_1) & \ldots & \sigma^{m_1-1}(j_i) \end{pmatrix} \cdot \ldots \cdot \begin{pmatrix} s_v+1 & s_v+2 & \cdots & s_v+m_v \\ j_v & \sigma(j_v) & \ldots & \sigma^{m_v-1}(j_v) \end{pmatrix},$$

and let $p = p_\tau$ in $\operatorname{Mat}_m(C)$ be the permutation matrix attached to τ, i.e., with $p_{i,j} = 0$ for $i \neq \tau(j)$ and $p_{\tau(j),j} = 1_C$ for all j. Then one verifies that $p_\tau c p_\tau^{-1}$ is a block diagonal matrix in $\operatorname{Mat}_m(C)$ with v blocks on the diagonal, the l^{th} block lies in $\operatorname{Mat}_{m_l}(C)$ and is of the form

$$\begin{pmatrix} 0 & 0 & & \cdots & c_{j_l,\sigma^{m_l-1}(j_l)} \\ c_{\sigma(j_l),j_l} & 0 & \ddots & & 0 \\ 0 & c_{\sigma^2(j_l),\sigma(j_l)} & \ddots & & \ddots \\ \vdots & \ddots & & & \\ 0 & \ldots & 0 & c_{\sigma^{m_l-1}(j_l),\sigma^{m_l-2}(j_l)} & 0 \end{pmatrix}.$$

We leave it as a simple exercise in matrix manipulations to complete the result in this case. $\square$

Lemma 4.6.6. *Let the hypotheses be as in Lemma 4.6.1. Fix $g' \in G$ and denote by m' its order in the group G/H. Then one has*

$$\chi_{\operatorname{Ind}_H^G \rho(g')}(t) = \prod_{g \in G/H\langle g' \rangle} \chi_{\rho^{(g^{-1})}((g')^{m'})}(t^{m'}).$$

If $G/H\langle g' \rangle$ is a group (and not only a coset), then the inversion in $\rho^{(g^{-1})}$ can be omitted.

Recall that the conjugate $\rho^{(g^{-1})}$ as defined in Definition 2.1.1 also applies to the present situation. Note also that $(g'h)^{m'}$ lies in H so that the above formula is well-defined, since $H \subset G$ is a normal subgroup and $m' = \operatorname{ord}_{G/H}(g')$.

Proof. Let the notation be as in the proof of Lemma 4.6.1, and set $v = m/m'$. Define $\sigma_l \in \mathfrak{S}_m$ as the (unique) permutation such that $g_l g_j \in g_{\sigma_l(j)} H$ for all $l \in \{1, \ldots, m\}$. Let $c \in \operatorname{Mat}_m(C)$ be the matrix with $c_{i,j} = 0$ for $i \neq \sigma_l(j)$ and $c_{\sigma_l(j),j} = \rho(g_{\sigma_l(j)}^{-1} g_l h g_j)$, so that $c = \rho^*(g_l h)$. Choose

$j_1, \ldots, j_v$ such that the elements g_{j_i} are representatives of the cosets of $G/H\langle g_j\rangle$, or, equivalently, such that the orbits of the j_i under σ_l are in bijection with the orbits in $\{1, \ldots, m\}$ under σ_l. Now c is monomial, and by Lemma 4.6.5 its characteristic polynomial is given by

$$\chi_{\rho^*(g_l h)}(t) = \prod_{s=1}^{v} \chi_{\rho(g_{j_s}^{-1} g_l h g_{\sigma_l^{m'-1}(j_s)})\rho(g_{\sigma_l^{m'-1}(j_s)}^{-1} g_l h g_{\sigma_l^{m'-2}(j_s)})\cdot\ldots\cdot\rho(g_{\sigma_l(j_s)}^{-1} g_l h g_{j_s})}(t^{m'}),$$

where we use our explicit shape of c, so that in particular $j \mapsto m_j$ is constant with value m'. Next one one uses the multiplicativity of ρ as a representation to combine its arguments as a product in which cancellations occur. Using also the conjugate of ρ by some $g \in G$ defined in Definition 2.1.1, we obtain

$$\chi_{\rho^*(g_l h)}(t) = \prod_{s=1}^{v} \chi_{\rho^{(g_{j_s}^{-1})}((g_l h)^{m'})}(t^{m'}),$$

Now up to isomorphy we can replace g_{j_s} in $\rho^{(g_{j_s}^{-1})}$ by any other representative of the class $g_{j_s}H\langle g_l\rangle$. To conclude the proof of the formula in the lemma note that we may from that start assume that the g_i are chosen in such a way that g' is among them. $\square$

In the remainder of this subsection we assume G to be a profinite group and $H \subset G$ to be open and normal in G. Let

$$D_H : H \longrightarrow B$$

be a pseudorepresentation of dimension n with values in a commutative ring B. Denote by $\mathrm{Min}(B)$ the set of minimal primes of B. For a local ring A denote by A^{sh} its strict henselization.

In order to define an induction of D_H, we formulate the following conditions on (B, D_H).

Assumption 4.6.7 (Basic assumptions on B and D_H). (a) The ring B is Noetherian.

(b) The natural map $B \to \prod_{\mathfrak{p}\in\mathrm{Min}(B)} B_{\mathfrak{p}}$ is injective.

(c) For each $\mathfrak{p} \in \mathrm{Min}(B)$ there is an n-dimensional representation $\rho_{\mathfrak{p}}$ of H over $B_{\mathfrak{p}}^{\mathrm{sh}}$ such that $D_H \otimes_B B_{\mathfrak{p}}^{\mathrm{sh}}$ is the determinant attached to $\rho_{\mathfrak{p}}$.

Remark 4.6.8. (a) For instance by going through the proof of [Sta18, § 031Q], one see that the map in Assumption 4.6.7(b) is injective if and only if B satisfies Serre's condition (S_1).

(b) It is likely that one can relax our hypothesis and is still able to prove Theorem 4.6.10.[7]

Lemma 4.6.9. *The following two conditions on a Noetherian ring B suffice for Assumption 4.6.7 to hold.*

(b′) The ring B is reduced.

(c′) For all $\mathfrak{p} \in \mathrm{Min}(B)$ the pseudorepresentation $D_{H,\mathfrak{p}}$ is irreducible.

Proof. Condition (b') clearly implies condition (b) of Assumption 4.6.7. Let us see that (c') implies Assumption 4.6.7(c): By Proposition 4.1.22(i), the pseudorepresentation $D := D_H \otimes_B B_{\mathfrak{p}}^{\mathrm{sh}}$ factors via

$$B_{\mathfrak{p}}^{\mathrm{sh}}[G] \to B_{\mathfrak{p}}^{\mathrm{sh}}[G]/\operatorname{CH}(D) \to B_{\mathfrak{p}}^{\mathrm{sh}}.$$

The irreducibility of $D_{H,\mathfrak{p}}$ together with Theorem 4.3.10(i) show that $B_{\mathfrak{p}}^{\mathrm{sh}}[G]/\operatorname{CH}(D)$ is isomorphic to a matrix algebra $C := M_n(B_{\mathfrak{p}}^{\mathrm{sh}})$ and that under this isomorphism the map $B_{\mathfrak{p}}^{\mathrm{sh}}[G]/\operatorname{CH}(D) \to B_{\mathfrak{p}}^{\mathrm{sh}}$ is equal to $\det_C$ and hence arises from a representation. $\square$

[7] Here's one idea: define induction for universal pseudorepresentations; use that they are irreducible so that (ii) holds; then use universal induction to specialize to arbitrary D; problem: why should (i) hold - say even if we know that the universal ring is equidimensional and regular except for codimension at least 3?

Theorem 4.6.10. *Suppose Assumption 4.6.7 holds. Then there exists a unique pseudorepresentation $D_G\colon G \to B$ whose characteristic polynomial on a coset $g'H$ is given by*

$$\chi_{D_G,B}(g'h,t) = \prod_{g\in G/H\langle g'\rangle} \chi_{D_H^{(g^{-1})},B}((g'h)^{m'},t^{m'}), \tag{19}$$

where m' denotes the order of $g'H$ in G/H. It has the following properties.

(i) For any geometric point $\overline{x} \to \operatorname{Spec} B$ the representations $\rho_{D_{G,\overline{x}}}$ and $\operatorname{Ind}_H^G \rho_{D_{H,\overline{x}}}$ are isomorphic.

(ii) If D_H is continuous, then so is D_G.

(iii) One has

$$\operatorname{Res}_H^G D_G \cong \bigoplus_{g\in G/H} D_H^g.$$

(iv) For $i \in \{0,\ldots,nm\}$, $g' \in G$ and $m' = \operatorname{ord}_{G/H}(gH)$ one has $\Lambda_{D_G,i}(g') = 0$ if $m' \nmid i$.

(v) One has $D = D \otimes \chi$ for any 1-dimensional character $\chi\colon G/H \to A^\times$.

(vi) The formation of D_G commutes with base change, i.e., the following holds: Let $B \to B'$ be any homomorphism. Set $D'_H := D_H \otimes_B B'$ and $D'_G := D_G \otimes_B B'$. Then (19) holds with D_H and D_G replaced by D'_H and D'_G, respectively. [8]

(vii) Let $U = \operatorname{Spec} B' \subset \operatorname{Spec} B$ be an affine open subset such that $D_{H,x}$ is irreducible for all $x \in U$. Let $C := B'[G]/\operatorname{CH}(D'_H)$ for $D'_H = D_H \otimes_B B'$ and let ψ be the natural homomorphism $G \to C^\times$. From Proposition 4.1.25 we see that C is a B'-Azumaya algebra and that $\det_C \circ \psi = D'_H$. In addition we have $D_G \otimes_B B' = \det_{M_m(C)} \circ \operatorname{Ind}_H^G \psi$.

Proof. By the Cohen structure theorem we have $B_\mathfrak{p} \cong \kappa(\mathfrak{p})[x_1,\ldots,x_h]/I$ for some $h \in \mathbb{N}_{\geq 1}$ and some ideal I such that a power of $(x_1,\ldots,x_h)$ is a subset of I. Then $B_\mathfrak{p}^{\mathrm{sh}} = \kappa(\mathfrak{p})^{\mathrm{sep}}[X_1,\ldots,x_n]/I$ with the canonical inclusion $B_\mathfrak{p} \to B_\mathfrak{p}^{\mathrm{sh}}$ is the strict henselization of $B_\mathfrak{p}$. It follows from Assumption 4.6.7(b) that the ring homomorphism $\iota\colon B \to B_\eta^{\mathrm{sh}} := \prod_{\mathfrak{p}\in\operatorname{Min}(B)} B_\mathfrak{p}^{\mathrm{sh}}$ is injective. Hence we shall regard B as a subring of B_η^{sh} via ι, and by Assumption 4.6.7(c) there exists a representation $\rho_\eta\colon H \to \mathrm{GL}_n(B_\eta^{\mathrm{sh}})$ such that $\det\circ\rho_\eta = D_H \otimes_B B_\eta^{\mathrm{sh}}$. Define D_G as $\det \circ \operatorname{Ind}_H^G \rho_\eta\colon G \to \mathrm{GL}_{nm}(B_\eta^{\mathrm{sh}})$. Then (19) holds for D_G by Lemma 4.6.6. Obviously the right hand side of (19) has coefficients in B. Thus by Proposition 4.1.10 the pseudorepresentation D_G is already defined over B, and by the same result D_G is uniquely determined by the coefficients of χ_{D_G}. It remains to prove the properties listed in (i)–(vi).

To see (i) note first that formula (19) is preserved under base change to $\kappa(\overline{x})$, i.e., the formula still holds if we replace simultaneously D_G by $D_{G,\overline{x}}$ and D_H by $D_{H,\overline{x}}$. By its definition, $\rho_{D_{H,\overline{x}}}$ has characteristic polynomial $\chi_{D_{H,\overline{x}}}$, and by Lemma 4.6.6, the right hand side of (19) over $\kappa(\overline{x})$ is equal to $\chi_{\operatorname{Ind}_H^G \rho_{\overline{x}}}$. This proves (i). Part (v) follows from Lemma 4.6.6 and Corollary 4.5.11.

Part (ii) follows immediately from (19): it suffices to verify the continuity of the characteristic polynomial coefficients, and this may be done on the open cover gH, $g \in G$. On each open of this cover, (19) describes these coefficients. Since D_H and hence the $D_H^{(g')}$ are continuous and since $gh \mapsto (gh)^{m'}$ is continuous, the result follows.

Next, the formula in (iii) clearly holds over B_η^{sh} since there $\operatorname{Res}_H^G \operatorname{Ind}_H^G \rho_\eta = \oplus_{g\in G/H}\rho_\eta^g$. Because ι is injective, taking characteristic polynomials, formula (iii) holds. Part (iv) is immediate from (19), since the right hand side is a polynomial in $t^{m'}$.

[8] Observe that (B', D'_H) needs not satisfy Assumption 4.6.7. So we have no direct method to construct a solution for D'_H satisfying (19). Nevertheless D'_G is such a solution (and it is clearly unique).

Part (vi) is clear, since we may tensor 19 over B with B' and since a pseudorepresentation is uniquely determined by its characteristic polynomial by Amitsur's formula; see Proposition 4.1.10(i). Concerning (vii), note that our hypotheses and the assumption on B' imply that $B' \to \prod_{\mathfrak{p}\in \mathrm{Min}(B')} B'_{\mathfrak{p}}$ is injective. So it suffices to verify $D_G \otimes_B B' = \det_{M_m(C)} \circ \operatorname{Ind}_H^G \psi$ at the generic points of $\operatorname{Spec} B'$, i.e., at a subset of the generic points of $\operatorname{Spec} B$. This follows, for instance, from (ii). □

Definition 4.6.11. *We call the pseudorepresentation D_G from Theorem 4.6.10 the* induced pseudorepresentation of D_H under $H \subset G$ *and write* $\operatorname{Ind}_H^G D_H$ *for it.*

For later use, we formulate the following simple finiteness result related to induction.

Lemma 4.6.12. *Let k be a field, let $\chi\colon G \to k^\times$ be a character of finite order m with kernel $H := \ker\chi$, and let D be in $\mathcal{P}s\mathcal{R}^n_{G_K}(k^{\mathrm{alg}})$. Define*

$$\mathcal{S}_D := \{D' \in \mathcal{P}s\mathcal{R}^{n/m}_H(k^{\mathrm{alg}}) \,:\, \operatorname{Ind}_H^G D' = D\}.$$

Then the following hold:

(i) $\mathcal{S}_D$ is finite.

(ii) $\mathcal{S}_D$ is nonempty if and only if $D = D \otimes \chi$.

If moreover G is profinite, k^{alg} carries the discrete topology and D is continuous, then there is a finite extension of k in k^{alg} over which all $D' \in \mathcal{S}_D$ are defined and split.

Proof. By Theorem 4.1.19 and Corollary 4.4.6, the map $\rho \to D_\rho$ from semisimple representations of G over k^{alg} to pseudorepresentations of G over k^{alg} is a bijection; and the same holds over H. We also have $D_\rho \otimes \chi = D_{\rho\otimes\chi}$ by Remark 4.5.9(a). Thus (i) and (ii) are really assertions on semisimple representations. Now if ρ is a representation and if $\rho = \operatorname{Ind}_H^G \rho'$ for some representation ρ', then ρ' is a direct summand of the semisimple representation $\rho|_H$ by Lemma 2.1.4. Since up to isomorphism there are only finitely many such summands (i) follow. Part (ii) is now immediate from Corollary 2.2.2. The last assertion is immediate from Corollary 4.4.6 since $\mathcal{S}_D$ is finite. □

4.7 Pseudodeformations and their universal rings

This subsection recalls in Proposition 4.7.4 the main object of our interest, the universal pseudodeformation ring of a residual pseudorepresentation $\overline{D}$. Here continuity plays a major role. We state basic results relevant to the present work. In addition to the usual treatment, we also give some special attention to functors $\widehat{\mathcal{A}r}_\kappa \to \mathit{Sets}$ where κ is a local field. The subsection also contains some results on deformations over formal schemes and on the locus of irreducibility.

We let $\mathbb{F}$ be either a finite or a local field; in the former case Λ is a complete Noetherian local commutative $W(\mathbb{F})$-algebra with residue field $\mathbb{F}$, in the latter case $\Lambda = \mathbb{F}$. Recall the categories $\mathcal{A}r_\Lambda$ and $\widehat{\mathcal{A}r}_\Lambda$ from Subsection 3.1 and the topological conditions we impose on there objects and morphisms. By A we denote a ring in $\widehat{\mathcal{A}r}_{\mathbb{F}}$; its maximal ideal is $\mathfrak{m}_A$ and it comes with a natural reduction map $\pi_A\colon A \to A/\mathfrak{m}_A = \mathbb{F}$. We let G be a profinite group and we denote by $\overline{D}\colon G \to \mathbb{F}$ a continuous pseudorepresentation of dimension n.

Definition 4.7.1 ([WE13, § 3.1.4.3])**.** *(i) A* pseudodeformation of $\overline{D}$ to A *is a continuous pseudorepresentation $D\colon G \to A$ such that $D \otimes_A \mathbb{F} = \pi_A \circ D\colon G \to \mathbb{F}$ is equal to $\overline{D}$.*

(ii) The functor

$$\mathcal{P}sD_{\overline{D}}\colon \widehat{\mathcal{A}r}_\Lambda \to \mathit{Sets}, \quad A \longmapsto \{D\colon G \longrightarrow A \text{ is a pseudodeformation of } \overline{D}\},$$

is called the pseudodeformation functor *of the* residual pseudorepresentation $\overline{D}$*.*

Note that unlike in parts of [WE13] for us all pseudodeformations will be continuous.

Definition 4.7.2. *Let $\pi\colon B \to \mathbb{F}$ be a morphism in $\mathcal{CA}lg_\Lambda$ and let $D\colon G \to B$ be a pseudorepresentation, such that $D \otimes_B \mathbb{F} = \overline{D}$.*

An ideal I of B is called D-open *if the following conditions hold:*

(a) The map π factors via B/I and B/I is a local Artin ring.

(b) $D_I := D \otimes_B B/I$ is continuous if we equip B/I with the topology of an object in $\mathcal{A}r_\Lambda$.

Lemma 4.7.3. *With the notation from Definition 4.7.2, the D-open ideals form a basis of a topology on B.*

Proof. (Cf. [WE13, Thm. 3.1.4.6]) One has to show that if I, I' are D-open ideals, then so is $I \cap I'$. Consider the injective homomorphism

$$\iota\colon B/(I \cap I') \longrightarrow B/I \times B/I'.$$

For both Λ that we consider, it is straightforward to see that ι is a topological isomorphism onto its image. Now a pseudorepresentation is continuous if and only if this holds for its characteristic polynomial functions; cf. Definition 4.4.1. Since both I and I' are D-open, it is now immediate that $I \cap I'$ is D-open. ◻

The following result is proved in [Che14, Prop. 3.3] for $\Lambda = W(\mathbb{F})$ and in [WE13, Thm. 3.1.4.6] for $\Lambda \in \widehat{\mathcal{A}r}_{W(\mathbb{F})}$.

Proposition 4.7.4. *The pseudodeformation functor $\mathcal{P}sD_{\overline{D}}$ is pro-representable a topological Λ-algebra $R^{\mathrm{univ}}_{\Lambda,\overline{D}}$ that is a filtered inverse limit of objects in $\mathcal{A}r_\Lambda$, together with a universal pseudodeformation*

$$D^{\mathrm{univ}}_{\Lambda,\overline{D}}\colon G \longrightarrow R^{\mathrm{univ}}_{\Lambda,\overline{D}}.$$

Proof. We recall a sketch of the proof from [WE13, Thm. 3.1.4.6] to indicate that it also applies to the case when $\Lambda = \kappa$ is a local field. Consider the universal ring $R^{\mathrm{univ}}_{\Lambda[G],n}$ from Definition 4.2.2 with its universal pseudorepresentation $D^{\mathrm{univ}}_{\Lambda[G]}\colon G \longrightarrow R^{\mathrm{univ}}_{\Lambda[G],n}$. By definition $R^{\mathrm{univ}}_{\Lambda[G],n}$ is a Λ-algebra. The map $\overline{D}$ induces a Λ-algebra homomorphism $\pi\colon R^{\mathrm{univ}}_{\Lambda[G],n} \to \mathbb{F}$. By Lemma 4.7.3, the D-open ideals of $R^{\mathrm{univ}}_{\Lambda[G],n}$ form the basis of a topology on $R^{\mathrm{univ}}_{\Lambda[G],n}$, and one defines $R^{\mathrm{univ}}_{\Lambda,\overline{D}}$ as the completion of $R^{\mathrm{univ}}_{\Lambda[G],n}$ with respect to this topology. It is then straightforward to establish the asserted properties for $R^{\mathrm{univ}}_{\Lambda,\overline{D}}$ together with the pseudorepresentation $D^{\mathrm{univ}}_{\Lambda,\overline{D}} := D^{\mathrm{univ}}_{\Lambda[G]} \otimes_{R^{\mathrm{univ}}_{\Lambda[G,]}n} R^{\mathrm{univ}}_{\Lambda,\overline{D}}$, by verifying it for the restriction of $\mathcal{P}sD_{\overline{D}}$ to $\mathcal{A}r_\Lambda$. ◻

Definition 4.7.5. *The ring $R^{\mathrm{univ}}_{\Lambda,\overline{D}}$ from Proposition 4.7.4 is called the* universal (Λ-)pseudodeformation ring *of $\overline{D}$, the pseudorepresentation $D^{\mathrm{univ}}_{\Lambda,\overline{D}}\colon G \to R^{\mathrm{univ}}_{\Lambda,\overline{D}}$ is called the* universal (Λ-)pseudodeformation *of $\overline{D}$ and the space $X^{\mathrm{univ}}_{\Lambda,\overline{D}} := \operatorname{Spec} R^{\mathrm{univ}}_{\Lambda,\overline{D}}$ the* universal (Λ-)pseudodeformation space *of $\overline{D}$; we write $R^{\mathrm{univ}}_{G,\Lambda,\overline{D}}$ if there is a need to indicate G; we often drop the index Λ if it is clear from context.*

The ring $R^{\mathrm{univ}}_{\Lambda,\overline{D}}$ behaves well under change of coefficient ring.

Proposition 4.7.6 (Cf. [Wil95, p. 457])**.** *Let $\bar{f}\colon \kappa \to \kappa'$ be a homomorphism between either two finite or two local fields, and let $f\colon \Lambda \to \Lambda'$ be a local homomorphisms of complete local Noetherian commutative rings that reduces on residue fields to $\bar{f}$. Define $\overline{D}' := \overline{D} \otimes_\kappa \kappa'\colon G \to \kappa \to \kappa'$. Then one has a natural isomorphism*

$$R^{\mathrm{univ}}_{\Lambda',\overline{D}'} \longrightarrow R^{\mathrm{univ}}_{\Lambda,\overline{D}} \hat{\otimes}_\Lambda \Lambda'.$$

Proof. The proof is as in [Wil95, p. 457] for deformation rings: If $\bar{f}$ is the identity, one can proceed as follows. Any $A \in \widehat{\mathcal{A}r}_{\Lambda'}$ can be regarded as a ring in $\widehat{\mathcal{A}r}_{\Lambda}$ vie the action induced from f; the residue fields of A, Λ and Λ' are the same. Then the assertion follows rapidly by using the isomorphism $\mathrm{Hom}_{\widehat{\mathcal{A}r}_{\Lambda}}(A,B) \cong \mathrm{Hom}_{\Lambda'}(A \otimes_\Lambda \Lambda', B)$ for $A \in \widehat{\mathcal{A}r}_{\Lambda}$ and $B \in \widehat{\mathcal{A}r}_{\Lambda'}$ together with the universal properties of $R^{\mathrm{univ}}_{\Lambda',\overline{D}'}$ and $R^{\mathrm{univ}}_{\Lambda,\overline{D}}$.

