A Twist Conjecture For Certain CM Elliptic Curves

By ULRICH SCHMITT*
address: Blutenburgstraße 98, 80636 Munich, Germany
e-mail: ulrich.schmitt@cantab.net

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Abstract. In this paper we study three conjectures of K. Kato. The first two concern the existence of universal global and local elements $L_{p,u}$ and $\mathcal{E}_{p,u'}$, which satisfy a prescribed interpolation property and are characteristic elements of certain universal Iwasawa modules. Under a torsion assumption, we prove that $L_{p,u}$ exists in certain abelian CM elliptic curves cases over quadratic imaginary fields. For an element $\mathcal{E}_{p,u'}$ constructed by O. Venjakob that is a characteristic element of the desired universal Iwasawa modules, we prove, for certain abelian p-adic Lie extensions, that $\mathcal{E}_{p,u'}$ also satisfies the prescribed interpolation property. The third conjecture concerns the possibility of expressing p-adic L-functions of motives, up to certain base change elements $\Omega_{p,u,u'}$, as twists of $L_{p,u}$ and $\mathcal{E}_{p,u'}$ by representations related to the motive. Under a torsion assumption, we prove a result in this direction for certain CM elliptic curves $E/\mathbb{Q}$ and good ordinary primes p.

1 Introduction

The search for p-adic L-functions and their relation to arithmetic objects such as ideal class groups and Selmer groups has been central to Iwasawa theory. In this paper we study three conjectures that were stated by K. Kato, concerning the possibility of expressing p-adic L-functions of motives by twisting certain universal local and global elements. We note that T. Fukaya and K. Kato [21] expect twisting principles to hold in much greater generality in the context of (global and local) non-commutative Tamagawa number conjectures.

The first two conjectures are stated independently of any motive and depend just on a global, resp., local p-adic Lie extension F_∞/F for which they predict the existence of a global, resp., local universal element. In this paper we focus mainly on the special case which arises from the study of an elliptic curve over $\mathbb{Q}$ with complex multiplication by the ring of integers $\mathcal{O}_K$ of a quadratic imaginary number field K and a prime p at which E has good ordinary reduction. Our results build on K. Rubin's [34] two variable main conjecture and K. Kato's [23] and O. Venjakob's [44] work on (commutative) local ϵ-isomorphisms. Let us describe the conjectures.

The first conjecture is of global nature and will be studied in section 3. Let us fix a prime p and a compact p-adic Lie extension $F_\infty/\mathbb{Q}$ containing $\mathbb{Q}(\mu_{p^\infty})$, where μ_{p^∞} denotes the group of all p-power roots of unity. Moreover, we assume that $F_\infty = \bigcup_{n\geq1} F_n$, where $\dots F_{n-1} \subseteq F_n \subseteq F_{n+1} \dots$ is a tower of finite Galois extensions F_n of $\mathbb{Q}$. We write $\mathcal{G} = \mathrm{Gal}(F_\infty/\mathbb{Q})$ and $\mathcal{H} = G(F_\infty/\mathbb{Q}^{cyc})$, where $\mathbb{Q}^{cyc}$ is the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$, such that $\mathcal{G}/\mathcal{H} \cong \mathbb{Z}_p$. We write $\mathcal{S}$ and $\mathcal{S}^*$ for the canonical left and right Ore sets of the Iwasawa algebra $\Lambda(\mathcal{G})$ from [9] associated to $\mathcal{H}$, see section 2 for a definition. Moreover, let Σ be the set of primes of $\mathbb{Q}$ consisting of the archimedean prime ν_∞ of $\mathbb{Q}$, the prime (p) and those primes that ramify in $F_\infty/\mathbb{Q}$ and assume that Σ is finite. The first conjecture predicts the existence of a universal element

$$L_{p,u} \in K_1(\Lambda(\mathcal{G})_{\mathcal{S}^*}),$$

depending on a global unit $u \in \varprojlim_n (\mathcal{O}_{F_n}^\times \otimes \mathbb{Z}_p)$ which is a $\Lambda(\mathcal{G})_S$-basis of $S^{-1}(\varprojlim_n (\mathcal{O}_{F_n}^\times \otimes \mathbb{Z}_p))$ - in general, such a unit is only conjectured to exist. $L_{p,u}$ is characterized by two properties. Firstly, its values at Artin representations ρ of $\mathcal{G}$ are supposed to interpolate the leading coefficient of the Artin L-function $L_\Sigma(\rho, s)$ of ρ divided by a regulator $R(u, \rho)$ depending on ρ and u, i.e.,

$$L_{p,u}(\rho) = \lim_{s \to 0} \frac{s^{-r_\Sigma(\rho)} L_\Sigma(\rho, s)}{R(u, \rho)}, \qquad (L_{p,u}\text{-values})$$

where $r_\Sigma(\rho)$ is the order of vanishing of $L_{\Sigma_f}(\rho, s)$ at $s = 0$. Secondly, $L_{p,u}$ is supposed to map to a prescribed element under the connecting homomorphism $\partial : K_1(\Lambda(\mathcal{G})_{S^*}) \longrightarrow K_0(\mathfrak{M}_\mathcal{H}(\mathcal{G}))$ from K-theory, where $\mathfrak{M}_\mathcal{H}(\mathcal{G})$ denotes the category of finitely generated $\Lambda(\mathcal{G})$-modules that are $\mathcal{S}^*$-torsion. To be more concrete, let us define

$$\mathbb{H}_\Sigma^m := \varprojlim_n H_{\text{ét}}^m\big(\mathcal{O}_{F_n}[\frac{1}{\Sigma_f}], \mathbb{Z}_p(1)\big) \cong \varprojlim_n H^m\big(G_\Sigma(F_n), \mathbb{Z}_p(1)\big)$$

for $m \geq 1$. The Kummer sequence gives an isomorphism $\varprojlim_n (\mathcal{O}_{F_n,\Sigma}^\times \otimes_\mathbb{Z} \mathbb{Z}_p) \xrightarrow{\sim} \mathbb{H}_\Sigma^1$ so that we can consider the image of u in $\mathbb{H}_\Sigma^1$. We consider $\mathbb{H}_\Sigma^1$ and $\mathbb{H}_\Sigma^2$ as *universal* Iwasawa modules attached to the extension $F_\infty/\mathbb{Q}$. Now, the second defining property of $L_{p,u}$ is given by

$$\partial(L_{p,u}) = [\mathbb{H}_\Sigma^2] - [\mathbb{H}_\Sigma^1/\Lambda(\mathcal{G})u]. \qquad (\partial\text{-image } L_{p,u})$$

While K. Kato formulated the global conjecture for the base field $\mathbb{Q}$ and, in general, non-abelian extensions $F_\infty/\mathbb{Q}$, we are interested in an analogue of his conjecture for certain abelian extensions K_∞/K over a quadratic imaginary base field K which arise as follows. Let

(i) E/K be an elliptic curve with complex multiplication by the ring of integers $\mathcal{O}_K$ and conductor (over K) divisible by one prime of K only,

(ii) p, $p \neq 2, 3$, be a prime above which E/K has good ordinary reduction so that p splits in K into distinct primes $\mathfrak{p} = (\pi)$ and $\bar{\mathfrak{p}} = (\bar{\pi})$ with generators π and $\bar{\pi}$, respectively.

Now, let us write $K_{k,n} = K(E[\pi^n \bar{\pi}^k])$, $k, n \geq 0$, for the field obtained by adjoining the coordinates of $\pi^n \bar{\pi}^k$-division points of E to K and set $K_\infty = \bigcup_{k,n \geq 1} K_{k,n}$. The Galois group $G = G(K_\infty/K)$ is abelian. Then, under the assumption that $\varprojlim_{k,n} Cl(K_{k,n})\{p\}$ is S^*-torsion, we prove in Theorem 3.9 the analogue of Kato's conjecture for the abelian p-adic Lie extension K_∞/K, where $Cl(K_{k,n})\{p\}$ denotes the p-primary part of the ideal class group of $K_{k,n}$ and S^* denotes the canonical Ore set in $\Lambda(G)$. The proof uses Rubin's work [34] on the two variable main conjecture and a result about the structure of certain modules of elliptic units from the author's paper [37].

It is more than likely that K. Kato knew - or at least expected - that an analogous conjecture holds in this abelian setting when he formulated the conjecture for the base field $\mathbb{Q}$. As for the generality of the above setting, we note that all elliptic curves defined over $\mathbb{Q}$ that are listed in Appendix A, §3 of J. Silverman's book [39] satisfy assumptions (i) and (ii).

Next, we discuss the second of K. Kato's conjectures, which is of local nature and will be studied in section 4. For a fixed prime p, we let F'_∞ be a p-adic Lie extension of $\mathbb{Q}_p$ containing $\mathbb{Q}_p(\mu_{p^\infty})$. We write $\mathcal{G}' = \mathrm{Gal}(F'_\infty/\mathbb{Q}_p)$ and $\mathcal{H}' = \mathrm{Gal}(F'_\infty/\mathbb{Q}_p^{cyc})$. Later, in the CM setting considered above, we will be interested in the case $F'_\infty = K_{\infty,\nu}$ such that $\mathcal{G}' = G_\nu$ is the decomposition group in G of a place ν of K_∞ above p. Let us write $\widehat{\mathbb{Z}}_p^{ur}$ for the ring of Witt vectors $W(\bar{\mathbb{F}}_p)$ of a fixed algebraic closure $\bar{\mathbb{F}}_p$ of $\mathbb{F}_p$. Moreover, we write $\mathcal{S}'^*$ and $\tilde{\mathcal{S}}'^*$ for the canonical Ore sets in $\Lambda(\mathcal{G}')$ and $\widehat{\mathbb{Z}}_p^{ur}[[\mathcal{G}']]$, respectively. The local conjecture predicts the existence of a universal element

$$\mathcal{E}_{p,u'} \in K_1(\widehat{\mathbb{Z}}_p^{ur}[[\mathcal{G}']]_{\tilde{\mathcal{S}}'^*}),$$

depending on a local $\Lambda(\mathcal{G}')_{\mathcal{S}'}$-basis u' of $\mathcal{U}'(F'_\infty)_{\mathcal{S}'}$ belonging to $\mathcal{U}'(F'_\infty) := \varprojlim_{L,m} \mathcal{O}_L^\times/(\mathcal{O}_L^\times)^{p^m}$, where the limit is taken over all finite subextensions $L/\mathbb{Q}_p$ of $F'_\infty/\mathbb{Q}_p$, with respect to norm maps, and all $m \in \mathbb{N}$. The element $\mathcal{E}_{p,u'}$, similar to $L_{p,u}$ above, is characterized by an interpolation property and by the requirement to map to a prescribed element under the connecting homomorphism $\partial : K_1(\widehat{\mathbb{Z}}_p^{ur}[[\mathcal{G}']]_{\tilde{\mathcal{S}}'^*}) \to K_0(\mathfrak{M}_{\widehat{\mathbb{Z}}_p^{\mathrm{ur}},\mathcal{H}'}(\mathcal{G}'))$ from K-theory, where $\mathfrak{M}_{\widehat{\mathbb{Z}}_p^{\mathrm{ur}},\mathcal{H}'}(\mathcal{G}')$ denotes the category of finitely generated $\widehat{\mathbb{Z}}_p^{ur}[[\mathcal{G}']]$-modules that are $\tilde{\mathcal{S}}'^*$-torsion. In order to state the conjecture let us define the local universal cohomology groups

$$\mathbb{H}^m_{\mathrm{loc}} = \varprojlim_{L'} H^m(L', \mathbb{Z}_p(1))$$

for $m \geq 1$, where L' ranges through the finite subextensions of $F'_\infty/\mathbb{Q}_p$. We note that Kummer theory gives a map $\mathcal{U}'(F'_\infty) \longrightarrow \mathbb{H}^1_{\mathrm{loc}}$.

Local Main Conjecture. *There exists $\mathcal{E}_{p,u'} \in K_1(\widehat{\mathbb{Z}}_p^{ur}[[\mathcal{G}']]_{\tilde{\mathcal{S}}'^*})$ such that for any Artin representation $\rho : \mathcal{G}' \longrightarrow \mathrm{Aut}_{\mathbb{C}_p}(V)$, we have*

$$\mathcal{E}_{p,u'}(\rho) = \frac{\epsilon_p(\rho)}{R_p(u',\rho)} \qquad (\mathcal{E}_{p,u'}\text{-values})$$

whenever $R_p(u',\rho) \neq 0$, where $\epsilon_p(\rho) = \epsilon_p(V)$ is the local constant attached to V and $R_p(u',\rho)$ is a p-adic regulator associated to ρ and u'. Moreover, the image of $\mathcal{E}_{p,u'}$ under the connecting homomorphism from K-theory is given by

$$\partial(\mathcal{E}_{p,u'}) = \left[\widehat{\mathbb{Z}}_p^{ur}\hat{\otimes}_{\mathbb{Z}_p}\mathbb{H}^2_{\mathrm{loc}}\right] - \left[\widehat{\mathbb{Z}}_p^{ur}\hat{\otimes}_{\mathbb{Z}_p}\left(\mathbb{H}^1_{\mathrm{loc}}/\Lambda(\mathcal{G}')u'\right)\right] \quad \text{in} \quad K_0(\mathfrak{M}_{\widehat{\mathbb{Z}}_p^{\mathrm{ur}},\mathcal{H}'}(\mathcal{G}')). \quad (\partial\text{-image } \mathcal{E}_{p,u'})$$

Based on the existence of an ϵ-isomorphism, O. Venjakob [44] constructs an element $\mathcal{E}_{p,u'}$ satisfying (∂-image $\mathcal{E}_{p,u'}$) for abelian p-adic Lie extensions $F'_\infty/\mathbb{Q}_p$ of the form $F'_\infty = K'(\mu_{p^\infty})$, where K' is an infinite unramified extension of $\mathbb{Q}_p$. In this setting a local $\Lambda(\mathcal{G}')_{\mathcal{S}'}$-basis u' of $\mathcal{U}'(F'_\infty)_{\mathcal{S}'}$ exists. We will prove that for $F'_\infty = K'(\mu_{p^\infty})$, Venjakob's element $\mathcal{E}_{p,u'}$ (multiplied by -1) has the desired interpolation property ($\mathcal{E}_{p,u'}$-values) for Artin characters. To be more concrete, $\mathcal{E}_{p,u'}^{-1}$ belongs to $\widehat{\mathbb{Z}}_p^{ur}[[\mathcal{G}']] \cap (\widehat{\mathbb{Z}}_p^{ur}[[\mathcal{G}']]_{\tilde{\mathcal{S}}'^*})^\times$ and considering $\mathcal{E}_{p,u'}^{-1}$ as a $\widehat{\mathbb{Z}}_p^{ur}$-valued measure on $\mathcal{G}'$, we prove the following theorem.

