

Cone theta functions and spherical polytopes with rational volumes

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Abstract

We study a class of polyhedral functions called cone theta functions, which are closely related to classical theta functions. Each polyhedral cone $\mathcal{K} \subset \mathbb{R}^d$ has an associated cone theta function, and we show that they encode information about the rationality of the spherical volume of K . We show that if K is a Weyl chamber for any finite Weyl group, then its cone theta function lies in a graded ring of classical theta functions and in this sense is ‘almost’ modular. Conversely, in the case that the spherical volume is irrational, it is natural to ask whether the cone theta functions are themselves modular, and we prove that in general they are not.

1 Introduction.

We study the relationship between volumes of spherical polytopes, and ‘almost’ modular cone theta functions associated to them, by considering some connections between the following two apriori different problems:

Problem 1. *Which lattice polyhedral cones K give rise to spherical polytopes with a rational volume?*

Problem 2. *Analyzing a certain cone theta function Φ_K attached to a polyhedral cone K , how ‘close’ is Φ_K to being modular?*

The present investigations arose from studying the volume of a spherical polytope, also known as a solid angle, and extending the so-called Gram relations of a Euclidean polytope by use of cone theta functions [Rob]. A strong motivation for this work comes from the rational simplex conjecture of Jeff Cheeger and James Simons [DuSa], who proposed the following conjecture at a Stanford conference in 1973: “Given a geodesic simplex in the spherical 3-space so that all of its interior dihedral angles are rational multiples of π , is it true that its volume is a rational multiple of the volume of the 3-sphere?” Although the answer is positive in all known examples, Cheeger and Simons conjectured that the answer should be “almost always” negative, and their conjecture remains widely open.

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In this paper, we associate certain graded rings of modular forms to each polyhedral cone, and consequently to each spherical polytope. The main idea is to translate the problem of rationality of a spherical polytope to the problem of an associated cone theta function being included in this ring.

We first recall some of the basic definitions from the combinatorial geometry of cones and the theory of modular forms, and then we combine ideas from the geometry of polyhedral cones with some modern analysis of theta functions. To begin, suppose we are given a d -dimensional (simple) **polyhedral cone**, defined by

$$K := \left\{ \sum_{j=1}^d \lambda_j w_j \mid \text{all } \lambda_j \geq 0 \right\},$$

where the edges of the cone are some fixed set of d linearly independent vectors $w_j \in \mathbb{R}^d$. Such a cone K is called a **pointed cone**, and in the present work every cone has the origin as its vertex. One important special case of a polyhedral cone is the **positive orthant**, defined by $K_0 := \{(x_1, \dots, x_d) \in \mathbb{R}^d \mid \text{each } x_j \geq 0\} := \mathbb{R}_{\geq 0}^d$. For each pointed cone K , and each full rank lattice $\mathcal{L} \subset \mathbb{R}^d$, we define its **cone theta function** by:

$$\Phi_{K,\mathcal{L}}(\tau) := \sum_{m \in \mathcal{L} \cap K} e^{\pi i \tau \|m\|^2}, \quad (1)$$

where τ lies in the complex upper half plane $\mathbb{H} := \{\tau := x + iy \mid x \in \mathbb{R}, y \in \mathbb{R}^+\} \subset \mathbb{C}$. If \mathcal{L} is clear from the context, we will only write Φ_K . The cone theta function $\Phi_{K,\mathcal{L}}(\tau)$ given in (1) is reminiscent of the modular theta function

$$\theta(\tau) := \sum_{n \in \mathbb{Z}} e^{\pi i \tau n^2}, \quad (2)$$

a statement that we will make more precise in what follows. The function $\theta(\tau)$ is a classical example of a **modular form**. Due to the fact that this paper combines several different fields of expertise, in particular discrete geometry and modular forms, we briefly outline the language of each field for the reader. Loosely speaking, a holomorphic modular form of integer weight k on a suitable subgroup $\Gamma \subseteq \text{SL}_2(\mathbb{Z})$ is any holomorphic function $f : \mathbb{H} \rightarrow \mathbb{C}$ satisfying $f(\gamma\tau) = (c\tau + d)^k f(\tau)$ for all $\gamma := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, as well as a suitable growth condition in the cusps of Γ . (See [Sh], e.g., for a more precise definition.) It is well known that the **modular group** $\text{SL}_2(\mathbb{Z})$ acts on \mathbb{H} by fractional linear transformation $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau := \frac{a\tau + b}{c\tau + d}$, and so modular forms can be thought of as complex analytic functions that obey a certain symmetry with respect to this action. Modular forms are also defined for half-integral weights k (indeed, the theta function $\theta(\tau)$ is of weight $1/2$), however the definition in this case is slightly more complicated due to the presence of a generalized Jacobi symbol. We point the interested reader to [Sh2] for more detail.

To see where the cone theta function (1) naturally comes from, we let S^{d-1} be the unit sphere centered at the origin. We define the **solid angle** ω_K at the vertex of K (which is the origin) by:

$$\omega_K := \frac{\text{vol}(K \cap S^{d-1})}{\text{vol}(S^{d-1})}. \quad (3)$$

In other words, ω_K is the normalized volume of a $(d-1)$ -dimensional spherical polytope. With this normalization, we note that $0 \leq \omega_K \leq 1$, and that in two dimensions we have $\omega_K = \theta/2\pi$, where θ is the usual 2-dimensional angle, measured in radians, at the vertex of the 2-dimensional cone K . It is an elementary fact that

$$\omega_K = \int_K e^{-\pi\|x\|^2} dx, \quad (4)$$

and for the sake of completeness we run through the proof of this equivalence here.