In the general case, define for any $B' \in \widehat{\mathcal{A}r}_{\Lambda'}$ the ring B'' as the subring of B' of elements whose reduction to κ' lies in the subfield κ, so that $B'' \in \widehat{\mathcal{A}r}_{\Lambda''}$. The argument just given applies to $\Lambda \to \Lambda''$. For $\Lambda'' \to \Lambda'$ note first that any $D' \in \mathcal{P}sD_{\Lambda',\overline{D}'}(B')$ takes values in B'' because $\overline{D}'$ takes values in κ, so that D' defines a $D'' \in \mathcal{P}sD_{\Lambda'',\overline{D}}(B'')$. Conversely, if such a D'' is given, we may form $D'' \otimes_{\Lambda''} \Lambda'$ and compose it with the natural Λ'-homomorphism $B'' \otimes_{\Lambda''} \Lambda' \to B'$ to get back to D'. This yields the following chain of isomorphisms

$$\begin{aligned} \mathrm{Hom}_{\Lambda'}(R^{\mathrm{univ}}_{\Lambda',\overline{D}'}, B') &\cong \mathcal{P}sD_{\Lambda',\overline{D}'}(B') \cong \mathcal{P}sD_{\Lambda'',\overline{D}}(B'') \\ &\cong \mathrm{Hom}_{\Lambda''}(R^{\mathrm{univ}}_{\Lambda'',\overline{D}}, B'') \cong \mathrm{Hom}_{\Lambda''}(R^{\mathrm{univ}}_{\Lambda'',\overline{D}}, B') \\ &\cong \mathrm{Hom}_{\Lambda'}(R^{\mathrm{univ}}_{\Lambda'',\overline{D}} \otimes_{\Lambda''} \Lambda', B') \end{aligned}$$

We deduce $R^{\mathrm{univ}}_{\Lambda',\overline{D}'} \cong R^{\mathrm{univ}}_{\Lambda'',\overline{D}} \otimes_{\Lambda''} \Lambda'$ because any $B' \in \widehat{\mathcal{A}r}_{\Lambda'}$ can occur as test objects. ☐

The previous proposition justifies the following definition.

Definition 4.7.7. *If $\mathbb{F}$ is finite, we call $\overline{R}^{\mathrm{univ}}_{\overline{D}} := R^{\mathrm{univ}}_{\mathbb{F},\overline{D}}$ the* universal mod p pseudodeformation ring *of $\overline{D}$ and we call $\overline{X}^{\mathrm{univ}}_{\overline{D}} := X^{\mathrm{univ}}_{\mathbb{F},\overline{D}}$ the* special fiber *of the universal pseudodeformation space of $\overline{D}$.*

We shall also need to consider the Cayley-Hamilton quotient of $R^{\mathrm{univ}}_{\Lambda,\overline{D}}$. Recall from Remark 4.4.4 that $D^{\mathrm{univ}}_{\Lambda,\overline{D}}$ induces a continuous pseudorepresentation (for which we shall use the same name)

$$D^{\mathrm{univ}}_{\Lambda,\overline{D}} \colon R^{\mathrm{univ}}_{\Lambda,\overline{D}}[\![G]\!] \longrightarrow R^{\mathrm{univ}}_{\Lambda,\overline{D}}.$$

Let the following be the diagram induced from (14)

$$R^{\mathrm{univ}}_{\Lambda,\overline{D}}[\![G]\!] \xrightarrow{\rho^{\mathrm{CH}}_{\Lambda,\overline{D}}} S^{\text{CH-univ}}_{\Lambda,\overline{D}} := (R^{\mathrm{univ}}_{G,\overline{D}}[[G]])^{\mathrm{CH}}_{D^{\mathrm{univ}}_{\Lambda,\overline{D}}} \xrightarrow{D^{\text{CH-univ}}_{\Lambda,\overline{D}}} R^{\mathrm{univ}}_{\Lambda,\overline{D}}. \tag{20}$$

Definition 4.7.8. *For 'object' the algebra $S^{\text{CH-univ}}_{\Lambda,\overline{D}}$, the pseudorepresentation $D^{\text{CH-univ}}_{\Lambda,\overline{D}}$ or the CH-representation $\rho^{\mathrm{CH}}_{\Lambda,\overline{D}}$, respectively, we use the term* universal Cayley-Hamilton object *attached to $\overline{D}$.*

Remark 4.7.9. As explained in [Che14, Prop. 1.23], the factorization in (20) has indeed a universal property.

It is interesting to give a criterion for $R^{\mathrm{univ}}_{\Lambda,\overline{D}}$ to be Noetherian, and that implies some finiteness properties for $S^{\text{CH-univ}}_{\Lambda,\overline{D}}$.

Definition 4.7.10 (Cf. [WE13, 3.1.5])**.** *Suppose $\mathbb{F}$ is finite. Then we define condition $\Phi_{\overline{D}}$ to be condition $\Phi_{\rho_{\overline{D}\otimes_{\mathbb{F}}\mathbb{F}^{\mathrm{alg}}}}$ from Definition 3.2.2.*

Proposition 4.7.11 ([WE18, Props. 3.2 and 3.6])**.** *Suppose that $\mathbb{F}$ is finite and $\Phi_{\overline{D}}$ is satisfied. Then the following hold:*

(i) *The topological* Λ*-algebra* $R^{\mathrm{univ}}_{\Lambda,\overline{D}}$ *lies in* $\widehat{\mathcal{A}r}_\Lambda$.
(ii) *The CH-representation* $\rho^{\mathrm{CH}}_{\Lambda,\overline{D}}$ *is a continuous homomorphism.*
(iii) *The ring* $S^{\text{CH-univ}}_{\Lambda,\overline{D}}$ *is module-finite as an* $R^{\mathrm{univ}}_{\Lambda,\overline{D}}$*-algebra, and therefore Noetherian.*
(iv) *On* $S^{\text{CH-univ}}_{\Lambda,\overline{D}}$ *the profinite topology, the* $\mathfrak{m}_{\overline{D}}$*-adic topology, and the quotient topology from the surjection* $\rho^{\mathrm{CH}}_{\Lambda,\overline{D}}$ *are equivalent.*

Remark 4.7.12. Suppose $G = G_K$ for K a p-adic field. Then by Proposition 3.2.3 and Proposition 4.7.11, the ring $R^{\mathrm{univ}}_{\Lambda,\overline{D}}$ is Noetherian.

Corollary 4.7.13. *Suppose* $\mathbb{F}$ *is finite and* G *satisfies* $\Phi_{\overline{D}}$*. Let* A *be a quotient of* $R := R^{\mathrm{univ}}_{W(\mathbb{F}),\overline{D}}$ *and let* $D_A = D^{\mathrm{univ}}_{\overline{D}} \otimes_R A$*. Then* A *is the ring of definition over* $W(\mathbb{F})$ *of* D_A *in the sense of Definition 4.4.10.*

Proof. Let $C \subset A$ be the ring of definition of D_A over $W(\mathbb{F})$, and let D_C be the pseudorepresentation over C such that $D_C \otimes_C A = D_A$. By the universality of R we have a unique $W(\mathbb{F})$-algebra homomorphism $R \to C$ such that $D_C = D^{\mathrm{univ}}_{\overline{D}} \otimes_R C$. We deduce that the composition $R \to C \hookrightarrow A$ is equal to the initially given quotient map. Hence $C \hookrightarrow A$ must be the identity. □

We shall also need the following result. Parts of it are used in the proof of Proposition 4.7.11 in [WE13].

Proposition 4.7.14. *Suppose that* $\mathbb{F}$ *is finite and that condition* $\Phi_{\overline{D}}$ *is satisfied. Then the following hold:*

(i) *For any* $\varphi\colon R^{\mathrm{univ}}_{\Lambda,\overline{D}} \to A$ *in* $\widehat{\mathcal{A}r}_\Lambda$ *giving rise to the pseudorepresentation* D_A*, the induced maps*

$$(\Lambda[[G]] \otimes_\Lambda A)^{\mathrm{CH}}_{D_A} \to (R^{\mathrm{univ}}_{\Lambda,\overline{D}}[[G]] \otimes_{R^{\mathrm{univ}}_{\Lambda,\overline{D}}} A)^{\mathrm{CH}}_{D_A} \to (A[[G]])^{\mathrm{CH}}_{D_A} \to S^{\text{CH-univ}}_{\Lambda,\overline{D}} \otimes_{R^{\mathrm{univ}}_{\Lambda,\overline{D}}} A$$

are isomorphisms.

(ii) *The* $\mathbb{F}$*-algebra* $(\mathbb{F}[[G]])^{\mathrm{CH}}_{\overline{D}}$ *has finite* $\mathbb{F}$*-dimension.*

Proof. For (i) consider the maps in

$$\Lambda[[G]] \otimes_\Lambda A \to R^{\mathrm{univ}}_{\Lambda,\overline{D}}[[G]] \otimes_{R^{\mathrm{univ}}_{\Lambda,\overline{D}}} A \to A[[G]]. \tag{21}$$

They are injective with dense image. By the definition of the Cayley-Hamilton ideal, this still holds after passing to Cayley-Hamilton quotients. By [WE13, Cor. 1.2.2.9 and Prop. 3.2.2.1] the A-algebra $(A[[G]])^{\mathrm{CH}}_{D_A}$ is a finitely generated A-module and hence Noetherian. It follows that its subrings $(\Lambda[[G]] \otimes_\Lambda A)^{\mathrm{CH}}_{D_A} \subset (R^{\mathrm{univ}}_{\Lambda,\overline{D}}[[G]] \otimes_{R^{\mathrm{univ}}_{\Lambda,\overline{D}}} A)^{\mathrm{CH}}_{D_A}$ are also finite A-modules. By completeness of A and their density in $(A[[G]])^{\mathrm{CH}}_{D_A}$, the inclusions must be equalities. By Proposition 4.1.22(iii), we also know that the formation of the Cayley-Hamilton quotient commutes with base change. Hence $(R^{\mathrm{univ}}_{\Lambda,\overline{D}}[[G]] \otimes_{R^{\mathrm{univ}}_{\Lambda,\overline{D}}} A)^{\mathrm{CH}}_{D_A} \to S^{\text{CH-univ}}_{\Lambda,\overline{D}} \otimes_{R^{\mathrm{univ}}_{\Lambda,\overline{D}}} A$ is an isomorphism, and this completes the proof of (i). Part (ii) follows from [WE13, Thm. 1.3.3.2]; it is also a consequence of part (i) and Proposition 4.7.11. □

The next result concerns the reducible locus for multiplicity free $\overline{D}$.

Corollary 4.7.15. *Suppose* $\overline{D}$ *is split and multiplicity free over* $\mathbb{F}$ *and equal to* $\overline{D}_1 \oplus \overline{D}_2$*. Then the morphism* $\iota_{\overline{D}_1,\overline{D}_2}\colon X^{\mathrm{univ}}_{\overline{D}_1} \widehat{\times} X^{\mathrm{univ}}_{\overline{D}_1} \to X^{\mathrm{univ}}_{\overline{D}}, (D_1, D_2) \mapsto D_1 \oplus D_2$ *is a closed immersion.*

Proof. We need to show that the ring homomorphism

$$R_{\overline{D}}^{\mathrm{univ}} \longrightarrow R_{\overline{D}_1}^{\mathrm{univ}} \hat{\otimes}_{\mathbb{F}'} R_{\overline{D}_2}^{\mathrm{univ}}$$

corresponding to $\iota_{\overline{D}_1,\overline{D}_2}$ is surjective. Since both sides are complete Noetherian local rings with isomorphic residue field, it suffices to show the surjectivity for the induced map of the duals of their tangent spaces; i.e., the injectivity of

$$\mathcal{P}sD_{\overline{D}_1}(\mathbb{F}[\varepsilon]) \times \mathcal{P}sD_{\overline{D}_2}(\mathbb{F}[\varepsilon]) \longrightarrow \mathcal{P}sD_{\overline{D}}(\mathbb{F}[\varepsilon]), \quad (D_1, D_2) \longmapsto D_1 \oplus D_2. \tag{22}$$

Consider n_i-dimensional pseudodeformations $D_i, D_i' \in \mathcal{P}sD_{\overline{D}_i}(\mathbb{F}[\varepsilon])$ for $i = 1, 2$ such that $D := D_1 \oplus D_2 = D_1' \oplus D_2'$. By hypothesis, $\overline{D}_1 \oplus \overline{D}_2$ is split and multiplicity free so that we have isomorphisms

$$\mathbb{F}[\varepsilon][G]/\ker(\overline{D}_i) \cong \prod_{j=1}^{s_i} \mathrm{Mat}_{n_{i,j}}(\mathbb{F}) \qquad \text{with } \sum_{j=1}^{s_i} n_{i,j} = n_i \text{ for } i = 1, 2.$$

As discussed in the proof of [Che14, Thm. 2.22], we can lift the canonical family of central orthogonal idempotents of $\mathbb{F}[\varepsilon][G]/\ker(\overline{D}_i)$ to a family of orthogonal idempotents $e_{i,1} + \ldots + e_{i,s_i} = 1$ in $\mathbb{F}[\varepsilon][G]/\mathrm{CH}(D)$, and we further have a family of A-algebra isomorphisms

$$\psi_{i,j} : e_{i,j}\mathbb{F}[\varepsilon][G]/\mathrm{CH}(D)e_{i,j} \xrightarrow{\sim} \mathrm{Mat}_{n_{i,j}}(\mathbb{F}[\varepsilon])$$

for $j = 1, \ldots, s_i$ and $i = 1, 2$. Putting this together, we obtain by Theorem 4.3.10 applied to D_i and D_i' that $(\mathbb{F}[\varepsilon][G], \mathcal{E}_i)$ is a generalized matrix algebra with data of idempotents $\mathcal{E}_i := \{e_{i,j}, \psi_{i,j}\}_{j=1,\ldots,r}$ and determinant $D_i = \det_{(\mathbb{F}[\varepsilon][G],\mathcal{E}_i)} = D_i'$ for $i = 1, 2$, which implies the assertion on the map (22). ☐

The locus of irreducible points shall be of special importance.

Definition 4.7.16. *The* irreducible locus of $X_{\overline{D}}^{\mathrm{univ}}$ *is defined as*

$$(X_{\overline{D}}^{\mathrm{univ}})^{\mathrm{irr}} := \{x \in X_{\overline{D}}^{\mathrm{univ}} \, : \, (D_{\overline{D}}^{\mathrm{univ}})_x \text{ is irreducible}\}$$

and its reducible locus $(X_{\overline{D}}^{\mathrm{univ}})^{\mathrm{red}}$ *as the topological space* $X_{\overline{D}}^{\mathrm{univ}} \smallsetminus (X_{\overline{D}}^{\mathrm{univ}})^{\mathrm{irr}}$*. We overline the notation for the corresponding subsets of* $\overline{X}_{\overline{D}}^{\mathrm{univ}}$*.*

The argument in [Che14, Example 2.20] also proves.

Proposition 4.7.17. *The subset* $(X_{\overline{D}}^{\mathrm{univ}})^{\mathrm{irr}}$ *of* $X_{\overline{D}}^{\mathrm{univ}}$ *is Zariski open.*

By Proposition 4.7.11(iii), we can associate to $S_{\Lambda,\overline{D}}^{\text{CH-univ}}$ a sheaf of coherent $\mathcal{O}_{X_{\Lambda,\overline{D}}^{\mathrm{univ}}}$-algebras $\mathcal{S}_{\Lambda,\overline{D}}^{\text{CH-univ}}$ under the finiteness condition $\Phi_{\overline{D}}$. The next result is not stated verbatim in [Che14]; however its proof is that of [Che14, Cor. 2.23], with a continuity requirement added.

Proposition 4.7.18. *Over* $(X_{\Lambda,\overline{D}}^{\mathrm{univ}})^{\mathrm{irr}}$*, the sheaf* $\mathcal{S}_{\Lambda,\overline{D}}^{\text{CH-univ}}$ *is an Azumaya* $\mathcal{O}_{X_{\Lambda,\overline{D}}^{\mathrm{univ}}}$*-algebra of rank* n^2 *equipped with its reduced norm.*

Over affine open subsets of $(X_{\Lambda,\overline{D}}^{\mathrm{univ}})^{\mathrm{irr}}$, Proposition 4.7.18 is a variant of Proposition 4.1.25 under some continuity constraints.

4.8 Pseudodeformations over local fields

In this subsection we develop some results analogous to Subsection 3.3. We shall prove various results on continuous pseudodeformations $D\colon G \to \kappa$ where κ is a local field. At the end of the subsection, we gather same basic but elementary properties of pseudodeformation that follow from the results presented so far. The results herein are presumably known to experts. Proofs that we could not locate in the literature are given with much detail.

Lemma 4.8.1. *Let κ be a local field with valuation ring $\mathcal{O}_\kappa$, and let $D\colon G \to \kappa$ be a continuous n-dimensional pseudorepresentation. Then the following hold:*

(i) There exists $D_{\mathcal{O}} \in \mathcal{P}s\mathcal{R}^n_G(\mathcal{O}_\kappa)$ such that $D_{\mathcal{O}} \otimes_{\mathcal{O}_\kappa} \kappa = D$.

Let $C \subset \mathcal{O}_\kappa$ be the admissible profinite subring of $\mathcal{O}_\kappa$ from Lemma 4.4.8 and let $D_C \in \mathcal{P}s\mathcal{R}^n_G(C)$ be such that $D_C \otimes_C \mathcal{O}_\kappa = D_{\mathcal{O}}$. Then furthermore:

(ii) C is local, its residue field $\kappa(C)$ is finite, either C is a finite field, or κ is a finite extension of the fraction field of C, and $D_{\mathcal{O}}$ is residually equal to $\overline{D} := D_C \otimes_C \kappa(C)$.

Proof. Let $\rho_{D\otimes_\kappa \kappa^{\mathrm{alg}}}$ be the representation from Theorem 4.1.19. For (i) observe first that the characteristic polynomial coefficients $\Lambda_{D,i}$ of $\chi_D(g,\,\cdot\,)$ are continuous for $1 \le i \le n$, and hence the sets $\Lambda_{D,i}(G)$ are compact in κ. Assume that for some $g \in G$, $\Lambda_{D,i}(g)$ does not lie in $\mathcal{O}_\kappa$. Then at least one eigenvalue of $\rho_{D\otimes_\kappa \kappa^{\mathrm{alg}}}(g)$ has valuation different from 0, and, since we can pass to g^{-1}, we may assume that this valuation is negative. Let $\lambda_1, \dots, \lambda_n \in \kappa^{\mathrm{alg}}$ denote the eigenvalues of $\rho_{D\otimes_\kappa \kappa^{\mathrm{alg}}}(g)$ and index them so that $\lambda_1, \dots, \lambda_j$ are precisely those with negative valuation. Then for $n > 0$, the valuation of $\Lambda_{D,j}(g^n)$ is the valuation of $(\lambda_1 \cdot \ldots \cdot \lambda_j)^n$. The latter valuations are unbounded. This contradicts the compactness of $\Lambda_{D,j}(G)$ and thus proves (i).

We now prove (ii). By Lemma 4.4.8 the ring C is a finite product $\prod_i C_i$ of local admissible profinite $W(\mathbb{F})$-algebras C_i and the residue field of each C_i is finite. Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}_\kappa$. Then $C \cap \mathfrak{m}$ is topologically nilpotent and $C/(C \cap \mathfrak{m})$ is a finite field that surjects onto the product of the residue fields of the C_i. It follows that C is local with finite residue field $\kappa(C)$. It remains to show the assertion on the fraction field of C, since the last part of (ii) follows from Lemma 4.4.8. For this we may assume that C is infinite.

Let κ' be the fraction field of C. If κ' is a p-adic field, the assertion is obvious, since then κ and κ' are finite extensions of $\mathbb{Q}_p$. Thus we assume that $\mathcal{O}_\kappa \cong \mathbb{F}[[t]]$ for a finite extension $\mathbb{F}$ of $\mathbb{F}_p$. Let $b_1, \dots, b_m$ be an $\mathbb{F}_p$-basis of $\mathbb{F}$ and let $f \in C$ be a non-zero element of valuation $n > 0$. Then $\{t^i b_j \mid i = 0, \dots, n-1, j = 1, \dots, m\}$ is a basis of $\mathcal{O}_\kappa$ over $\mathbb{F}_p[[f]] \subset C$, From this it is straightforward to see that $[\kappa : \kappa']$ is finite. ∎

The following result is a generalization of Corollary 4.4.6.

Corollary 4.8.2. *Let κ be a local field, let A be in $\mathcal{A}r_\kappa$ and let $D \in \mathcal{P}s\mathcal{R}^n_G(A)$ be continuous. Define $\overline{D}$ as in Lemma 4.8.1, and assume that condition $\Phi_{\overline{D}}$ holds. Then the following hold:*

(i) If $A = \kappa$ then $\rho_{D\otimes_\kappa \kappa^{\mathrm{alg}}}$ is continuous.

(ii) If D is split and irreducible, then $\rho_D = \rho_D^{\mathrm{CH}}$ from Proposition 4.1.25 is a continuous representation to $M_n(A)$.

Proof. We first prove (i), and so here we assume $A = \kappa$. Set $\Lambda := \mathcal{O}_\kappa$ and consider the diagram

$$R^{\mathrm{univ}}_{\Lambda,\overline{D}}[[G]] \xrightarrow{\rho^{\mathrm{CH}}_{\Lambda,\overline{D}}} S^{\text{CH-univ}}_{\Lambda,\overline{D}} \xrightarrow{\mathrm{id}\otimes\varphi} S^{\text{CH-univ}}_{\Lambda,\overline{D}} \otimes_{R^{\mathrm{univ}}_{\Lambda,\overline{D}}} \mathcal{O}_\kappa \xrightarrow{\mathrm{id}\otimes\iota} S^{\text{CH-univ}}_{\Lambda,\overline{D}} \otimes_{R^{\mathrm{univ}}_{\Lambda,\overline{D}}} \kappa^{\mathrm{alg}},$$

where $\varphi\colon R^{\mathrm{univ}}_{\Lambda,\overline{D}} \to \mathcal{O}_\kappa$ is the map induced from the universal property of $R^{\mathrm{univ}}_{\Lambda,\overline{D}}$, and where $\iota\colon \mathcal{O}_\kappa \to \kappa^{\mathrm{alg}}$ is the natural inclusion. The first map is continuous by Proposition 4.7.11(ii), the second by Proposition 4.7.11(iv), which says that $S^{\text{CH-univ}}_{\Lambda,\overline{D}}$ carries the $\mathfrak{m}_{\overline{D}}$-adic topology, By Proposition 4.7.11(iv), the ring $S^{\text{CH-univ}}_{\Lambda,\overline{D}} \otimes_{R^{\text{univ}}_{\Lambda,\overline{D}}} \mathcal{O}_\kappa$ is finitely generated as an $\mathcal{O}_F$-module, and hence the $\mathfrak{m}_{\overline{D}}$-topology also coincides with the topology inherited from $S^{\text{CH-univ}}_{\Lambda,\overline{D}} \otimes_{R^{\text{univ}}_{\Lambda,\overline{D}}} \kappa \in \mathcal{A}r_\kappa$; it follows that also the last map is continuous and that $S^{\text{CH-univ}}_{\Lambda,\overline{D}} \otimes_{R^{\text{univ}}_{\Lambda,\overline{D}}} \kappa^{\text{alg}}$ has finite κ^{alg}-dimension. But the also the map

$$S^{\text{CH-univ}}_{\Lambda,\overline{D}} \otimes_{R^{\text{univ}}_{\Lambda,\overline{D}}} \kappa^{\text{alg}} \xrightarrow[4.7.14]{\simeq} (\kappa^{\text{alg}}[[G]])^{\text{CH}}_D \longrightarrow \kappa^{\text{alg}}[[G]]/\ker(D)$$

is continuous. Hence in the factorization of $D\colon G \to \kappa^{\text{alg}}$ via $\kappa^{\text{alg}}[[G]]/\ker(D)$ given in Proposition 4.1.16, the first map is continuous. From Theorem 4.1.17 we know that $\kappa^{\text{alg}}[[G]]/\ker(D)$ is semisimple and finite-dimensional over κ^{alg} and that its determinants are given by determinants of the simple matrix algebra factors of $\kappa^{\text{alg}}[[G]]/\ker(D)$. Hence the second map in the factorization given in Proposition 4.1.16 is continuous, and thus so is the composition ρ.