Theorem (see Theorem 4.10). *Let F'_∞ be of the form $K'(\mu_{p^\infty})$ as above. Then, for an Artin character $\chi : \mathcal{G}' \longrightarrow \mathbb{C}_p^\times$ we always have*

$$\left(\int_{\mathcal{G}'} \chi \, d(\mathcal{E}_{p,u'}^{-1})\right) \cdot \varepsilon_p(\chi, \psi_{\epsilon^{-1}}, dx) = -R_p(u',\chi),$$

regardless of whether $R_p(u',\chi) \neq 0$.

Note that multiplying by -1 does not change the image of $\mathcal{E}_{p,u'}$ under ∂.

The third of K. Kato's conjectures, which will be studied in section 7, brings together the elements $L_{p,u} \in K_1(\Lambda(G)_{S^*})$ from the commutative main theorem and $\mathcal{E}_{p,u'} \in K_1(\widehat{\mathbb{Z}}_p^{ur}[[\mathcal{G}']]_{\tilde{\mathcal{S}}'^*})$ from the local main conjecture for the following G and $\mathcal{G}'$. Let $E/\mathbb{Q}$ be one of the elliptic curves from ([39], Appendix A, §3) with complex multiplication by $\mathcal{O}_K$, K quadratic imaginary. These curves have bad reduction at one prime (l) of $\mathbb{Z}$ only and this prime ramifies in $K/\mathbb{Q}$. As before, we set $K_\infty = \bigcup_n K(E[p^n])$ for some prime p, $p \neq 2,3$, at which E has good ordinary reduction, which implies that p splits in K into disctinct primes $\mathfrak{p} = (\pi)$ and $\bar{\mathfrak{p}} = (\bar{\pi})$. Let us write $\mathcal{G}$ for the non-abelian Galois group $G(K_\infty/\mathbb{Q})$, $\mathcal{H} = G(K_\infty/\mathbb{Q}^{cyc})$ and, as before, $G = G(K_\infty/K)$. Moreover, we fix some prime ν of K_∞ above $\mathfrak{p}$ and consider the extension $K_{\infty,\nu}/\mathbb{Q}_p$, which is abelian because $\mathcal{G}' = G(K_{\infty,\nu}/\mathbb{Q}_p)$ embeds into G by the assumption that p splits in K, and of the form $K'(\mu_{p^\infty})$ for some infinite unramified extension $K'/\mathbb{Q}_p$. In particular, the local main conjecture holds for $K_{\infty,\nu}/\mathbb{Q}_p$, i.e., $\mathcal{E}_{p,u'}$ exists. We write $T_{\bar{\pi}}E = \varprojlim_n E[\bar{\pi}^n]$ and $T_pE = \varprojlim_n E[p^n]$ for the $\bar{\pi}$-adic and p-adic Tate modules of E and $T_p\hat{E} = \varprojlim_n \hat{E}[p^n]$ for the p-adic Tate module of the formal group $\hat{E}$ associated to a model of E. Let us write $\tilde{\mathcal{S}}^*$ for the canonical Ore set in $\hat{\mathbb{Z}}_p^{\mathrm{ur}}[[\mathcal{G}]]$. Under the assumption that the dual Selmer group $\mathrm{Sel}(K_\infty, T_pE^*(1))^\vee$ from section 5 is $\mathcal{S}^*$-torsion, we define $\Omega_{p,u,u'} \in \Lambda(\mathcal{G})^\times_{\mathcal{S}^*}$ in section 6 as a base change between two bases determined by u and u', respectively, of

$$\left(\mathrm{Ind}_{\mathcal{G}}^{\mathcal{G}'}\left(T_pE/T_p\hat{E}(-1) \otimes_{\mathbb{Z}_p} \mathbb{H}^1_{\mathrm{loc}}\right)\right)_{\mathcal{S}^*},$$

which is a free $\Lambda(\mathcal{G})_{\mathcal{S}^*}$-module of rank one. Then, we define

$$\mathcal{L}_{p,u,E} := \frac{\tau_{E_{\bar{\pi}}(-1)}(L_{p,u})}{\tau_{E/\hat{E}(-1)}(\mathcal{E}_{p,u'})} \cdot \Omega_{p,u,u'} \cdot \frac{1}{12} \in K_1(\hat{\mathbb{Z}}_p^{\mathrm{ur}}[[\mathcal{G}]]_{\tilde{\mathcal{S}}^*}),$$

where $\tau_{E_{\bar{\pi}}(-1)}(L_{p,u})$ and $\tau_{E/\hat{E}(-1)}(\mathcal{E}_{p,u'})$ are twists of $L_{p,u}$ and $\mathcal{E}_{p,u'}$ induced by the G-module $T_{\bar{\pi}}E(-1)$ and the $\mathcal{G}'$-module $(T_pE/T_p\hat{E})(-1)$, respectively, (-1) denoting the -1-th Tate twist. Twist operators are defined in section 2 based on work from [43]. Let us write $\mathfrak{M}_{\hat{\mathbb{Z}}_p^{\mathrm{ur}},\mathcal{H}}(\mathcal{G})$ for the category of finitely generated $\hat{\mathbb{Z}}_p^{\mathrm{ur}}[[\mathcal{G}]]$-modules which are $\tilde{\mathcal{S}}^*$-torsion. Our main theorems are the following.

Twist Theorem (see theorem 7.5). *Assume that* $\mathrm{Sel}(K_\infty, T_pE^*(1))^\vee$ *is* $\mathcal{S}^*$*-torsion. Then, up to a twisted Euler factor,* $\mathcal{L}_{p,u,E}$ *is a characteristic element of* $\hat{\mathbb{Z}}_p^{\mathrm{ur}}\hat{\otimes}_{\mathbb{Z}_p}\mathrm{Sel}(K_\infty, T_pE^*(1))^\vee$*, i.e., we have*

$$\partial(\mathcal{L}_{p,u,E}) = [\hat{\mathbb{Z}}_p^{\mathrm{ur}}\hat{\otimes}_{\mathbb{Z}_p}\mathrm{Sel}(K_\infty, T_pE^*(1))^\vee] + [\hat{\mathbb{Z}}_p^{\mathrm{ur}}\hat{\otimes}_{\mathbb{Z}_p}\mathrm{c\text{-}Ind}_{\mathcal{G}}^{\mathcal{G}_{\nu_l}} T_pE(-1)] \quad \text{in} \quad K_0(\mathfrak{M}_{\hat{\mathbb{Z}}_p^{\mathrm{ur}},\mathcal{H}}(\mathcal{G})),$$

where l *is the unique prime at which* $E/\mathbb{Q}$ *has bad reduction and* $\mathcal{G}_{\nu_l}$ *is the decomposition group of some place of* K_∞ *above* l*.*

In order to determine the interpolation property of $\mathcal{L}_{p,u,E}$ we then first prove the following theorem.

Theorem (see theorem 7.7). *The element* $\frac{\Omega_{p,u,u'}}{\tau_{E/\hat{E}(-1)}(\mathcal{E}_{p,u'})}$ *is equal to the* $\tau_{E_{\bar{\pi}}(-1)}$*-twist of the image of* u *under the semi-local version* $\mathcal{L}_{semi-loc}$ *of the Coleman map for* $\mathbb{G}_m$*, i.e.,*

$$\frac{\Omega_{p,u,u'}}{\tau_{E/\hat{E}(-1)}(\mathcal{E}_{p,u'})} = \tau_{E_{\bar{\pi}}(-1)}(\mathcal{L}_{semi-loc}(u)),$$

which shows that the left side does not depend on u'. In particular, $\mathcal{L}_{p,u,E}$ is independent of u'.

From this theorem we deduce that $\mathcal{L}_{p,u,E}$ is just a different guise of a well-known element studied in [14] for which an interpolation formula exists.

Corollary (see corollary 7.8). *We have an equality of elements in $\hat{\mathbb{Z}}_p^{\mathrm{ur}}[[\mathcal{G}]]_{\mathcal{S}^*}$*

$$\mathcal{L}_{p,u,E} = \tau_{\psi^{-1}}(\lambda),$$

where $\tau_{\psi^{-1}}(\lambda)$ denotes the twist of de Shalit's element $\lambda \in \Lambda(G)$ from definition 3.6 by the G-module $(T_\pi E)^*$. The action of G on $(T_\pi E)^*$ is given by ψ^{-1}, where ψ denotes the Größencharacter of E/K. For an Artin character χ of $\mathcal{G}$ we have

$$\frac{1}{\Omega_p}\cdot\int_G \mathrm{Res}_G^{\mathcal{G}}\chi \; d\mathcal{L}_{p,u,E} \;=\; \frac{1}{\Omega}\cdot G\big(\psi\cdot\mathrm{Res}\chi\big)\cdot\Big(1-\frac{(\psi\cdot\mathrm{Res}\chi)(\mathfrak{p})}{p}\Big)\cdot L_{\mathfrak{f}\bar{\mathfrak{p}}}((\psi\cdot\mathrm{Res}\chi)^{-1},0), \quad (1.1)$$

where we refer to ([14], p. 80) for the definition of $G\big(\psi\cdot\mathrm{Res}\chi\big)$ which is related to a local constant. In the expression $(\psi\cdot\mathrm{Res}\chi)(\mathfrak{p})$ we consider $\psi\cdot\mathrm{Res}\chi$ as a map on ideals of K prime to $\mathfrak{f}$. Ω is a complex period and Ω_p is a p-adic period determining an isomorphism of formal groups $\mathbb{G}_m \cong \hat{E}$.

We note that large parts of this paper are modified versions of parts of the author's doctoral dissertation [36].

2 Twist operators on K_0 and K_1 groups

In this section we collect some results about twist operators induced by p-adic Galois representations on K-groups. For a unital ring R we define $GL(R)$ to be the infinite general linear group, i.e., the inductive limit $GL(R) := \varinjlim_n GL_n(R)$, with the usual inclusion of $GL_n(R)$ into $GL_{n+1}(R)$ as the upper left block matrices. Similarly, we define the subgroup of infinite elementary matrices $E(R) := \varinjlim_n E_n(R)$, which is equal to the commutator subgroup $[GL(R), GL(R)]$ of $GL(R)$, see ([33], 2.1.4. Proposition (Whitehead's Lemma)). We then set

$$K_1(R) = GL(R)/E(R),$$

which is the definition of first K-group found in loc. cit. We see that $K_1(R)$ is the abelianization of $GL(R)$.

Now, let G be a p-adic Lie group containing a closed normal subgroup H such that

$$G/H = \Gamma \cong \mathbb{Z}_p.$$

Moreover, in the following $\mathcal{O}$ will stand for either

$$\mathcal{O} = \mathbb{Z}_p \qquad \text{or} \qquad \mathcal{O} = \hat{\mathbb{Z}}_p^{ur}$$

and we will write $\Lambda_{\mathcal{O}}(G) = \mathcal{O}[[G]]$ and $\Lambda_{\mathcal{O}}(H) = \mathcal{O}[[H]]$ for the Iwasawa algebra with coefficients in $\mathcal{O}$ of G and H, respectively. Whenever we omit $\mathcal{O}$ from the notation, we refer to $\mathbb{Z}_p$-coefficients, i.e., $\Lambda(G) = \Lambda_{\mathbb{Z}_p}(G)$ and $\Lambda(H) = \Lambda_{\mathbb{Z}_p}(H)$. We write S and S^* for the two canonical Ore sets of $\Lambda_{\mathcal{O}}(G)$ as defined in [9] by

$$S = \{f \in \Lambda_{\mathcal{O}}(G) \mid \Lambda_{\mathcal{O}}(G)/\Lambda_{\mathcal{O}}(G)f \text{ is finitely generated as a } \Lambda_{\mathcal{O}}(H)-\text{module}\} \tag{2.1}$$

and

$$S^* = \bigcup_{n\geq 1} p^n S. \tag{2.2}$$

Now, let $\rho : G \longrightarrow \mathrm{Aut}_{\mathbb{Z}_p}(T)$ be a continuous $\mathbb{Z}_p$-linear representation, where T is a free $\mathbb{Z}_p$-module of finite rank r. We fix an isomorphism

$$\phi_T : T \cong \mathbb{Z}_p^r, \tag{2.3}$$

which is equivalent to choosing a $\mathbb{Z}_p$-basis of T, and for any $g \in G$ we shall denote by

$$\rho_{\phi_T}(g) \;\in\; M_r(\mathbb{Z}_p)$$

the $(r \times r)$-matrix with coefficients in $\mathbb{Z}_p$ associated to $\phi_T \circ \rho(g) \circ \phi_T^{-1}$ and the standard basis of $\mathbb{Z}_p^r$. For any matrix A let us write A^{t} for the transpose of A. The homomorphism

$$G \longrightarrow \Big(M_r\big(\Lambda_{\mathcal{O}}(G)\big)\Big)^{\times}, \qquad g \longmapsto (\rho_{\phi_T}(g^{-1}))^{\mathrm{t}} g,$$

induces an $\mathcal{O}$-algebra map

$$\tau_\rho : \Lambda_{\mathcal{O}}(G) \longrightarrow M_r\big(\Lambda_{\mathcal{O}}(G)\big).$$

which extends to the localized Iwasawa algebra.

Proposition 2.1. *The homorphism τ_ρ extends to a ring homomorphism*

$$\tau_\rho : \Lambda_{\mathcal{O}}(G)_{S^*} \longrightarrow M_r\big(\Lambda_{\mathcal{O}}(G)_{S^*}\big).$$

Proof. The proof can be found in ([36], proposition 1.1.9) and uses similar arguments as the proof of Lemma 7.6 in [43].

Using the fact ([27], §17B, (17.20) Theorem) that $\Lambda_{\mathcal{O}}(G)_{S^*}$ and $M_r\big(\Lambda_{\mathcal{O}}(G)_{S^*}\big)$, $r \geq 1$, are Morita-equivalent and the Morita invariance of K_1 ([46], III §1, Proposition 1.6.4), we immediately get the following result.

Corollary 2.2. *The homomorphism τ_ρ induces an operator τ_ρ on $K_1(\Lambda_{\mathcal{O}}(G)_{S^*})$.*

Next we turn to $K_0(\mathfrak{M}_{\mathcal{O},H}(G))$ the K_0 group of the category $\mathfrak{M}_{\mathcal{O},H}(G)$ of finitely generated $\Lambda_{\mathcal{O}}(G)$-modules that are S^*-torsion. If $\mathcal{O} = \mathbb{Z}_p$ we also write $\mathfrak{M}_H(G) = \mathfrak{M}_{\mathcal{O},H}(G)$.