First, we change the integral (4) into spherical coordinates on S^{d-1} by writing $x = rs$, where $r = \|x\|$ and $s \in S^{d-1}$. The Jacobian of this change of variables is r^{d-1} , allowing us to write the Euclidean volume element as $dx = r^{d-1} dr dS$ with dS being the $d-1$ -dimensional spherical volume element on S^{d-1} . It is well known that $\int_{\mathbb{R}^d} e^{-\pi\|x\|^2} dx = 1$, so that the integral (4) now becomes

$$\omega_K = \frac{\int_K e^{-\pi\|x\|^2} dx}{\int_{\mathbb{R}^d} e^{-\pi\|x\|^2} dx} = \frac{\int_0^\infty e^{-\pi r^2} r^{d-1} dr \int_{S^{d-1} \cap K} dS}{\int_0^\infty e^{-\pi r^2} r^{d-1} dr \int_{S^{d-1}} dS} = \frac{\int_{S^{d-1} \cap K} dS}{\int_{S^{d-1}} dS} = \frac{\text{vol}(K \cap S^{d-1})}{\text{vol}(S^{d-1})}.$$

We note that, as in the argument above, when K is replaced by all of Euclidean space, this integral (4) becomes $\int_{\mathbb{R}^d} e^{-\pi\|x\|^2} dx = 1$, confirming that we do indeed have the proper normalization $0 \leq \omega_K \leq 1$. For more information about rational pointed cones, and connections between discrete volumes of polytopes and local spherical angle contributions, the reader may consult [Sta] and [DeRo]. The papers [McM1], [McM2], [PeSh], provide further background for solid angles and their relations.

Thus, the foregoing discussion shows that a strong motivation for defining the cone theta function $\Phi_K(\tau)$ is that it is essentially a discrete, Riemann sum approximation to the integral definition of the volume ω_K of a spherical polytope, as defined by (4). We will make precise sense of this intuition in section 2, where Lemma (7) is proved, and which will be used later to consider carefully the putative expansion of $\Phi_K(\tau)$ at the cusp $\tau = 0$.

We define R to be the ring of all finite, rational linear combinations of theta functions $\Theta_{\mathcal{L}}$, for any d -dimensional even integral lattice $\mathcal{L} \subset \mathbb{R}^d$, varying over all dimensions d . The ring R has a natural grading, namely it is graded by the weight $k = \frac{d}{2}$ of the relevant theta functions $\Theta_{\mathcal{L}}$, for each rank d lattice $\mathcal{L} \subset \mathbb{R}^d$. Equivalently, we may also grade R by the dimension d of the lattices $\mathcal{L} \subset \mathbb{R}^d$, as d varies over the positive integers.

Theorem 3. *If the polyhedral cone K is the Weyl chamber of a finite reflection group W , then the cone theta function $\Phi_{K, 2\mathcal{L}_{root}}(\tau)$ is in the graded ring R .*

The spirit of this result is that enough symmetry of the integer cone K will be reflected in some functional relations between the associated cone theta functions Φ_{K, \mathcal{L}_j} , for various j -dimensional lattices \mathcal{L}_j which lie on the boundaries of $K \cap \mathcal{L}$. It is in this sense that Φ_K is ‘almost modular’ - it is a linear combination of modular forms of different weights. We note that the definition we use here for the term “almost modular form” is different from the definition of these words in [BGHZ].

However, our feeling is that no amount of symmetry of K can ever allow the cone theta function to lie in a single grading of this ring R , namely to be a modular form. Theorem 4 and Theorem 5 offer partial solutions to this set of queries.

Theorem 4. *Suppose that the polyhedral cone $K \subset \mathbb{R}^d$ has the solid angle ω_K at its vertex, located at the origin, and that $\mathcal{L} := A(\mathbb{Z}^d)$ is an even integral lattice of full rank. If $\frac{\omega_K}{|\det A|}$ is irrational, then $\Phi_{K,\mathcal{L}}(\tau)$ is not a modular form of weight k on any congruence subgroup, and for any $k \in \frac{1}{2}\mathbb{Z}$, $k \geq \frac{1}{2}$.*

Theorem 5. *Suppose we are given an integer cone $K \subset \mathbb{R}^2$, with integer edge vectors $w_1, w_2 \in \mathbb{Z}^2$. Then $\Phi_{K,\mathbb{Z}^2}(\tau)$ is not a modular form of weight 1 for any congruence subgroup.*

The point of Theorem 5 is that, in dimension 2, we can give a finite list of integer cones that have an irrational angle, using standard Galois theory, and outside this finite list every integer cone must have an irrational angle, allowing Theorem 4 to kick into effect. Whether such a list (finite or infinite) can be easily described in higher dimensions remains a fascinating open question.

Cone theta functions are also related to the representation numbers of quadratic forms, a link which we explicate here. We use a lattice $\mathcal{L} := A(\mathbb{Z}^d)$, where A is a d by d integer matrix. From the above definitions, it is immediate that

$$\Phi_{K,\mathcal{L}}(\tau) = \sum_{m \in \mathcal{L} \cap K} e^{\pi i \tau \|m\|^2} = \sum_{m \in \mathbb{Z}^d \cap K} e^{\pi i \tau (m^t (A^t A) m)} = \sum_{k=0}^{\infty} a(k) q^{k/2}, \quad (5)$$

where $q = e^{2\pi i \tau}$, and $a(k) := \#\{m \in \mathbb{Z}^d \mid m^t (A^t A) m = k, \text{ and } m \in K\}$. This combinatorial interpretation of the Fourier coefficients tells us that the k 'th Fourier coefficient is the number of ways to represent the integer k by the quadratic form $m^t (A^t A) m$, while m is simultaneously constrained to satisfy the finite system of linear inequalities defined by the cone K .