The proof of (ii) is analogous. One has to replace κ by A in most places and one substitute Theorem 4.1.17 by Theorem 4.3.10. ☐

Remark 4.8.3. In an abstract setting, $\Phi_{\overline{D}}$ in Corollary 4.8.2 seems hard to check. In our applications we mostly know $\overline{D}$, so then the formulation is useful. A more natural condition to require would be Φ_D; we do suspect that this condition also suffices. Also, we wonder if the conclusion of Corollary 4.8.2 might hold without assuming $\Phi_{\overline{D}}$, and without invoking Lemma 4.8.1; just because $A \in \mathcal{A}r_\kappa$.

Corollary 4.8.4. *Let κ be a local field, and let $D \in \mathcal{P}s\mathcal{R}^n_G(A\kappa$ be continuous. Define $\overline{D}$ as in Lemma 4.8.1, and assume that condition $\Phi_{\overline{D}}$ holds. Then there exists a finite extension κ' of κ and split irreducible continuous $D_i \in \mathcal{P}s\mathcal{R}^{n_i}_G(\kappa')$, $i = 1, \ldots, r$ such that*

$$D \otimes_{\mathcal{O}_\kappa} \mathcal{O}_{\kappa'} = D_1 \oplus \ldots \oplus D_r \tag{23}$$

Proof. By Theorem 4.1.17 the κ-algebra $S := \kappa[[G]]/\ker(D)$ has finite κ-dimension. Hence Lemma A.2.3 allows us to find a finite extension κ' of κ such that $S \otimes_\kappa \kappa'/\operatorname{Rad}(S \otimes_\kappa \kappa')$ is a product of matrix rings over κ'. It follows that $\kappa'[[G]]/\ker(D \otimes_\kappa \kappa')$ is a product of matrix algebras over κ'. Hence we have $D \otimes_\kappa \kappa' = \oplus_{i=1}^r D_i$ for split irreducible $D_i \in \mathcal{P}s\mathcal{R}^{n_i}_G(\kappa')$. We find that $(\oplus_{i=1}^r \rho_{D_i}) \otimes_{\kappa'} \kappa^{\text{alg}} \cong \rho_{D \otimes_\kappa \kappa^{\text{alg}}}$. Since the latter is continuous by Corollary 4.8.2, so are the ρ_{D_i}. By Lemma 4.8.1, the D_i can be defined over $\mathcal{O}_{\kappa'}$. Equation (23) is immediate from the isomorphism of representations we just stated by taking characteristic polynomials. ☐

We also need the following continuity result.

Proposition 4.8.5. *Suppose $\Phi_{\overline{D}}$ holds. Let A be a quotient of $R^{\text{univ}}_{W(\mathbb{F}),\overline{D}}$ with corresponding pseudodeformation D_A. Suppose A is a domain with fraction field $\mathbb{K}$ and that $D_\mathbb{K} := D_A \otimes_A \mathbb{K}$ is multiplicity free. Then there exist*

(a) a finite extension $\mathbb{K}'$ of $\mathbb{K}$ with integral closure A' of A in $\mathbb{K}'$, and

(b) continuous irreducible pseudorepresentations $D'_i\colon G \to A'$,

such that A' lies in $\mathcal{A}\text{dm}_{W(\mathbb{F})}$ and $D_A \otimes_A A' = \oplus_i D'_i$. If moreover $\overline{D}$ is split, then the ring of definition A_i of each D'_i lies in $\widehat{\mathcal{A}r}_{W(\mathbb{F})}$ and one has $\overline{D} = \oplus_i (D'_i \otimes_{A_i} \kappa(A_i))$ over κ.

Proof. Define the rings

$$S_A := S^{\text{CH-univ}}_{\Lambda,\overline{D}} \otimes_{R^{\text{univ}}_{\Lambda,\overline{D}}} A \text{ and } S_\mathbb{K} := S_A \otimes_A \mathbb{K}.$$

Then by Proposition 4.7.11, the A-algebra S_A is finitely generated as an A-module, and the induced homomorphism $G \to S_A^\times$ is continuous if S_A is equipped with the $\mathfrak{m}_A$-adic topology as an A-module. In particular the $\mathbb{K}$-algebra $S_\mathbb{K}$ is of finite $\mathbb{K}$-dimension.

Now Lemma A.2.3 gives a finite extension $\mathbb{K}'$ of $\mathbb{K}$ so that $S' := S_\mathbb{K} \otimes_\mathbb{K} \mathbb{K}' / \operatorname{Rad}(S_\mathbb{K} \otimes_\mathbb{K} \mathbb{K}')$ is a product of matrix algebras $S' = \prod_i M_{n_i}(\mathbb{K}')$. Since $D_{\mathbb{K}'} := D_\mathbb{K} \otimes_\mathbb{K} \mathbb{K}'$ factors via S' and since it $D_{\mathbb{K}'}$ multiplicity free, it is the composition of $G \to S_A^\times \to (S')^\times$ with $\prod_i \det_{n_i}$ where $\det_{n_i}$ is the determinant of $M_{n_i}(\mathbb{K}')$. Because D_A is multiplicity free, the number of factors of S' is the number of summands of $D_\mathbb{K} \otimes_\mathbb{K} \mathbb{K}^{\mathrm{alg}}$.

Let A' be the integral closure of A in $\mathbb{K}'$. Because of Lemma A.1.1(i), it is finite over A and hence lies in $\mathcal{A}\mathrm{dm}_{W(\mathbb{F})}$. Let $S_{A'} := S_A \otimes_A A'$ and write $(S_A)'$ for the image of $S_{A'}$ in S'. Then the induced map $G \to ((S_A)')^\times$ is clearly continuous, where $(S_A)'$ carries the topology induced from A' on finitely generated A' modules. Moreover one has $(S_A)' \hookrightarrow (S_A)' \otimes_{A'} \mathbb{K}' = S'$. Hence we can find $f' \in A'$ non-zero so that $(S_A)'[\frac{1}{f'}] \cong \prod_i M_{n_i}(A'[\frac{1}{f'}])$, because after localizing at a sufficiently divisible element, the ring $(S_A)'[\frac{1}{f'}]$ contains all $E_i^{k.l}$ in the sense of Notation 4.3.2. The ring $A'[\frac{1}{f'}]$ is a topological ring with open neighborhood system of zero being given by the sets $(f')^n I$ for $n \in \mathbb{Z}$ and I an open ideal of A'.

Define $D_i' \colon G \to A'[\frac{1}{f'}]$ as the continuous pseudorepresentation that is the composition of

$$G \to ((S_A)')^\times \hookrightarrow \Big((S_A)'[\frac{1}{f'}]\Big)^\times$$

with $\det_{n_i}$. It is clear that $\bigoplus_i D_i' = D \otimes_A A'[\frac{1}{f'}]$. Since the characteristic polynomials satisfy

$$\prod_i \chi_{D_i'}(\cdot, t) = \chi_{D_{\mathbb{K}'}}(\cdot, t) \in \mathbb{K}'[t]$$

the coefficients of the $\chi_{D_i'}(\cdot, t)$ lie in A' by [Mat89, Thm. 9.2]. Hence the D_i' take values in A' by Proposition 4.1.10, i.e., Amitsur's formula. From the topology on $A'[\frac{1}{f'}]$ it now follows that D_i' considered as a pseudorepresentation $G \to A'$ also continuous and that for these D_i' we have $\bigoplus_i D_i' = D \otimes_A A'$. It remains to prove the last assertion.

Let $A_i \subset A'$ be the ring of definition of D_i', denote by $D_i \colon G \to A_i$ the corresponding pseudorepresentation and let $\overline{D}_i := D_i \otimes_{A_i} \kappa(A_i)$. Note that the $\kappa(A_i)$ are the rings of definition of $\overline{D}_i$. Let κ be the smallest extension of κ that contains all $\kappa(A_i)$. Then $\overline{D} \otimes_\kappa \kappa \cong \bigoplus_i \overline{D}_i \otimes_{\kappa(A_i)} \kappa$. However $\overline{D}$ is split over κ and so all $\overline{D}_i$ are defined over κ, and this shows $\kappa(A_i) = \kappa$ for all i. We deduce $A_i \in \widehat{\mathcal{A}r}_{W(\mathbb{F})}$. ☐

Corollary 4.8.6. *Let κ be a finite or a local field, and let $\rho\colon G \to \mathrm{GL}_n(\kappa)$ be a continuous absolutely irreducible homomorphism with associated pseudorepresentation D. Suppose that $\Phi_{\overline{D}}$ holds for $\overline{D}$ attached to D as in Lemma 4.8.1. Then the natural map $R^{\mathrm{univ}}_{\Lambda,\rho} \to R^{\mathrm{univ}}_{\Lambda,D}$ induced from $\rho_A \mapsto D_{\rho_A}$ for $A \in \mathcal{A}r_\kappa$ is an isomorphism.*

Proof. If κ is finite, the assertion is [Che14, Exmp. 3.4]. For local κ, we need to show that natural transformation of functors $\mathcal{A}r_\kappa \to \mathit{Sets}$ defined by

$$A \mapsto \{\text{continuous deformations } \rho_A \text{ of } \rho \text{ to } A\} \xrightarrow{\ \rho_A \mapsto D_{\rho_A}\ } \{\text{continuous pseudodeformations } D_A \text{ of } D \text{ to } A\}$$

is an isomorphism. Well-definedness is clear. Injectivity follows from Theorem 4.3.10(i) since ρ_D is absolutely irreducible. To prove surjectivity, consider a pseudodeformation $D_A\colon G \to A$ of D

and note that by Theorem 4.3.10(i) there exists a deformation ρ_A of ρ_D to A with $D_A = D_{\rho_A}$. The continuity of ρ_A follows from Corollary 4.8.2(ii). □

We now give an analog of Theorem 3.3.1 for pseudorepresentations.

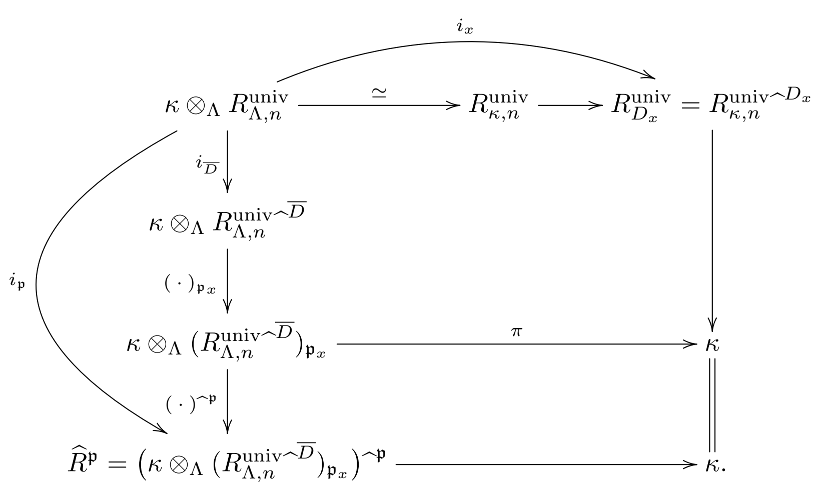
Corollary 4.8.7. *Let $\mathbb{F}$ be finite and let $\overline{D} \in \mathcal{P}s\mathcal{R}^n_G(\mathbb{F})$ be continuous. Let $x \in X^{\mathrm{univ}}_{\Lambda,\overline{D}}$ be a point of dimension 1 with residue homomorphism $\pi'\colon R_x := R^{\mathrm{univ}}_{\Lambda,\overline{D}} \otimes_\Lambda \kappa(x) \to \kappa(x)$ and $\mathfrak{p} := \ker \pi'$ and with attached pseudorepresentation $D_x\colon G \to \kappa(x)$. To x one naturally attaches:*

(a) The universal pseudodeformation $D^{\mathrm{univ}}_{\Lambda,D_x}\colon G \longrightarrow R^{\mathrm{univ}}_{\Lambda,D_x}$ from Proposition 4.7.4.

(b) The completion of $D^{\mathrm{univ}}\widehat{\otimes}_{R^{\mathrm{univ}}_{\Lambda,\overline{D}}} R_x\colon R_x[[G]] \to R_x$ at $\mathfrak{p}$ which we denote by $\hat{D}_\mathfrak{p}\colon G \to \widehat{R}^\mathfrak{p}$.

The induced map $\varphi\colon R^{\mathrm{univ}}_{\Lambda,D_x} \to \widehat{R}^\mathfrak{p}$ is an isomorphism.

Proof. We write κ for $\kappa(x)$. For any commutative ring A, in this proof we abbreviate $R^{\mathrm{univ}}_{A,n} := R^{\mathrm{univ}}_{A[G,]}n$, which is naturally isomorphic to $A \otimes_\mathbb{Z} R^{\mathrm{univ}}_{G,n}$. The symbol ${}^{\frown\overline{D}}$ denotes the completion of a ring at its $\overline{D}$-open ideals, cf. Definition 4.7.2, and similarly ${}^{\frown D_x}$ for the completion at the D_x-open ideals – when this makes sense. Then $R^{\mathrm{univ}}_{\Lambda,\overline{D}} = R^{\mathrm{univ}\frown\overline{D}}_{\Lambda,n}$, and the universal pseudodeformation ring for continuous pseudodeformations of D_x is $R^{\mathrm{univ}}_{D_x} = R^{\mathrm{univ}\frown D_x}_{\kappa,n}$. Let $\mathfrak{p}_x$ denote the kernel of the homomorphism $R^{\mathrm{univ}}_{\Lambda,\overline{D}} \to \kappa$ corresponding to D_x, and $\mathfrak{p}$ the kernel of $\pi\colon R := \kappa \otimes_\Lambda (R^{\mathrm{univ}\frown\overline{D}}_{\Lambda,n})_{\mathfrak{p}_x} \to \kappa$[9] or of π restricted to $\kappa \otimes_\Lambda R^{\mathrm{univ}}_{\Lambda,\overline{D}}$, and write ${}^{\frown\mathfrak{p}}$ for the completion at $\mathfrak{p}$ and similarly ${}^{\frown\mathfrak{p}_x}$ for that at $\mathfrak{p}_x$. Consider the diagram:



Let $\varphi_\kappa\colon \kappa \otimes_\Lambda R^{\mathrm{univ}}_{\Lambda,n} \to \kappa$ be the diagonal homomorphism from the top left to the bottom right. To show the present corollary, let A be in $\mathcal{A}r_\kappa$ with residue homomorphism $\psi\colon A \to \kappa$, and let $\varphi_A\colon \kappa \otimes_\Lambda R^{\mathrm{univ}}_{\Lambda,n} \to A$ be a surjective homomorphism with $\psi \circ \varphi_A = \varphi_\kappa$. Let $D_A\colon G \to A$ be the induced pseudorepresentation. We need to show that φ_A factors via i_x if and only if it factors via $i_\mathfrak{p}$.

Note first that from the definition of ${}^{\frown\mathfrak{p}}$ it is clear that φ_A factors via $i_\mathfrak{p}$ if and only if it factors via $(\,\cdot\,)_{\mathfrak{p}_x} \circ i_{\overline{D}}$. Since φ_κ maps the elements of $R^{\mathrm{univ}}_{\Lambda,n} \smallsetminus \mathfrak{p}_x$ to units in κ, so does φ_A since A is local with residue field κ. Hence we need to show that φ_A factors via $i_\mathfrak{p}$ if and only if it factors via $i_{\overline{D}}$. Let I be the kernel of the compositum of φ_A with $R^{\mathrm{univ}}_{\Lambda,n} \to \kappa \otimes_\Lambda R^{\mathrm{univ}}_{\Lambda,n}, r \mapsto 1 \otimes r$, and

[9] The index $\mathfrak{p}_x$ refers to localization at $\mathfrak{p}_x$.

let $D_I \colon G \to R^{\mathrm{univ}}_{\Lambda,n}/I$ be the induced pseudorepresentation. Because of $\psi \circ \varphi_A = \varphi_\kappa$ we have $I \subset \mathfrak{p}_x$, and because A is Artinian it follows that $\mathfrak{p}_x$ is the radical of I. Let $\mathfrak{m}_{\overline{D}}$ be the kernel of $R^{\mathrm{univ}}_{\Lambda,n} \to \kappa_{\overline{D}}$ given by $\overline{D}$. That φ_A factors via $i_{\overline{D}}$ means that the ideals $I_n = I + \mathfrak{m}^n_{\overline{D}}$ are $\overline{D}$-open for $n \in \mathbb{N}_{\geq 1}$ and that $I = \bigcap_n I_n$. For each I_n (and for I) let $D_{I_n} \colon G \to R^{\mathrm{univ}}_{\Lambda,n}/I_n$ (and D_I, resp.,) be the induced pseudorepresentation. That all D_{I_n} are continuous is therefore equivalent to D_I being continuous. On the other hand, that φ_A factors via i_x means that $\ker \varphi_A$ is D_x-open.

Define $\widehat{R}^{\mathrm{univ}}_{\Lambda,n}/I$ as the completion of $R^{\mathrm{univ}}_{\Lambda,n}/I$ with respect to the I_n (i.e., with respect to $\mathfrak{m}^n_{\overline{D}}$). Hence we need to show that the following two conditions are equivalent:

(a) $D_A \colon G \to A$ is continuous, i.e., all of its characteristic polynomial coefficients are;

(b) $D_I \colon G \to \widehat{R}^{\mathrm{univ}}_{\Lambda,n}/I$ is continuous with respect to the profinite topology on $\widehat{R}^{\mathrm{univ}}_{\Lambda,n}/I$.

By definition of D_x, as a pseudodeformation of $\overline{D}$, it is continuous as a pseudorepresentation $G \to \kappa$ with respect to the natural topology on κ, and continuous as a pseudorepresentation $G \to \widehat{R}^{\mathrm{univ}}_{\Lambda,n}/\mathfrak{p}_x$ with the profinite topology on the latter. We also note that by construction, we start from the map φ_A, the pseudorepresentation D_A factors as D_I composed with the homomorphism $\widehat{R}^{\mathrm{univ}}_{\Lambda,n}/I \to A$. So we need to show that $\widehat{R}^{\mathrm{univ}}_{\Lambda,n}/I \to A$, which by construction is injective, identifies $\widehat{R}^{\mathrm{univ}}_{\Lambda,n}/I$ with a compact open subring of A (in the topology of A).

We shall induct over the length of A to show that $\widehat{R}^{\mathrm{univ}}_{\Lambda,n}/I \subset A$ is a compact open subring. As observed in the previous paragraph, by hypothesis we know that $R/\mathfrak{p}_x \subset \kappa$ is a compact open subring that is contained in the valuation ring of κ. This completes the case where A has length 1. In the induction step, let $J \subset A$ be an ideal with quotient $A' = A/J$ such that $\dim_\kappa J = 1$. Let I' be the corresponding ideal of $R^{\mathrm{univ}}_{\Lambda,n}$, and consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J & \longrightarrow & A & \longrightarrow & A' & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & I/I' & \longrightarrow & \widehat{R}^{\mathrm{univ}}_{\Lambda,n}/I & \longrightarrow & \widehat{R}^{\mathrm{univ}}_{\Lambda,n}/I' & \longrightarrow & 0. \end{array}$$

By the surjectivity of φ_A, it is clear that A is the κ-span of it subring $\widehat{R}^{\mathrm{univ}}_{\Lambda,n}/I$. By induction hypothesis, the right hand inclusion identifies $\widehat{R}^{\mathrm{univ}}_{\Lambda,n}/I'$ with a compact open subring of A' that spans A' over κ. Denoting by $\mathcal{O}$ the ring of integers of κ, this is equivalent to $\mathcal{O} \cdot \widehat{R}^{\mathrm{univ}}_{\Lambda,n}/I'$ being an $\mathcal{O}$-lattice in A' and to $(\mathcal{O} \cdot \widehat{R}^{\mathrm{univ}}_{\Lambda,n}/I')/(\widehat{R}^{\mathrm{univ}}_{\Lambda,n}/I')$ being finite. We need to show the analog for A and I.

We know that $J \cong \kappa$ as a κ-module and that I/I' is a finitely generated $\widehat{R}^{\mathrm{univ}}_{\Lambda,n}/\mathfrak{p}_x$-submodule. Since $\widehat{R}^{\mathrm{univ}}_{\Lambda,n}/\mathfrak{p}_x \subset \kappa$ is compact open, we find that $\mathcal{O} \cdot I/I'$ is an $\mathcal{O}$-lattice in J and that $\mathcal{O} \cdot (I/I')/(I/I')$ is finite. Let $b_0 \in I/I'$ be an $\mathcal{O}$-basis of $\mathcal{O} \cdot I/I'$. Choose an $\mathcal{O}$-basis of $\mathcal{O} \cdot \widehat{R}^{\mathrm{univ}}_{\Lambda,n}/I'$ in $\widehat{R}^{\mathrm{univ}}_{\Lambda,n}/I'$ (this is possible by Nakayama's Lemma by first working in the reduction modulo $\mathfrak{m}_{\mathcal{O}}$) and lift these basis elements to elements $b_1, \ldots, b_t$ in $\widehat{R}^{\mathrm{univ}}_{\Lambda,n}/I$. Then one verifies that the $\mathcal{O}$-span of $\{b_0, \ldots, b_t\}$ contains $\widehat{R}^{\mathrm{univ}}_{\Lambda,n}/I$, and that $(\mathcal{O} \cdot \widehat{R}^{\mathrm{univ}}_{\Lambda,n}/I)/(\widehat{R}^{\mathrm{univ}}_{\Lambda,n}/I)$ is finite. This completes the induction step and the proof of the proposition. □

Remark 4.8.8. We think that [Che14, Cor. 2.23](ii) has to be formulated in a way similar to Corollary 4.8.7; only if $\kappa(x)$ is a p-adic field, one can simply complete $(R^{\mathrm{univ}}_{\Lambda,\overline{D}})_{\mathfrak{p}_x}$ to obtain a universal pseudodeformation ring for D_x. In Corollary 4.8.7 we have only verified this for dimension 1 points.

Corollary 4.8.9. *Let κ be a local field and let $D \in \mathcal{P}s\mathcal{R}^n_G(\kappa)$ be continuous. Suppose that condition $\Phi_{\overline{D}}$, for $\overline{D}$ from Lemma 4.8.1, is satisfied. Then the following hold:*

(i) *The ring* $R^{\mathrm{univ}}_{\kappa,D}$ *is Noetherian.*

(ii) *Suppose that* D *is irreducible and that* $H^2(G,\mathrm{ad}_\rho)=0$ *for* $\rho:=\rho_{D\otimes_\kappa\kappa^{\mathrm{alg}}}$. *Then* $R^{\mathrm{univ}}_{\kappa,D}$ *is formally smooth over* κ *of relative dimension* $\dim_{\kappa^{\mathrm{alg}}} H^1(G,\mathrm{ad}_\rho)$.

Proof. Let C and $\overline{D}$ be as in Lemma 4.8.1, let $\overline{D}' := \overline{D}\otimes_{\kappa(C)}\kappa$. The hypotheses guarantee that $R^{\mathrm{univ}}_{W(\kappa(C)),\overline{D}'}$ is Noetherian by Proposition 4.7.4. Part (i) follows by choosing $x\in X^{\mathrm{univ}}_{W(\kappa(C)),\overline{D}'}$ as the point corresponding to D, and by applying Corollary 4.8.7 with this x; note that $\widehat{R}^{\mathfrak{p}}$ (in Corollary 4.8.7) is Noetherian as the completion of a Noetherian local ring.