Definition 2.3 (Twist operator on $K_0(\mathfrak{M}_{\mathcal{O},H}(G))$). *We define a twist operator $\tilde{\tau}_\rho$ on $K_0(\mathfrak{M}_{\mathcal{O},H}(G))$ by*

$$\tilde{\tau}_\rho : K_0(\mathfrak{M}_{\mathcal{O},H}(G)) \longrightarrow K_0(\mathfrak{M}_{\mathcal{O},H}(G)), \qquad [M] \longmapsto [T \otimes_{\mathbb{Z}_p} M],$$

where the action of $\Lambda_{\mathcal{O}}(G)$ on $T \otimes_{\mathbb{Z}_p} M$ is induced by the diagonal G-action.

For the facts that the diagonal G-action on $T \otimes_{\mathbb{Z}_p} M$ extends to an action of $\Lambda_{\mathcal{O}}(G)$ and that $T \otimes_{\mathbb{Z}_p} M$, with this action, belongs to $\mathfrak{M}_{\mathcal{O},H}(G)$ whenever M does, see ([36], lemmata 1.1.4 and 1.1.14). With these definitions, we have a commutative diagram

$$\begin{array}{ccc} K_1(\Lambda_{\mathcal{O}}(G)_{S^*}) & \xrightarrow{\partial} & K_0(\mathfrak{M}_{\mathcal{O},H}(G)) \\ \Big\downarrow \tau_\rho & & \Big\downarrow \tilde{\tau}_\rho \\ K_1(\Lambda_{\mathcal{O}}(G)_{S^*}) & \xrightarrow{\partial} & K_0(\mathfrak{M}_{\mathcal{O},H}(G)), \end{array} \tag{2.4}$$

where ∂ is the connecting homomorphism from K-theory, see subsection 1.1.4 of loc. cit.

3 Global Theorem

In this section we study the first of the three conjectures of Kato that were stated in the introduction, regarding the existence of a global universal element $L_{p,u}$. While Kato stated the conjecture for an, in general, non-abelian p-adic Lie extension $F_\infty/\mathbb{Q}$, we will prove, under a torsion assumption, an analogue of the conjecture in theorem 3.9 for any extension K_∞/K such that

(i) K is a quadratic imaginary number field,

(ii) there exists an elliptic curve E/K with complex multiplication by the ring of integers $\mathcal{O}_K$ and conductor over K divisible by one prime of K only,

(iii) $K_\infty = K(E[p^\infty])$ is the field obtained by adjoining all coordinates of p-power division points of E to K for some prime $p \in \mathbb{Z}$, $p \neq 2,3$, above which E/K has good ordinary reduction.

First, in subsection 3.2, we collect some results about the structure of certain modules of elliptic units from the author's [37], [36] and in subsection 3.3 we then prove theorem 3.9 using Rubin's work on the two variable main conjecture.

3.1 Setting

Elliptic curves. Let K be a quadratic imaginary number field and let E be an elliptic curve defined over K with complex multiplication by $\mathcal{O}_K$. Then, K has class number one. Assume that E/K has good ordinary reduction above a prime $p \in \mathbb{Z}$, $p \neq 2,3$, so that p splits in K into distinct primes $\mathfrak{p} = (\pi)$ and $\bar{\mathfrak{p}} = (\bar{\pi})$. We write $\mathfrak{f} = \mathfrak{f}_\psi \subset \mathcal{O}_K$ for the conductor of the Größencharacter ψ attached to E/K and L for the period lattice associated to a fixed global minimal Weierstraß equation for E. Moreover, we fix $\Omega \in L$ such that

$$\mathcal{O}_K \Omega = L.$$

Later, we impose the following condition on E/K.

Assumption 3.1. *We assume that the conductor $\mathfrak{f}$ of the Größencharacter ψ over K is a prime power*

$$\mathfrak{f} = \mathfrak{l}^r$$

for some prime ideal $\mathfrak{l}$ and some $r \geq 1$.

Remark 3.2. If E is already defined over $\mathbb{Q}$ and E is a representative with minimal discriminant and conductor in its $\bar{\mathbb{Q}}$-isomorphism class as in ([39], Appendix A, §3), then, compare ([36], theorem A.6.8 and proposition A.6.9) we have

$$\mathfrak{f} = \mathfrak{l}^r, \quad r \geq 1,$$

for some prime ideal $\mathfrak{l}$ so that assumption 3.1 holds.

Fields and Galois groups. Let us fix an embedding $\bar{\mathbb{Q}} \subset \mathbb{C}_p$, where $\mathbb{C}_p$ denotes the completion $\hat{\bar{\mathbb{Q}}}_p$ of an algebraic closure $\bar{\mathbb{Q}}_p$ of $\mathbb{Q}_p$. We will write

$$F_{k,n} := K(\mathfrak{f}\bar{\mathfrak{p}}^k\mathfrak{p}^n) \quad \text{and} \quad L_{k,n} := K(\bar{\mathfrak{p}}^k\mathfrak{p}^n), \qquad k, n \geq 0,$$

for the ray class fields of K of modulus $\mathfrak{f}\bar{\mathfrak{p}}^k\mathfrak{p}^n$ and $\bar{\mathfrak{p}}^k\mathfrak{p}^n$, respectively, and

$$K_{k,n} := K(E[\bar{\mathfrak{p}}^k\mathfrak{p}^n]), \qquad k, n \geq 0,$$

for the fields obtained by adjoing to K the coordinates of $\bar{\mathfrak{p}}^k\mathfrak{p}^n$-division points of E. The conductors of $K_{k,n}$ over K are $\mathfrak{f}\bar{\mathfrak{p}}^k\mathfrak{p}^n$ whenever $k, n \geq 0$, $(k,n) \neq (0,0)$, see ([36], lemma 2.4.17), and we have $L_{k,n} \subset K_{k,n} \subset F_{k,n}$. We will write

$$L_\infty = \bigcup_{k,n} L_{k,n}, \quad K_\infty = \bigcup_{k,n} K_{k,n} \quad \text{and} \quad F_\infty = \bigcup_{k,n} F_{k,n},$$

and define G to be the abelian Galois group $G = G(K_\infty/K)$. We also define a closed normal subgroup H of G by $H = G(K_\infty/K^{cyc})$, where K^{cyc}/K is the cyclotomic $\mathbb{Z}_p$-extension, so that $G/H \cong \mathbb{Z}_p$.

Generally, if F is a number field, $\bar{F}$ an algebraic closure and Σ' a set of places of F containing the infinite places, we write $F_{\Sigma'}$ for the maximal extension of F inside $\bar{F}$ that is unramified outside Σ' and $G_{\Sigma'}(F) = G(F_{\Sigma'}/F)$ for the Galois group of $F_{\Sigma'}/F$. Moreover, we will write $\mathcal{O}_{F,\Sigma'}$ for the ring of Σ'-integers of F. In the setting described above, let us write $\Sigma = \{\mathfrak{p}, \bar{\mathfrak{p}}, \mathfrak{l}_1, \dots, \mathfrak{l}_s, \nu_\infty\}$, where $\mathfrak{l}_1, \dots, \mathfrak{l}_s$ are the primes dividing the conductor $\mathfrak{f}$ and ν_∞ is the complex archimedean place of K.

Iwasawa modules. The Σ just defined is a finite set of places of K. If F is an extension of K we will denote by Σ_F the places of F above the places in Σ. If there is no danger of confusion, we will drop the subscript F in Σ_F. In particular, we write $F_\Sigma = F_{\Sigma_F}$, $G_\Sigma(F) = G_{\Sigma_F}(F)$ and $\mathcal{O}_{F,\Sigma}$ and $\mathcal{O}_{F,\Sigma_F}$. For $m = 1, 2$ we then set

$$\mathbb{H}^m_\Sigma := \varprojlim_{k,n} H^m(G_\Sigma(K_{k,n}), \mathbb{Z}_p(1)) \cong H^m(G_\Sigma(K), \Lambda(G)^\#(1)),$$

where the isomorphism on the right is explained in (loc. cit. section A.3) and $\Lambda(G)^\#$ denotes the left $\Lambda(G)$-module $\Lambda(G)$ equipped with a left $G_\Sigma(K)$-action defined by $g.\lambda := \lambda \cdot \bar{g}^{-1}$, $g \in G_\Sigma(K), \lambda \in \Lambda(G)$, where $\bar{g}$ denotes the image of g in G.

We will write $\mathcal{E}_\infty := \varprojlim_{k,n}(\mathcal{O}^\times_{K_{k,n}} \otimes_\mathbb{Z} \mathbb{Z}_p)$ for the global units and use the identifications

$$\mathcal{E}_\infty \cong \varprojlim_{k,n}(\mathcal{O}^\times_{K_{k,n},\Sigma} \otimes_\mathbb{Z} \mathbb{Z}_p) \cong \mathbb{H}^1_\Sigma, \tag{3.1}$$

where the second isomorphism comes from the Kummer sequence and the first one holds by ([42], proposition 3.1.3 (ii)) for which we note that $\bar{\mathfrak{p}}$ (resp. $\mathfrak{p}$) is unramified in $\cup_{n\geq 1} K(E[\pi^n])$ (resp. $\cup_{k\geq 1} K(E[\bar{\pi}^k])$) and that the cyclotomic $\mathbb{Z}_p$-extension is contained in K_∞/K.

We also define $\mathcal{A}_\infty = \varprojlim_{k,n}(Cl(K_{k,n})\{p\})$, where $Cl(K_{k,n})\{p\}$ denotes the p-primary part of the ideal class group $Cl(K_{k,n})$ of $K_{k,n}$, $k,n \geq 1$. The composite of the natural maps

$$\mathcal{A}_\infty \twoheadrightarrow \varprojlim_{k,n}\big(\mathrm{Pic}(\mathcal{O}_{K_{k,n}}[1/p])\{p\}\big) \cong \varprojlim_{k,n}\big(\mathrm{Pic}(\mathcal{O}_{K_{k,n},\Sigma})\{p\}\big), \tag{3.2}$$

is a surjection with a kernel which is a finitely generated $\mathbb{Z}_p$-module. In fact, it is clear that the maps are surjective. The second map is also injective since the cyclotomic $\mathbb{Z}_p$-extension of K is contained in K_∞, see ([25], section 2.5). The first map has a kernel that is finitely generated as a $\mathbb{Z}_p$-module because the decomposition groups of $\mathfrak{p}$ and $\bar{\mathfrak{p}}$ have finite index in G - see also ([36], section A.3.4). The Kummer sequence also gives an exact sequence

$$1 \longrightarrow \varprojlim_{k,n}\big(\mathrm{Pic}(\mathcal{O}_{K_{k,n},\Sigma})\{p\}\big) \longrightarrow \mathbb{H}^2_\Sigma \longrightarrow \bigoplus_{\nu\in\Sigma_f} \Lambda(G)\otimes_{\Lambda(G_\nu)} \mathbb{Z}_p \longrightarrow \mathbb{Z}_p \longrightarrow 0, \tag{3.3}$$

where Σ_f denotes the subset of finite primes in Σ and G_ν is the decomposition group of ν in G.

L-functions. For an Artin character $\chi : G_K \longrightarrow \mathbb{C}^\times$ factoring through $G(K_{k,n}/K)$ we write

$$L_{\Sigma_f}(\chi, s) = \sum_{\substack{\mathfrak{b}\subset\mathcal{O}_K \\ (\mathfrak{b},\Sigma_f)=1}} \frac{\chi(\mathfrak{b})}{N(\mathfrak{b})^s} \qquad s\in\mathbb{C},\ \mathfrak{R}(s) > 1$$

for the L-function attached to χ, where $\chi(\mathfrak{b}) = \chi((\mathfrak{b}, K_{k,n}/K))$.

3.2 Elliptic Units

In this subsection we briefly recall some definitions of elliptic units and state a comparison result, see theorem 3.5, about modules of elliptic units considered by Rubin [34] and Yager [48]. In proposition 3.7 we cite a description of the image of one of the modules under the (two-variable) semi-local version $\mathbb{L}$ of the Coleman map for the formal group $\hat{E}$ associated to the fixed Weierstraß equation of E.

For any integral ideal $\mathfrak{a} \subset \mathcal{O}_K$, $(\mathfrak{a}, 6) = 1$, consider the meromorphic function

$$\Theta_0(z;\mathfrak{a}) = \Theta_0(z;L,\mathfrak{a}) = \Big(\frac{\Delta(L)^{N\mathfrak{a}}}{\Delta(\mathfrak{a}^{-1}L)}\Big)^{1/12} \prod_{u\in(\mathfrak{a}^{-1}L/L)/\pm 1} \big(\wp(z;L) - \wp(u;L)\big)^{-1},$$

where Δ is the Ramanujan Δ-function, a twelfth root of $\frac{\Delta(L)^{N\mathfrak{a}}}{\Delta(\mathfrak{a}^{-1}L)}$ is fixed and $\wp(z;L)$ is the Weierstraß $\wp$-function for the lattice L.

For any integral ideal $\mathfrak{m}$ of K such that $\mathcal{O}_K^\times \longrightarrow \mathcal{O}_K/\mathfrak{m}$ is injective - recall that $\mathcal{O}_K^\times$ is a finite group - and any element $\tau \in \mathbb{C}/L$ of order exactly $\mathfrak{m}$, $\Theta_0(\tau;\mathfrak{a})$ belongs to $K(\mathfrak{m})$, see ([3], Proposition 2.2).

Definition 3.3. *Let F be a finite abelian extension of K. We define* *(i)* C_F *as the group generated by the various elements*

$$\left(N_{FK(\mathfrak{m})/F}\ \Theta_0(\tau;\mathfrak{a})\right)^{\sigma-1}, \tag{3.4}$$

where σ *ranges through* $\mathrm{Gal}(F/K)$, $\mathfrak{m}$ *through the integral ideals of* K *such that* $\mathcal{O}_K^\times \longrightarrow \mathcal{O}_K/\mathfrak{m}$ *is injective,* $\mathfrak{a}$ *through integral ideals such that* $(\mathfrak{a}, 6\mathfrak{m}) = 1$ *and* τ *through primitive* $\mathfrak{m}$*-division points.* $N_{FK(\mathfrak{m})/F}$ *denotes the norm map from the composite field* $FK(\mathfrak{m})$ *of* F *and* $K(\mathfrak{m})$ *to* F *and we note that the elements* $\sigma - 1$ *generate the augmentation ideal* $I(F/K)$*, i.e., the kernel of the augmentation map* $\mathrm{aug} : \mathbb{Z}[G(F/K)] \to \mathbb{Z}$.