To put the q -series above in a more general context, any combinatorial q -series may be thought of as a series of the form $\sum_k \alpha(k) q^k$, where the coefficients $\alpha(k)$ enumerate certain combinatorial structures. For example, let $p(n) := \#\{\text{integer partitions of } n\}$, and $b(n) := p(n \mid \text{even rank}) - p(n \mid \text{odd rank})$, where the rank of a partition is defined to be the largest part of the partition minus the number of parts. Consider the two generating functions, both of which are well known:

$$P(q) := \sum_{n \geq 0} p(n) q^n = \sum_{n \geq 0} \frac{q^{n^2}}{(q; q)_n^2}, \quad f(q) := \sum_{n \geq 0} b(n) q^n = \sum_{n \geq 0} \frac{q^{n^2}}{(-q; q)_n^2}, \quad (6)$$

where the q -Pochhammer symbol is defined for $n \in \mathbb{N}$ by $(a; q)_n := (1 - a)(1 - aq) \cdots (1 - aq^{n-1})$, and $(a; q)_0 := 1$. The functions $P(q)$ and $f(q)$ both admit a combinatorial interpretation, and exhibit a very similar q -series expansion. However, their subtle differences are enough to disrupt modular properties: the function $q^{-1/24} P(q)$ is modular (when $q = e^{2\pi i \tau}$, $\tau \in \mathbb{H}$), while $f(q)$ is an example of one of Ramanujan's "mock" theta functions [Zwe]. In general, the question of when a q -series, a priori combinatorial or not, is modular, continues to be an actively researched area, and additionally inspires our investigation of the modular properties of cone theta functions. For example, recent work of Zagier [Zag1] and Vlasenko-Zwegers [VZ] examined the modular properties of a general family of q -series as conjectured by Nahm [Nah]. Another very recent analysis of cones from a different perspective takes place in [GPZ]. The authors of [GPZ] define certain zeta functions attached

to polyhedral cones and analyze conical zeta values as a geometric generalization of the celebrated multiple zeta values.

In order to give the reader a more concrete feeling for some cone theta functions, we end this section with the simplest family of examples of cone theta functions, arising from the positive orthant in each dimension.

Example 6. For the cone theta function of the positive orthant $K_0 := \mathbb{R}_{\geq 0}^d$ (and $\mathcal{L} = \mathbb{Z}^d$), we claim that

$$\Phi_{K_0}(\tau) = \frac{1}{2^d} (\theta(\tau) + 1)^d, \quad (7)$$

where $\theta(\tau) := \sum_{n \in \mathbb{Z}} e^{\pi i \tau n^2}$, the classical weight $1/2$ modular form. In particular,

$$\Phi_{K_0}(\tau) = \frac{1}{2^d} \sum_{k=0}^d \binom{d}{k} \theta^k(\tau), \quad (8)$$

a linear combination of modular forms of distinct weights, with nonzero coefficients, and hence Φ_{K_0} is not a modular form.

To see (8), we begin with the case in which the dimension of K_0 equals 1, so that here $K_0 := \mathbb{R}_{\geq 0}$. We note that $1 + \theta(\tau) = 1 + \sum_{n \in \mathbb{Z}} e^{\pi i \tau n^2} = 2 + 2 \sum_{n \geq 1} e^{\pi i \tau n^2}$. Therefore, $1 + \theta(\tau) = 2\Phi_{\mathbb{R}_{\geq 0}}(\tau)$. Finally, the relation $\Phi_{K_0}(\tau) = \Phi_{\mathbb{R}_{\geq 0}}^d(\tau)$ gives us the desired expansion (7) above. \square

We notice that the positive orthant possesses a lot of symmetry, so it is natural to ask if other cones with less symmetry might not be modular, and for which cones K might we get a phenomenon similar in spirit to the example above, in the sense that Φ_K may be written as a linear combination of “classical” theta functions attached to lower-dimensional lattices. An answer to such a query is given precisely by Theorem 3.

2 The solid angle ω_K

The defining integral (4) for the volume of a spherical polytope, namely ω_K , has a nice discretization which is essentially the cone theta function $\Phi_{K, \mathcal{L}}(it)$, for $t > 0$. We now make this intuition precise. Throughout, we use a full rank lattice defined by $\mathcal{L} = A(\mathbb{Z}^d)$, for some $A \in GL_d(\mathbb{R})$. We follow the standard convention of denoting the volume of a fundamental parallelepiped of \mathcal{L} by $|\det(A)|$.

Lemma 7. *Let $\mathcal{L} \subset \mathbb{R}^d$ be any full rank lattice. Then*

$$t^{\frac{d}{2}} \Phi_{K, \mathcal{L}}(it) \sim \frac{\omega_K}{|\det A|},$$

as $t \rightarrow 0^+$.