To see part (ii), let $\kappa'\supset\kappa$ be a finite extension over which D is split. Let $\rho:=\rho_{D\otimes_\kappa\kappa'}\colon G\to \mathrm{GL}_n(\kappa')$ be the continuous and absolutely irreducible representation with $D_{\rho'}.=D\otimes_\kappa\kappa'$. Our hypotheses gives $H^2(G,\mathrm{ad}_{\rho'})=0$ and it will suffice to show that $R^{\mathrm{univ}}_{\kappa',D}$ is formally smooth over κ' of dimension $\dim_{\kappa'} H^1(G,\mathrm{ad}_{\rho'})$. This follows from Corollary 4.8.6 and Theorem 3.2.4 (d). □

For later use we also need variants of Corollary 4.8.6 and Corollary 4.8.7 for deformations of pairs of representations and pseudorepresentations. Let $\overline{D}_1,\overline{D}_2\colon G\to\mathbb{F}$ be continuous pseudorepresentations of dimensions n_1 and n_2, such that $\Phi_{\overline{D}_i}$ holds for $i=1,2$. Consider the functor

$$\mathcal{P}s D_{(\overline{D}_1,\overline{D}_2)}\colon \widehat{\mathcal{A}r}_\Lambda\to Sets,\quad A\longmapsto\{(D_1,D_2)\mid D_i\colon G\longrightarrow A \text{ is a pseudodeformation of } \overline{D}_i\}.$$

It is straightforward to see that $\mathcal{P}sD_{(\overline{D}_1,\overline{D}_2)}$ is represented by $R^{\mathrm{univ}}_{(\Lambda,\overline{D}_1,\overline{D}_2)} := R^{\mathrm{univ}}_{\Lambda,\overline{D}_1}\widehat{\otimes}R^{\mathrm{univ}}_{\Lambda,\overline{D}_2}$ and that the ring $R^{\mathrm{univ}}_{(\Lambda,\overline{D}_1,\overline{D}_2)}$ is Noetherian, using Proposition 4.7.4 and Proposition 4.7.11.

Let $x\in X^{\mathrm{univ}}_{(\overline{D}_1,\overline{D}_2)} := \operatorname{Spec} R^{\mathrm{univ}}_{(\overline{D}_1,\overline{D}_2)}$ be a point of dimension 1 such that $D_{i,x}$ is irreducible for $i=1,2$ for the corresponding pair $(D_{1,x},D_{2,x})$. As above one can define a deformation functor for this pair an $\mathcal{A}r_{\kappa(x)}$. It is representable by $R^{\mathrm{univ}}_{\kappa(x),(D_{1,x},D_{2,x})} := R^{\mathrm{univ}}_{\kappa(x),D_{1,x}}\widehat{\otimes}_{\kappa(x)}R^{\mathrm{univ}}_{\kappa(x),D_{2,x}}$, which is again complete local Noetherian. Let $\pi\colon R_x := R^{\mathrm{univ}}_{\Lambda,(\overline{D}_1,\overline{D}_2)}\otimes_\Lambda\kappa(x)\to\kappa(x)$ be the homomorphism induced from x, and let

$$\varphi\colon R^{\mathrm{univ}}_{\kappa(x),(D_{1,x},D_{2,x})}\to\widehat{R}_x$$

be the natural homomorphism constructed as in Corollary 4.8.7, where $\widehat{R}_x$ denotes the completion of R_x at $\mathfrak{p}_x := \operatorname{Ker}\pi$.

Let finally L be a finite extension of $\kappa(x)$ over which there exist absolutely irreducible representations $\rho_i\colon G\to\mathrm{GL}_{n_i}(L)$ such that $D_{\rho_i}=D_{i,x}\otimes_{\kappa(x)}L$ for $i=1,2$. Define the functor

$$\mathcal{D}_{(\rho_1,\rho_2)}\colon \mathcal{A}r_\Lambda\longrightarrow Sets,\quad A\longmapsto\{(\rho_{1,A},\rho_{2,A})\mid \rho_{i,A}\colon G\longrightarrow\mathrm{GL}_n(A)\ :\ \rho\text{ is a deformation of }\rho_i\},$$

Since the ρ_i are absolutely irreducible, it is represented by $R^{\mathrm{univ}}_{L,(\rho_1,\rho_2)} := R^{\mathrm{univ}}_{L,\rho_1}\widehat{\otimes}_L R^{\mathrm{univ}}_{L,\rho_2}$, and the latter ring is Noetherian local by Proposition 3.2.3 and Theorem 3.2.4 since we assume $\Phi_{\overline{D}_i}$, $i=1,2$. As in Corollary 4.8.6 one has a natural homomorphism

$$\psi\colon R^{\mathrm{univ}}_{L,(\rho_1,\rho_2)}\to R^{\mathrm{univ}}_{\kappa(x),(D_{1,x},D_{2,x})}\otimes_{\kappa(x)}L.$$

Proposition 4.8.10. *The following hold:*

(i) *The maps* ψ *and* φ *are isomorphisms.*

(ii) *Suppose* $G=G_K$ *and* $H^0(G,\overline{\mathrm{ad}}_{\rho_i})=0$ *for* $i=1,2$. *Then* $R^{\mathrm{univ}}_{L,(\rho_1,\rho_2),\mathrm{red}}$ *is formally smooth over* L *of dimension* $d(n_1^2+n_2^2)+2$ *and hence* x *is a smooth point on* $X^{\mathrm{univ}}_{(\overline{D}_1,\overline{D}_2),\mathrm{red}}$ *with tangent space dimension* $d(n_1^2+n_2^2)+1$.

Proof. The two assertions in (i) are proved exactly as Corollary 4.8.6 and Corollary 4.8.7, and we omit the details. The first assertion in (ii) follows from our description of $R^{\mathrm{univ}}_{L,(\rho_1,\rho_2),\mathrm{red}}$ as a completed tensor product and from Corollary 3.4.3. The second assertion now is a consequence of (i), of Proposition 4.7.6 and of Lemma 3.3.3. □

5 Equidimensionality of special fibers and Zariski density of the regular locus

This section proves the main result of this work, the equidimensionality of the special fiber of universal pseudodeformation rings of expected dimension. The proof follows Chenevier's proof of the equidimensionality of the generic fiber of the universal pseudorepresentation space from [Che11]. The main contribution is to overcome all complications that arise in the special fiber.

There are certain points in the special fiber that have no counterpart in the generic fiber. We call them *special points* and describe them in Subsection 5.1; see Definition 5.1.2. Non-special (irreducible) points will take the role of irreducible points in Chenevier's work. Subsection 5.1 also contains some technical result, Lemma 5.1.6, on the comparison of universal pseudodeformation and universal deformation rings over local fields where the residual pseudorepresentation is a sum of two irreducible ones.

In Subsection 5.2 we prove the inductive theorem Theorem 5.2.1 to obtain our main result. If the non-special irreducible points are Zariski open dense in universal pseudodeformation spaces for $\overline{D}$ of dimension less than n, then irreducible points are Zariski dense for $\overline{D}$ of dimension n. The main point of Subsection 5.3 is to show in Theorem 5.3.1 that for n the non-special irreducible points are dense open in the irreducible points. This uses induction of pseudorepresentations from Subsection 4.6 as a main tool.

Then it is straightforward in Subsection 5.4 to complete the proof of our main theorem, Theorem 5.4.1. In Theorem 5.4.5 we determine the singular locus of $\overline{R}^{\mathrm{univ}}_{\overline{D}}$ when $\zeta_p \notin K$. This allows us in Theorem 5.4.7 to establish Serre's condition (R_2) for $\overline{R}^{\mathrm{univ}}_{\overline{D}}$ except for one single $\overline{D}$.

In this section, we use the notation $K \supset \mathbb{Q}_p$, d, G_K, ζ_p, $\overline{D}\colon G_K \to \mathbb{F}$ (continuous) with $\mathbb{F}$ finite, as before. Often we write n for $\dim \overline{D}$. To emphasize K in universal objects, we sometimes write $\overline{R}^{\mathrm{univ}}_{K,\overline{D}}$ for $R^{\mathrm{univ}}_{G_K,\mathbb{F},\overline{D}}$ and $\overline{X}^{\mathrm{univ}}_{K,\overline{D}}$ for $\operatorname{Spec} \overline{R}^{\mathrm{univ}}_{K,\overline{D}}$. All results of this section only concern the special fiber of pseudodeformation spaces.

5.1 Special Points

Let $\chi_{\mathrm{cyc}}\colon G_K \to \mathbb{Z}_p^\times$ denote the p-adic cyclotomic character. Let A be in $\widehat{\mathcal{A}r}_{W(\mathbb{F})}$ (or a localization of such a ring), let $\rho\colon G_K \to \mathrm{GL}_n(A)$ be a continuous representation and $D\colon G_K \to A$ be a continuous pseudorepresentation. For $i \in \mathbb{Z}$, we shall denote by $\rho(i)$ and $D(i)$ the twist by χ^i_{cyc} of ρ and D, respectively. A crucial if elementary observation in [Che11] was that $H^2(G_K, \mathrm{ad}_\rho) = 0$ whenever a $\rho\colon G_K \to \mathrm{GL}_n(E)$ is a continuous representation into a p-adic field E that satisfies $E = \mathrm{End}_{E[G_K]}(\rho)$; the reason is Tate local duality in the form Theorem 3.4.1 from which one has $H^2(G_K, \mathrm{ad}_\rho)^* = \mathrm{Hom}_{G_K}(\rho, \rho(1))$ and that χ_{cyc} has infinite order. For representations into local (or finite) fields of positive characteristic this is no longer true.

Lemma 5.1.1. *Let E be a finite or local field of characteristic p and let $\rho\colon G_K \to \mathrm{GL}_n(E)$ be a continuous absolutely irreducible representation. Then the following hold:*

Suppose that $\zeta_p \notin K$ (Case I). *Then the following are equivalent:*

(a) $H^2(G_K, \mathrm{ad}_\rho)$ is non-zero.

(b) The G_K-representations ρ and $\rho(1)$ are isomorphic.

(c) The exists a finite separable extension E' of E such that $\rho \otimes_E E'$ is induced from a continuous representation τ of $G_{K'}$ over E' for $K' = K(\zeta_p)$.

Suppose that $\zeta_p \in K$ (Case II). *Then $H^2(G_K, \mathrm{ad}_\rho)$ surjects onto $H^2(G_K, E) \cong E$, and the following are equivalent:*

(a') $H^2(G_K, \mathrm{ad}^0_\rho)$ is non-zero.

(b') $H^0(G_K, \overline{\mathrm{ad}}_\rho)$ is non-zero.

(c') The exists a finite extension E' of E and a Galois extension K' of K of degree p such that $\rho \otimes_E E'$ is induced from a continuous representation τ over E' of $G_{K'}$.

(d') The restriction $\rho \otimes_E E^{\mathrm{alg}}|_{G_{K'}}$ is reducible for some Galois extension K' of K of degree p.

In both cases, if τ exists, then it is absolutely irreducible, and in particular $\mathrm{End}_{G_{K'}}(\tau) = E$.

Proof. The equivalence of (a) and (b) follows from Tate local duality as given in Theorem 3.4.1 and the absolute irreducibility of ρ. This duality also yields the equivalence of (a') and (b'). In all cases, the continuity and absolute irreducibility of τ, if it exists, is implied by Lemma 2.1.4(b) and (f).

The equivalence of (b) and (c) now follows from Theorem 2.2.1. The equivalence of (c') and (d') is a consequence of Lemma 2.3.1. The implication (b')⇒(c') follows from Lemma 2.3.2(f) and (i), and the implication (c')⇒(b') is shown in Example 4.6.4. □

Definition 5.1.2. *We call $x \in (\overline{X}^{\mathrm{univ}}_{K,\overline{D}})^{\mathrm{irr}}$* special *if one of the following two conditions holds*

(a) $\zeta_p \notin K$ and $D_x = D_x(1)$,

(b) $\zeta_p \in K$ and $D_x|_{G_{K'}}$ is reducible for some degree p Galois extension K' of K;

otherwise x is called non-special. *We write $(\overline{X}^{\mathrm{univ}}_{K,\overline{D}})^{\mathrm{spcl}}$ for $\{x \in (\overline{X}^{\mathrm{univ}}_{K,\overline{D}})^{\mathrm{irr}} \mid$ is special$\}$ and $(\overline{X}^{\mathrm{univ}}_{K,\overline{D}})^{\text{n-spcl}}$ for $(\overline{X}^{\mathrm{univ}}_{K,\overline{D}})^{\mathrm{irr}} \smallsetminus (\overline{X}^{\mathrm{univ}}_{K,\overline{D}})^{\mathrm{spcl}}$.*

Lemma 5.1.3. *The set $(\overline{X}^{\mathrm{univ}}_{K,\overline{D}})^{\mathrm{spcl}}$ is closed in $(\overline{X}^{\mathrm{univ}}_{K,\overline{D}})^{\mathrm{irr}}$.*

Proof. If $\zeta_p \notin K$, then the condition $D = D(1)$ is a closed condition in $\overline{X}^{\mathrm{univ}}_{K,\overline{D}}$ by Corollary 4.5.11, and this concludes the argument.

If $\zeta_p \in K$, then note first that the set of Galois extensions K' of K of degree p is finite Since by class field theory $(G^{\mathrm{ab}}_K)/(G^{\mathrm{ab}}_K)^{\times p}$ is finite if K is a p-adic field. By [Che14, 2.20] the reducibility of a pseudorepresentation over a field can be detected by the vanishing of certain determinants whose entries are traces of the pseudorepresentation, evaluated at certain elements of the group in question. If $n = \dim D$, then $x \in (\overline{X}^{\mathrm{univ}}_{K,\overline{D}})^{\mathrm{spcl}}$ if and only if for all Galois extensions K' over K of degree p and all n^2 tuples $(g_i) \in G^{n^2}_{K'}$ one has

$$\det \Big(\Lambda_{D,1}(g_i g_j)_{i,j=1,\ldots,n^2} \Big) = 0.$$

Hence $(\overline{X}^{\mathrm{univ}}_{K,\overline{D}})^{\mathrm{spcl}}$ is Zariski closed in $(\overline{X}^{\mathrm{univ}}_{K,\overline{D}})^{\mathrm{irr}}$. □

For $X \subset \overline{X}^{\mathrm{univ}}_{\overline{D}}$ locally closed, we set $\dot{X} := X \smallsetminus \{\mathfrak{m}_{\overline{R}^{\mathrm{univ}}_{\overline{D}}}\}$. The following holds:

Facts 5.1.4. *(a) $\mathfrak{m}_{\overline{R}^{\mathrm{univ}}_{\overline{D}}}$ is the unique closed point of $\overline{X}^{\mathrm{univ}}_{\overline{D}}$.*

(b) For $i \geq 1$, the dimension i points of $\overline{X}^{\mathrm{univ}}_{\overline{D}}$ are the dimension $i-1$ points of $\dot{\overline{X}}^{\mathrm{univ}}_{\overline{D}}$.

(c) The dimension 0 points of $\dot{X}$ are very dense in $\dot{X}$; see Lemma A.1.8.

We call $x \in \overline{X}_{\overline{D}}^{\mathrm{univ}}$ regular, if $\overline{R}_{\overline{D}}^{\mathrm{univ}}$ is regular at x, and singular otherwise.

Notation 5.1.5. *Let X be a locally closed subset of $\overline{X}_{\overline{D}}^{\mathrm{univ}}$.*

(a) We use the superscripts irr, red, reg, sing *on X to denote the subset of irreducible, reducible, regular and singular points, respectively; cf. Definition 4.7.16.*

(b) We write X_{red} (subscript!) for X with its induced reduced subscheme structure.

(c) For attributes a, b, c of X, if they apply, we write $X^{a,b}$ for $X^a \cap X^b$, X_b^a for $X^a \cap X_b$, $X_{a,b}$ for $X_a \cap X_b$ etc.

The remaining results in this section concern dimension 1 points on $\overline{X}_{\overline{D}}^{\mathrm{univ}}$.

Lemma 5.1.6. *Let x be a closed point of $U := (\dot{\overline{X}}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{irr}}$, let D_x' be the pseudorepresentation $G \to \kappa(x), g \mapsto 1 \otimes_{W(\mathbb{F})} D_x(g)$ and let $\widehat{R}^{\mathfrak{p}}$ be the universal pseudodeformation ring for D_x' from Corollary 4.8.7. Then the following hold:*

(i) Suppose that $\zeta_p \notin K$ and that x is non-special. Then $\widehat{R}^{\mathfrak{p}}$ is regular of dimension dn^2+1. If in addition $U^{\text{n-spcl}}$ is non-empty, it is regular and equidimensional of dimension dn^2.

(ii) Suppose that $\zeta_p \notin K$ and that x is special. Then $\widehat{R}^{\mathfrak{p}}$ is a complete intersection ring with $\dim \widehat{R}^{\mathfrak{p}} \in \{dn^2+1, dn^2+2\}$. Moreover, U is of dimension at most dn^2+1.

(iii) Suppose that $\zeta_p \in K$ and that x is non-special. Then $\widehat{R}^{\mathfrak{p}}_{\mathrm{red}}$ is regular of dimension dn^2+1. If in addition $U^{\text{n-spcl}}_{\mathrm{red}}$ is non-empty, it is regular and equidimensional of dimension dn^2.

Proof. Consider the Galois representation $\rho_x \colon G_K \to \mathrm{GL}_n(L)$ with $D_{\rho_x} = D_x'$ from Theorem 4.3.10 that is defined over a finite extension L of $\kappa(x)$. For (i) note that we have $H^2(G_K, \mathrm{ad}_{\rho_x}) = 0$ by Lemma 5.1.1, Case I, and the definition of special. The Euler characteristic formula of Theorem 3.4.1 now yields

$$\dim \widehat{R}^{\mathfrak{p}} = \dim_L H^1(G_K, \mathrm{ad}_{\rho_x}) = dn^2 + \dim_L H^0(G_K, \mathrm{ad}_{\rho_x}) = dn^2+1.$$

It follows from Lemma 3.3.3 that x is a regular point of $\overline{X}_{\overline{D}}^{\mathrm{univ}}$ of dimension $dn^2+1-1 = dn^2$. Since x lies on U, it is also a regular point of U. To see that U is regular, let $Y \subset U$ be the closed subscheme of singular points. We know that points of dimension at most 1 will be dense in the constructible set Y. Since the unique closed point of $\overline{X}_{\overline{D}}^{\mathrm{univ}}$ is not in U, points of dimension 1 are dense in $Y \subset U$. However as we just saw, such points are regular and cannot lie in Y. Therefore Y must be empty. And again by the density of dimension 1 points in U, it follows that U is regular and equidimensional of dimension dn^2, proving (i).

For (ii), we observe $H^2(G_K, \mathrm{ad}_{\rho_x})^\vee = H^0(G_K, \mathrm{ad}_{\rho_x}(1)) = \mathrm{Hom}_{G_K}(\rho_x, \rho_x(1)) \cong L$ using Theorem 3.4.1, and in the last step that $\rho_x \cong \rho_x(1)$ and that ρ_x is absolutely irreducible. This time, the Euler characteristic formula provides a presentation

$$0 \to I \to \kappa(x)[\![X_1, \ldots, X_{dn^2+2}]\!] \to \widehat{R}_{\mathfrak{p}} \to 0,$$

where the ideal I is generated by at most one element over $\kappa(x)[\![X_1, \ldots, X_{dn^2+2}]\!]$. This proves the claims on $\widehat{R}_{\mathfrak{p}}$. The remaining assertion follows from the density of dimension 1 points in U and Lemma 3.3.3.

For (iii), it follows from the non-specialness of D_x and from Corollary 3.4.3 that $(\overline{R}_{\rho_x}^{\mathrm{univ}})_{\mathrm{red}}$ is regular local of dimension dn^2+1. From Proposition 4.7.6 and Corollary 4.8.6 we deduce $\overline{R}_{D_x}^{\mathrm{univ}} \otimes_{\kappa(x)} L \cong \overline{R}_{\rho_x}^{\mathrm{univ}}$, and the assertion on $\widehat{R}^{\mathfrak{p}}_{\mathrm{red}}$ follows. The remaining assertion follows from the density of dimension 1 points in U and Lemma 3.3.3. □

We also need a similar result in certain reducible cases. It is adapted from [Che11, Lem. 2.2].

Lemma 5.1.7. *For $i = 1, 2$, let $\overline{D}_i\colon G_K \to \mathbb{F}$ be pseudorepresentations over a finite field $\mathbb{F}$, let $x_i \in \overline{X}^{\mathrm{univ}}_{\overline{D}_i}$ be irreducible non-special dimension 1 points, and let L be a finite extension of both $\kappa(x_i)$ over which there exist absolutely irreducible representations $\rho_i\colon G_K \to \mathrm{GL}_{n_i}(L)$ with $D_{\rho_i} = D_{x_i} \otimes_{\kappa(x)} L$. Let $\rho\colon G_K \to \mathrm{GL}_n(L)$ be a nontrivial extension of ρ_2 by ρ_1. Assume that $D_{x_1} \neq D_{x_2}(m)$ for any $m \in \{1, \dots, p-1\}$. Then the following hold:*

(i) The representation ρ exists; it satisfies $L = \mathrm{End}_{G_K}(\rho)$; one has $D_\rho = D_{\rho_1} \oplus D_{\rho_2}$ as pseudorepresentations into L; the functor $\mathcal{D}_\rho\colon \mathcal{A}r_L \to Sets$ is pro-representable.

In the following, we write R_ρ for the representing universal ring of $\mathcal{D}_\rho$, $\rho^{\mathrm{univ}}_\rho\colon G_K \to \mathrm{GL}_n(R^{\mathrm{univ}}_\rho)$ for a universal deformation and X^{univ}_ρ for $\mathrm{Spec}\, R_\rho$. Denote by $\widehat{R}^{\mathfrak{p}}$ the universal pseudodeformation ring for D_ρ to $\mathcal{A}r_L$, by $\varphi\colon X^{\mathrm{univ}}_\rho \to \widehat{X} := \mathrm{Spec}\,\widehat{R}^{\mathfrak{p}}$ the map of L-schemes induced by sending $\rho^{\mathrm{univ}}_\rho$ to its associated pseudorepresentation $D_{\rho^{\mathrm{univ}}_\rho}$, and by $\mathrm{d}\,\varphi\colon \mathbf{t}_{X^{\mathrm{univ}}_\rho,\rho} \to \mathbf{t}_{\widehat{X},x}$ the induced L-linear map on tangent spaces.

(ii) Suppose that $\rho' \in \ker \mathrm{d}\varphi \subset \mathbf{t}_{X^{\mathrm{univ}}_\rho,\rho} \cong \mathcal{D}_\rho(L[\varepsilon])$, i.e., that $D_{\rho'} = D_\rho$. Then with respect to a suitable basis ρ' has constant diagonal blocks and is upper triangular.

(iii) If $\zeta_p \notin K$, then R^{univ}_ρ is formally smooth over L of dimension $\dim_L \mathbf{t}_{X^{\mathrm{univ}}_\rho,\rho} = dn^2 + 1$,

$$\dim_L \ker \mathrm{d}\varphi = dn_1n_2 - 1 \quad \text{and} \quad \dim_L \mathrm{im}\, \mathrm{d}\varphi = dn^2 - dn_1n_2 + 2.$$

(iv) If $\zeta_p \in K$, then $R^{\mathrm{univ}}_{\rho,\mathrm{red}}$ is formally smooth over L of relative dimension $h - 1$ for $h := \dim_L \mathbf{t}_{X^{\mathrm{univ}}_\rho,\rho} = dn^2 + 2$. Denoting by $\varphi_{\mathrm{red}}\colon (X^{\mathrm{univ}}_\rho)_{\mathrm{red}} \to (\widehat{X})_{\mathrm{red}}$ the morphism on reduced L-schemes associated to φ and by $\mathrm{d}\,\varphi_{\mathrm{red}}\colon \mathbf{t}_{(X^{\mathrm{univ}}_\rho)_{\mathrm{red}},\rho} \to \mathbf{t}_{(\widehat{X})_{\mathrm{red}},x}$ the induced map on tangent spaces, there furthermore exists $\delta \in \{0, 1\}$ such that

$$\dim_L \ker \mathrm{d}\varphi_{\mathrm{red}} = dn_1n_2 - 1 - \delta \quad \text{and} \quad \dim_L \mathrm{im}\, \mathrm{d}\varphi_{\mathrm{red}} = dn^2 - dn_1n_2 + 2 + \delta.$$

Proof. We begin with (i). By Theorem 3.4.1, the assumptions imply that

$$\dim_L H^2(G_K, \rho_{x_1} \otimes \rho_{x_2}^\vee) = \dim_L H^0(G_K, \rho_{x_1} \otimes \rho_{x_2}(1)^\vee) = \dim_L \mathrm{Hom}_{G_K}(\rho_{x_1}, \rho_{x_2}(1)) = 0.$$

The Euler characteristic formula in Theorem 3.4.1 now gives

$$\begin{aligned}\dim_L \mathrm{Ext}^1_{G_K}(\rho_{x_2}, \rho_{x_1}) &= \dim_L H^1(G_K, \rho_{x_1} \otimes \rho_{x_2}^\vee) \\ &= dn_1n_2 + \dim_L H^0(G_K, \rho_{x_1} \otimes \rho_{x_2}^\vee) + \dim_L H^2(G_K, \rho_{x_1} \otimes \rho_{x_2}^\vee) \\ &= dn_1n_2.\end{aligned}$$

Thus there exists an nonzero element $c \in \mathrm{Ext}^1_{G_K}(\rho_{x_2}, \rho_{x_1})$. Setting $\rho = \begin{pmatrix} \rho_{x_1} & c \\ 0 & \rho_{x_2} \end{pmatrix}$ and applying Lemma 3.4.4 and Theorem 3.2.4 completes the proof of (i).