(ii) $\mathcal{C}(F)$*, the group of elliptic units of* F *considered by Rubin in [34], as*

$$\mathcal{C}(F) = \mu_\infty(F) C_F,$$

where $\mu_\infty(F)$ *is the group of all roots of unity in* F.

(iii) for any integral ideal $\mathfrak{n}$ *of* K, $\Theta_\mathfrak{n}$ *to be the subgroup of* $K(\mathfrak{n})^\times$ *generated by*

$$\Theta(1;\mathfrak{n},\mathfrak{a}) = \Theta(\Omega/a_\mathfrak{n}; L, \mathfrak{a}), \tag{3.5}$$

where $\Theta(z; L, \mathfrak{a}) = \Theta_0(z;\mathfrak{a})^{12}$, $a_\mathfrak{n}$ *is an* $\mathcal{O}_K$*-generator of* $\mathfrak{n}$*, i.e.,* $\mathfrak{n} = (a_\mathfrak{n})$*, and* $\mathfrak{a}$ *ranges through the integral ideals of* K *such that* $(\mathfrak{a}, 6\mathfrak{n}) = 1$.

We define two $\Lambda(G)$-submodules of elliptic units $\mathcal{C}_\infty$ and $\mathcal{D}_\infty$ of the global units $\mathcal{E}_\infty = \varprojlim_{k,n}(\mathcal{O}^\times_{K_{k,n}} \otimes_\mathbb{Z} \mathbb{Z}_p)$, where the limit is taken with respect to the norm maps, by

$$\mathcal{C}_\infty = \varprojlim_{k,n}(\mathcal{C}(K_{k,n}) \otimes_\mathbb{Z} \mathbb{Z}_p), \quad \mathcal{D}_\infty = I \varprojlim_{k,n}\left((N_{F_{k,n}/K_{k,n}}\Theta_{\mathfrak{f}\mathfrak{p}^n\bar{\mathfrak{p}}^k}) \otimes_\mathbb{Z} \mathbb{Z}_p\right)$$

where I denotes the augmentation ideal $I = I(K_\infty/K)$ in $\Lambda(G)$. It follows from the definitions that $\mathcal{D}_\infty \subset \mathcal{C}_\infty$ and we refer to ([36], theorem 2.4.16) for the fact that the quotient $\mathcal{C}_\infty/\mathcal{D}_\infty$ is S-torsion, where $S \subset \Lambda(G)$ was defined in (2.1).

Remark 3.4. $\mathcal{D}_\infty$ coincides with a module of elliptic units considered by Yager [48].

We write $S-\mathrm{tor}$ for the category of finitely generated $\Lambda(G)$-modules that are S-torsion and note that any object of $S-\mathrm{tor}$ is also an object in $\mathfrak{M}_H(G)$ so that we have a natural map $K_0(S\text{–tor}) \to K_0(\mathfrak{M}_H(G))$.

Theorem 3.5 *Assume that assumption 3.1 holds, i.e.,* $\mathfrak{f} = \mathfrak{l}^r$ *for some prime ideal* $\mathfrak{l}$ *of* K *and some* $r \geq 1$*. In* $K_0(S\text{–tor})$*, and hence in* $K_0(\mathfrak{M}_H(G))$*, we then have an equality*

$$[\mathcal{C}_\infty/\mathcal{D}_\infty] = [\Lambda(G/D_\mathfrak{l})],$$

where we write $D_\mathfrak{l}$ *for the decomposition group of* $\mathfrak{l}$ *in* $G = G(K_\infty/K)$.

Proof. See [37] or ([36], theorem 2.4.33) from which it also becomes apparent that $\Lambda(G/D_\mathfrak{l})$ is S-torsion.

Next, we turn to the semi-local version of the Coleman map. For any prime ideal $\mathfrak{P}$ of $K_{k,n}$ above the prime $\mathfrak{p}$ of K we write $K_{k,n,\mathfrak{P}}$ for the completion of $K_{k,n}$ at $\mathfrak{P}$ and $\mathcal{O}^1_{K_{k,n,\mathfrak{P}}}$ for the subgroup of principal units in the group of units $\mathcal{O}^\times_{K_{k,n,\mathfrak{P}}}$. We write $\hat{\mathcal{O}}^\times_{K_{k,n,\mathfrak{P}}}$ for the p-adic completion of $\mathcal{O}^\times_{K_{k,n,\mathfrak{P}}}$ and note that we canonically have $\mathcal{O}^1_{K_{k,n,\mathfrak{P}}} \cong \hat{\mathcal{O}}^\times_{K_{k,n,\mathfrak{P}}}$. We write

$$U_{k,n} = \prod_{\mathfrak{P}|\mathfrak{p}} \mathcal{O}^1_{K_{k,n,\mathfrak{P}}} \cong \prod_{\mathfrak{P}|\mathfrak{p}} \hat{\mathcal{O}}^\times_{K_{k,n,\mathfrak{P}}}$$

for the subgroup of principal units in the group of semi-local units $\prod_{\mathfrak{P}|\mathfrak{p}} \mathcal{O}^\times_{K_{k,n,\mathfrak{P}}}$ and define

$$\mathcal{U}_\infty = \varprojlim_{k,n} U_{k,n}$$

as the projective limit with respect to the norm maps. Since Leopoldt's conjecture holds for finite abelian extensions of K, we have an embedding

$$\mathcal{E}_\infty \hookrightarrow \mathcal{U}_\infty.$$

We denote by $\Lambda(G, \widehat{\mathbb{Z}}_p^{ur})$ the Iwasawa algebra of G with coefficients in $\widehat{\mathbb{Z}}_p^{ur}$, the ring of Witt vectors $W(\bar{\mathbb{F}}_p)$ of a fixed algebraic closure $\bar{\mathbb{F}}_p$ of $\mathbb{F}_p$. We note that there is a semi-local version

$$\mathbb{L} : \mathcal{U}_\infty \hookrightarrow \Lambda(G, \widehat{\mathbb{Z}}_p^{ur}) \tag{3.6}$$

of the Coleman map for the formal group $\hat{E}$ associated to the fixed Weierstraß equation of E, see ([36], theorem 2.4.25 and corollary 2.4.26) for its construction and its injectivity. $\mathbb{L}$ is obtained as the limit $\varprojlim_k \mathbb{L}_k$ of maps $\mathbb{L}_k : \varprojlim_n U_{k,n} \to \Lambda(G_k, \widehat{\mathbb{Z}}_p^{ur})$ for $k \geq 1$, where $G_k = \mathrm{Gal}(K_{k,\infty}/K)$, $K_{k,\infty} = \bigcup_n K_{k,n}$.

Definition 3.6. *Let $(p,6) = 1$ and fix an $\mathcal{O}_K$-generator f of the conductor $\mathfrak{f}$ of E, so that $\mathfrak{f} = (f)$. For $k,n \geq 1$ and an integral ideal $\mathfrak{a}$ of K, $(\mathfrak{a}, \mathfrak{f}\bar{\mathfrak{p}}\mathfrak{p}) = 1$, we put*

$$e'_{k,n}(\mathfrak{a}) := \Theta\Big(\frac{\Omega}{f\bar{\pi}^k\pi^n}, L, \mathfrak{a}\Big) \in \mathcal{O}^\times_{F_{k,n}} \quad \text{and} \quad e_{k,n}(\mathfrak{a}) := N_{F_{k,n}/K_{k,n}}(e'_{k,n}) \in \mathcal{O}^\times_{K_{k,n}},$$

which defines a norm-compatible system $e(\mathfrak{a}) = (e_{k,n}(\mathfrak{a}))_{k,n} \in \varprojlim_{k,n} \mathcal{O}^\times_{K_{k,n}}$ of global units. Let us denote by $u(\mathfrak{a})$ the image of $e(\mathfrak{a})$ in $\mathcal{E}_\infty$ and also write $u(\mathfrak{a})$ for the image in $\mathcal{U}_\infty = \varprojlim_{k,n} U_{k,n}$ under $\mathcal{E}_\infty \hookrightarrow \mathcal{U}_\infty$. We write

$$\lambda_\mathfrak{a} := \lambda^0_{u(\mathfrak{a})} = \mathbb{L}(u(\mathfrak{a}))$$

for the p-adic integral measure on G corresponding to $u(\mathfrak{a})$ under $\mathbb{L}$. We also set

$$\lambda := \frac{1}{12} \cdot \frac{\lambda_\mathfrak{a}}{x_\mathfrak{a}} \quad \in \; Q(\Lambda(G, \hat{\mathbb{Z}}_p^{ur})),$$

where $x_\mathfrak{a} := \sigma_\mathfrak{a} - N\mathfrak{a}$, $\sigma_\mathfrak{a} = (\mathfrak{a}, K_\infty/K) \in G$. λ is independent of $\mathfrak{a}$ and an integral measure, i.e., $\lambda \in \Lambda(G, \hat{\mathbb{Z}}_p^{ur})$, compare ([14], II proof of Theorem 4.12) where this is proven at the level of each $\Lambda(G_k, \widehat{\mathbb{Z}}_p^{ur})$, $k \geq 1$.

Proposition 3.7. *The map $\mathbb{L}$ induces an isomorphism of $\Lambda(G)$-modules*

$$\varprojlim_{n,k}\big((N_{F_{n,k}/K_{n,k}}\Theta_{\mathfrak{f}\mathfrak{p}^n\bar{\mathfrak{p}}^k})\otimes_{\mathbb{Z}}\mathbb{Z}_p\big)\cong J\lambda,$$

where J is the annihilator in $\Lambda(G)$ of $\mu_{p^\infty}(K_\infty)$, the module of p-power roots of unity in K_∞. Hence, $\mathbb{L}$ induces $\mathcal{D}_\infty \cong IJ\lambda$.

Proof. See ([36], theorem 2.4.25 and corollary 2.4.26).

3.3 Results

We assume that assumption 3.1 holds and write, as before, $\Sigma = \{\mathfrak{p},\bar{\mathfrak{p}},\mathfrak{l},\nu_\infty\}$, where $\mathfrak{l}$ is the unique prime dividing the conductor of E/K and ν_∞ is the complex archimedean place of K. We write $\Lambda = \Lambda(G)$. Moreover, for any integral ideal $\mathfrak{b}$ of K prime to $6p\mathfrak{f}$ we define

$$x_{\mathfrak{b}} = \sigma_{\mathfrak{b}} - N(\mathfrak{b}) \in \Lambda(G),$$

where $\sigma_{\mathfrak{b}} = (\mathfrak{b}, K_\infty/K) \in G$ is the image of $\mathfrak{b}$ under the Artin map. Next, we fix an integral auxiliary ideal $\mathfrak{q}$ of K, $(\mathfrak{q}, 6p\mathfrak{f}) = 1$, $\mathfrak{q} \neq \mathcal{O}_K$ such that

$$N(\mathfrak{q}) \equiv 1 \bmod p.$$

Note that by Dirichlet's theorem on arithmetic progressions infinitely many such prime ideals exist, compare ([31], VII, (5.14) p. 490). Henceforth, we will write q for the prime of $\mathbb{Q}$ below $\mathfrak{q}$.

Remark 3.8. By ([36], lemma A.9.2) the element $x_{\mathfrak{q}} = \sigma_{\mathfrak{q}} - N(\mathfrak{q})$ belongs to the Ore set S of Λ.

Theorem 3.9 (Commutative main theorem) *Assume that $\mathfrak{f} = \mathfrak{l}^r$ for some prime $\mathfrak{l}$ of K. Moreover, assume that $\mathcal{A}_\infty = \varprojlim_{k,n}(Cl(K_{k,n})\{p\})$ is S^*-torsion. Then, the following holds for $L_{p,u(\mathfrak{q})} := [1/x_{\mathfrak{q}}] \in K_1(\Lambda_{S^*})$, where $[1/x_{\mathfrak{q}}]$ is the class of $\frac{1}{x_{\mathfrak{q}}} \in \Lambda_S^\times$ in $K_1(\Lambda_{S^*})$.*

(i) under the connecting homomorphism $K_1(\Lambda_{S^*}) \xrightarrow{\partial} K_0(\mathfrak{M}_H(G))$, $L_{p,u(\mathfrak{q})}$ maps to

$$\partial\big([1/x_{\mathfrak{q}}]\big) = -[\Lambda/\Lambda x_{\mathfrak{q}}] = [\mathbb{H}^2_\Sigma] - [\mathbb{H}^1_\Sigma/\Lambda u(\mathfrak{q})] \quad \text{in} \quad K_0(\mathfrak{M}_H(G)), \tag{3.7}$$

where $u(\mathfrak{q})$ was defined in definition 3.6.

(ii) moreover, $\frac{1}{x_{\mathfrak{q}}} = \frac{1}{(\sigma_{\mathfrak{q}} - N\mathfrak{q})}$ satisfies the following interpolation property. Let χ be a complex Artin character $\chi : G_K \longrightarrow \mathbb{C}^\times$ such that the fixed field of the kernel is equal to $\bar{K}^{\ker(\chi)} = K_{k,n}$, $k, n \geq 1$. Then, χ has conductor $\mathfrak{f}_\chi = \mathfrak{f}\bar{\mathfrak{p}}^k\mathfrak{p}^n$ and we have (Kronecker's second limit formula)

$$\frac{d}{ds}L_{\Sigma_f}(\chi, s)\,|_{s=0} = \frac{1}{\chi(\sigma_{\mathfrak{q}}) - N\mathfrak{q}} \cdot \frac{1}{12\omega_{\mathfrak{f}\bar{\mathfrak{p}}^k\mathfrak{p}^n}} \cdot \sum_{\sigma\in G(K_{k,n}/K)} \log|\,\sigma(e_{k,n}(\mathfrak{q}))\,|^2\,\chi(\sigma), \tag{3.8}$$

where $\omega_{\mathfrak{f}\bar{\mathfrak{p}}^k\mathfrak{p}^n}$ denotes the number of roots of unity in K congruent to 1 modulo $\mathfrak{f}\bar{\mathfrak{p}}^k\mathfrak{p}^n$.

Before proceeding with the proof, let us explain that the assumption of $\mathcal{A}_\infty$ being S^*-torsion implies that $\mathbb{H}^2_\Sigma$ and $\mathbb{H}^1_\Sigma/\Lambda u(\mathfrak{q})$ are S^*-torsion.