Proof. We first note that

$$\begin{aligned} \Phi_{K, \mathcal{L}}(\tau) &= \sum_{n \in A(\mathbb{Z}^d) \cap K} e^{\pi i \|n\|^2 \tau} \\ &= \sum_{n \in \mathbb{Z}^d \cap A^{-1}(K)} e^{\pi i \|An\|^2 \tau}. \end{aligned}$$

We let $f(x) := e^{-\pi\|Ax\|^2}$ and consider the Riemann sum definition of its integral:

$$\int_{A^{-1}(K)} f(x)dx = \lim_{\Delta x \rightarrow 0} \sum_{n \in \mathbb{Z}^d \cap A^{-1}(K)} (\Delta x)^d f(n \Delta x). \quad (9)$$

Thus $(\Delta x)^d$ is the d -dimensional volume of a small cube of side length Δx . These cubes intersect in sets of measure zero only, and they cover $\mathbb{Z}^d \cap A^{-1}(K)$. Letting $\Delta x = t^{\frac{1}{2}}$, we obtain

$$\begin{aligned} \int_{A^{-1}(K)} f(x)dx &= \lim_{t \rightarrow 0^+} \sum_{n \in \mathbb{Z}^d \cap A^{-1}(K)} t^{\frac{d}{2}} f(t^{\frac{1}{2}}n) \\ &= \lim_{t \rightarrow 0^+} \sum_{n \in \mathbb{Z}^d \cap A^{-1}(K)} t^{\frac{d}{2}} e^{-\pi t \|An\|^2} \\ &= \lim_{t \rightarrow 0^+} t^{\frac{d}{2}} \Phi_{K, \mathcal{L}}(it). \end{aligned}$$

On the other hand, letting $y = Ax$ in the integral defined by 9 and observing that the Jacobian is $\det A$, we find that

$$\begin{aligned} \int_{A^{-1}(K)} f(x)dx &= \frac{1}{|\det A|} \int_K e^{-\pi\|y\|^2} dy \\ &= \frac{\omega_K}{|\det A|}. \end{aligned}$$

□

3 The cone theta function $\Phi_K(\tau)$ is ‘almost’ modular when K is a Weyl chamber

It is interesting to note that in the usual arithmetic theory of modular forms, it does not in general appear natural to combine forms of different weights in the same equation, because we usually grade forms by weight. However, as we already see from the example above, it is quite natural from a combinatorial perspective to combine theta functions of different weights in the same equation, due to the structure of polyhedral cones.

Let $Q(x) := x^t(A^t A)x$ be a positive definite quadratic form, arising from any real matrix $A \in GL_d(\mathbb{R})$. We recall that the Gram matrix $A^t A$ is defined to be **even integral** if all of its diagonal elements lie in $2\mathbb{Z}$ and all of its off-diagonal elements lie in \mathbb{Z} . In this case, we call the associated lattice $\mathcal{L} := A(\mathbb{Z}^d)$ an **even integral lattice**. If we are given any basis $\{e_j\}$ of \mathcal{L} , we note that according to these definitions we have $\langle e_i, e_j \rangle \in \mathbb{Z}$ when $i \neq j$, and $\langle e_i, e_i \rangle \in 2\mathbb{Z}$, for all basis vectors e_i, e_j . This justifies the motivation of our definition for an even integral lattice, and there is an even stronger motivation for such lattices, arising from their modular properties, which we now recall.

For each even integral lattice \mathcal{L} , we define its usual theta function by:

$$\Theta_{\mathcal{L}}(\tau) := \sum_{n \in \mathcal{L}} e^{\pi i \tau \|n\|^2},$$

where τ lies in the upper half plane H .

We quote the standard fact that when \mathcal{L} is an even integral lattice, the theta function $\Theta_{\mathcal{L}}(\tau)$ turns out to be a modular form, of weight $\frac{d}{2}$ and level N , where N is the smallest positive integer M such that $M(A^t A)^{-1}$ is also even integral. It is also a theorem that the level N divides $|\det(A)|$ (See [Sch], Chapter 9, for this fact in a more general context).

We now recall briefly the definition of a **root system** and some of its properties, which we need in our context, and then define their corresponding theta functions, using the definitions above. The interested reader may consult the book by [GrBe], for example, for more information on finite root systems. Let S be a finite set of nonzero vectors in \mathbb{R}^d , called **roots**, and for each $\alpha \in S$, define the linear transformation $s_{\alpha} : \mathbb{R}^d \rightarrow \mathbb{R}^d$ by

$$s_{\alpha}(x) := x - 2 \frac{\langle x, \alpha \rangle}{\langle \alpha, \alpha \rangle} \alpha.$$

This is a reflection associated to each α in S , about the hyperplane orthogonal to α . Suppose that for each $\alpha \in S$ we have $s_{\alpha}(S) = S$, and also suppose that for all $\alpha, \beta \in S$ we have $2 \frac{\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}$. Then S is called a finite **root system**.

The finite collection of reflections s_{α} generate, by composition, a finite group W which is called a finite **Weyl group**. The collection of hyperplanes orthogonal to the roots decompose \mathbb{R}^d into convex chambers, called **Weyl chambers**. It is a theorem that the Weyl group acts transitively on these chambers, so that any chamber is a fundamental domain for the action of W on \mathbb{R}^d . In particular, each Weyl chamber is a polyhedral cone, whose edges are some roots of S .

When $K \subset \mathbb{R}^d$ is a Weyl chamber of any finite Weyl group W , we define the **root lattice** \mathcal{L}_{root} as the integer span of all the roots of S , and it is a theorem that they do indeed form a rank d lattice.