For the proof of (ii), we use the canonical identifications (see [Maz97, Prop., p. 271])

$$\mathcal{D}_\rho(L[\varepsilon]) \cong \mathbf{t}_{X^{\mathrm{univ}}_\rho,\rho} \quad \text{and} \quad \mathcal{P}sD_{D_\rho}(L[\varepsilon]) \cong \mathbf{t}_{\widehat{X},D_\rho} \tag{24}$$

to identify $\ker \mathrm{d}\varphi$ with the L-subspace of $\mathcal{D}_\rho(L[\varepsilon])$, which consists of the deformations of ρ to $L[\varepsilon]$ that map under $\mathrm{d}\,\varphi$ to the trivial pseudodeformation to $L[\varepsilon]$ of the residual pseudorepresentation D_ρ associated with ρ. Let ρ' be a deformation of ρ whose associated pseudorepresentation satisfies $D_{\rho'} = D_\rho$. The linearization of ρ' gives a continuous homomorphism

$$L[\varepsilon][G_K] \longrightarrow \begin{pmatrix} \mathrm{Mat}_{n_1}(L[\varepsilon]) & \mathrm{Mat}_{n_1n_2}(\mathcal{A}_{12}) \\ \mathrm{Mat}_{n_2n_1}(\mathcal{A}_{21}) & \mathrm{Mat}_{n_2}(L[\varepsilon]) \end{pmatrix},$$

which when composed with the determinant gives $D_{\rho'}$, i.e., ρ' factors via a GMA. By hypothesis we must have $\mathcal{A}_{12} = L[\varepsilon]$ and $\mathcal{A}_{21} \subset \varepsilon L$. Also by hypothesis, the residual pseudorepresentation D_ρ is multiplicity free and split, so that by Proposition 4.3.9(ii) the ideal of total reducibility $\mathcal{A}_{12}\mathcal{A}_{21}$ vanishes, and hence $\mathcal{A}_{21} = 0$, and ρ' is upper triangular. Let D_1 and D_2 be the pseudorepresentations described by the upper left and lower right diagonal blocks of ρ'. then again by Proposition 4.3.9(ii) (and by the non-splitness of ρ) we have $D_i = D_{\rho_i}$, $i = 1, 2$, and hence by Theorem 4.3.10(i), ρ' is constant on its diagonal blocks and upper triangular.

For (iii) and (iv) we first compute $\mathbf{t}_{X_\rho^{\mathrm{univ}},\rho} = \dim_L H^1(G_K, \mathrm{ad}_\rho)$. Since $H^0(G_K, \mathrm{ad}_\rho) \cong L$, formula Theorem 3.4.1(c) yields

$$\dim_L H^1(G_K, \mathrm{ad}_\rho) = dn^2 + 1 + \dim_L H^2(G_K, \mathrm{ad}_\rho),$$

and by Theorem 3.4.1(b) we have $\dim_L H^2(G_K, \mathrm{ad}_\rho) = \dim_L \mathrm{Hom}_{G_K}(\rho, \rho(1))$. The claimed expressions for $\dim_L \mathbf{t}_{X_\rho^{\mathrm{univ}},\rho}$ now follow from Lemma 3.4.4 with $\chi = \mathbb{F}(1)$ and our hypotheses. The claim on R_ρ^{univ} in (iii) now follows from Theorem 3.2.4. The claim on R_ρ^{univ} in (iv) follows from Corollary 3.4.3 provided that we show that $H^0(G_K, \overline{\mathrm{ad}}_\rho) = 0$. But under our hypotheses this follows from Corollary 2.3.3.

For the assertions on $\mathrm{d}\,\varphi$ and $\mathrm{d}\,\varphi_{\mathrm{red}}$ in (iii) and (iv), we first give a formula for $\dim_L \ker \mathrm{d}\varphi$ in either case. We consider lifts ρ_1, ρ_2 of ρ to $L[\varepsilon]$ whose associated deformation classes satisfy $[\rho_1] = [\rho_2] \in \ker \mathrm{d}\varphi \subset \mathbf{t}_{X_\rho^{\mathrm{univ}}} \cong \mathcal{D}_\rho(L[\varepsilon])$. By assertion (ii) we have $\rho_i = \rho + \varepsilon \begin{pmatrix} 0 & c_i \\ 0 & 0 \end{pmatrix}$ for some cocycle $c_i \in Z^1(G_K, \rho_{x_1} \otimes \rho_{x_2}^\vee)$. In order to obtain $\dim_L \ker \mathrm{d}\varphi$, we determine when ρ_1 is equivalent to ρ_2. In this case there exists a matrix $U \in \mathrm{Mat}_n(L)$ such that

$$\begin{aligned} \rho + \varepsilon \begin{pmatrix} 0 & c_2 \\ 0 & 0 \end{pmatrix} &= \rho_2 \\ &= (1 + \varepsilon U)\rho_1(1 - \varepsilon U) \\ &= (1 + \varepsilon U)(\rho + \varepsilon \begin{pmatrix} 0 & c_1 \\ 0 & 0 \end{pmatrix})(1 - \varepsilon U) \\ &= \rho + \varepsilon(U\rho - \rho U + \begin{pmatrix} 0 & c_1 \\ 0 & 0 \end{pmatrix}). \end{aligned}$$

If we write $U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$ with matrices $U_{ij} \in \mathrm{Mat}_{n_i \times n_j}(L)$ for $1 \le i, j \le 2$, then the above equality is equivalent to

$$\begin{pmatrix} 0 & c_2 - c_1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} U_{11}\rho_{x_1} & U_{11}c + U_{12}\rho_{x_2} \\ U_{21}\rho_{x_1} & U_{21}c + U_{22}\rho_{x_2} \end{pmatrix} - \begin{pmatrix} \rho_{x_1}U_{11} + cU_{21} & \rho_{x_1}U_{12} + cU_{22} \\ \rho_{x_2}U_{21} & \rho_{x_2}U_{22} \end{pmatrix}$$

Because $\dim_L H^0(G_K, \rho_{x_i} \otimes \rho_{x_j}^\vee) = 0$ and $\dim_L H^0(G_K, \rho_{x_i} \otimes \rho_{x_i}^\vee) = 1$ for $1 \le i, j \le 2$ and $i \ne j$, we deduce that $U_{21} = 0$ and that U_{11} and U_{22} are scalar matrices. Finally, the map

$$-\rho_{x_1}U_{12} + U_{12}\rho_{x_2} \in B^1(G_K, \rho_{x_1} \otimes \rho_{x_2}^\vee)$$

is a coboundary. Therefore, $c_2 = (U_{11} + U_{22})c + c_1 \in H^1(G_K, \rho_{x_1} \otimes \rho_{x_2}^\vee)$ and

$$\dim_L \ker \mathrm{d}\varphi = \dim_L \mathrm{Ext}^1_{G_K}(\rho_{x_2}, \rho_{x_1}) - 1 = \dim_L H^1(G_K, \rho_{x_1} \otimes \rho_{x_2}^\vee) - 1 = dn_1n_2 - 1, \tag{25}$$

by the computation for (i). Using $\dim V = \dim \ker \psi + \dim \mathrm{im}\, \psi$ for a vector space V and a linear map ψ with domain V, and the already computed dimension of $\mathbf{t}_{X_\rho^{\mathrm{univ}},\rho}$, the proof of (iii) is complete.

For (iv) consider the following diagram with left exact rows and where the middle and right vertical arrows are injective (by definition of $\mathbf{t}$):

$$\begin{array}{ccccccc} 0 & \longrightarrow & \ker\varphi & \longrightarrow & \mathbf{t}_{X_\rho^{\mathrm{univ}},\rho} & \xrightarrow{\mathrm{d}\,\varphi} & \mathbf{t}_{\widehat{X},D_\rho} \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & \ker\varphi_{\mathrm{red}} & \longrightarrow & \mathbf{t}_{(X_\rho^{\mathrm{univ}})_{\mathrm{red}},\rho} & \xrightarrow{\mathrm{d}\,\varphi_{\mathrm{red}}} & \mathbf{t}_{(\widehat{X})_{\mathrm{red}},D_\rho}. \end{array}$$

By a simple diagram chase one deduces $\ker\varphi_{\mathrm{red}} = \ker\varphi \cap \mathbf{t}_{(X_\rho^{\mathrm{univ}})_{\mathrm{red}},\rho} \subset \mathbf{t}_{X_\rho^{\mathrm{univ}},\rho}$. Next consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker \mathrm{d}\varphi & \longrightarrow & \mathbf{t}_{X_\rho^{\mathrm{univ}},\rho} & \longrightarrow & \mathrm{im}\,\mathrm{d}\,\varphi & \longrightarrow & 0 \\ & & \alpha\uparrow & & \beta\uparrow & & \gamma\uparrow & & \\ 0 & \longrightarrow & \ker \mathrm{d}\varphi_{\mathrm{red}} & \longrightarrow & \mathbf{t}_{(X_\rho^{\mathrm{univ}})_{\mathrm{red}},\rho} & \longrightarrow & \mathrm{im}\,\mathrm{d}\,\varphi_{\mathrm{red}} & \longrightarrow & 0 \end{array}$$

with exact rows and where the left and middle vertical arrows are injective. Because of $\ker\varphi_{\mathrm{red}} = \ker\varphi \cap \mathbf{t}_{(X_\rho^{\mathrm{univ}})_{\mathrm{red}},\rho}$ the map γ is injective, and we deduce from the 9-Lemma that $\dim\mathrm{coker}\,\alpha + \dim\mathrm{coker}\,\gamma = \dim\mathrm{coker}\,\beta$. From the tangent space computations for (iv) made so far, we deduce $\dim\mathrm{coker}\,\beta = 1$. Letting $\delta := \dim\mathrm{coker}\,\alpha$, we must have $0 \le \delta \le 1$ and $\dim\mathrm{coker}\,\beta = 1-\delta$. Arguing as for (iii) and using $\dim_L \mathbf{t}_{X_{\rho,\mathrm{red}}^{\mathrm{univ}},\rho} = dn^2+1$, the proof of (iv) is complete, as well. □

5.2 Zariski density of the irreducible locus

The aim of this subsection is to formulate an inductive procedure to prove Zariski density of the irreducible locus the special fibers of universal pseudodeformation spaces, and to establish some key steps. Our procedure is an adaption of an analogous result of Chenvier for the generic fiber; see [Che11, Thme. 2.1]. The following is the main result of this subsection. Its prove will be given at the end of it.

Theorem 5.2.1. *Let $n \ge 2$ be an integer. Suppose that for all pseudorepresentations $\overline{D}'\colon G_K \to \mathbb{F}$ of dimension $n' < n$ the following hold:*

(i) $\overline{X}_{\overline{D}'}^{\mathrm{univ}}$ is equidimensional of dimension $[K:\mathbb{Q}_p](n')^2+1$,

(ii) $(\overline{X}_{\overline{D}'}^{\mathrm{univ}})^{\text{n-spcl}}$ is Zariski dense in $\overline{X}_{\overline{D}'}^{\mathrm{univ}}$.

Then for all n-dimensional pseudorepresentations $\overline{D}\colon G_K \to \mathbb{F}$ the subspace $(\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{irr}} \subset \overline{X}_{\overline{D}}^{\mathrm{univ}}$ is Zariski dense.

Let us begin with some preparations. Let $n_1, n_2 \ge 1$ be integers such that $n = n_1+n_2$. Let $\overline{D}_i\colon G_K \to \mathbb{F}$ be residual pseudorepresentations of dimension n_i. Addition $(D_1,D_2)\mapsto D_1\oplus D_2$ of pseudorepresentations yields a morphism

$$\overline{X}_{\overline{D}_1}^{\mathrm{univ}} \hat{\times}_{\mathbb{F}} \overline{X}_{\overline{D}_2}^{\mathrm{univ}} \longrightarrow \overline{X}_{\overline{D}}^{\mathrm{univ}} \tag{26}$$

for $\overline{D} := \overline{D}_1 \oplus \overline{D}_2$. If $\overline{D}_1 \ne \overline{D}_2$, we define $\overline{X}_{\overline{D}_1,\overline{D}_2}^{\mathrm{univ}} := \overline{X}_{\overline{D}_1}^{\mathrm{univ}} \hat{\times}_{\mathbb{F}} \overline{X}_{\overline{D}_2}^{\mathrm{univ}}$ and write $\iota_{\overline{D}_1,\overline{D}_2}$ for the above morphism. In the other case we let $\mathbb{Z}/2$ act on $\overline{X}_{\overline{D}_1}^{\mathrm{univ}} \hat{\times}_{\mathbb{F}} \overline{X}_{\overline{D}_1}^{\mathrm{univ}}$ by exchanging the factors; it preserves the diagonal, which we denote by $\Delta_{\overline{D}_1}^{\mathrm{univ}}$, and one has an induced morphism

$$\iota_{\overline{D}_1,\overline{D}_1}\colon \overline{X}_{\overline{D}_1,\overline{D}_1}^{\mathrm{univ}} := (\overline{X}_{\overline{D}_1}^{\mathrm{univ}} \hat{\times}_{\mathbb{F}} \overline{X}_{\overline{D}_1}^{\mathrm{univ}})/(\mathbb{Z}/2) \longrightarrow \overline{X}_{\overline{D}}^{\mathrm{univ}} \tag{27}$$

Note that away from the $\Delta_{\overline{D}_1}^{\mathrm{univ}}$, the morphism $\overline{X}_{\overline{D}_1}^{\mathrm{univ}} \hat{\times}_{\mathbb{F}} \overline{X}_{\overline{D}_1}^{\mathrm{univ}} \to \overline{X}_{\overline{D}_1,\overline{D}_1}^{\mathrm{univ}}$ is an étale Galois cover with monodromy group $\mathbb{Z}/2$.

Lemma 5.2.2 ([Che11, Lem. 1.1.]). *Let $x_i \in \overline{X}^{\mathrm{univ}}_{\overline{D}_i}$ be irreducible one-dimensional points, and let L be a finite common extension of the residue fields $\kappa(x_i)$. If $\overline{D}_1 = \overline{D}_2$, assume also that $x_1 \neq x_2$. Let $x \in \overline{X}^{\mathrm{univ}}_{\overline{D}}$ be the 1-dimensional point with $D_x \otimes_{\kappa(x)} L = D_{x_1} \otimes_{\kappa(x_1)} L \oplus D_{x_2} \otimes_{\kappa(x_1)} L$. Let $\overline{x}\colon \operatorname{Spec} \kappa(x) \to \overline{X}^{\mathrm{univ}}_{\overline{D}}$ be a geometric point over x.*

Then there is an étale neighborhood $(U, \overline{u}, \varphi_U \colon \overline{u} \to U)$ of $\overline{x}$, such that the base change

$$U' := U \times_{\varphi_U, \overline{X}^{\mathrm{univ}}_{\overline{D}}, \iota_{\overline{D}_1, \overline{D}_2}} \overline{X}^{\mathrm{univ}}_{\overline{D}_1, \overline{D}_2} \xrightarrow{\iota_U} U$$

of $\iota_{\overline{D}_1, \overline{D}_2}$ along φ_U is a closed immersion with image $U^{\mathrm{red}} = \{u \in U \mid u \text{ is irreducible}\}$. Moreover if $\overline{D}_1 = \overline{D}_2$, then we may choose U such that $\varphi_U(U)$ is disjoint from $\iota_{\overline{D}_1, \overline{D}_1}(\Delta^{\mathrm{univ}}_{\overline{D}_1})$.

Proof. As recalled above Definition 4.7.8, the universal pseudodeformation $D^{\mathrm{univ}}_{\overline{D}}$ factors via the universal Cayley-Hamilton pseudodeformation and CH-representation

$$\overline{R}^{\mathrm{univ}}_{\overline{D}}[\![G_K]\!] \xrightarrow{\rho^{\mathrm{CH}}_{\overline{D}}} \overline{S}^{\mathrm{CH\text{-}univ}}_{\overline{D}} \xrightarrow{D^{\mathrm{CH\text{-}univ}}_{\overline{D}}} \overline{R}^{\mathrm{univ}}_{\overline{D}}.$$

Consider the strictly local ring $\mathcal{O}^{\mathrm{sh}}_{\overline{x}} := \operatorname{colim}_{(V,\overline{v})} \mathcal{O}(V)$ for $\mathcal{O}(V) := \mathcal{O}_{\overline{X}^{\mathrm{univ}}_{\overline{D}}}(V)$, where $(V, \overline{v})$ runs over all connected étale neighborhoods of $\overline{x}$ in $\overline{X}^{\mathrm{univ}}_{\overline{D}}$ [Sta18, Lem. 04HX]. Since by Proposition 4.1.22 the formation of the Cayley-Hamilton quotient $\overline{S}^{\mathrm{CH\text{-}univ}}_{\overline{D}}$ commutes with arbitrary base change, for any étale neighborhood $(V, \overline{v})$ of $\overline{x}$ there is an isomorphism

$$\mathcal{O}(V)[[G_K]]/\operatorname{CH}(D^{\mathrm{univ}}_{\overline{D}} \otimes_{\overline{R}^{\mathrm{univ}}_{\overline{D}}} \mathcal{O}(V)) \xrightarrow{\sim} \overline{S}^{\mathrm{CH\text{-}univ}}_{\overline{D}} \otimes_{\overline{R}^{\mathrm{univ}}_{\overline{D}}} \mathcal{O}(V) =: \overline{S}_V.$$

From Theorem 4.3.10 it follows that $\overline{S}_{\overline{x}} := \operatorname{colim}_{(V,\overline{v})} \overline{S}_V$ is a GMA of type (n_1, n_2). In particular, there exists idempotents $e_1, e_2 \in \overline{S}_{\overline{x}}$ with $e_1 + e_2 = 1$ and for $i = 1, 2$ an isomorphism $\psi_{\overline{x},i} \colon e_i \overline{S}_{\overline{x}} e_i \to \operatorname{Mat}_{n_i}(\mathcal{O}^{\mathrm{sh}}_{\overline{x}})$. Denote by $\mathcal{E}_{\overline{x}} := (e_i, \psi_{\overline{x},i})_{i=1,2}$, then also the induced pseudorepresentation to $\mathcal{O}^{\mathrm{sh}}_{\overline{x}}$ factors via the natural Cayley-Hamilton pseudorepresentation $D_{\overline{S}_{\overline{x}}, \mathcal{E}_{\overline{x}}}$ from Proposition 4.3.5.

By Proposition 4.7.11, the ring $\overline{S}^{\mathrm{CH\text{-}univ}}_{\overline{D}}$ is module-finite as an $\overline{R}^{\mathrm{univ}}_{\overline{D}}$-algebra and Noetherian. Note also that we constructed $\overline{S}_{\overline{s}}$ and $\mathcal{O}^{\mathrm{sh}}_{\overline{x}}$ as direct limits over étale neighborhoods. Using spreading out principles from [Gro66, § 8.5], we can thus find a connected affine étale neighborhood $(\overline{u}, U, \varphi \colon U \to \overline{X}^{\mathrm{univ}}_{\overline{D}})$ of $\overline{x}$, such that the e_i can be defined over $\overline{S}_U$ and are idempotents therein with $e_1 + e_2 = 1$, and such that one has isomorphism

$$\psi_{U,i} \colon e_i \overline{S}_U e_i \to \operatorname{Mat}_{n_i}(\mathcal{O}(U)),$$

whose base change under $\mathcal{O}(U) \to \mathcal{O}^{\mathrm{sh}}_{\overline{x}}$ is $\psi_{\overline{x},i}$. Hence $\overline{S}_U$ together with $\mathcal{E}_U := (e_i, \psi_{U,i})_{i=1,2}$ is a GMA. By choosing U sufficiently large, we may also assume that the pseudorepresentation $D_U \colon G \to \mathcal{O}(U)$ induced from $D^{\mathrm{univ}}_{\overline{D}}$ factors via the induced CH-representation $G \to (\overline{S}_U)^\times$ composed with the natural Cayley-Hamilton pseudorepresentation $D_{\overline{S}_U, \mathcal{E}_U}$.

Let us write

$$\overline{S}_U \cong \begin{pmatrix} \operatorname{Mat}_{n_1}(\mathcal{O}(U)) & \operatorname{Mat}_{n_1,n_2}(\mathcal{A}_{12}) \\ \operatorname{Mat}_{n_2,n_1}(\mathcal{A}_{21}) & \operatorname{Mat}_{n_2}(\mathcal{O}(U)) \end{pmatrix}.$$

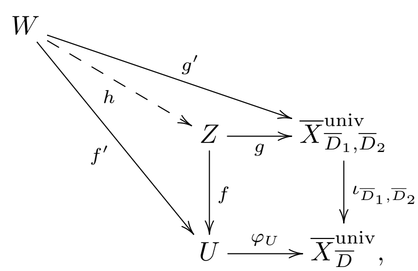
with finitely generated $\mathcal{O}(U)$-modules $\mathcal{A}_{12}$ and $\mathcal{A}_{21}$ together with the structure of a GMA described in Lemma 4.3.3. Let $I = \mathcal{A}_{12}\mathcal{A}_{21} + \mathcal{A}_{21}\mathcal{A}_{12} = \mathcal{A}_{12}\mathcal{A}_{21}$ be the ideal of total reducibility. From Proposition 4.3.9(i) we deduce that there exist unique pseudorepresentations $D_i \colon e_i \overline{S}_U e_i \to \mathcal{O}(U)/I$ for $i = 1, 2$ such that

$$(D_U \mod I) = D_1 \oplus D_2.$$

Denote by $Z := \operatorname{Spec}(\mathcal{O}(U)/I)$ the locus of total reducibility, by $f\colon Z \to U$ the induced closed immersion and by $g\colon Z \to \overline{X}^{\mathrm{univ}}_{\overline{D}_1,\overline{D}_2}$ the morphism corresponding to the $\mathcal{O}(Z)$-valued pseudorepresentations (D_1, D_2). Then the morphism $\varphi_U \circ f$ corresponds to the $\mathcal{O}(U)/I$-valued pseudorepresentation $D_U \mod I$ and there is a commutative diagram

$$\begin{array}{ccc} Z & \xrightarrow{\;g\;} & \overline{X}^{\mathrm{univ}}_{\overline{D}_1,\overline{D}_2} \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle \iota_{\overline{D}_1,\overline{D}_2}} \\ U & \xrightarrow{\;\varphi_U\;} & \overline{X}^{\mathrm{univ}}_{\overline{D}} \end{array} \tag{28}$$

since $\varphi_U \circ f$ and $\iota_{\overline{D}_1,\overline{D}_2} \circ g$ both correspond to $D_U \mod I = D_1 \oplus D_2$. We need to show that this diagram is cartesian; then $\iota_U = f$ is a closed immersion, by construction. I.e., by [Sta18, Def. 01JP] given any connected affine scheme W together with morphisms $f'\colon W \to U$ and $g'\colon W \to \overline{X}^{\mathrm{univ}}_{\overline{D}_1,\overline{D}_2}$ such that in the following diagram the solid square commutes



we need to check that there exists a unique dashed arrow h making the diagram commute.