Remark 3.10 (S^*-torsion modules). (i) Rubin's main theorem 4.1 (i) in [34], stating that $\mathcal{A}_\infty$ and $\mathcal{E}_\infty/\mathcal{C}_\infty$ have the same characteristic ideal, implies that $\mathcal{A}_\infty$ being S^*-torsion is equivalent to $\mathcal{E}_\infty/\mathcal{C}_\infty$ being S^*-torsion. To see this, one can use the fact that for our G $((\cong \mathbb{Z}_p^\times)^2)$ a finitely generated pseudo-null (in the sense of Rubin [34]) $\Lambda(G)$-module M is S^*-torsion. We note that the module $\mathcal{E}_\infty/\mathcal{C}_\infty$ is always $\Lambda(G)$-torsion, see ([34], Corollary 7.8).

(ii) under the assumption that $\mathcal{E}_\infty/\mathcal{C}_\infty$ is S^*-torsion, $\mathbb{H}^1_\Sigma/\Lambda u(\mathfrak{q})$ is also S^*-torsion. In fact, let us fix an auxilliary prime ideal $\mathfrak{c}$ of K prime to $6p\mathfrak{f}$. Then $y_\mathfrak{c} := 1-(\mathfrak{c}, K_\infty/K) \in I$ belongs to S, see ([36], lemma A.9.3). Proposition 3.7 shows that $\mathbb{L}$ induces an isomorphism

$$\mathcal{D}_\infty\big/\Lambda y_\mathfrak{c} u(\mathfrak{q}) \cong IJ\lambda\big/\Lambda y_\mathfrak{c} x_\mathfrak{q}\lambda,$$

and these modules are S-torsion since $y_\mathfrak{c} x_\mathfrak{q}$ belongs to S, see (loc. cit., lemmata A.9.2 and A.9.3). We have note before that $\mathcal{C}_\infty/\mathcal{D}_\infty$ is S-torsion, see (loc. cit., theorem 2.4.16). Hence, $\mathcal{C}_\infty/\Lambda y_\mathfrak{c} u(\mathfrak{q})$ is S-torsion. Under the assumption that $\mathcal{E}_\infty/\mathcal{C}_\infty$ is S^*-torsion, we see that $\mathcal{E}_\infty/\Lambda y_\mathfrak{c} u(\mathfrak{q})$ is S^*-torsion. Since $\Lambda u(\mathfrak{q})\big/\Lambda y_\mathfrak{c} u(\mathfrak{q})$ is S-torsion, we conclude that $\mathcal{E}_\infty/\Lambda u(\mathfrak{q}) \cong \mathbb{H}^1_\Sigma/\Lambda u(\mathfrak{q})$ is S^*-torsion. See (3.1) for the last isomorphism.

(iii) the module $\mathcal{A}_\infty = \varprojlim_{k,n} \big(Cl(K_{k,n})\{p\}\big)$ is also always $\Lambda(G)$-torsion, see ([34], Theorem 5.4). We note that $\mathcal{A}_\infty$ being S^*-torsion is equivalent to $\mathbb{H}^2_\Sigma$ being S^*-torsion, which follows from (3.3), (3.2) (finitely generated $\mathbb{Z}_p$-modules are S-torsion, see ([9], proposition 2.3)) and the fact that $\bigoplus_{\nu\in\Sigma_f} \Lambda(G)\otimes_{\Lambda(G_\nu)} \mathbb{Z}_p \cong \bigoplus_{\nu\in\Sigma_f} \Lambda(G/G_\nu)$ is S-torsion. As for the latter fact, for $\nu = \mathfrak{p}, \bar{\mathfrak{p}}$ the module $\Lambda(G/G_\nu)$ is S-torsion because G_ν is of finite index in G so that $\Lambda(G/G_\nu)$ is finitely generated as a $\mathbb{Z}_p$-module. Moreover, as we have remarked before in theorem 3.5 (where we used the notation $G_\mathfrak{l} = D_\mathfrak{l}$), $\Lambda(G/G_\mathfrak{l})$ is S-torsion, see ([36], corollary 2.4.34).

(iv) in conclusion, if we assume that either $\mathcal{A}_\infty$ or $\mathcal{E}_\infty/\mathcal{C}_\infty$ is S^*-torsion, then the other is S^*-torsion and by (ii) and (iii) $\mathbb{H}^1_\Sigma/\Lambda u(\mathfrak{q})$ and $\mathbb{H}^2_\Sigma$ are S^*-torsion.

Proof (of theorem 3.9). We first consider the relations in $K_0(\mathfrak{M}_H(G))$. We will repeatedly use the fact that classes $[M] \in K_0(\mathfrak{M}_H(G))$ of modules M that are finitely generated as $\mathbb{Z}_p$-modules are equal to the zero class, i.e., $[M] = 0$ in $K_0(\mathfrak{M}_H(G))$, which can be derived from results of G. Zábrádi [49] and K. Ardakov and S. Wadsley [1], [2], see ([36], remark 2.4.36).

Let us fix an auxilliary prime ideal $\mathfrak{c}$ of K prime to $6p\mathfrak{f}$. Then $y_\mathfrak{c} := 1-(\mathfrak{c}, K_\infty/K)$ is not a zero-divisor in Λ and, in fact, belongs to S as noted in remark 3.10 (ii). We then have

$$\begin{aligned}[\mathcal{E}_\infty/\mathcal{C}_\infty] - [\mathcal{E}_\infty/\Lambda y_\mathfrak{c} u(\mathfrak{q})] &= -[\mathcal{C}_\infty/\Lambda y_\mathfrak{c} u(\mathfrak{q})]\\ &= -[\mathcal{D}_\infty/\Lambda y_\mathfrak{c} u(\mathfrak{q})] - [\mathcal{C}_\infty/\mathcal{D}_\infty]\\ &= -[IJ\lambda/\Lambda y_\mathfrak{c} x_\mathfrak{q}\lambda] - [\Lambda(G/D_\mathfrak{l})]\\ &= -[IJ/\Lambda y_\mathfrak{c} x_\mathfrak{q}] - [\Lambda(G/D_\mathfrak{l})]\\ &= -[\Lambda/\Lambda y_\mathfrak{c} x_\mathfrak{q}] - [\Lambda(G/D_\mathfrak{l})],\end{aligned} \tag{3.9}$$

where the first two equations follow from the exact sequences

$$0 \to \mathcal{C}_\infty/\Lambda y_\mathfrak{c} u(\mathfrak{q}) \to \mathcal{E}_\infty/\Lambda y_\mathfrak{c} u(\mathfrak{q}) \to \mathcal{E}_\infty/\mathcal{C}_\infty \to 0$$

and

$$0 \to \mathcal{D}_\infty/\Lambda y_\mathfrak{c} u(\mathfrak{q}) \to \mathcal{C}_\infty/\Lambda y_\mathfrak{c} u(\mathfrak{q}) \to \mathcal{C}_\infty/\mathcal{D}_\infty \to 0,$$

the third equation follows from proposition 3.7 and theorem 3.5, the fourth equation from the fact that λ is not a zero-divisor, see ([36], proposition 2.4.28), and the last equation from the fact that

$$[\Lambda/IJ] = [\Lambda/(I \cap J)] = 0,$$

where the first equality holds since I and J are coprime, see (loc. cit., lemma 2.4.39), and the second stems from the fact that $\Lambda/(I \cap J)$ embeds into the finitely generated $\mathbb{Z}_p$-module $\mathbb{Z}_p \bigoplus \mathbb{Z}_p(1)$.

Recall that $\mathbb{L}(u(\mathfrak{q})) = 12x_\mathfrak{q}\lambda$. Using the fact that $x_\mathfrak{q}\lambda$ is not a zero-divisor, one sees that

$$[\mathcal{E}_\infty/\Lambda y_\mathfrak{c} u(\mathfrak{q})] = [\mathcal{E}_\infty/\Lambda u(\mathfrak{q})] + [\Lambda/\Lambda y_\mathfrak{c}].$$

Similarly, using that $x_\mathfrak{q}$ is not a zero-divisor, we see that

$$[\Lambda/\Lambda y_\mathfrak{c} x_\mathfrak{q}] = [\Lambda/\Lambda x_\mathfrak{q}] + [\Lambda/\Lambda y_\mathfrak{c}].$$

It follows from (3.9) that we have an equation

$$[\mathcal{E}_\infty/\mathcal{C}_\infty] - [\mathcal{E}_\infty/\Lambda u(\mathfrak{q})] = -[\Lambda/\Lambda x_\mathfrak{q}] - [\Lambda(G/D_\mathfrak{l})], \tag{3.10}$$

in which the auxilliary element $y_\mathfrak{c}$ no longer appears. Now, we use Rubin's main result on the two variable main conjecture, see ([34], theorem 4.1 (i)), stating that

$$[\mathcal{E}_\infty/\mathcal{C}_\infty] = [\mathcal{A}_\infty]. \tag{3.11}$$

It follows that

$$-[\Lambda/\Lambda x_\mathfrak{q}] = [\mathcal{A}_\infty] + [\Lambda(G/D_\mathfrak{l})] - [\mathcal{E}_\infty/\Lambda u(\mathfrak{q})]. \tag{3.12}$$

The map from (3.2) has a kernel which is a finitely generated $\mathbb{Z}_p$-module, showing that

$$[\mathcal{A}_\infty] = [\varprojlim_{k,n}(\mathrm{Pic}(\mathcal{O}_{K_{k,n},\Sigma})\{p\})].$$

The modules $\varprojlim_{k,n}(\mathrm{Pic}(\mathcal{O}_{K_{k,n},\Sigma})\{p\})$ and $\Lambda(G/D_\mathfrak{l}) \cong \Lambda(G) \otimes_{\Lambda(D_\mathfrak{l})} \mathbb{Z}_p$ appear in the exact sequence (3.3), yielding

$$[\mathbb{H}^2_\Sigma] = [\varprojlim_{k,n}(\mathrm{Pic}(\mathcal{O}_{K_{k,n},\Sigma})\{p\})] + [\Lambda(G/D_\mathfrak{l})], \tag{3.13}$$

where we used that $\mathrm{Ind}_G^{G_\mathfrak{p}}\mathbb{Z}_p \cong \Lambda(G) \otimes_{\Lambda(G_\mathfrak{p})} \mathbb{Z}_p$ and $\mathrm{Ind}_G^{G_{\bar{\mathfrak{p}}}}\mathbb{Z}_p \cong \Lambda(G) \otimes_{\Lambda(G_{\bar{\mathfrak{p}}})} \mathbb{Z}_p$ are finitely generated over $\mathbb{Z}_p$. Together with (3.1) we can rewrite (3.12) as

$$-[\Lambda/\Lambda x_\mathfrak{q}] = [\mathbb{H}^2_\Sigma] - [\mathbb{H}^1_\Sigma/\Lambda u(\mathfrak{q})],$$

which is what we wanted to prove.

Next, we determine the interpolation property of $x_{\mathfrak{a}}$ for $\mathfrak{a}$ prime to $6p\mathfrak{f}$, which, as noted above, is derived from Kronecker's (second) limit formula. The latter has already been stated in the form in which we want to use it in an article by Flach, see ([16], Lemma 2.2 e), p. 265f). For a lattice $L \subset \mathbb{C}$, $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, Flach defines a Theta-function $\varphi(z,\tau)$, where $\tau = \frac{\omega_1}{\omega_2}$, and using such Theta-functions, he also defines a function $\psi(z, L, \mathfrak{a}^{-1}L)$ for integral ideals $\mathfrak{a}$ of K, $(\mathfrak{a}, 6) = 1$. In ([14], II, 2.1), de Shalit defines a Theta-function $\theta(z, L)$ and thanks to the product expansion of $\theta(z, L/\omega_2)$ for the normalized lattice L/ω_2 one immediately derives an equality

$$\theta(z, L/\omega_2) = \varphi(z,\tau)^{12}.$$

Using the monogeneity property of θ, one can now show for an integral ideal $\mathfrak{a}$ of K, $(\mathfrak{a}, 6) = 1$, that the 12-th power of Flach's function $\psi(z, L, \mathfrak{a}^{-1}L)$ is given by

$$\psi(z, L, \mathfrak{a}^{-1}L)^{12} = \Theta(z, L, \mathfrak{a}).$$

For an integral ideal $\mathfrak{g}$ of K, considered as a lattice in $\mathbb{C}$ and generated over $\mathcal{O}_K$ by g, and $\mathfrak{a}$ such that $(\mathfrak{a}, 6\mathfrak{g}) = 1$ it follows that

$$\psi(1, \mathfrak{g}, \mathfrak{a}^{-1}\mathfrak{g})^{12} = \Theta(1, \mathfrak{g}, \mathfrak{a}) = \Theta\Big(\frac{\Omega}{g}, L, \mathfrak{a}\Big)$$

and we see that Flach's element

$${}_{\mathfrak{a}}z_{\mathfrak{f}\bar{\mathfrak{p}}^k\mathfrak{p}^n} := \psi(1, \mathfrak{f}\bar{\mathfrak{p}}^k\mathfrak{p}^n, \mathfrak{a}^{-1}\mathfrak{f}\bar{\mathfrak{p}}^k\mathfrak{p}^n), \qquad (\mathfrak{a}, 6\mathfrak{f}p) = 1,$$

for $k, n \geq 1$, is a twelfth root of our $e'_{k,n}(\mathfrak{a}) = \Theta\Big(\frac{\Omega}{f\bar{\pi}^k\pi^n}, L, \mathfrak{a}\Big) \in \mathcal{O}^{\times}_{F_{k,n}}$, recall that we write $F_{k,n} = K(\mathfrak{f}\bar{\mathfrak{p}}^k\mathfrak{p}^n)$.

Now, let χ be an Artin character as in the statement of the theorem. The claim that χ has conductor $\mathfrak{f}_\chi = \mathfrak{f}\bar{\mathfrak{p}}^k\mathfrak{p}^n$ follows from a fact from the theory of Artin conductors, see ([31], VII, §11, (11.10) Satz), and from the fact that $K_{k,n}$, $k, n \geq 1$, has conductor $\mathfrak{f}\bar{\mathfrak{p}}^k\mathfrak{p}^n$, see ([36], lemma 2.4.17).