Throughout, we let \mathcal{L}_{root} be a root lattice for any of the standard finite Weyl groups A_n, B_n, C_n, D_n , or BC_n , for any $n \geq 1$. We notice that for each of these finite Weyl groups, there is a representation of the root lattice in Euclidean space, for which all of the basis vectors $\{e_j\}$ of \mathcal{L}_{root} are integer vectors. Hence $\langle e_i, e_i \rangle \in \mathbb{Z}$ for each basis vector e_i of \mathcal{L}_{root} . Moreover, from the definition of a finite Weyl group given above, we see that $2\langle e_i, e_j \rangle \in \langle e_i, e_i \rangle \mathbb{Z} \subset \mathbb{Z}$. Thus, for each pair of basis vectors e_i, e_j of \mathcal{L}_{root} , we conclude that $\langle e_i, e_j \rangle \in \frac{1}{2}\mathbb{Z}$, showing that $2\mathcal{L}_{root}$ is an even integral lattice. Thus, we may finally conclude from the foregoing discussion that

$$\Theta_{2\mathcal{L}_{root}}(\tau) \text{ is always a modular form,} \tag{10}$$

a fact we will require in the proof of Theorem 3.

We also note that, almost by definition, every cone K which is a Weyl chamber necessarily has a *rational* solid angle $\omega_K = \frac{1}{|W|}$, because the Weyl group W tiles \mathbb{R}^d with isometric copies of the cone K .

Example 8. Consider the 2-dimensional root system defined by

$$S := \{(1, 1), (-1, 1), (1, -1), (-1, -1), (2, 0), (-2, 0), (0, 2), (0, -2)\},$$

so that we have the root lattice $\mathcal{L}_{root} := \{m(1, 1) + n(2, 0) \mid m, n \in \mathbb{Z}\}$. Here the root lattice L_{root} is already an even integral lattice. The finite Weyl group W here consists of 8 elements, and this root system is known as C_2 . One fundamental domain for this group action on \mathbb{R}^2 is the polyhedral cone K whose edge vectors are the roots $(1, 1)$ and $(2, 0)$, and whose (solid) angle is $\omega = \frac{1}{8}$. Here, the cone theta function is

$$\Phi_{K, \mathcal{L}_{root}}(\tau) = \sum_{m \geq 0, n \geq 0} e^{\pi i \tau \|m(2,0) + n(1,1)\|^2} \quad (11)$$

$$= \sum_{m \geq 0, n \geq 0} e^{\pi i \tau (4m^2 + 4mn + 2n^2)}. \quad (12)$$

We note that for this root system, the reflection $s_{(1,1)}(K)$ is the cone K_2 defined by the non-negative real span of the vectors $(1, 1)$ and $(0, 2)$. As noted earlier, we also have $\Phi_{K_2, \mathcal{L}_{root}}(\tau) = \Phi_{K, \mathcal{L}_{root}}(\tau)$, so that, modulo the intersection of these two cones K and K_2 , their union is the positive orthant $\mathbb{R}_{\geq 0}^2$. Gluing together all 8 copies of K , and taking all of their one-dimensional intersections into account, we see that

$$8\Phi_{K, \mathcal{L}_{root}}(\tau) - 4 \sum_{k \geq 0} e^{\pi i \tau (2k^2)} - 4 \sum_{k \geq 0} e^{\pi i \tau (4k^2)} + 1 = \sum_{(m,n) \in \mathcal{L}_{root}} e^{\pi i \tau (m^2 + n^2)}.$$

Thus, we arrive at the following representation of $\Phi_{K, \mathcal{L}_{root}}(\tau)$ as a nontrivial rational linear combination of classical theta functions:

$$\Phi_{K, \mathcal{L}_{root}}(\tau) = \frac{3}{8} + \frac{1}{4} \sum_{k \in \mathbb{Z}} e^{\pi i \tau (2k^2)} + \frac{1}{4} \sum_{k \in \mathbb{Z}} e^{\pi i \tau (4k^2)} + \frac{1}{8} \sum_{(m,n) \in \mathbb{Z}^2} e^{\pi i \tau (4m^2 + 4mn + n^2)}. \quad (13)$$

Therefore we see that for this example, $\Phi_{K, \mathcal{L}_{root}}(\tau)$ lies in the ring R , as a nontrivial linear combination of theta functions of different weights. In particular, $\Phi_{K, \mathcal{L}_{root}}(\tau)$ is not modular. \square

We now give the proof of Theorem 3.

Proof of Theorem 3. We proceed by induction on the dimension of K . By definition of a finite Weyl group, there is a finite collection of roots, each of which acts as a reflection about a hyperplane passing through a facet of K , also called a wall of K . These reflections act on \mathbb{R}^d by reflecting K , and subsequently tiling all of \mathbb{R}^d with copies of K . The only difficulty is that due to intersections along the boundary of K , we have to consider carefully how many times we have overcounted each j -dimensional face of K in the process of tiling \mathbb{R}^d with the reflected images of K .

A key element of the proof is the fact that each reflection about a wall of K happens to be an involution of the root lattice \mathcal{L}_{root} . Thus, the reflection of a root lattice point $m \in \mathcal{L}_{root}$ about a wall of K gives us another root lattice point (of the same norm), and therefore the corresponding theta function of any reflected copy of K is again equal to $\Phi_K(\tau)$. If we ignore the boundary issues, then we see that therefore $\Phi_K(\tau)$ times the order of the Weyl group equals $\sum_{n \in 2\mathcal{L}_{root}} e^{\pi i \tau \|n\|^2}$, a classical theta function of weight $k = \frac{d}{2}$, by the remark 10 above.

However, this last identity is not quite true, due to the boundary effects that occur when we reflect K and paste together all of the reflected images of K .