The morphism $\varphi_U \circ f' = \iota_{\overline{D}_1,\overline{D}_2} \circ g'$ defines an $\mathcal{O}(W)$-valued pseudorepresentation D'_W, and the morphism g' a pair (D'_1, D'_2) of $\mathcal{O}(W)$-valued pseudorepresentations of dimension n_j for $j = 1, 2$. By Lemma 4.3.4 the base change $\overline{S}_W$ of $\overline{S}_U$ along f' is a generalized matrix algebra over $\mathcal{O}(W)$ of type (n_1, n_2). The definition of $\iota_{\overline{D}_1,\overline{D}_2}$ implies that $D'_W = D'_1 \oplus D'_2$, and from Proposition 4.3.9(ii) we conclude that the ideal

$$I' := I \otimes_{\mathcal{O}(U),(f')^*} \mathcal{O}(W) = \mathcal{A}_{12}\mathcal{A}_{21} \otimes_{\mathcal{O}(U),(f')^*} \mathcal{O}(W)$$

of total reducibility of $\overline{S}_W$ vanishes. Hence there exists a unique morphism $h\colon W \to Z$ such that $(f')^*$ factors as $\mathcal{O}(U) \xrightarrow{f^*} \mathcal{O}(Z) \xrightarrow{h^*} \mathcal{O}(W)$. Note the $g^* \circ h^*$ determines a pair (D''_1, D''_2) of pseudorepresentations $G \to \mathcal{O}(W)$. From Proposition 4.3.9(ii) we deduce $\{D'_1, D'_2\} = \{D''_1, D''_2\}$. The universal property of $\overline{X}^{\mathrm{univ}}_{\overline{D}_1} \hat{\times} \overline{X}^{\mathrm{univ}}_{\overline{D}_2}$, and our definition of $(\iota_{\overline{D}_1,\overline{D}_2}, \overline{X}^{\mathrm{univ}}_{\overline{D}_1,\overline{D}_2})$ implies that $(h')^* = g^* \circ h^*$.

Next we prove $Z = U^{\mathrm{red}}$ under the closed immersion f. By the definition of $\iota_{\overline{D}_1,\overline{D}_2}$ the inclusion $\subseteq$ is obvious. Let therefore y be any point of U^{red}. To see that y lies on $f(Z)$, let D_y be the reducible pseudorepresentation corresponding $\overline{R}^{\mathrm{univ}}_{\overline{D}} \to \mathcal{O}(U) \to \kappa(y)$. By Lemma 4.3.4 the base change $S_y := \overline{S}_U \otimes_{\mathcal{O}(U)} \kappa(y)^{\mathrm{alg}}$ of $\overline{S}_U$ is also a generalized matrix algebra of type (n_1, n_2). Since D_y is reducible, there exists pseudorepresentations $D_1, D_2\colon G_K \to \kappa(y)^{\mathrm{alg}}$ such that $D_y = D_1 \oplus D_2$. By again applying Proposition 4.3.9 we find that the ideal of total reducibility of the generalized matrix algebra S_y vanishes. Hence $\mathcal{O}(U) \to \kappa(y)$ factors via $\mathcal{O}(W)$ as was to be shown.

For the final assertion, suppose from now on that $\overline{D}_1 = \overline{D}_2$, so that $m := n_1 = n_2$. Consider the maps

$$\Lambda^j_i\colon G \longrightarrow \overline{S}_U \xrightarrow{\psi_{U,i}} \operatorname{End}_{\mathcal{O}(U)}(\mathcal{O}(U)^m) \xrightarrow{\Lambda^j} \operatorname{End}_{\mathcal{O}(U)}(\mathcal{O}(U)^{\binom{m}{j}}) \xrightarrow{\mathrm{tr}} \mathcal{O}(U)$$

for $i = 1, 2$ and $j = 1, \ldots, m$, where $\bigwedge^j$ denotes the exterior power map on endomorphisms. For every $g \in G$, the vanishing locus of $\Lambda_1^j(g) - \Lambda_2^j(g) \in \mathcal{O}(U)$ is a closed subscheme Y_g of U, and hence the intersection $Y := \bigcap_{g \in G} Y_g$ is closed in U. Since $x_1 \neq x_2$, the closed subset Y is properly contained in U. By replacing U with an affine connected Zariski open neighborhood of $\overline{u}$ disjoint from Y, the last assertion is proved as well. □

Proof of Theorem 5.2.1. We suppose to the contrary that there exists a nonempty open affine $V \subset \overline{X}_{\overline{D}}^{\mathrm{univ}}$ such that $(\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{irr}} \cap V = \varnothing$. Since $V \neq \operatorname{Spec} \mathbb{F}$ and the one-dimensional points are very dense in $\overline{X}_{\overline{D}}^{\mathrm{univ}}$ by Lemma A.1.8, there exists a 1-dimensional point $x \in V$ that defines a reducible pseudodeformation

$$D_x \colon G_K \longrightarrow \kappa(x)$$

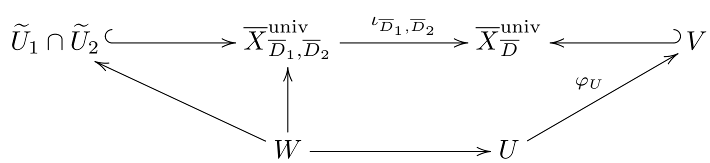
of $\overline{D}$ such that $\kappa(x)$ is a local field. By **??** there exist a finite extension $L'/\kappa(x)$ with finite residue field $\mathbb{F}' \supset \mathbb{F}$, residual pseudorepresentation $\overline{D}_i \colon G_K \to \mathbb{F}'$ of dimension n_i for some $n_i \in \mathbb{N}_0$ with $n_1 + n_2 = n$, and pseudorepresentations $D_1, D_2 \colon G_K \to \mathcal{O}_{L'}$ corresponding to points $(x_1, x_2) \in \overline{X}_{\overline{D}_1}^{\mathrm{univ}} \widehat{\times} \overline{X}_{\overline{D}_2}^{\mathrm{univ}}$ such that $D_x \otimes_{\kappa(x)} L' = (D_1 \oplus D_2) \otimes_{\mathcal{O}_{L'}} L'$. By Lemma 3.2.6, we may and will assume $\mathbb{F} = \mathbb{F}'$.

The inverse image of V under $\overline{X}_{\overline{D}_1}^{\mathrm{univ}} \widehat{\times} \overline{X}_{\overline{D}_2}^{\mathrm{univ}} \to \overline{X}_{\overline{D}}^{\mathrm{univ}}, (D_1, D_2) \mapsto D_1 + D_2$ is an open neighborhood of (x_1, x_2). By hypotheses Theorem 5.2.1(ii) we may within this neighborhood replace the initially chosen pair by (x_1, x_2) such that both are non-special, and by Lemma A.1.7 we may further assume that D_1 is not isomorphic to any of the finitely many $D_2(m)$, $m \in \{1, \ldots, p-1\}$, since $\dim \overline{X}_{\overline{D}_i}^{\mathrm{univ}} \geq 2$. Let $U_i := (\overline{X}_{\overline{D}_i}^{\mathrm{univ}})^{\text{n-spcl}}$. Then we observe that by Lemma 5.1.6 the schemes $(U_i)_{\mathrm{red}}$ are regular, and if $\zeta_p \notin K$, then $U_i = (U_i)_{\mathrm{red}}$.

Let $\overline{x}$ be a geometric point above x. By Lemma 5.2.2 there exists an étale neighborhood $(U, \bar{u}, \varphi_U \colon U \to \overline{X}_{\overline{D}}^{\mathrm{univ}})$ of $\overline{x}$, such that the pullback of $\iota_{\overline{D}_1, \overline{D}_2}$ along φ_U

$$W := U \times_{\varphi_U, \overline{X}_{\overline{D}}^{\mathrm{univ}}, \iota_{\overline{D}_1, \overline{D}_2}} \overline{X}_{\overline{D}_1, \overline{D}_2}^{\mathrm{univ}} \longrightarrow U$$

is a closed immersion with image U^{red}. We may replace U by $\varphi_U^{-1}(V)$, which is nonempty since $x \in V$, and is étale over V, and we may shrink W accordingly. By further replacing U by an open subset (and accordingly W), we can assume that U is connected and affine. Since $W \to U$ is a closed immersion, the scheme W is affine. But we also have that $W \to U$ is surjective as a map of topological spaces, since all points of V are reducible. Hence the nil-reduction of $W \to U$ is an isomorphism of schemes $W_{\mathrm{red}} \to U_{\mathrm{red}}$, and as a map of topological spaces $W \to U$ is a homeomorphism. Since the base change of étale morphisms is étale, so is the map $W \to \overline{X}_{\overline{D}_1, \overline{D}_2}^{\mathrm{univ}}$ that is the base change of φ_U under $\iota_{\overline{D}_1, \overline{D}_2}$. Let $\widetilde{U}_i$ be the preimage of U_i under the i-th projection $\overline{X}_{\overline{D}_1, \overline{D}_2}^{\mathrm{univ}} \to \overline{X}_{\overline{D}_i}^{\mathrm{univ}}$. We shrink W (and hence U) to a connected affine open so that the image of W in $\overline{X}_{\overline{D}_1, \overline{D}_2}^{\mathrm{univ}}$ lies in $\widetilde{U}_1 \cap \widetilde{U}_2$.[10] We display the situation in the following diagram:



Note also that $\varphi_U(U)$ intersects trivially with $\iota_{\overline{D}_1, \overline{D}_1}(\Delta_{\overline{D}_1}^{\mathrm{univ}})$ if $\overline{D}_1 = \overline{D}_2$. Hence in all cases, the morphism $\overline{X}_{\overline{D}_1}^{\mathrm{univ}} \widehat{\times}_{\mathbb{F}} \overline{X}_{\overline{D}_2}^{\mathrm{univ}} \to \overline{X}_{\overline{D}_1, \overline{D}_2}^{\mathrm{univ}}$ is an étale Galois cover above $\widetilde{U}_1 \cap \widetilde{U}_2$ with group $\mathbb{Z}/2$ or trivial group.

[10] The intersection $\widetilde{U}_1 \cap \widetilde{U}_2$ is strictly bigger than $U_1 \widehat{\times} U_2$. If for instance $X_i = \operatorname{Spec} L[[T_i]]$, $i = 1, 2$ and $U_i = \operatorname{Spec} L((T_i))$. Then $\widetilde{U}_1 \cap \widetilde{U}_2 = \operatorname{Spec} L[[T_1, T_2]][\frac{1}{T_1 T_2}]$ contains all but 3 points of $\operatorname{Spec} L[[T_1, T_2]]$.

Let $w \in W$ be the point corresponding to $u \in U$ under the homeomorphism $W \to U$. We complete at w and its images and pass to nil reductions. This gives

$$\widehat{\mathcal{O}}_{\widetilde{U}_1 \cap \widetilde{U}_2,(x_1,x_2),\mathrm{red}} \xrightarrow{\ \alpha\ } \widehat{\mathcal{O}}_{W,w,\mathrm{red}} \xrightarrow{\ \simeq\ } \widehat{\mathcal{O}}_{U,u,\mathrm{red}} \xleftarrow{\ \beta\ } \widehat{\mathcal{O}}_{V,x,\mathrm{red}}.$$

By Lemma A.1.14, the maps α and β are finite étale. The completion $\widehat{\mathcal{O}}_{\widetilde{U}_1 \cap \widetilde{U}_2,(x_1,x_2),\mathrm{red}}$ can be compared with the deformation ring $R^{\mathrm{univ}}_{L,(\rho_1,\rho_2)}$; using Proposition 4.8.10, it follows that $\widehat{\mathcal{O}}_{\widetilde{U}_1 \cap \widetilde{U}_2,(x_1,x_2),\mathrm{red}}$ is formally smooth over L of dimension $d(n_1^2+n_2^2)+1$, because by Lemma 5.1.6 and by hypothesis Theorem 5.2.1(i) the rings $R^{\mathrm{univ}}_{\rho_1,\mathrm{red}}$ are formally smooth over L of dimension dn_i^2+1. Hence by Lemma A.1.14 all local rings in the above diagram will be formally smooth over L of dimension $d(n_1^2+n_2^2)+1$.

Let now $\rho\colon G_K \to \mathrm{GL}_n(L)$ be a non-trivial extension of ρ_2 by ρ_1 as constructed in Lemma 5.1.7 (i) for $n = n_1+n_2$. It possesses a universal deformation ring R^{univ}_ρ for deformation to $\mathcal{A}r_L$, because $L = H^0(G_K, \mathrm{ad}_\rho)$. Let also $\widehat{R}^{\mathfrak{p}}$ be the universal pseudodeformation ring for D_ρ, and write φ for the natural morphism between associated space:

$$\varphi\colon X^{\mathrm{univ}}_\rho := \operatorname{Spec} R^{\mathrm{univ}}_\rho \longrightarrow \widehat{X} := \operatorname{Spec} \widehat{R}^{\mathfrak{p}}$$

The relation to the above is given by the following isomorphism obtained by combining Corollary 4.8.7 and Lemma 3.3.3

$$\widehat{R}^{\mathfrak{p}} = \widehat{\mathcal{O}}_{V,x}[\![T]\!]. \tag{29}$$

We now consider the map $\mathrm{d}\,\varphi\colon \mathbf{t}_{X^{\mathrm{univ}}_\rho,\rho} \to \mathbf{t}_{\widehat{X},x}$ induced from φ on tangent spaces at ρ and D_ρ, respectively, or rather the induced map on nil-reductions

$$\mathrm{d}\,\varphi_{\mathrm{red}}\colon \mathbf{t}_{X^{\mathrm{univ}}_{\rho,\mathrm{red}},\rho} \to \mathbf{t}_{\widehat{X}_{\mathrm{red}},x}.$$

By Lemma 5.1.7 (iii) and (iv) we have $\delta \in \{0,1\}$ (and $\delta = 0$ if $\zeta_p \notin K$), such that

$$dn^2 - dn_1n_2 + 2 + \delta = \dim_L \mathrm{Im}(\mathrm{d}\,\varphi_{\mathrm{red}}\ )$$

From (29) and the dimension found for $\widehat{\mathcal{O}}_{V,x}$, we have $\dim \mathbf{t}_{\widehat{X}_{\mathrm{red}},x} = 1 + d(n_1^2+n_2^2)$. This gives the inequality

$$dn^2 - dn_1n_2 + 2 + \delta \le d(n_1^2+n_2^2) + 2$$

Using $n = n_1+n_2$, we deduce $dn_1n_2 + \delta \le 0$, which is absurd since both $n_i > 0$. □

5.3 A dimension bound for the special locus

As before, we denote by $\overline{D}\colon G_K \to \mathbb{F}$ a residual pseudorepresentation, and we let n be its dimension. Theorem 5.2.1 of the previous subsection provided part of an inductive procedure to prove the equidimensionality of $\overline{X}^{\mathrm{univ}}_{K,\overline{D}}$ for the dimension $[K:\mathbb{Q}_p]\cdot n^2+1$. What remains to be proved is that $(\overline{X}^{\mathrm{univ}}_{K,\overline{D}})^{\text{n-spcl}} \subset (\overline{X}^{\mathrm{univ}}_{K,\overline{D}})^{\mathrm{irr}}$ is Zariski dense. In this subsection, we shall prove the following results.

Theorem 5.3.1. *Let $n \ge 2$ be an integer. Suppose that for all pseudorepresentations $\overline{D}'\colon G_{K'} \to \mathbb{F}$ of dimension $n' < n$ with K' a p-adic field the Krull dimension of the space $\overline{X}^{\mathrm{univ}}_{K',\overline{D}'}$ is bounded by $[K':\mathbb{Q}_p](n')^2+1$, Then for all n-dimensional pseudorepresentations $\overline{D}\colon G_K \to \mathbb{F}$ one has:*

(i) The Zariski closure of $(\overline{X}^{\mathrm{univ}}_{K,\overline{D}})^{\mathrm{spcl}}$ has dimension at most $\frac{1}{2}[K:\mathbb{Q}_p]n^2+1$.

(ii) $(\overline{X}^{\mathrm{univ}}_{K,\overline{D}})^{\text{n-spcl}} \subset (\overline{X}^{\mathrm{univ}}_{K,\overline{D}})^{\mathrm{irr}}$ is Zariski dense.

Before giving the proof, we need the following auxiliary result.

Lemma 5.3.2. *Let $\overline{R}_{G,W(\mathbb{F}),\overline{D}} \to A$ be a surjective homomorphism such that A is a domain with field of fractions $\mathbb{K}$, and set $D_A := D^{\mathrm{univ}}_{\overline{D}} \otimes_{\overline{R}_{G,W(\mathbb{F}),\overline{D}}} A$. Let $H \subset G$ be an open normal subgroup and suppose the following hold:*

(a) $\overline{D}|_H$ is split over $\mathbb{F}$ and condition $\Phi_{\overline{D}_H}$ is satisfied.

(b) $D_{\mathbb{K}} := D_A \otimes_A \mathbb{K}$ is irreducible and $\rho := \rho_{D_{\mathbb{K}} \otimes \mathbb{K}^{\mathrm{alg}}}$ is induced from H.

Then there exist a domain $A' \in \widehat{\mathcal{A}r}_{W(\mathbb{F})}$ that contains A and is finite over A, and a continuous irreducible pseudorepresentation $D' \colon H \to A'$ that is residually equal to a direct summand $\overline{D}'$ of $\overline{D}_H$, such that the following hold:

(i) $\operatorname{Ind}_H^G D' = D_A \otimes_A A'$

(ii) The homomorphism $\overline{R}_{H,W(\mathbb{F}),\overline{D}'} \to A'$ that results from D' is surjective.

In particular $\dim A = \dim A' \le \dim \overline{R}_{H,W(\mathbb{F}),\overline{D}'}$.

Proof. Note first by Lemma 2.1.4(b) and (f) that $\rho = \operatorname{Ind}_H^G \rho'$ for some irreducible representation $\rho' \colon H \to \mathrm{GL}_{n'}(\mathbb{K}^{\mathrm{alg}})$ such that the representations $(\rho')^g$, $g \in G/H$, are pairwise non-isomorphic, and that $\operatorname{Res}_H^G \rho = \bigoplus_{g \in G/H} (\rho')^g$. Hence $D_{\mathbb{K}}|_H$ is multiplicity free, so that we can apply Proposition 4.8.5 to it.

Note that by what was just observed conjugation by G/H acts simply transitively on the continuous pseudorepresentations D'_i from Proposition 4.8.5, and so the A_i from Proposition 4.8.5 are independent of i. Define A' as any of the A_i and let $D' \colon H \to A'$ be that pseudorepresentation D'_i for which $D'_i \otimes_{A'} \mathbb{K}^{\mathrm{alg}}$ is the pseudorepresentation attached to ρ'. Then $\operatorname{Ind}_H^G D' \otimes_{A'} \mathbb{K}^{\mathrm{alg}} = D_A \otimes_A \mathbb{K}^{\mathrm{alg}}$. Now $\operatorname{Ind}_H^G D'$ is defined over A' and A is the minimal field of definition of D_A by Corollary 4.7.13. Hence A is contained in A'. By Proposition 4.8.5 it is then clear that A' is finite integral over A and lies in $\widehat{\mathcal{A}r}_{W(\mathbb{F})}$, and moreover that $\oplus_{g \in G/H} (\overline{D}')^g = \overline{D}$ for $D' := D' \otimes_{A'} \kappa(A')$. Part (i) is also clear from what was just said.

It is also clear that D' is a deformation of $\overline{D}'$. Since $A' \in \widehat{\mathcal{A}r}_{W(\mathbb{F})}$ we have a corresponding homomorphism $\overline{R}_{H,W(\mathbb{F}),\overline{D}'} \to A'$, and the latter must be surjective by Corollary 4.7.13, since A' is the ring of definition of D'. Now by Lemma A.1.2 we have $\dim A' = \dim A$, and the inequality $\dim A' \le \dim \overline{R}_{H,W(\mathbb{F}),\overline{D}'}$ is trivial. □

Proof of Theorem 5.3.1. By Lemma 3.2.6, by possibly enlarging $\mathbb{F}$, we may assume that $\overline{D}$ is split over $\mathbb{F}$. Since the number of Galois extensions K' of K of degree p is finite, we may, by the same reasoning, also assume that $D|_{G'_K}$ is split for any such K' and for $K' = K(\zeta_p)$. It is also clear that Mazur's condition Φ_p holds over any such K' and hence $\Phi_{\overline{D}|_{G_{K'}}}$ holds.

To prove (i), let η be any generic point of $(\overline{X}^{\mathrm{univ}}_{\overline{D}})^{\mathrm{spcl}}$. Let

$$\varphi \colon \overline{R}_{K,\mathbb{F},\overline{D}} \to A$$

be the corresponding surjective ring homomorphism, so that $\eta = \operatorname{Ker}(\varphi)$. Because D_η is irreducible, $\rho := \rho_{D_\eta \otimes_{\kappa(\eta)} \kappa(\eta)^{\mathrm{alg}}}$ is defined. Since η is special, there exists a Galois extension K' of K such that either $K' = K(\zeta_p)$ or K' has degree p over K and such that ρ is induced from $G_{K'}$. From Lemma 5.3.2 we deduce

$$\dim A \le \dim \overline{R}_{K',W(\mathbb{F}),\overline{D}'} = [K' : \mathbb{Q}_p](n/[K' : K])^2 + 1 = \frac{1}{[K' : K]}\big([K : \mathbb{Q}_p]n^2\big) + 1.$$

As the schemes $\operatorname{Spec} A$ cover $(\overline{X}^{\mathrm{univ}}_{K,\overline{D}})^{\mathrm{spcl}}$ and as $[K' : K] \ge 2$, the proof of (i) is complete.

To prove (ii), we argue by contradiction and assume that there exists an open subset $V \subset (\overline{X}^{\mathrm{univ}}_{K,\overline{D}})^{\mathrm{spcl}}$ the is entirely contained in $(\overline{X}^{\mathrm{univ}}_{K,\overline{D}})^{\mathrm{spcl}}$. Then $\dim \overline{V} \leq \frac{1}{2}[K:\mathbb{Q}_p]n^2+1$ by (ii), for $\overline{V}$ the Zariski closure of V. Let x be any dimension 1 point of V and let $\rho\colon G_K \to \mathrm{GL}_n(L)$ be an absolutely irreducible representation over a local field L containing $\kappa(x)$ such that $D_\rho = D_x \otimes_{\kappa(x)} L$. Let R^{univ}_ρ be the universal ring for deformations of ρ to $\mathcal{A}r_L$. Then $\widehat{\mathcal{O}}_{\overline{V},x}[[T]] \cong R^{\mathrm{univ}}_\rho$ by Corollary 4.8.7 and Lemma 3.3.3. On the other hand $\dim R^{\mathrm{univ}}_\rho \geq [K:\mathbb{Q}_p]n^2+1$ by a standard argument using Theorem 3.4.1. It follows that

$$\frac{1}{2}[K:\mathbb{Q}_p]n^2+1+1 \geq [K:\mathbb{Q}_p]n^2+1,$$

and hence $2 \geq [K:\mathbb{Q}_p]n^2$, which implies $n=1$. But then x cannot be induced, and hence not special, and we reach a contradiction. ☐

5.4 Main results

Let K be a p-adic field, let $\overline{D}\colon G_K \to \mathbb{F}$ be a residual pseudorepresentation, and set $n := \dim \overline{D}$.

Theorem 5.4.1 (Theorem 1)**.** *The following assertions hold:*

(i) $\overline{X}^{\mathrm{univ}}_{\overline{D}}$ *is equidimensional of dimension* $[K:\mathbb{Q}_p]n^2+1$.

(ii) $(\overline{X}^{\mathrm{univ}}_{\overline{D}})^{\text{n-spcl}} \subset \overline{X}^{\mathrm{univ}}_{\overline{D}}$ *is open and Zariski dense.*

(iii) If $\zeta_p \notin K$, *then* $(\overline{X}^{\mathrm{univ}}_{\overline{D}})^{\text{n-spcl}}$ *is regular.*

(iv) If $\zeta_p \in K$, *then* $(\overline{X}^{\mathrm{univ}}_{\overline{D}})^{\text{n-spcl}}_{\mathrm{red}}$ *is regular, and* $(\overline{X}^{\mathrm{univ}}_{\overline{D}})^{\mathrm{reg}}$ *is empty.*

Proof. Part (i) follows from Corollary 3.4.3, Theorem 5.2.1 and Theorem 5.3.1 by induction on $\dim \overline{D}$ and $[K:\mathbb{Q}_p]$. The same results also prove (ii). Parts (iii) and (iv) follow from Lemma 5.1.6; the last part of (iv) uses Corollary 3.4.3(a). ☐

Lemma 5.4.2. *One has the following estimates:*

(i) If $n>1$, *then*

$$\dim(\overline{X}^{\mathrm{univ}}_{\overline{D}})^{\mathrm{red}} = \dim \overline{X}^{\mathrm{univ}}_{\overline{D}} - 2[K:\mathbb{Q}_p](n-1)+1,$$

and in particular $\dim(\overline{X}^{\mathrm{univ}}_{\overline{D}})^{\mathrm{red}} \leq \dim \overline{X}^{\mathrm{univ}}_{\overline{D}} - 2$ *unless* $n=2$ *and* $K=\mathbb{Q}_p$. *In the latter case* $\dim(\overline{X}^{\mathrm{univ}}_{\overline{D}})^{\mathrm{red}} = \dim \overline{X}^{\mathrm{univ}}_{\overline{D}} - 1$.