Kronecker's second limit formula, as stated in ([16], Lemma 2.2 e), p. 265f), says

$$\begin{aligned}\frac{d}{ds}L_{\Sigma_f}(\chi, s)\,|_{s=0} &= -\frac{1}{N\mathfrak{a} - \chi(\sigma_{\mathfrak{a}})} \cdot \frac{1}{\omega_{\mathfrak{f}\bar{\mathfrak{p}}^k\mathfrak{p}^n}} \cdot \sum_{\sigma \in G(F_{k,n}/K)} \log|\,\sigma({}_{\mathfrak{a}}z_{\mathfrak{f}\bar{\mathfrak{p}}^k\mathfrak{p}^n})\,|^2\, \chi(\sigma) \\ &= -\frac{1}{N\mathfrak{a} - \chi(\sigma_{\mathfrak{a}})} \cdot \frac{1}{12\omega_{\mathfrak{f}\bar{\mathfrak{p}}^k\mathfrak{p}^n}} \cdot \sum_{\sigma \in G(F_{k,n}/K)} \log|\,\sigma(e'_{k,n}(\mathfrak{a}))\,|^2\, \chi(\sigma) \qquad (3.14)\end{aligned}$$

where $\omega_{\mathfrak{f}\bar{\mathfrak{p}}^k\mathfrak{p}^n}$ denotes the number of roots of unity in K congruent to 1 modulo $\mathfrak{f}\bar{\mathfrak{p}}^k\mathfrak{p}^n$. Note that $\omega_{\mathfrak{f}\bar{\mathfrak{p}}^k\mathfrak{p}^n}$ divides 12. Since χ factors through $G(K_{k,n}/K)$, we immediately conclude that

$$\frac{d}{ds}L_{\Sigma_f}(\chi, s)\,|_{s=0} = \frac{1}{\chi(\sigma_{\mathfrak{a}}) - N\mathfrak{a}} \cdot \frac{1}{12\omega_{\mathfrak{f}\bar{\mathfrak{p}}^k\mathfrak{p}^n}} \cdot \sum_{\sigma \in G(K_{k,n}/K)} \log|\,\sigma(e_{k,n}(\mathfrak{a}))\,|^2\, \chi(\sigma), \qquad (3.15)$$

where $e_{k,n}(\mathfrak{a}) = N_{F_{k,n}/K_{k,n}}(e'_{k,n}(\mathfrak{a})) \in \mathcal{O}^{\times}_{K_{k,n}}$. Compare also the limit formulas in ([14], II, §5, p. 88) and ([24], section 15, p. 252).

We want to conclude this section by making a remark about the interpolation property stated in theorem 3.9.

Remark 3.11. (i) the requirement on χ that $\bar{K}^{\ker(\chi)} = K_{k,n},\ k,n \geq 1$ can be slightly relaxed by requiring only that χ has conductor $\mathfrak{f}_\chi = \mathfrak{f}\bar{\mathfrak{p}}^k\mathfrak{p}^n$ (which follows from the stronger requirement). One then gets the interpolation property as in (3.14) with the units $e'_{k,n}(\mathfrak{q}) \in \mathcal{O}^\times_{F_{k,n}}$ instead of $e_{k,n}(\mathfrak{q}) \in \mathcal{O}^\times_{K_{k,n}}$.

(ii) a question that immediately arises from (3.8) is: when are both sides of the equation unequal to 0? An answer is provided by the following formula for the order of vanishing $r_\Sigma(\chi)$ of $L_{\Sigma_f}(\chi, s)$ at $s = 0$, see ([13], equation (1.11)), which is given by

$$r_\Sigma(\chi) = \begin{cases} \#\{\nu \in \Sigma \mid \chi(G_\nu) = 1\} & \text{if } \chi \neq 1, \\ \#\Sigma - 1 & \text{if } \chi = 1, \end{cases}$$

where G_ν denotes the decomposition group at a place $\nu \in \Sigma = \Sigma_f \cup \{\nu_\infty\}$. Since the complex archimedean place ν_∞ of K has trivial decomposition group, i.e., $G(K_{k,n}/K)_{\nu_\infty} = 1$, we see that $r_\Sigma(\chi) \geq 1$ for all χ. On the other hand, if χ is ramified at the other primes, i.e., at the primes in Σ_f, then $r_\Sigma(\chi) = 1$. Moreover, we see that (3.8) always holds for the trivial character χ, since both sides are equal to 0; for the left hand side, note that $\#\Sigma = 4$ and for the right hand side note that $e_{k,n}(\mathfrak{q})$ is a unit so that

$$\sum_{\sigma \in G(K_{k,n}/K)} \log \mid \sigma(e_{k,n}(\mathfrak{a})) \mid^2 = \log \mid N_{K_{k,n}/K}\sigma(e_{k,n}(\mathfrak{a})) \mid^2 = 0$$

since $N_{K_{k,n}/K}\sigma(e_{k,n}(\mathfrak{a}))$ is a unit in K, i.e., a root of unit, and therefore has absolute value 1.

4 Local Theorem

In this section we study a local conjecture due to Kato, concerning a p-adic Lie extensions $F_\infty/\mathbb{Q}_p$ and the *universal case* $T = \mathbb{Z}_p(1)$. The idea is to prove the universal case and then derive analogous results for more general representations T through twisting (and induction). For general conjectures concerning local ϵ-isomorphisms see [23] and [21].

Venjakob's article [44], based on Kato's work [23], contains a proof of the existence of ϵ-isomorphisms in certain abelian cases (arising from twists of the universal case). In the same paper Venjakob shows that the existence of ϵ-isomorphisms implies the algebraic part of the (abelian version of the) local main conjecture stated in this section. We will build on these results and prove the analytic part of certain cases of the abelian local main conjecture, i.e., we determine the interpolation property of $\mathcal{E}_{u'} = \mathcal{E}_{p,u'}$, see conjecture 4.4 for the precise statement.

4.1 Setting and Notation

Fields and general definitions. Fix a prime number p and an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$. Let $F_\infty/\mathbb{Q}_p$ be a not necessarily abelian p-adic Lie extension containing $\mathbb{Q}_p(\mu_{p^\infty})$. We write $\mathcal{G}' = \mathrm{Gal}(F_\infty/\mathbb{Q}_p)$ and $\mathcal{H}' = \mathrm{Gal}(F_\infty/\mathbb{Q}_p^{cyc})$. Let us write $\mathcal{S}' \subset \mathcal{S}'^* \subset \Lambda(\mathcal{G}')$ and $\tilde{\mathcal{S}}' \subset \tilde{\mathcal{S}}'^* \subset \widehat{\mathbb{Z}}_p^{ur}[[\mathcal{G}']]$ for the canonical Ore sets, as defined in (2.1) and (2.2). We fix a generator $\epsilon = (\epsilon_n)_n = (\zeta_{p^n})_n$ of $\mathbb{Z}_p(1) = \varprojlim_n \mu_{p^n}$, where μ_{p^n}, $n \geq 1$, denotes the group of p^n-th roots of unities in $\overline{\mathbb{Q}}_p$. This choice determines a unique homomorphism $\psi_\epsilon : \mathbb{Q}_p \longrightarrow \mathbb{C}_p^\times$ with kernel $\ker(\psi_\epsilon) = \mathbb{Z}_p$, such that $\psi_\epsilon(1/p^n) = \zeta_{p^n}$, $n \geq 1$.

Iwasawa modules. For an arbitrary, possibly infinite algebraic extension $L/\mathbb{Q}_p$ we set

$$\mathcal{U}'(L) = \varprojlim_{L',m} \mathcal{O}_{L'}^\times/(\mathcal{O}_{L'}^\times)^{p^m},$$

where the limit is taken over all finite subextensions $L'/\mathbb{Q}_p$ of $L/\mathbb{Q}_p$ with respect to norm maps and all $m \in \mathbb{N}$ with respect to the natural projections. Here, $\mathcal{O}_{L'}^\times$ is the unit group of the ring of integers $\mathcal{O}_{L'}$ of L'. For any extension F/L, there is a canonical projection map $\mathcal{U}'(F) \longrightarrow \mathcal{U}'(L)$. We also consider the cohomology groups

$$\mathbb{H}^i_{\mathrm{loc}} = \mathrm{H}^i(\mathbb{Q}_p, \mathbb{T}_{un}), \quad i = 1, 2,$$

with $\mathbb{T}_{un} = \mathbb{T}_{un}(F_\infty)$, compare [44], defined as

$$\mathbb{T}_{un} = \Lambda(\mathcal{G}')^{\#}(1), \tag{4.1}$$

where (1) denotes the Tate twist and $\Lambda(\mathcal{G}')^{\#}$ is just $\Lambda(\mathcal{G}')$ as a $\Lambda(\mathcal{G}')$-module, but has the following action of $G_{\mathbb{Q}_p}$. An element $g \in G_{\mathbb{Q}_p}$ acts on $\lambda \in \Lambda(\mathcal{G}')$ by $g.\lambda = \lambda \overline{g}^{-1}$, where $\overline{g}$ is the image of g in $\mathcal{G}'$. The Kummer sequence for local fields induces the isomorphism on the right of

$$\mathbb{H}^1_{\mathrm{loc}} \cong \varprojlim_{\mathbb{Q}_p \subset_f F \subset F_\infty} H^1(F, \mathbb{Z}_p(1)) \cong \varprojlim_{F,m} F^\times/(F^\times)^{p^m},$$

where $\subset_f$ means that $F/\mathbb{Q}_p$ is a finite extension. For the isomorphism on the left see ([36], (A.3.12)). For extensions $F_\infty/\mathbb{Q}_p$ of infinite residue degree, we have

$$\mathcal{U}'(F_\infty) \cong \mathbb{H}^1_{\mathrm{loc}},$$

compare ([44], section 2.1). We also have

$$\mathbb{H}^2_{\mathrm{loc}} = \mathrm{H}^2(\mathbb{Q}_p, \mathbb{T}_{un}) = \varprojlim_{\mathbb{Q}_p \subseteq_f L \subset F_\infty} \mathrm{H}^2(L, \mathbb{Z}_p(1)) \cong \mathbb{Z}_p$$

by local Tate duality.

Class field theory. We use the following convention.

Convention 4.1. *For the reciprocity map from local class field theory for a general non-archimedean local field F we assume that a prime element π_F of F corresponds to a geometric Frobenius element φ_{geo}, i.e. a map that, on an algebraic closure $\overline{k}_F$ of the residue field k_F of F, corresponds to $x \longmapsto x^{\frac{1}{q_F}}$ where q_F is the number of elements of k_F. Writing $W(\bar{F}/F)$ for the Weil group and I_F for the inertia group, as in [15], we then have a commutative diagram*

$$\begin{array}{ccccc} I_F & \longrightarrow & W(\bar{F}/F) & \longrightarrow & \mathbb{Z} \\ \downarrow & & \downarrow {\scriptstyle \varphi_{geo} \mapsto \pi_F} & & \downarrow {\scriptstyle \cdot(-1)} \\ \mathcal{O}_F^\times & \longrightarrow & F^\times & \xrightarrow{v_F} & \mathbb{Z} \end{array} \tag{4.2}$$

where v_F is the valuation of F and the upper horizontal map is defined as the composite $G_F \twoheadrightarrow G_{\overline{k}_F} \to \hat{\mathbb{Z}}$ where the second map sends the arithmetic Frobenius to 1.

Remark 4.2. Convention 4.1 is in line with the conventions in Deligne's article [15], compare ([15], p. 523), Tate's article [41], compare ([41], p. 6), and the conventions in the book [8] by Bushnell and Henniart, compare ([8], p. 186).

Torsion categories. Let us write $\Lambda' = \Lambda(\mathcal{G}')$ and $\tilde{\Lambda}' = \Lambda(\mathcal{G}', \hat{\mathbb{Z}}_p^{ur})$ for the Iwasawa algebra of $\mathcal{G}'$ with coefficients in $\hat{\mathbb{Z}}_p^{ur}$ and note that $\tilde{\Lambda}' \cong \hat{\mathbb{Z}}_p^{ur} \hat{\otimes}_{\mathbb{Z}_p} \Lambda'$. It is explained in section 3.1 of [36] that for a finitely generated Λ'-module M we have a natural isomorphism

$$\tilde{\Lambda}' \otimes_{\Lambda'} M \cong \hat{\mathbb{Z}}_p^{ur} \hat{\otimes}_{\mathbb{Z}_p} M \tag{4.3}$$

induced by the universal property of $- \otimes_{\Lambda'} -$, and also that $\tilde{\Lambda}' \otimes_{\Lambda'} -$ induces a map

$$K_0(\mathfrak{M}_{\mathcal{H}'}(\mathcal{G}')) \longrightarrow K_0(\mathfrak{M}_{\hat{\mathbb{Z}}_p^{ur}, \mathcal{H}'}(\mathcal{G}')), \quad [M] \longrightarrow [\tilde{\Lambda}' \otimes_{\Lambda'} M] = [\hat{\mathbb{Z}}_p^{ur} \hat{\otimes}_{\mathbb{Z}_p} M].$$

4.2 Statement of the Local Main Conjecture

We make the following assumption, which is satisfied by the isomorphism from (4.13) in the setting considered later in section 4.3, i.e., the setting considered by Venjakob in [44].

Assumption 4.3. *There exists $u' \in \mathcal{U}'(F_\infty)$ such that*

$$\Lambda(\mathcal{G}')_{\mathcal{S}'} \to \mathcal{U}'(F_\infty)_{\mathcal{S}'}, \qquad 1 \longmapsto u',$$

is an isomorphism of $\Lambda(\mathcal{G}')_{\mathcal{S}'}$-modules. We will say that u' is a local generator of $\mathcal{U}'(F_\infty)$.

We now state the local main conjecture.

Conjecture 4.4 (Local Main Conjecture). *There exists $\mathcal{E}_{p,u'} \in K_1(\tilde{\Lambda}'_{\tilde{\mathcal{S}}'^*})$ such that*

(i) for any Artin representation $\rho : \mathcal{G}' \longrightarrow \mathrm{Aut}_{\mathbb{C}_p}(V)$,

$$\mathcal{E}_{p,u'}(\rho) = \frac{\epsilon_p(\rho)}{R_p(u', \rho)} \tag{4.4}$$

if $R_p(u', \rho) \neq 0$, where $\epsilon_p(\rho) = \epsilon_p(V)$ is the local constant attached to V and $R_p(u', \rho)$ is the p-adic regulator associated to ρ and u', see subsection 4.3.3,

(ii) the image of $\mathcal{E}_{p,u'}$ under the connecting homomorphism $\partial : K_1(\tilde{\Lambda}'_{\tilde{\mathcal{S}}'^*}) \to K_0(\mathfrak{M}_{\widehat{\mathbb{Z}}_p^{\mathrm{ur}}, \mathcal{H}'}(\mathcal{G}'))$ from K-theory is given by

$$\partial(\mathcal{E}_{p,u'}) = \left[\tilde{\Lambda}' \otimes_{\Lambda'} \mathbb{H}^2_{\mathrm{loc}}\right] - \left[\tilde{\Lambda}' \otimes_{\Lambda'} \left(\mathbb{H}^1_{\mathrm{loc}} / \Lambda' u'\right)\right] \quad \text{in} \quad K_0(\mathfrak{M}_{\widehat{\mathbb{Z}}_p^{\mathrm{ur}}, \mathcal{H}'}(\mathcal{G}')). \tag{4.5}$$

4.3 Interpolation Property in Certain Abelian Cases

At the end of section 2 in [44] an element $\mathcal{E}_{p,u'}$ satisfying (4.5) is given explicitly under the assumption that $F_\infty/\mathbb{Q}_p$ is an abelian p-adic Lie extension of the form $K'(\mu_{p^\infty})$, where K' is an infinite unramified extension of $\mathbb{Q}_p$. We will prove that in this case $\mathcal{E}_{p,u'}$ has the desired interpolation property (4.4) for Artin characters. In order to determine the values of $\mathcal{E}_{p,u'}$ at Artin characters in the abelian setting, let us recall its construction. In particular, we need to review Coleman's interpolation theory for the multiplicative formal group $\hat{\mathbb{G}}_m$, see

Coleman's article [11] for general Lubin-Tate groups and also [12] for applications; there is also a summary contained in [14] and for the theory over $\mathbb{Q}_p$ compare [10].