We can, however, correct for the intersections of the reflected images of K along the boundary of K , by invoking the inclusion-exclusion principle. From the definition of a root system, we also know that any subspace of \mathbb{R}^d intersects the root system in a lower-dimensional root system, so that we may conclude that each wall of K is a $(d-1)$ -dimensional chamber for some $(d-1)$ -dimensional finite Weyl group. Proceeding with this inclusion-exclusion process to the next, $(d-2)$ -dimensional faces of K , we can add and subtract various lower-dimensional Weyl chambers, and we know by the induction hypothesis that each of them gives us a corresponding cone theta function that already lies in R . \square

4 The q -expansion principle and the non-modularity of the cone theta function $\Phi_{K,\mathcal{L}}$

The “ q -expansion principle”, due to Deligne and Rapoport [DeRa, Théorème 3.9, p.304], tells us that if an integer weight modular form f has *rational* Fourier coefficients at the cusp $i\infty$, then the Fourier expansion of f at all other cusps must also have rational coefficients. In general, a “generic” polyhedral cone K should intuitively have an irrational solid angle ω at its vertex. We therefore work under this generic assumption of an irrational solid angle, and we will obtain a nice contradiction with the q -expansion principle.

Although it is straightforward, given the q -expansion principle, to prove that Φ_K is not a modular form of weight $\frac{d}{2}$, it appears to be a much more subtle question of whether it may be a form of higher weight. To handle all possible weights, we need a preliminary Lemma.

Lemma 9. *Suppose that $\Phi_{K,\mathcal{L}}(\tau)$ is a modular form on $\Gamma_0(N)$ for some N , with Dirichlet character χ , and of (integral or half integral) weight k . Then necessarily $k = \frac{d}{2}$.*

Proof. We have, by Lemma 7:

$$\lim_{t \rightarrow 0} t^{\frac{d}{2}} \Phi_{K,\mathcal{L}}(it) = \frac{\omega_K}{|\det A|} > 0, \quad (14)$$

where $\mathcal{L} := A(\mathbb{Z}^d)$. Since $\Phi_{K,\mathcal{L}}$ is by hypothesis modular of weight k , it must be holomorphic at the cusp 0, so in particular

$$b_0 := \lim_{t \rightarrow 0} t^k \Phi_{K,\mathcal{L}}(it) \quad (15)$$

exists and is finite.

Case 1. If $k < \frac{d}{2}$, then by (14) we conclude that $b_0 = \infty$, a contradiction.

Case 2. Now suppose that $k > \frac{d}{2}$. The remainder of the proof is devoted to deriving a contradiction for this case. From (14) we deduce that

$$b_0 = 0. \quad (16)$$

We choose a large even positive integer h such that $F := \Phi_{K,\mathcal{L}}^h$ has trivial character and is of even integral weight $k' := kh \geq 2$. We then have, by (16), that

$$c_0 := \lim_{t \rightarrow 0} t^{k'} F(it) = 0. \quad (17)$$

Note that c_0 (up to a non-zero constant) is the constant term of the modular form

$$F|_{k'}W_N(\tau) := N^{-k'/2}\tau^{-k'}F(-1/N\tau),$$

where $W_N := \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$ is the Fricke involution. Let us write $F(z) = \sum_{n \geq 0} a_n e^{\pi i n z}$ and observe that $a_0 = 1$ because K is a *closed* polyhedral cone. We denote by $L_F(s) = \sum_{n \geq 1} a_n n^{-s}$ ($\operatorname{Re}(s) \gg 1$), the Hecke L-series of F . By a classical result of Hecke (see [Ogg]), we know that $L_F(s)$ has meromorphic continuation to all of \mathbb{C} and is holomorphic except for a possible simple pole at $s = k'$ with residue equal to $c_0 = 0$.

On the other hand, $L_F(s)$ is a Dirichlet series with non-negative coefficients, hence by a well-known theorem of Landau must converge up to the first singularity, i.e. must converge for all $s \in \mathbb{C}$. In particular it follows that $a_n = O(n^c)$ for any $c \leq (k-1)/2$. It was proven in [Koh] and [Scm] that if the Fourier coefficients of a modular form F on $\Gamma_0(N)$ of even integral weight greater than or equal to 2 satisfy Deligne's bound $O_\epsilon \left(n^{\frac{k-1}{2} + \epsilon} \right)$ for any $\epsilon > 0$, then F must already be a cusp form. Thus we deduce that F is cuspidal, so we conclude that $a_0 = 0$, a contradiction. \square

Proof of Theorem 4. Suppose first that $d \geq 3$, and $\Phi_{K,\mathcal{L}}(\tau)$ is a modular form of integer weight $k \geq 2$ on a congruence subgroup. All of the coefficients of $\Phi_{K,\mathcal{L}}(\tau)$ at $i\infty$ are, by definition, rational numbers. Then by Lemma 9 we must have that the weight of $\Phi_{K,\mathcal{L}}(\tau)$ equals $\frac{d}{2}$. If we now consider the expansion of $\Phi_{K,\mathcal{L}}$ at the cusp 0, we have by Lemma 7 that the constant term is $\lim_{t \rightarrow 0} t^{d/2} \Phi_K(it) = \frac{\omega_K}{|\det A|}$. Because the constant term $\frac{\omega_K}{|\det A|}$ is an irrational real number, we obtain a contradiction, by the q -expansion principle.