(ii) $\dim(\overline{X}^{\mathrm{univ}}_{\overline{D}})^{\mathrm{spcl}} \leq \dim \overline{X}^{\mathrm{univ}}_{\overline{D}} - 2$.

Proof. Since $(\overline{X}^{\mathrm{univ}}_{\overline{D}})^{\mathrm{spcl}}$ is empty for $n=1$, because non-trivially induced representations have dimension at least 2, part (ii) is immediate from Theorem 5.3.1. For part (i), we may assume that $\overline{D}$ is split by Lemma 3.2.6. Then $(\overline{X}^{\mathrm{univ}}_{\overline{D}})^{\mathrm{red}} \subset \bigcup_{\overline{D}_1 \oplus \overline{D}_2 = \overline{D}} \iota_{\overline{D}_1,\overline{D}_2}(\overline{X}^{\mathrm{univ}}_{\overline{D}_1,\overline{D}_2})$, and now Theorem 5.4.1(i) yields

$$\dim(\overline{X}^{\mathrm{univ}}_{\overline{D}})^{\mathrm{red}} = \max_{\substack{n_1+n_2=n \\ n_1,n_2>0}} \dim \overline{X}_{n_1} + \dim \overline{X}_{n_2} = \max_{\substack{n_1+n_2=n \\ n_1,n_2>0}} d(n_1^2+n_2^2)+2 = d((n-1)^2+1)+2.$$

The wanted estimate in (i) is immediate. For the remaining assertion note that $(\overline{X}^{\mathrm{univ}}_{\overline{D}})^{\mathrm{red}}$ is empty when $n=1$. ☐

Corollary 5.4.3. *Suppose that* $\zeta_p \notin K$ *and that* $(\overline{X}^{\mathrm{univ}}_{\overline{D}})^{\mathrm{spcl}}$ *is non-empty, so that* $e := [K':K]$ *divides* n, *for* $K' = K(\zeta_p)$. *Then the Zariski closure of* $(\overline{X}^{\mathrm{univ}}_{\overline{D}})^{\mathrm{spcl}}$ *has dimension* $\frac{1}{e}[K:\mathbb{Q}_p]n^2+1$.

Proof. Let $\overline{D}'\colon G_{K'} \to \mathbb{F}$ be a pseudorepresentation with $\mathrm{Ind}_{G_{K'}}^{G_K} \overline{D}' = \overline{D}$ – perhaps after enlarging $\mathbb{F}$ by a finite amount. Let $\mathfrak{p}$ be a minimal prime of $\overline{R}_{K',\mathbb{F},\overline{D}'}$ and let $D'\colon G_{K'} \to A := \overline{R}_{K,\mathbb{F},\overline{D}}/\mathfrak{p}$ be the induced pseudorepresentation. Then $\dim A = (\frac{n}{e})^2[K' : \mathbb{Q}_p] + 1$ and $D' \otimes_A \mathrm{Frac}(A)$ is irreducible by Lemma 5.4.2. Let $\rho'\colon G_{K'} \to \mathrm{GL}_n(L)$ be an absolutely irreducible representation over a finite extension L of $\mathrm{Frac}(A)$ such that $D_{\rho'} = D' \otimes_{\mathrm{Frac}(A)} L$. Let $H = \{\sigma \in G_K \mid {D'}^h = D'\}$ where ${D'}^h(g) = D'(ghg^{-1})$ for $g \in G$, so that $G_{K'} \subset H$ and let $K'' \subset K$ be the corresponding field. Then by [Hid00, Cor. 4.37] the representation ρ' can be extended to a representation $\rho''\colon G_{K''} \to \mathrm{GL}_n(L')$ for a finite extension L' of L. It is clear that the attached pseudorepresentation $D'' := D_{\rho''}$ is continuous, and takes values of the integral closure A' of A in L'; for $f = [H : G_{K'}]$ and $g \in G_{K'}$, thes characteristic polynomial coefficients of $D'(g^f)$ lie in A. The latter is finite over A by Lemma A.1.1(i), and so by Lemma A.1.2, we have $\dim A = \dim A'$. It follows that there is a homomorphism $\overline{R}_{K'',\mathbb{F},\overline{D}} \to A'$ that induces D''. Let $A'' \subset A'$ be the image of this homomorphism. Then D'' and hence also D' are defined over A'' by Corollary 4.7.13. But $\dim A''$ is bounded by $\dim \overline{R}_{K'',\mathbb{F},\overline{D}} = [K'' : \mathbb{Q}_p](\frac{n}{e})^2 + 1 < [K' : \mathbb{Q}_p](\frac{n}{e})^2 + 1 = \dim A$ unless H is trivial. We deduce that D' is not equal to any twist ${D'}^h$ for $h \in G_K \smallsetminus G_{K'}$. Hence by Lemma 2.1.4, the representation $\mathrm{Ind}_{G_{K'}}^{G_K} \rho'$ is absolutely irreducible, and A is a ring of definition of the pseudorepresentation $\mathrm{Ind}_{G_{K'}}^{G_K} D'$. The proof of Lemma 5.3.2 finally shows that the ring of definition of $\mathrm{Ind}_{G_{K'}}^{G_K} D'$ has the same dimension as A, i.e., $(\frac{n}{e})^2[K' : \mathbb{Q}_p] + 1$. □

Lemma 5.4.4. *Let κ be a local or a finite field. Suppose $p > 2$. Let $D_i\colon G_{\mathbb{Q}_p} \to \kappa$, $i = 1, 2$, be continuous pseudorepresentations of dimension 1, and let $D = D_1 \oplus D_2$. Then*

- *(i) If $D_1 \neq D_2(m)$ for $m \in \{0, \pm 1\}$, then*
  - *(1) there exists a unique non-trivial extension $\rho\colon G_{\mathbb{Q}_p} \to \mathrm{GL}_2(\kappa)$ of D_2 by D_1,*
  - *(2) the natural map $\overline{R}_D^{\mathrm{univ}} \to \overline{R}_\rho^{\mathrm{univ}}$ is an isomorphism,*
  - *(3) and both rings are formally smooth over κ of dimension 5.*
- *(ii) If $D_1 = D_2(m)$ for some $m \in \{\pm 1\}$, then $\overline{R}_D^{\mathrm{univ}}$ is not regular.*
- *(iii) If $D_1 = D_2$, then $\overline{R}_D^{\mathrm{univ}}$ is regular.*

Proof. The idea for (i) stems from the proof of [Che11, Cor. 4.4] and goes back to Kisin. We regard the D_i exchangeably as pseudorepresentations or as representations, because they are of dimension 1. Lemma 5.1.7(i) guarantees the existence of ρ as in (1). Since $D_1 \notin \{D_2, D_2(\pm 1)\}$, Theorem 3.4.1 yields $\dim \mathrm{Ext}^1_{G_{\mathbb{Q}_p}}(D_i, D_j) = 1$ for $i \neq j$, and this implies the uniqueness of ρ up to isomorphism. Note that once (2) is proved, part (3) follows from Lemma 5.1.7(iii). To see (2), let $X_\rho := \mathrm{Spec}\, \overline{R}_\rho^{\mathrm{univ}}$, $X_D := \overline{R}_D^{\mathrm{univ}}$ for $D := D_1 \oplus D_2$, write φ for the map in part (1), and denote by

$$\mathrm{d}\,\varphi\colon \mathbf{t}_{X_\rho,\rho} \to \mathbf{t}_{X_D,D}$$

the induced map on tangent spaces. By the formula in Lemma 5.1.7(iii), the kernel of $\mathrm{d}\,\varphi$ is zero. Because $p > 2$ we also have $\dim \mathrm{Ext}^1_{G_{\mathbb{Q}_p}}(D_i, D_i) = 2$ for $i = 1, 2$. Consider now the following exact sequence from [Bel12, Thm. 2] with $\rho_i = D_i$

$$\begin{aligned} 0 \longrightarrow \textstyle\bigoplus_{i=1,2} \mathrm{Ext}^1_{G_K}(\rho_i, \rho_i) \longrightarrow \dim \mathbf{t}_{X_D,D} \otimes_{\kappa(x)} L \\ \longrightarrow \mathrm{Ext}^1_{G_K}(\rho_1, \rho_2) \otimes \mathrm{Ext}^1_{G_K}(\rho_2, \rho_1) \xrightarrow{h} \textstyle\bigoplus_{i=1,2} \mathrm{Ext}^2_{G_K}(\rho_i, \rho_i). \end{aligned} \tag{30}$$

It implies $\dim \mathbf{t}_{X_D,D} \leq 5$. Hence $\mathrm{d}\,\varphi$ must be an isomorphism and $\dim \mathbf{t}_{X_D,D} = 5$. This implies that φ must be surjective, and hence an isomorphism since the target is formally smooth over κ.

To prove (ii), note that we have $\mathrm{Ext}^2(\rho_i,\rho_i) = 0$ and $\mathrm{Ext}^1(\rho_i,\rho_i)$ is of dimension 1, while $\mathrm{Ext}^1(\rho_i,\rho_i(m))$ is 2 for $m = 1$ and 1 for $m = -1$. Hence (30) yields $\dim_\kappa \mathbf{t}_{X_D,D} = 6$. However $\dim X_D = 5$ by Theorem 5.4.1, and hence R_D^{univ} is not regular.

Finally we show (iii). Because $p > 2$, we may apply [Che11, Thm. 3.1] in exactly the same way, as done in [Che11, Lem. 2.5]: Using that the mod p reduction of $G_{\mathbb{Q}_p}^{\mathrm{ab}}$ is isomorphic to $(\mathbb{Z}/p)^2$, one has $\dim_\kappa \mathrm{Hom}(G_{\mathbb{Q}_p},\kappa) = 2$. $\dim_\kappa \mathrm{Sym}(G_{\mathbb{Q}_p},\kappa) = 3$. and $\dim_\kappa \mathrm{Alt}(G_{\mathbb{Q}_p},\kappa) = 0$, and hence $\dim_\kappa \mathbf{t}_{X_D,D} = 5$. We now conclude using $\dim X_D = 5$ by Theorem 5.4.1. □

We now characterize the singular locus when $\zeta_p \notin K$.

Theorem 5.4.5 (Theorem 2, [Che11, Thm. 2.3])**.** *If $\zeta_p \notin K$, then the following hold:*

(i) The closure of $X_1 := (\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{spcl}}$ in $\overline{X}_{\overline{D}}^{\mathrm{univ}}$ lies in $(\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{sing}}$.

(ii) If $n > 2$ or $[K : \mathbb{Q}_p] > 1$, then $X_2 := (\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{red}} \subset (\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{sing}}$.

(iii) If $n = 2$, $K = \mathbb{Q}_p$, and $x \in X_2$ corresponds to a pair (D_1, D_2) of 1-dimensional pseudorepresentations, then $x \in (\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{sing}}$ if and only if $D_2 = D_1(m)$ for $m \in \{\pm 1\}$.

Proof. We know from Proposition 4.7.11 that $\overline{X}_{\overline{D}}^{\mathrm{univ}}$ is a complete Noetherian local ring so that by Lemma A.1.1(i), $(\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{sing}}$ is closed in $\overline{X}_{\overline{D}}^{\mathrm{univ}}$. Observe that X_i is non-empty, then its Zariski closure $\overline{X}_i$ has dimension at least 2: for X_2, this is clear from Lemma 5.4.2(i), for X_1 by Corollary 5.4.3. Hence Proposition A.1.11 shows that the points of X_i of dimension 1 are dense in X_i.

To prove (i), let $x \in X_1$ be of dimension 1. A standard computation of tangent spaces as in the proof of Lemma 5.1.7 (iii) shows $\dim H^1(G_K, \mathrm{ad}_{\rho_x}) = dn^2 + 2$, using $[\dim_L H^2(G_K, \mathrm{ad}_{\rho_x}) = \dim_L H^0(G_K, \mathrm{ad}_{\rho_x}(1)) = 1$. while $\dim R_{\rho_x}^{\mathrm{univ}} = dn^2 + 1$. It follows from Lemma 3.3.3 that x is not regular on $\overline{X}_{\overline{D}}^{\mathrm{univ}}$.

For the proof of (ii), we assume without loss of generality that $\overline{D}$ is split. Then $(\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{red}}$ is the image of the maps $\iota_{\overline{D}_1,\overline{D}_2}$ from (27) for all $\overline{D}_1$, $\overline{D}_2$ such that $\overline{D} = \overline{D}_1 \oplus \overline{D}_2$. Fix such a pair and let n_i be the dimension of $\overline{D}_i$. Because of Theorem 1 it suffices to consider pairs $x = (x_1, x_2)$ with $x_i \in (\overline{X}_{\overline{D}_i}^{\mathrm{univ}})^{\text{n-spcl}}$; and we may also assume that D_{x_1} is distinct from the finitely many $D_{x_2}(m)$, $m \in \{1, \dots, p-1\}$. We compute the tangent space dimension of $\overline{R}_{D_x}^{\mathrm{univ}}$ this time, using (30) from [Bel12, Thm. A] which also holds for ρ_{x_i} in place of ρ_i. We conclude as in the proof of [Che11, Lem. 2.4]: $\dim_L H^1(G_K, \mathrm{ad}_{\rho_{x_i}}) \geq 1 + dn_i^2$, $\dim_L \mathrm{Ext}^1_{G_K}(\rho_{x_i}, \rho_{x_{3-i}}) = dn_1n_2$, and the second extension groups vanish, since the D_{x_i} satisfy $D_{x_i} \neq D_{x_i}(1)$. Hence

$$\mathbf{t}_{\mathrm{Spec}\, \overline{R}_{D_x}^{\mathrm{univ}}} = d(n_1^2 + n_2^2) + 2 + d^2 n_1^2 n_2^2 \geq dn^2 + 1 + (dn_1n_2 - 1)^2.$$

This dimension is strictly larger than $dn^2 + 1$, unless $dn_1n_2 = 1$, i.e., $n_1 = n_2 = 1$ and $K = \mathbb{Q}_p$. However $\dim \overline{R}_{D_x}^{\mathrm{univ}} = dn^2 + 1$ by Lemma 3.3.3 and Theorem 5.4.1, and it follows that x cannot be regular, proving (ii).

Concerning (iii), note that if $x = (D_1, D_2)$ is any point of dimension at most 1, then the assertion follows from Lemma 5.4.4. Since such points are Zariski dense in the closure of any point of dimension at least 2, the assertion in (iii) follows in general. □

Remark 5.4.6. Note that Theorem 5.4.5 reproves a result of Paškūnas, namely [Paš13, Prop. B.17]: suppose that $n = 2$, $p > 2$, $K = \mathbb{Q}_p$, and $\overline{D} \colon G_{\mathbb{Q}_p} \to \mathbb{F}$ is a direct sum $\overline{D}_1 \oplus \overline{D}_2$ of 1-dimensional pseudocharacters $\overline{D}_i$ sucht that $\overline{D}_2 \neq \overline{D}_1(m)$ for $m = 0, \pm 1$. Then $\overline{R}_{\overline{D}}^{\mathrm{univ}} = \mathbb{F}_q[[X_1, \dots, X_5]]$.

Theorem 5.4.7 (Theorem 3)**.** *The ring $\overline{R}_{\overline{D},\mathrm{red}}^{\mathrm{univ}}$ satisfies Serre's condition (R_2), unless $n = 2$, $K = \mathbb{Q}_2$ and $\overline{D}$ is trivial.*

Proof. By Theorem 5.4.1 and Lemma 5.1.6, the subset $(\overline{X}^{\mathrm{univ}}_{\overline{D},\mathrm{red}})^{\text{n-spcl}}$ is regular, open and Zariski dense in $\overline{X}^{\mathrm{univ}}_{\overline{D},\mathrm{red}}$. Thus Lemma 5.4.2 implies the theorem unless $n = 2$ and $K = \mathbb{Q}_p$. Also, if $\overline{D}$ is irreducible, then so is any lift, and so $(\overline{X}^{\mathrm{univ}}_{\overline{D},\mathrm{red}})^{\mathrm{red}}$ is empty. Now again we conclude by Lemma 5.4.2. Suppose from now on that $K = \mathbb{Q}_p$ and $\overline{D} = \overline{D}_1 \oplus \overline{D}_2$ for 1-dimensional pseudorepresentations $\overline{D}_i \colon G_{\mathbb{Q}_p} \to \mathbb{F}$, and suppose now also $p > 2$ which was excluded in this case.

The locus of $x \in X_2 := (\overline{X}^{\mathrm{univ}}_{\overline{D}})^{\mathrm{red}}$ corresponds to a pair (D_1, D_2) of 1-dimensional pseudorepresentations, such that $D_2 = D_1(m)$ for $m \in \{\pm 1\}$, can be realized as the image of $\overline{X}^{\mathrm{univ}}_{\overline{D}_1}$. Hence it has dimension at most 2 because of Corollary 3.4.3. Outside this locus points are smooth by Theorem 5.4.5 (and the density of 1-dimensional points). It follows that $(\overline{X}^{\mathrm{univ}}_{\overline{D}})^{\mathrm{red,sing}}$ has dimension at most 2 which is less than $5 - 2 = 3$, so that then $\overline{X}^{\mathrm{univ}}_{\overline{D}}$ satisfies (R_2), also. ☐

A Appendix. Auxiliary results on rings, algebras and representations

In this appendix we collect some results used in various parts of this work. We also prove some minor facts that could not be found directly in the literature.

A.1 Commutative Algebra

Complete local rings, integral extensions and regularity

A domain B with quotient field $\mathbb{K}$ is said to satisfy N-2 if for any finite field extension L of $\mathbb{K}$, the integral closure of B in L is a finite over B. A ring A is called a *Nagata ring* if A is Noetherian and for every prime ideal $\mathfrak{p}$ of A the ring $A/\mathfrak{p}$ satisfies N-2, see [Sta18, § 032E].

Lemma A.1.1. *If A is complete Noetherian local ring, then the following hold:*

(i) A is a Nagata ring, and hence the set of regular points of $\operatorname{Spec} A$ *is open in* $\operatorname{Spec} A$.

(ii) If A is a domain with fraction field $\mathbb{K}$ and perfect residue field, then $[\mathbb{K} : \mathbb{K}^p] < \infty$.

Proof. Part (i) is [Sta18, § 032W] combined with [Gro65, Thm. (6.12.7)]. Part (ii) is proved in [Hoc07, Prop. (d),(g)]. ☐

Lemma A.1.2 ([Mat80, 13.C, Thm. 20])**.** *If B is a domain, and if $B' \subset B$ is a subring such that B if finite over B', then* $\dim B = \dim B'$.

Recall that for a prime $\mathfrak{p}$ of A, the *height of* $\mathfrak{p}$ is defined as $\operatorname{ht}\mathfrak{p} = \dim R_{\mathfrak{p}}$.

Definition A.1.3. *A commutative ring A is said to* satisfy (Serre's) condition (R_i)*, if A is* regular in codimension at most i*, i.e., if the local ring $A_{\mathfrak{p}}$ is regular for every prime $\mathfrak{p}$ of height $\leq i$.*

Density of points of dimension one

The next series of results stems from [Gro66, §10.1–10.5], except in one case where we give a direct reference. Let X be a topological space. It is called *Noetherian* if every descending chain of closed subsets becomes stationary. It is called *irreducible* if it is not the union of two proper closed subsets. If X is Noetherian, a subset is called *constructible* if it is a finite union of *locally closed* subsets of X, i.e., of subsets that are the intersection of an open and a closed subset of X. The closure of a subset $Z \subset X$ is denoted by $\overline{Z}$. For a subset Z of X its dimension $\dim Z \in \mathbb{N} \cup \{\infty\}$ is the maximal length n of a chain $Y_0 \subsetneq Y_1 \subsetneq \ldots \subsetneq Y_n \subset \overline{Z}$ of irreducible closed subsets Y_i in X.

Definition A.1.4. *A subset X_0 of X is called* very dense in X *if every nonempty locally closed subset $Z \subset X$ satisfies $Z \cap X_0 \neq \varnothing$.*

If X_0 is very dense in X, it is clearly dense in X.

Lemma A.1.5. *If X_0 is very dense in X, then $X_0 \cap Z$ is very dense in Z and dense in $\overline{Z}$ for any locally closed set Z in X.*

Proposition A.1.6. *For a subset X_0 of X the following conditions are equivalent:*

(i) X_0 is very dense in X;

(ii) Under $X' \mapsto X_0 \cap X'$ the open subsets in X are in bijection to those in X_0.

(iii) Under $X' \mapsto X_0 \cap X'$ the closed subsets in X are in bijection to those in X_0.

In the following we set $X_{\leq 1} := \{x \in X : \dim x \leq 1\}$. Since the union of finitely many irreducible subsets of dimension at most i has dimension at most i. we find:

Lemma A.1.7. *If $U \subset X$ satisfies $\dim U \geq 2$, then no finite subset of $U_{\leq 1}$ is dense in U.*

An important source for very dense subset of schemes comes from the following result:

Lemma A.1.8 ([Mat80, (33.F) Lem. 5])**.** *Let $X = \operatorname{Spec} A$ for a Noetherian ring A. Then the set $X_{\leq 1}$ is very dense in X.*

From Lemma A.1.8 and Lemma A.1.5 one deduce:

Corollary A.1.9. *Let $X = \operatorname{Spec} A$ for a Noetherian ring A, and let $Z \subset X$ be constructible. Then $X_{\leq 1} \cap Z$ is very dense in Z and dense in $\overline{Z}$.*

Definition A.1.10. *The space X is called* Jacobson *if $\{x \in X : \dim x = 0\}$ is very dense in X.*

A scheme is called *Jacobson* if the underlying topological space is Jacobson; A ring A is called *Jacobson* if the scheme $\operatorname{Spec} A$ is Jacobson. For us the following result is of importance:

Proposition A.1.11. *For a Noetherian local ring with A maximal ideal $\mathfrak{m}_A$ the scheme $\operatorname{Spec} A \smallsetminus \{\mathfrak{m}_A\}$ is Jacobson.*

Étale morphisms and étale neighborhoods

We recall some terminology and a result on étale morphisms to be used in Section 5.

Definition A.1.12 ([Sta18, § 00U0 and Def. 02GI])**.** *(i) A ring map $A \to B$ is called étale if it is a smooth ring map of relative dimension zero.*

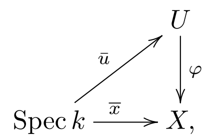
(ii) A morphism $f : X \to Y$ of schemes is called étale at $x \in X$ if there is an affine open neighborhood $\operatorname{Spec}(B) = U \subset X$ of x and an affine open $\operatorname{Spec}(A) = V \subset Y$ with $f(U) \subset V$ so that the corresponding ring map $A \to B$ is étale. We say that f is étale if it is étale at each point $x \in X$.