For any $L/\mathbb{Q}_p$ we set $L_n = L(\mu_{p^n})$ and $L_\infty = \bigcup_n L_n$. We assume throughout the rest of this section that $K'/\mathbb{Q}_p$ is an unramified Galois extension of infinite degree and that $\mathcal{G}' = \mathrm{Gal}(K'_\infty/\mathbb{Q}_p)$ is an abelian p-adic Lie group. As noted in [44], these assumptions imply that $\mathcal{G}' \cong \mathbb{Z}_p^2 \times \Delta'$, where Δ' is a finite group of order prime to p. There is also a decomposition of $\mathcal{G}'$

$$\mathcal{G}' \cong \Gamma \times H \tag{4.6}$$

into the ramified part $\Gamma = G(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p) \cong \mathbb{Z}_p^\times$ and the unramified part $H = G(K'/\mathbb{Q}_p)$. We will use the notation introduced in the previous subsections for general F_∞ from now on for $F_\infty = K'_\infty$.

4.3.1 Coleman map

In this subsection we follow [44] and explain that there is an exact sequence

$$0 \longrightarrow \mathcal{U}'(K'_\infty) \overset{\mathcal{L}_{K',\epsilon}}{\longrightarrow} \mathbb{T}_{\mathrm{un}}(K'_\infty) \otimes_{\Lambda'} \Lambda'_{\varphi_p} \longrightarrow \mathbb{Z}_p(1) \longrightarrow 0 \tag{4.7}$$

which arises as the projective limit of the compatible exact sequences (4.8) below and where Λ'_{φ_p} is a Λ' submodule of $\tilde{\Lambda}'$ defined as follows. First, we define ϕ to be the arithmetic Frobenius (given by $a \mapsto a^p$ on residue fields) in $G(\mathbb{Q}_p^{ur}/\mathbb{Q}_p)$. Note that the action of ϕ extends to $\widehat{\mathbb{Z}}_p^{ur}$. Next, we define an element $\varphi_p \in \mathcal{G}'$ that, in the decomposition (4.6), corresonds to

$$\varphi_p = (id_\Gamma, \phi_{|K'}),$$

i.e., φ_p is the element that acts trivially on $\mathbb{Q}_p(\mu_{p^\infty})$ and as the arithmetic Frobenius on K'. Now, we let ϕ act on $\tilde{\Lambda}'$ through its action on the coefficients and denote this action by $\phi(x)$, $x \in \tilde{\Lambda}'$. We set

$$\Lambda'_{\varphi_p} := \{x \in \tilde{\Lambda}' \mid \phi(x) = \varphi_p \cdot x\},$$

where $\varphi_p \cdot x$ is multiplication in the Iwasawa algebra $\tilde{\Lambda}'$. Since ϕ acts trivially on the coefficients of any element in Λ' it is clear that Λ'_{φ_p} is a Λ'-module.

We note that for any quotient of $\mathcal{G}'$ of the form $(\Gamma/U) \times H_L$, where $H_L = G(L/\mathbb{Q}_p)$, $L \subset K'$, we can define a $\Lambda((\Gamma/U) \times H_L)$-submodule $\Lambda((\Gamma/U) \times H_L)_{\bar{\varphi}_p}$ of $\widehat{\mathbb{Z}}_p^{ur}[[(\Gamma/U) \times H_L]]$ in the same manner

$$\Lambda((\Gamma/U) \times H_L)_{\bar{\varphi}_p} := \{x \in \widehat{\mathbb{Z}}_p^{ur}[[(\Gamma/U) \times H_L]] \mid \phi(x) = \bar{\varphi}_p \cdot x\},$$

where $\bar{\varphi}_p$ denotes the image of φ_p in $\widehat{\mathbb{Z}}_p^{ur}[[(\Gamma/U) \times H_L]]$. If $(\Gamma/U) \times H_L$ is a finite quotient of $\mathcal{G}'$ then we have

$$\widehat{\mathbb{Z}}_p^{ur}[[(\Gamma/U) \times H_L]] = \widehat{\mathbb{Z}}_p^{ur}[(\Gamma/U) \times H_L] = \widehat{\mathbb{Z}}_p^{ur}[\Gamma/U][H_L]$$

and we will also write $\mathbb{Z}_p[\Gamma/U][H_L]_{\bar{\varphi}_p}$ for $\Lambda((\Gamma/U) \times H_L)_{\bar{\varphi}_p}$.

Let $L/\mathbb{Q}_p$ be a finite unramified extension contained in K', i.e., $L \subset K'$. Then, as in (4.6), we have a decomposition of the Galois group

$$G(L_\infty/\mathbb{Q}_p) \cong \Gamma \times H_L,$$

into a ramified and an unramified part, where $H_L = G(L/\mathbb{Q}_p)$. Writing $\Lambda'(L) = \Lambda\big(\Gamma \times H_L\big)$ there is an exact sequence

$$0 \longrightarrow \mathbb{Z}_p(1) \longrightarrow \mathcal{U}'(L_\infty) \xrightarrow{\mathcal{L}_{L,\epsilon}} \mathbb{T}_{\mathrm{un}}(L_\infty) \otimes_{\Lambda'(L)} \Lambda'(L)_{\bar{\varphi}_p} \longrightarrow \mathbb{Z}_p(1) \longrightarrow 0, \tag{4.8}$$

where $\mathcal{L}_{L,\epsilon}$ is given by the composite map

$$\mathcal{U}'(L_\infty) \xrightarrow{\mathrm{Col}_{\epsilon,L}} \mathcal{O}_L[[\Gamma]] \xrightarrow{\sim} \Lambda'(L)_{\bar{\varphi}_p} \xrightarrow{\sim} \mathbb{T}_{\mathrm{un}}(L_\infty) \otimes_{\Lambda'(L)} \Lambda'(L)_{\bar{\varphi}_p}, \tag{4.9}$$

which is defined as follows. The third map is given by $x \longmapsto (1 \otimes \epsilon) \otimes x$, where $(1 \otimes \epsilon) \in \mathbb{T}_{un}(L_\infty) = \Lambda(\Gamma \times H_L)^{\#} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1)$. The second map in (4.9), which is an isomorphism of $\Lambda'(L)$-modules, not of algebras, is given as follows. If $d = [L : \mathbb{Q}_p]$ is the degree of L over $\mathbb{Q}_p$, then H_L is a cyclic group $H_L = \{\mathrm{id}, \bar{\varphi}_p, \ldots, \bar{\varphi}_p{}^{d-1}\}$ generated by the image of φ_p in H_L. In ([44], Proposition 2.1), Venjakob shows that for all open normal subgroups U of Γ there is an isomorphism given by

$$\mathcal{O}_L[\Gamma/U] \xrightarrow{\sim} \mathbb{Z}_p[\Gamma/U][H_L]_{\bar{\varphi}_p} \subset \widehat{\mathbb{Z}}_p^{\mathrm{ur}}[\Gamma/U][H_L], \quad a \longmapsto \sum_{i=0}^{d-1} \phi^{-i}(a)\bar{\varphi}_p{}^i.$$

Passing to the limit with respect to open normal subgroups U and the canonical projection maps one obtains the second map in (4.9). The first map in (4.9) is the Coleman map, which we want to review in order to introduce some notation. First recall that the p-adic completion $\varprojlim_m \mathcal{O}_{L'}^\times/(\mathcal{O}_{L'}^\times)^{p^m}$ of the units $\mathcal{O}_{L'}^\times$ in a finite extension L' of $\mathbb{Q}_p$ can be naturally identified with the principal units $\mathcal{O}_{L'}^1$ in $\mathcal{O}_{L'}^\times$. In particular, we have

$$\mathcal{U}'(L_\infty) \overset{\mathrm{def}}{=} \varprojlim_{L',m} \mathcal{O}_{L'}^\times/(\mathcal{O}_{L'}^\times)^{p^m} \cong \varprojlim_{L'} \mathcal{O}_{L'}^1 \subset \varprojlim_{L'} \mathcal{O}_{L'}^\times,$$

where we let L' range through the finite subextensions of $L_\infty/\mathbb{Q}_p$, or, equivalently, through the (cofinal subsystem of) finite subextensions of L_∞/L, note that we still assume $[L : \mathbb{Q}_p] < \infty$. For our fixed generator $\epsilon = (\zeta_{p^n})_n$ of the Tate module $\mathbb{Z}_p(1) = \varprojlim_n \mu_{p^n}$ Coleman's theory gives a map

$$\varprojlim_n \mathcal{O}_{L_n}^\times \hookrightarrow \mathcal{O}_L[[T]]^\times, \quad u = (u_n)_n \longrightarrow f_u$$

such that

$$(f_u^{\phi^{-n}})(\zeta_{p^n} - 1) = u_n, \qquad \forall n \geq 1,$$

where ϕ acts on the coefficients of f_u. The elements $\zeta_{p^n} - 1$ are the p^n-torsion points in $\hat{\mathbb{G}}_m(\mathfrak{m})$, where $\mathfrak{m}$ is the valuation ideal of $\mathbb{C}_p$, since multiplication by p^n on the formal multiplicative group $\hat{\mathbb{G}}_m$ is given by

$$[p^n]_{\hat{\mathbb{G}}_m}(X) = (1 + X)^{p^n} - 1.$$

We define a $\mathbb{Z}_p$-algebra homomorphism φ by

$$\varphi : \mathcal{O}_L[[T]] \longrightarrow \mathcal{O}_L[[T]], \quad f(T) \longmapsto f^\phi((1 + T)^p - 1),$$

where f^ϕ means that ϕ acts on the coefficients of f. Let us fix the topological generator $\gamma = 1_{\mathbb{Z}_p}$ of $\mathbb{Z}_p$ considered as an additive profinite group. Then, there is an isomorphism

$$\mathcal{M} : \mathcal{O}_L[[\mathbb{Z}_p]] \xrightarrow{\sim} \mathcal{O}_L[[T]], \quad 1_{\mathbb{Z}_p} \longmapsto 1 + T,$$

which is non-canonical since it depends on the choice of γ, see ([45], Theorem 7.1). Here, $\mathcal{M}$ stands for Mahler transform, compare ([10], §3.3) and Mahler's article [28]. We note that there is a multiplicative norm operator $\mathcal{N}$ on $\mathcal{O}_L[[T]]$, see [14], such that for $f_u \in \mathcal{O}_L[[T]]^\times$, $u \in \varprojlim_n \mathcal{O}_{L_n}^\times$, we have $\mathcal{N} f_u = f_u^\phi$. Using property (loc. cit., I, §2.1, equation (1)) of $\mathcal{N}$ one gets an equality

$$\varphi(f_u) = (\mathcal{N} f_u)((1+T)^p - 1) = \prod_{\zeta \in \mu_p} f_u\big(T\ [+]_{\hat{\mathbb{G}}_m} (\zeta - 1)\big) = \prod_{\zeta \in \mu_p} f_u(\zeta(1+T) - 1).$$

Let $u \in \varprojlim_{L'} \mathcal{O}_{L'}^1$ now be a norm-compatible system of principal units. One can then show that $\frac{1}{p}\log\Big(\frac{f_u^p}{\varphi(f_u)}\Big)$ has integral coefficients, see (loc. cit., I, §3.3 Lemma). The integral measure $\mu_u \in \mathcal{O}_L[[\mathbb{Z}_p]]$ satisfying $\mathcal{M}(\mu_u) = \frac{1}{p}\log\Big(\frac{f_u^p}{\varphi(f_u)}\Big)$ is supported on $\mathbb{Z}_p^\times$, by which we mean that μ_u belongs to the image of the map

$$\iota : \mathcal{O}_L[[\mathbb{Z}_p^\times]] \hookrightarrow \mathcal{O}_L[[\mathbb{Z}_p]], \tag{4.10}$$

which sends a measure λ on $\mathbb{Z}_p^\times$ to the measure $\iota(\lambda) = \tilde{\lambda}$ extended by 0 to $p\mathbb{Z}_p$, see (loc. cit., §3.3).

Let us write $\kappa : \Gamma \xrightarrow{\sim} \mathbb{Z}_p^\times$ for the cyclotomic character. We define

$$\kappa_* : \mathcal{O}_L[[\Gamma]] \longrightarrow \mathcal{O}_L[[\mathbb{Z}_p^\times]]$$

to be the isomorphism of Iwasawa algebras induced by κ. Finally we can define the first map in (4.9)

$$\mathcal{U}'(L_\infty) \cong \varprojlim_{L'} \mathcal{O}_{L'}^1 \xrightarrow{\mathrm{Col}_{\epsilon,L}} \mathcal{O}_L[[\Gamma]], \quad u \longmapsto \kappa_*^{-1}\iota^{-1}\mathcal{M}^{-1}\Big(\frac{1}{p}\log\Big(\frac{f_u^p}{\varphi(f_u)}\Big)\Big), \tag{4.11}$$

where we note that only the element f_u depends on the choice of ϵ. We have now defined the map $\mathcal{L}_{L,\epsilon}$ from (4.8). The first non-trivial map of (4.8) is just the inclusion and the last non-trivial map (via the identifications from (4.9)) is given by the composite of $\mathcal{O}_L[[\Gamma]] \to \mathcal{O}_L(1)$ (induced by mapping $\gamma \in \Gamma$ to $\kappa(\gamma)$) and $\mathrm{Tr}_{L/\mathbb{Q}_p} \otimes \mathrm{id} : \mathcal{O}_L \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1) \longrightarrow \mathbb{Z}_p \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1)$. It is explained in [44] that, with respect to the norm maps and the natural projection maps, the sequences from (4.8) are compatible for extensions $\mathbb{Q}_p \subset_f L \subset_f L' \subset K'$. Passing to the projective limit one gets the sequence from (4.7).