The remaining cases to consider are the cases in which $d = 3$ and $\Phi_{K,\mathcal{L}}(\tau)$ is of half integer weight $k \geq 3/2$, and the case in which $d = 1$ or 2, so that $\Phi_{K,\mathcal{L}}(\tau)$ is of weight $k \in \frac{1}{2}\mathbb{Z}$, $k \geq 1/2$. Non-modularity follows exactly as in the argument given above, after considering the following remarks.

Remarks.

- i) In addition to [DeRa], for the q -expansion principle, in particular the half-integral weight case $\geq 3/2$, we refer to [Bru].
- ii) We note that the q -expansion principle is also valid in weight $1/2$ resp. weight 1. Indeed, we may just multiply with a normalized Eisenstein series of weight ≥ 4 of level 1 (which has rational Fourier coefficients) to immediately deduce the result from the result in the higher weight case.

\square

We remark that the hypothesis of Theorem 4 above is satisfied, for example, if \mathcal{L} is an even integral lattice, and ω_K is not a quadratic irrational.

5 Integer polyhedral cones in \mathbb{R}^2 and their cone theta functions

We exhibit a concrete class of cone theta functions in \mathbb{R}^2 , which in general do indeed have irrational solid angles, so that Theorem 4 applies to them. Namely, we consider cones in \mathbb{R}^2 which have integer edge vectors, called **integer cones**. First, we prove a diophantine-type Lemma concerning these cones, which we will make use of in the main theorem of this section.

Lemma 10.

i). Suppose we are given two integers m and n with $n \neq 0$ and even, and m odd. Then the equation

$$\frac{m}{n} = \frac{\langle w_1, w_2 \rangle}{\|w_1\| \|w_2\|} \quad (18)$$

has no solutions in non-zero integer vectors $w_1, w_2 \in \mathbb{Z}^2$.

ii). Suppose that m and n are integers with $n \neq 0$ and m odd. Then the equation

$$\frac{m}{2\sqrt{n}} = \frac{\langle w_1, w_2 \rangle}{\|w_1\| \|w_2\|} \quad (19)$$

has no solutions in non-zero integer vectors $w_1, w_2 \in \mathbb{Z}^2$.

Proof. i). Suppose that $w_1 = (a, b) \in \mathbb{Z}^2$ and $w_2 = (f, g) \in \mathbb{Z}^2$ are a solution to (18). Removing the greatest common divisors, we may assume without loss of generality that w_1 and w_2 are primitive vectors, i.e. $\gcd(a, b) = \gcd(f, g) = 1$. From (18) we get

$$m^2(a^2 + b^2)(f^2 + g^2) = n^2(af + bg)^2. \quad (20)$$

We know m is odd and n is even, so we have

$$4 \mid (a^2 + b^2)(f^2 + g^2). \quad (21)$$

If $4 \mid a^2 + b^2$, then we must have either $2 \mid a$ and $2 \mid b$, or else $4 \mid a$ and $4 \mid b$. In either case, this conclusion contradicts our assumption that a and b are coprime. The only possibility left, by (21), is that $2 \mid a^2 + b^2$ and $2 \mid f^2 + g^2$. In this case, we must have both a and b odd, and both f and g odd. But then $af + bg$ is even, so that we have $4 \mid (af + bg)^2$ and hence by 20 (using n even), we get $8 \mid (a^2 + b^2)(f^2 + g^2)$, which we already know is a contradiction.

ii). Suppose we do have a solution, which again we may assume to consist of primitive vectors. Then from (19) we find

$$m^2(a^2 + b^2)(f^2 + g^2) = 4n(af + bg)^2. \quad (22)$$

Since m is odd we again conclude that $4 \mid (a^2 + b^2)(f^2 + g^2)$, and the same argument as in case i) again gives us a contradiction. \square

Proof of Theorem 5. We first note that the solid angle ω_K of K in this case may be expressed as $\omega_K = \vartheta/2\pi$, where ϑ is a usual 2-dimensional angle. For ‘most’ K , we will argue as in the proof of Theorem 4 using the q -expansion principle, hence will show that $\vartheta \notin 2\pi\mathbb{Q}$ (hence $\omega_K \notin \mathbb{Q}$). For some K however, $\vartheta \in 2\pi\mathbb{Q}$ and we are not able to use the q -expansion principle. In these cases we shall use Lemma 10 and a similar argument as in Example 6, respectively. We begin by addressing these exceptional cases.

Case 1. Suppose $K \subset \mathbb{R}^2$ gives rise to $\vartheta/2\pi$ in the exceptional set

$$\mathbf{E} := \left\{ c_0, \frac{c_1}{2}, \frac{c_2}{4}, \frac{c_3}{8} \mid c_j \in \mathbb{Z}, 0 \leq j \leq 3, \text{ and } c_1, c_2, c_3 \text{ odd} \right\}.$$

We will consider three separate cases (i-iii) below.

i. ($\vartheta = 2\pi c_0$ or $\vartheta = 2\pi c_1/2, c_1$ odd) We are able to immediately dismiss these cases, as they imply that w_1 and w_2 are integer multiples of one another, hence, are not linearly independent.

ii. ($\vartheta = 2\pi c_2/4, c_2$ odd) In this case, w_1 and w_2 are orthogonal. The case in which w_1 and w_2 span the first quadrant do not give rise to a modular form as was discussed in Example 6. In more generality, suppose $w_1 = (a, b), w_2 = r(-b, a)$ where r is a positive integer, and $a, b \in \mathbb{Z}$ satisfy (without loss of generality) $\gcd(a, b) = 1$. In this case the cone theta function is equal to $\Phi_0(Nz)\Phi_0(Nrz)$, where Φ_0 is the cone function attached to the positive orthant in $d = 1$ and $N = a^2 + b^2$. As in Example 6, this cone theta function is in the graded ring, but is not modular.