Definition A.1.13 ([Sta18, Def. 03PO])**.** *Let X be a scheme.*

(i) A geometric point *of X is a morphism $\overline{x}\colon \operatorname{Spec} k \to X$ where k is an algebraically closed field.*

(ii) One says that $\overline{x}$ is lies over *$x \in X$ to indicate that x is the image of $\overline{x}$.*

(iii) An étale neighborhood $(U, \overline{u}, \varphi)$ *of a geometric point* $\overline{x} \in X$ *is a commutative diagram*



where φ *is an étale morphism of schemes and* $\overline{u}$ *is a geometric point of* U*.*

Lemma A.1.14. *Let* $\varphi\colon U \to X$ *be an étale morphism between schemes* U *and* X*. Let* u *be a point of* U *and denote by* x *its image* $\varphi(u)$*. Consider the local homomorphism* $\varphi_u\colon \mathcal{O}_{X,x} \to \mathcal{O}_{U,u}$ *induced from* φ*. Then*

(i) The completion $\widehat{\varphi}_u\colon \widehat{\mathcal{O}}_{X,x} \to \widehat{\mathcal{O}}_{U,u}$ *of* φ_u *is finite étale; its degree is equal to* $[\kappa(u) : \kappa(x)]$*.*

(ii) The ring $\widehat{\mathcal{O}}_{X,x}$ *is regular if and only if* $\widehat{\mathcal{O}}_{U,u}$ *is regular, and in this case both have the same dimension.*

Proof. Part (i) is [Sta18, Lem. 039M] and the remark following it. For part (ii) note that by étaleness the tangent spaces at the closed point have the same dimension, and by finite étaleness the ring $\widehat{\mathcal{O}}_{U,u}$ is free of finite rank over $\widehat{\mathcal{O}}_{X,x}$ and hence they have the same dimension. From this (ii) follows easily. □

A.2 Finite dimensional algebras and modules

Let $\mathbb{K}$ be a field. We gather some results, mostly from [CR62], on not necessarily commutative $\mathbb{K}$-algebras S and modules M over them, assuming that either the algebra or the module have finite $\mathbb{K}$-dimension. Our intended applications are to $S = \mathbb{K}[G]$ for a possibly infinite group G, or to G-modules of finite $\mathbb{K}$-dimension; note that if G is profinite, $\mathbb{K}$ is a topological field and M is a $\mathbb{K}[G]$-module of finite $\mathbb{K}$-dimension with a continuous G-action, then all G-subquotient of M carry a continuous action. So we need not worry about continuity in the following.

Let first S be a $\mathbb{K}$-algebra of finite $\mathbb{K}$-dimension. In this case, see [CR62, § 24], the sum of all nilpotent left ideals of S is a two-sided ideal of S; it is the maximal nilpotent two-sided ideal of S; is called the *radical* of S and denoted by $\mathrm{Rad}(S)$. The radical is zero if and only if S is semisimple; in this case are is the product of simple $\mathbb{K}$-algebras (of finite $\mathbb{K}$-dimension). If $\mathbb{K}'$ is any field extension of $\mathbb{K}$, then

$$\mathrm{Rad}(S) \otimes_{\mathbb{K}} \mathbb{K}' \subset \mathrm{Rad}(S \otimes_{\mathbb{K}} \mathbb{K}'). \tag{31}$$

Definition A.2.1. *We call a* $\mathbb{K}$*-algebra* S *of finite* $\mathbb{K}$*–dimension is* absolutely semisimple *if* $S \otimes_{\mathbb{K}} \mathbb{K}^{\mathrm{alg}}$ *is semisimple.*

Remark A.2.2. Suppose that S is absolutely semisimple. Then by (31) it is semisimple. By the Theorem of Artin-Wedderburn, the algebra $S \otimes_{\mathbb{K}} \mathbb{K}^{\mathrm{alg}}$ is a product of matrix algebras over $\mathbb{K}^{\mathrm{alg}}$. From this one deduces, by repeated application of (31) that $S \otimes_{\mathbb{K}} \mathbb{K}'$ is semisimple for any field extension $\mathbb{K}'$ of $\mathbb{K}$. By considering simple factors D_i of a semisimple ring S, one shows that S is absolutely semisimple over $\mathbb{K}$ if and only if the center of each D_i is separable over $\mathbb{K}$.

Lemma A.2.3. *Let* S *a* $\mathbb{K}$*-algebra of finite* $\mathbb{K}$*-dimension and write* S' *for* $S \otimes_{\mathbb{K}} \mathbb{K}'$ *and any field extension* $\mathbb{K}'$ *of* $\mathbb{K}$*.*

(i) There exists finite extension $\mathbb{K}'$ *of* $\mathbb{K}$ *such that* $S'/\mathrm{Rad}(S')$ *is absolutely semisimple.*

(ii) If $S/\mathrm{Rad}(S)$ *is absolutely semisimple over* $\mathbb{K}$*, then there exists an extension* $\mathbb{K}'$ *of* $\mathbb{K}$ *with* $[\mathbb{K}' : \mathbb{K}] \le (\dim_{\mathbb{K}} S)!$ *such that* $S'/\mathrm{Rad}(S')$ *is a product of matrix algebras over* $\mathbb{K}'$*.*

(iii) If $\mathbb{K}$ *is finite, and if we then write* $S/\operatorname{Rad}(S) \cong M_{d_i}(\mathbb{K}_i)$ *for* $d_i \geq 1$ *and* $\mathbb{K}_i$ *finite over* $\mathbb{K}$*, then we may find* $\mathbb{K}'$ *as in (b) so that* $[\mathbb{K}' : \mathbb{K}]$ *divides* $(\sum_i d_i[\mathbb{K}_i : \mathbb{K}])!$.

Proof. For (a) note first that for $S^{\mathrm{alg}} := S \otimes_{\mathbb{K}} \mathbb{K}^{\mathrm{alg}}$ the ring $S^{\mathrm{alg}}/\operatorname{Rad}(S^{\mathrm{alg}})$ is semisimple, and trivially absolutely semisimple. Let $\mathbb{K}'$ be a finite extension of $\mathbb{K}$ over which $S' := S \otimes_{\mathbb{K}} \mathbb{K}'$ contains a sub-$\mathbb{K}'$ vector space I with $I \otimes_{\mathbb{K}'} \mathbb{K}^{\mathrm{alg}} = \operatorname{Rad}(S^{\mathrm{alg}})$. Considering I inside $\operatorname{Rad}(S^{\mathrm{alg}})$, it follows that I is a nilpotent ideal of S', so that $I \subset \operatorname{Rad}(S')$. But then using (31) and the faithful flatness of $\mathbb{K}' \to \mathbb{K}^{\mathrm{alg}}$, it is straightforward that $I = \operatorname{Rad}(S')$ and that S'/I is absolutely simple.

To prove (b) note first that we may replace S by $S/\operatorname{Rad}(S)$, again by (31), so that we may assume that S is absolutely semisimple. Write S as a product of division algebras D_i, for i in a finite index set I, and write $\mathbb{K}_i$ for the center of D_i and let $d_i \in \mathbb{N}$ be such that $d_i^2 = \dim_{\mathbb{K}_i} D_i$. We consider all finite field extensions of $\mathbb{K}$ as subfields of a fixed algebraic closure $\mathbb{K}^{\mathrm{alg}}$ of $\mathbb{K}$. Let $\mathbb{K}' \subset \mathbb{K}^{\mathrm{alg}}$ be the join of the normal hull of all $\mathbb{K}_i$. By Remark A.2.2, $\mathbb{K}'$ is separable over $\mathbb{K}$ and for each i we have $\mathbb{K}_i \otimes_{\mathbb{K}} \mathbb{K}' \cong (\mathbb{K}')^{m_i}$ for some $m_i \in \mathbb{N}$. Note also that $[\mathbb{K}' : \mathbb{K}] \leq \prod_{i\in I}[\mathbb{K}_i : \mathbb{K}]!$. Let $\mathbb{E}_i \subset D_i$ be a maximal subfield over $\mathbb{K}_i$ so that $D \otimes_{\mathbb{K}_i} \mathbb{E}_i \cong M_{d_i}(\mathbb{E}_i)$. Let $\mathbb{E}' \supset \mathbb{K}^{\mathrm{alg}}$ be the join of $\mathbb{K}'$ and the fields $\mathbb{E}_i$, $i \in I$. Then

$$S \otimes_{\mathbb{K}} \mathbb{E}' \cong \prod_{i\in I}(D_i \otimes_{\mathbb{K}_i} (\mathbb{K}_i \otimes_{\mathbb{K}} \mathbb{K}') \otimes_{\mathbb{K}'} \mathbb{E}') \cong \prod_{i\in I}(D_i \otimes_{\mathbb{K}_i} \mathbb{E}')^{m_i} \overset{\mathbb{E}'\supset\mathbb{E}_i}{\cong} \prod_{i\in I}(M_{d_i}(\mathbb{E}'))^{m_i}. \tag{32}$$

Hence $\mathbb{E}'$ is a field as in (a). Moreover $[\mathbb{E}' : \mathbb{K}] \leq \prod_{i\in I}(d_i \cdot [\mathbb{K}_i : \mathbb{K}]!) \leq \prod_{i\in I}(d_i \cdot [\mathbb{K}_i : \mathbb{K}])!$. Since $\sum_{i\in I}(d_i[\mathbb{K}_i : \mathbb{K}]) \leq \sum_{i\in I}[\mathbb{K}_i : \mathbb{K}] \cdot d_i^2 \cdot m_i = n$, using that multinomials are integers, we deduce $[\mathbb{E} : \mathbb{K}] \leq n!$, and this proves (b).

To see (c) note that each $\mathbb{K}_i$ is normal over $\mathbb{K}$ and for each degree there is a unique extension of $\mathbb{K}$ of that degree in a fixed choice $\mathbb{K}^{\mathrm{alg}}$. Hence in the proof of (b) we find $[\mathbb{K}' : \mathbb{K}] \leq \operatorname{lcm}_{i\in I}[\mathbb{K}_i : \mathbb{K}]$. Moreover over $\mathbb{K}_i$ the ring D_i is already split, and so we can take $\mathbb{E}' = \mathbb{K}'$. The assertion now is clear. ☐

Remark A.2.4. (a) Note that the hypothesis in Lemma A.2.3(b) holds whenever $\mathbb{K}$ is perfect.

(b) A version of Lemma A.2.3(a) only under algebraicity hypotheses for S over $\mathbb{K}$ can be found in [Che14, Lem. 2.14].

(c) It is possible to give effective bounds in Lemma A.2.3(b) also without any separability hypotheses. But the proof is longer and we do not need the result.

Let now S be any $\mathbb{K}$-algebra, not necessarily of finite $\mathbb{K}$-dimension. Let M be an S-algebra of finite $\mathbb{K}$-dimension. If M is semisimple, the representation $M \otimes_{\mathbb{K}} \mathbb{K}^{\mathrm{alg}}$ need in general not be semisimple over $S^{\mathrm{alg}} := S \otimes_{\mathbb{K}} \mathbb{K}^{\mathrm{alg}}$.[11]

Definition A.2.5. *We call* M absolutely semisimple*, if* $M \otimes_{\mathbb{K}} \mathbb{K}^{\mathrm{alg}}$ *is semisimple over* S^{alg}.

We call M absolutely completely reducible *if it is semisimple and all its irreducible summands are absolutely irreducible.*

Remark A.2.6. If M is absolutely completely reducible, it is clearly absolutely semisimple. If M is absolutely semisimple, it is absolutely completely reducible if and only if for each irreducible summand N of M the natural map $\mathbb{K} \to \operatorname{End}_S(N)$ is an isomorphism, see [CR62, 29.13]; the latter condition is equivalent to $\operatorname{End}_S(M)$ being a product of matrix algebras over $\mathbb{K}$.

For the following note that if N is a second S-modulo of finite $\mathbb{K}$-dimension and $\mathbb{K}'$ is any field extension of $\mathbb{K}$, then by [CR62, 29.2] one has

$$\operatorname{Hom}_S(M, N) \otimes_{\mathbb{K}} \mathbb{K}' \cong \operatorname{Hom}_{S\otimes_{\mathbb{K}}\mathbb{K}'}(M \otimes_{\mathbb{K}} \mathbb{K}', N \otimes_{\mathbb{K}} \mathbb{K}'). \tag{33}$$

[11] If S is a purely inseparable finite field extension of $\mathbb{K}$ and $M = S$, then $S \otimes_{\mathbb{K}} S$ is not semisimple.

Lemma A.2.7. *Suppose M is absolutely semisimple. Then the following hold:*

(a) The $\mathbb{K}$-algebra $\mathrm{End}_S(M)$ is absolutely semisimple.

(b) If $\mathbb{K}' \supset \mathbb{K}$ is an extension such that $\mathrm{End}_S(M) \otimes_{\mathbb{K}} \mathbb{K}'$ is a product of matrix algebras, then $M \otimes_{\mathbb{K}} \mathbb{K}'$ is absolutely completely reducible.

Proof. To prove (a) it suffices to assume that M is irreducible. Then $D := \mathrm{End}_S(M)$ is a skew field of finite dimension over $\mathbb{K}$. By (33) we have

$$D \otimes_{\mathbb{K}} \mathbb{K}^{\mathrm{alg}} \cong \mathrm{End}_{S \otimes_{\mathbb{K}} \mathbb{K}^{\mathrm{alg}}}(M \otimes_{\mathbb{K}} \mathbb{K}^{\mathrm{alg}}).$$

By hypothesis $M \otimes_{\mathbb{K}} \mathbb{K}^{\mathrm{alg}}$ is semisimple over $S \otimes_{\mathbb{K}} \mathbb{K}^{\mathrm{alg}}$. By Remark A.2.6, $D \otimes_{\mathbb{K}} \mathbb{K}^{\mathrm{alg}}$ is then a product of matrix algebras over $\mathbb{K}^{\mathrm{alg}}$. This proves (a). Part (b) is immediate from (33), since it implies $\mathbb{K}' \cong \mathrm{End}_{S \otimes_{\mathbb{K}} \mathbb{K}'}(N)$ for every irreducible summand of $M \otimes_{\mathbb{K}} \mathbb{K}'$. □

Remark A.2.8. For $\mathbb{K}'$ as in Lemma A.2.7(b) one can bound $[\mathbb{K}' : \mathbb{K}]$ by $((\dim_{\mathbb{K}} M)^2)!$ using Lemma A.2.3(b).

A.3 Absolutely irreducible mod p representations of the absolute Galois group of a p-adic field

In this subsection we shall give a proof of the following result. Part (a) for $K = \mathbb{Q}_p$ first appeared in work of L. Berger and for general $K \supset \mathbb{Q}_p$ in [Mul13, Prop. 2.1.1]. We give a complete proof of (a), since it also serves to prove (b). Throughout A.3 we let k be the residue field of K.

Lemma A.3.1 (Cf. [Ber10, Cor. 2.1.5])**.** *Let $\overline{\rho}\colon G_K \to \mathrm{GL}_n(k^{\mathrm{alg}})$ be an n-dimensional irreducible continuous representation. Let $\mathbb{F} \subset k^{\mathrm{alg}}$ be a finite field that contains k_n. Then the following hold:*

(a) There exists $\lambda \in (k^{\mathrm{alg}})^\times$, $\tau \in \mathcal{P}_n$ and a primitive number $r \in \{1, 2, \ldots, q^n - 2\}$ such that

$$\overline{\rho} \cong \overline{\mu}_{K,\lambda} \otimes \mathrm{Ind}_{G_{K_n}}^{G_K} \widehat{\omega}_{n,\tau}^r.$$

(b) $\bar{\rho}$ can be defined over $\mathbb{F}$ if and only if $\lambda^n \in \mathbb{F}$.

In particular, given n and $\mathbb{F} \supset k_n$, there are only finitely many isomorphism classes of absolutely irreducible representations $G_K \to \mathrm{GL}_n(\mathbb{F})$.

We begin with some preparations and reminders: Recall the classification of tame characters of the inertia group I_K of G_K from [Ser72]: Let m denote some natural number. Let k^{alg} be the residue field of K^{alg} and set $q := |k|$. Let in the following $\sigma \in G_K$ be any element that maps to Frobenius in G_k. Let $K^{\mathrm{nr}} \subset K^{\mathrm{t}} \subset K^{\mathrm{alg}}$ denote the maximal unramified and maximal tamely ramified extensions of K, respectively. Denote by $K_m \subset K^{\mathrm{nr}}$ the unique extension of K of degree m and by $k_m \subset k^{\mathrm{alg}}$ its residue field. If ϖ is a fixed choice of uniformizer of K and $K_m^t = K^{\mathrm{nr}}(\sqrt[q^m-1]{\varpi})$, then $K^{\mathrm{t}} = \varinjlim_{m \in \mathbb{N}_{\geq 1}} K_m^t$. The characters

$$\omega_m\colon I_{\mathrm{t}} := \mathrm{Gal}(K^{\mathrm{t}}/K^{\mathrm{nr}}) \to \mathrm{Gal}(K_m^t/K^{\mathrm{nr}}) \xrightarrow{\sim} \mu_{q^m-1}(K^{\mathrm{nr}}) = \mu_{q^m-1}(k^{\mathrm{alg}}) = k_m^\times, \sigma \mapsto \frac{\sigma(\sqrt[q^m-1]{\varpi})}{\sqrt[q^m-1]{\varpi}},$$

form an inverse system, $I_{\mathrm{t}} \cong \varprojlim\{k_m^\times : m \in \mathbb{N}\}$ is pro-cyclic and $I_{\mathrm{t}}^p = I_{\mathrm{t}}$; see [Ser72, Props. 1 and 2].

A continuous character $\omega\colon I_{\mathrm{t}} \to (k^{\mathrm{alg}})^\times$ is called *of level m (with respect to k)* if m is the smallest integer such that ω factors as $\omega = \varphi \circ \omega_m$ for some homomorphism $\varphi\colon k_m^\times \to (\mathbb{F}_p^{\mathrm{alg}})^\times$; since I_{t} is pro-cyclic this is equivalent to ω having order a divisor of $q^m - 1$; in particular the number of such characters is finite. For any $m \geq 1$, let $\mathcal{P}_m := \mathrm{Hom}_k(k_m, \mathbb{F}_p^{\mathrm{alg}})$, and set $\omega_{m,\tau} := \tau \circ \omega_m$ for $\tau \in \mathcal{P}_m$. For any $\tau \in \mathcal{P}_m$ we have $\mathcal{P}_m = \{\tau^{q^i}] \mid i = 0, \ldots, m-1\}$. Moreover $\sigma \in G_K$ as fixed above satisfies $\sigma\tau\sigma^{-1} = \tau^q$.

If ω is of level dividing m, it can be written as $\omega = \omega^r_{m,\tau}$ for some $\tau \in \mathcal{P}_m$ and some $r \in \{1, \ldots, q^m - 2\}$. Call $r \in \{1, 2, \ldots, q^m - 2\}$ *primitive for m (and q)* if there is no proper divisor d of m such that r is a multiple of $(q^m - 1)/(q^d - 1)$; equivalently, r is primitive, if its base q expansion $r = [e_{m-1}e_{n-2} \ldots e_1 e_0]_q$, with digits $e_j \in \{0, \ldots, q-1\}$, is preserved under no cyclic digit permutation but the identity. Then the level m is minimal for $\omega = \omega^r_{m,\tau}$ if and only if r is primitive for m. In the latter case, the orbit of ω under conjugation by σ has exact length m.

To extend $\omega_{m,\tau}$ to G_{K_m}, recall that the local Artin map is an isomorphism $\widehat{K}^\times_m \xrightarrow{\sim} G^{\mathrm{ab}}_{K_m}$ that maps $\mathcal{O}^\times_{K_m}$ to the inertia subgroup of $G^{\mathrm{ab}}_{K_m}$; the latter surjects onto $I_{\mathrm{t}}/(I_{\mathrm{t}})^{q^m-1}$. The choice of ϖ gives an isomorphism $\widehat{K}^\times_m \cong \widehat{\mathbb{Z}} \times \mathcal{O}^\times_{K_m}$; it induces a homomorphism $\mathrm{pr}_2 \colon G_{K_m} \to I_{\mathrm{t}}/(I_{\mathrm{t}})^{q^m-1}$. We define

$$\hat{\omega}_{m,\tau} \colon G_{K_m} \xrightarrow{\mathrm{pr}_2} I_{\mathrm{t}}/(I_{\mathrm{t}})^{q^m-1} \xrightarrow{\omega_m} k^\times_m \xrightarrow{\tau} (k^{\mathrm{alg}})^\times.$$

Finally, for $\lambda \in (k^{\mathrm{alg}})^\times$ and a finite extension field $K' \supset K$, we write $\overline{\mu}_{K',\lambda} \colon G_{K'} \to (k^{\mathrm{alg}})^\times$ for the unramified character of $G_{K'}$ that sends a Frobenius automorphism to $\lambda^{-1} \in k^{\mathrm{alg}}$.

Proof of Lemma A.3.1. The proof of (a) is essentially that of [Ber10, Cor. 2.1.5] as extended in [Mul13, Prop. 2.1.1]. Note first that the last assertion is immediate from (a) and (b).

To prove (a), let $\overline{\rho} \colon G_K \to \mathrm{GL}_n(\mathbb{F}^{\mathrm{alg}})$ be irreducible. Then the wild ramification subgroup P_K of G_K acts trivially via $\bar{\rho}$: the group P_K is normal in G_K and a pro-p subgroup. If its action on $\mathbb{F}^n$ was not trivial, then the invariants $(\mathbb{F}^n)^{P_K}$ would be a non-trivial proper subrepresentation of G_K. But this is impossible, since $\bar{\rho}$ is irreducible.

We deduce that the restriction $\bar{\rho}|_{I_K}$ factors via I_{t}, and hence is a direct sum of 1-dimensional continuous characters of I_{t}. Fix one such character ω and write $\omega = \omega^r_{m,\tau}$ for m the minimal level of ω, some $\tau \in \mathcal{P}_m$ and $r \in \{1, \ldots, q^m - 2\}$ primitive, and let $\widehat{\omega} := \widehat{\omega}^r_{m,\tau}$. It follows that $0 \neq (\bar{\rho}|_{G_{K_m}} \otimes \widehat{\omega}^{-1})^{I_K}$, and hence we can find $\lambda' \in (k^{\mathrm{alg}})^\times$ such that $\bar{\mu}_{K_m,\lambda'} \otimes \widehat{\omega}$ is a subrepresentation of $\bar{\rho}|_{G_{K_m}}$. Let $\lambda \in k^{\mathrm{alg}}$ be such that $\lambda^m = \lambda'$ so that $\bar{\mu}_{K_m,\lambda'} = \bar{\mu}_{K,\lambda}|_{G_{K_m}}$. Then by Frobenius reciprocity

$$\mathrm{Ind}^{G_K}_{G_{K_m}}(\widehat{\omega} \otimes \bar{\mu}_{K_m,\lambda'}) \cong \left(\mathrm{Ind}^{G_K}_{G_{K_m}} \widehat{\omega}\right) \otimes \bar{\mu}_{K,\lambda}$$

admits a non-zero homomorphism to the irreducible representation $\bar{\rho}$. By the primitivity of r, the orbit of ω under conjugation by σ has length $m = [G_K : G_{K_m}]$, and it follows that $\mathrm{Ind}^{G_K}_{G_{K_m}} \widehat{\omega}$ is irreducible by the criterion of Mackey, see Lemma 2.1.4(e). This yields the isomorphism

$$\overline{\rho} \cong \left(\mathrm{Ind}^{G_K}_{G_{K_m}} \widehat{\omega}^r_{m,\tau}\right) \otimes \overline{\mu}_{K,\lambda},$$

and moreover that $m = n$, proving (a).

For (b) assume first that $\bar{\rho}$ is defined over $\mathbb{F}$. From our definitions and our hypothesis on $|\mathbb{F}|$ is it clear that $\bar{\rho}' := \mathrm{Ind}^{G_K}_{G_{K_m}} \widehat{\omega}^r_{m,\tau}$ is defined over $k_n \subset \mathbb{F}$. It follows that $\det \bar{\rho}'(\sigma), \det \bar{\rho}(\sigma) \in \mathbb{F}^\times$. Since $\det \bar{\rho}(\sigma) = \lambda^n \cdot \det \bar{\rho}'(\sigma)$, we deduce $\lambda^n \in \mathbb{F}$. For the converse, let $\lambda \in (\mathbb{F}^{\mathrm{alg}}_p)^\times$ satisfy $\lambda^n \in \mathbb{F}$. From Lemma 4.6.6 one deduces that the characteristic polynomial of any $\sigma \in G_K$ acting via $\bar{\rho}$ lies in $\mathbb{F}[t]$. It follows from the triviality of the Brauer group of a finite field and [CR62, Sect. 70] that the representation $\bar{\rho}$ can be defined over $\mathbb{F}$. ☐

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