iii. ($\vartheta = 2\pi c_3/8, c_3$ odd) For the case in which $c_3/8 = 1/8$, the cone K forms a Weyl chamber for the root system BC_2 , of Example 8, and as we showed in that case, the cone theta function Φ_K is not modular because it is explicitly exhibited as a nontrivial linear combination of theta functions of different weights. For the case in which $c_3/8 = 3/8$, the relevant cone K consists of three contiguous copies of a fundamental domain of the root system BC_2 , and is again not modular for the same reason as the case $c/d = 1/8$. The general situation in which $\vartheta = 2\pi c_3/8$ with c_3 odd follows similarly.

Case 2. Next we assume that $K \subset \mathbb{R}^2$ gives rise to $\vartheta/2\pi = c/d \in \mathbb{Q} \setminus \mathbf{E}$. We assume $d > 0$, for if $d < 0$ we write $c/d = (-c)/(-d)$ with $-d > 0$. We also use the 2-dimensional inner product to write

$$\vartheta = \cos^{-1} \left(\frac{\langle w_1, w_2 \rangle}{\|w_1\| \|w_2\|} \right),$$

where without loss of generality we take $\cos^{-1} : [-1, 1] \rightarrow [0, \pi]$, as \cos is ± 1 periodic with period π . Thus we have that $\vartheta = \cos^{-1}(n/\sqrt{m})$, with $n, m \in \mathbb{Z}, m > 0$, and also that $0 \leq c \leq d/2, d > 0$. We may further assume $c > 0$, for $0 \in \mathbf{E}$. Note that by definition of \mathbf{E} we must only consider $d \geq 3, d \neq 4, d \neq 8$.

First, we consider those $d \geq 3$ that are in the set

$$\mathbf{F} := \{3, 5, 6, 10, 12\}.$$

Without loss of generality, we may assume that for such $d \in \mathbf{F}$ and c satisfying $0 < c \leq d/2$, we have that $\gcd(c, d) = 1$. This follows from the fact that any c'/d' with $d' \in \mathbf{F}$ and $0 < c' \leq d'/2$ with $\gcd(c', d') > 1$ may be written as c/d where $c = c'/\gcd(c', d')$ and $d = d'/\gcd(c', d')$ so that $\gcd(c, d) = 1$. Moreover, because $0 < c' \leq d'/2$, we have $0 < c \leq d/2$, and it is easy to verify that either $d \in \mathbf{F}$ or $d = 2, 4$, the latter of which has been addressed in Case 1 above.

i. If $d = 5$ we must have $c = 1$, and if $d = 10$, we must have $c = 1$ or $c = 3$. For such c/d , we have $\cos(2\pi c/d) = \pm(\sqrt{5} - 1)/4$ or $(\sqrt{5} + 1)/4$, and these values are not of the form n/\sqrt{m} ($m > 0$).

ii. If $d = 3$ and $d = 6$ we must have $c = 1$, so that in these two cases, $\cos(2\pi c/d) = \pm 1/2$. By Lemma 10 i) we see there are no non-zero integer vector solutions w_1, w_2 .

iii. If $d = 12$, the acceptable values of c are $c = 1, 5$, and we have that $\cos(2\pi c/12) = \pm 3/(2\sqrt{3})$. By Lemma 10 ii), again we see there are no non-zero integer vector solutions w_1, w_2 .

Finally, we consider all integers $d \geq 3$, $d \notin \mathbf{F} \cup \{4, 8\}$. For $d \notin \mathbf{F} \cup \{4, 8\}$ and corresponding $0 < c \leq d/2$ with $\gcd(c, d) = 1$, cyclotomic theory shows that

$$\left[\mathbb{Q} \left(\frac{1}{2} (\zeta_d^c + \zeta_d^{-c}) \right) : \mathbb{Q} \right] = \frac{\varphi(d)}{2} > 2,$$

where $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ denotes Euler's φ function, and $\zeta_d := e^{2\pi i/d}$. Thus, for such d we have

$$\cos \left(\frac{2\pi c}{d} \right) = \frac{1}{2} (\zeta_d^c + \zeta_d^{-c}) \notin \mathbb{Q}(\sqrt{a})$$

for any non-negative integer a , hence $\vartheta = \cos^{-1}(n/\sqrt{m}) \neq 2\pi c/d$.

This exhausts all cases, and we now argue as in the proof of Theorem 4 with the q -expansion principle, using the irrationality of ω_K . The statement of the theorem now follows. \square

6 Open problems

Problem 3. What are the necessary and sufficient conditions on the geometry of the cones K whose cone theta function belongs to the graded ring R ?

Problem 4. For the case that $\frac{\omega_K}{|\det A|} \in \mathbb{Q}$, we don't yet have any proofs of non-modularity for $\Phi_{K, \mathcal{L}}$, where $\mathcal{L} := A(\mathbb{Z}^d)$, except in the very special cases treated in the proof of Theorem 9, for $d = 2$.

Problem 5. Although problem 1 above appears to be too difficult to solve for a general dimension at this point, can we answer it in dimension $d = 3$? In other words, which

integer 3-dimensional cones have a rational spherical volume? This is rather close to the Cheeger-Simons rational simplex conjecture, so it is most likely quite challenging.

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