Remark 4.5. We note that starting with the generator $\epsilon^{-1} = (\zeta_{p^n}^{-1})_n$ of $\mathbb{Z}_p(1)$ we get in an entirely similar fashion a map $\mathrm{Col}_{\epsilon^{-1},L}$ and an exact sequence

$$0 \longrightarrow \mathcal{U}'(K'_\infty) \xrightarrow{-\mathcal{L}_{K',\epsilon^{-1}}} \mathbb{T}_{\mathrm{un}}(K'_\infty) \otimes_{\Lambda'} \Lambda'_{\varphi_p} \longrightarrow \mathbb{Z}_p(1) \longrightarrow 0 \tag{4.12}$$

as in (4.7) for ϵ^{-1} and $-\mathcal{L}_{K',\epsilon^{-1}}$.

We fix a Λ'-basis of $\mathbb{T}_{\mathrm{un}}(K'_\infty) \otimes_{\Lambda'} \Lambda'_{\varphi_p}$ which is free of rank one, see (loc. cit., Proposition 2.1), i.e., we fix an isomorphism $\delta : \mathbb{T}_{\mathrm{un}}(K'_\infty) \otimes_{\Lambda'} \Lambda'_{\varphi_p} \cong \Lambda'$. Since $\mathbb{Z}_p(1)$ is $\mathcal{S}'$-torsion, the exact sequence from (4.12) induces an isomorphism

$$\mathcal{U}'(K'_\infty)_{\mathcal{S}'} \xrightarrow{-\mathcal{L}_{K',\epsilon^{-1}}} \big(\mathbb{T}_{\mathrm{un}}(K'_\infty) \otimes_{\Lambda'} \Lambda'_{\varphi_p}\big)_{\mathcal{S}'} \overset{\delta}{\cong} \Lambda'_{\mathcal{S}'}, \tag{4.13}$$

showing that assumption 4.3 is satisfied, for which one should also note that $\mathcal{S}'$ does not contain any zero-divisors, see ([9], Theorem 2.4). Let us deduce an exact sequence of $\tilde{\Lambda}'$-modules from (4.12). We first recall that the intersection

$$\Lambda'_{\varphi_p} \cap (\tilde{\Lambda}')^\times \neq \varnothing$$

is non-empty, see ([44], §2.2) or ([21], Proposition 3.4.5), where we note that $K_1(\tilde{\Lambda}') \cong (\tilde{\Lambda}')^\times$ for our abelian p-adic Lie group $\mathcal{G}'$. Now, we have a canonical isomorphism

$$\Lambda'_{\varphi_p} \otimes_{\Lambda'} \tilde{\Lambda}' \cong \tilde{\Lambda}', \quad a \otimes b \mapsto a \cdot b \tag{4.14}$$

as in ([44], explanation preceding equation (2.18)) with inverse $x \mapsto c \otimes c^{-1}x$, where c is an element belonging to $\Lambda'_{\varphi_p} \cap (\tilde{\Lambda}')^\times$ and which does not depend on c (recall that Λ'_{φ_p} is free of rank one as a Λ'-module). Next, tensoring the exact sequence (4.12) with $\otimes_{\Lambda'}\tilde{\Lambda}'$ and using the isomorphism from (4.14), we get the exact sequence

$$0 \longrightarrow \mathcal{U}'(K'_\infty) \otimes_{\Lambda'} \tilde{\Lambda}' \xrightarrow{-\mathcal{L}_{K',\epsilon^{-1}}} \mathbb{T}_{\mathrm{un}}(K'_\infty) \otimes_{\Lambda'} \tilde{\Lambda}' \longrightarrow \hat{\mathbb{Z}}_p^{ur}(1) \longrightarrow 0, \tag{4.15}$$

see ([36], proposition A.8.17 and remark 3.1.4) for the fact that $\otimes_{\Lambda'}\tilde{\Lambda}'$ is an exact functor.

4.3.2 Definition of $\mathcal{E}_{p,u'}$

Let $u' \in \mathcal{U}'(K'_\infty)$ be an element as in assumption 4.3, which exists by (4.13). We now define $\mathcal{E}_{p,u'} = \mathcal{E}_{u'} \in (\tilde{\Lambda}'_{\tilde{\mathcal{S}}'})^\times$ by the equation

$$-\mathcal{L}_{K',\epsilon^{-1}}(u' \otimes 1) = \mathcal{E}_{u'}^{-1} \cdot ((1 \otimes \epsilon) \otimes 1), \tag{4.16}$$

for which we recall that $\epsilon = (\zeta_{p^n})_n$ is our fixed generator of $\mathbb{Z}_p(1)$. Let us introduce some notation and then make a remark about $\mathcal{E}_{u'}$. For $L \subseteq K'$, $[L : \mathbb{Q}_p] < \infty$, we will write u_{L_∞} and u_{L_n} for the images of any $u \in \mathcal{U}'(K'_\infty)$ under the maps $\mathcal{U}'(K'_\infty) \longrightarrow \mathcal{U}'(L_\infty)$ and $\mathcal{U}'(K'_\infty) \longrightarrow \mathcal{U}'(L_n) = \varprojlim_m \mathcal{O}_{L_n}^\times/(\mathcal{O}_{L_n}^\times)^{p^m} \cong \hat{\mathcal{O}}_{L_n}^\times$, respectively. Moreover, we will write $\Gamma_n = G(\mathbb{Q}_p(\zeta_{p^n})/\mathbb{Q}_p)$, $n \geq 1$, and for an element $\lambda \in \mathcal{O}_L[[\Gamma]] \cong \varprojlim_n \mathcal{O}_L[\Gamma_n]$ we write $\lambda = (\lambda_n)_n$ where λ_n, $n \geq 1$, belongs to the group ring $\mathcal{O}_L[\Gamma_n]$. For example, we will write $\big(\mathrm{Col}_{\epsilon^{-1},L}(u_{L_\infty})_n\big)_n$. Let us write $\mathcal{O}_{K'} = \bigcup_L \mathcal{O}_L$ for the valuation ring of K', where L ranges through all finite subextensions of $K'/\mathbb{Q}_p$.

Remark 4.6. First note that $\mathcal{E}_{u'}$ is actually a unit in $\tilde{\Lambda}'_{\tilde{\mathcal{S}}'}$ since both $-\mathcal{L}_{K',-\epsilon}(u' \otimes 1)$ and $(1 \otimes \epsilon) \otimes 1$ are a basis of the free rank one $\tilde{\Lambda}'_{\tilde{\mathcal{S}}'}$-module $\mathbb{T}_{\mathrm{un}}(K'_\infty) \otimes_{\Lambda'} \tilde{\Lambda}'_{\tilde{\mathcal{S}}'}$. Moreover, $\mathcal{E}_{u'}^{-1}$ even belongs to $\mathcal{O}_{K'}[[\mathcal{G}']] \bigcap (\tilde{\Lambda}'_{\tilde{\mathcal{S}}'})^\times$ since for $L \subset K'$ a finite extension of $\mathbb{Q}_p$ of degree d_L we have

$$-\mathcal{L}_{L,\epsilon^{-1}}(u'_{L_\infty}) = (1 \otimes \epsilon) \otimes \Big(-\sum_{i=0}^{d_L-1} \phi^{-i}\big(\mathrm{Col}_{\epsilon^{-1},L}(u'_{L_\infty})_n\big)\bar{\varphi}_p^i\Big)_n \quad \in \; \mathbb{T}_{\mathrm{un}}(L_\infty) \otimes \varprojlim_n \mathbb{Z}_p[\Gamma_n][H_L]_{\bar{\varphi}_p},$$

where the elements $\phi^{-i}\big(\mathrm{Col}_{\epsilon^{-1},L}(u'_{L_\infty})_n\big)$ belong to $\mathcal{O}_L[\Gamma_n]$, i.e., have coefficients in $\mathcal{O}_L$. It follows that we get a compatible system of elements

$$\mathcal{E}_{u'_{L_\infty}}^{-1} := \Big(-\sum_{i=0}^{d_L-1} \phi^{-i}\big(\mathrm{Col}_{\epsilon^{-1},L}(u'_{L_\infty})_n\big)\bar{\varphi}_p^i\Big)_n \quad \in \; \varprojlim_n \mathbb{Z}_p[\Gamma_n][H_L]_{\bar{\varphi}_p} \subset \varprojlim_n \mathcal{O}_L[\Gamma_n][H_L]$$

for L ranging through all finite extensions of $\mathbb{Q}_p$ contained in K'. For the inclusion $\mathbb{Z}_p[\Gamma_n][H_L]_{\bar{\varphi}_p} \subset \mathcal{O}_L[\Gamma_n][H_L]$ see the proof of ([44], Proposition 2.1). This system gives $\mathcal{E}_{u'}^{-1} = (\mathcal{E}_{u'_{L_\infty}}^{-1})_L$ which has coefficients in $\mathcal{O}_{K'}$.

4.3.3 Local constants and p-adic regulators

Recall that $\Gamma = G(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p)$, $H = G(K'/\mathbb{Q}_p)$. Let $\chi : \mathcal{G}' \cong H \times \Gamma \longrightarrow \mathbb{C}_p^\times$, be an Artin character, i.e., a character with finite image. We fix χ for the rest of this section. Let us restrict χ to H and define L to be the fixed field of the kernel of $\chi_{|H}$, such that

$$\chi : G(L/\mathbb{Q}_p) \hookrightarrow \mathbb{C}_p^\times$$

is injective. Since χ is an Artin character, $L/\mathbb{Q}_p$ is a finite extension and we write d for the degree of $L/\mathbb{Q}_p$. Moreover, we write $H_L = G(L/\mathbb{Q}_p)$. Likewise, restrict χ to Γ and let n be the smallest integer such that $\chi_{|\Gamma}$ factors through $\Gamma_n = G(\mathbb{Q}_p(\zeta_{p^n})/\mathbb{Q}_p)$ but not through Γ_{n-1}. We will consider χ as a character of the finite group $H_L \times \Gamma_n \cong G(L(\zeta_{p^n})/\mathbb{Q}_p)$. Restricted to $\Gamma_n \cong (\mathbb{Z}_p/p^n)^\times$, χ can also be interpreted as a primitive Dirichlet-character modulo p^n.

Definition 4.7 (Local constants). *Let $\epsilon^{-1} = (\zeta_{p^m}^{-1})_m$ be the inverse of our fixed generator of $\mathbb{Z}_p(1)$. We write*

$$\psi_{\epsilon^{-1}} : \mathbb{Q}_p \longrightarrow \mathbb{C}_p^\times$$

for the map with kernel equal to $\mathbb{Z}_p$ and $\psi_{\epsilon^{-1}}(1/p^m) = \zeta_{p^m}^{-1}$. Moreover, let $dx = dx_1$ be the Haar measure on $\mathbb{Q}_p$ assigning the value 1 to $\mathbb{Z}_p$, see the discussion in ([41], section (3.6)) for implications of this convention. Let us write $a_{\mathbb{Q}_p} : \mathbb{Q}_p^\times \to W(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ for the reciprocity map satisfying our sign convention 4.1. We also interpret χ as a character

$$\chi : \mathbb{Q}_p^\times \xrightarrow{a_{\mathbb{Q}_p}} W(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \hookrightarrow G_{\mathbb{Q}_p} \twoheadrightarrow \mathcal{G}' \to \mathbb{C}_p^\times$$

of $\mathbb{Q}_p^\times$ and the integer $n \geq 0$ (which is the smallest integer $m \geq 0$ such that $\chi_{|\Gamma}$ factors through Γ_m) is then its conductor. We now define the local constant attached to χ by

$$\varepsilon_p(\chi, \psi_{\epsilon^{-1}}, dx) = \begin{cases} \sum_{m\in\mathbb{Z}} \int_{p^m\mathbb{Z}_p^\times} \chi(x)^{-1}\psi_{\epsilon^{-1}}(x)\ dx & \text{if } n \geq 1, \quad \text{(ramified case)} \\ 1 & \text{if } n = 0, \quad \text{(unramified case)}, \end{cases} \tag{4.17}$$

which is in line with [15] and [41].

Remark 4.8. In the ramified case, i.e., when $n \geq 1$, the local constant $\varepsilon(\chi, \psi_{\epsilon^{-1}}, dx)$ can be expressed as a Gauß sum. In fact, first note that our Haar measure $\mu = dx$ assigns to each residue class $a(1 + (p)^n) = a + (p^n)$ of $\mathbb{Z}_p^\times$ modulo (the n-th principal units) $(1 + (p)^n)$, $a \in \mathbb{Z}_p^\times$, the value $\mu(a + (p^n)) = \mu((p^n)) = 1/p^n$, see ([6], INT VII.18). Also recall the explicit description of the local reciprocity map for Γ_n, see ([30], I, §3, Example 3.13) or ([38], §3.1, Theorem 2, p.146), but note that the sign convention there is opposite to ours. Using the change of variables $x \mapsto x/p^n$, we then get, as in ([22], §8.5, Example of integrals, *Gauß sum*, p. 259), that

$$\begin{aligned}\varepsilon_p(\chi, \psi_{\epsilon^{-1}}, dx) &= \int_{p^{-n}\mathbb{Z}_p^\times} \chi(x)^{-1}\psi_{\epsilon^{-1}}(x)\ dx \\ &= \chi(\bar{\varphi}_p^n)^{-1} \sum_{\gamma\in\Gamma_n} \chi(\gamma)^{-1}\psi_{\epsilon^{-1}}(\kappa(\gamma)/p^n) \\ &= \chi(\bar{\varphi}_p^n)^{-1} \sum_{\gamma\in\Gamma_n} \chi(\gamma)^{-1}\gamma(\zeta_{p^n}^{-1}),\end{aligned} \tag{4.18}$$

where, for the first equation, see (loc. cit., (4b), p. 259). See also the discussion in ([44], Appendix A, p. 33, footnote 2) regarding the conventions made in [21].

We will now define the p-adic regulator associated to a character χ as above and a norm-compatible sequence of principle units $u \in \mathcal{U}'(K'_\infty)$.

Definition 4.9 (p-adic regulators). *Let χ be an Artin character as at the beginning of this subsection and let $u \in \mathcal{U}'(K'_\infty)$. Write u_{L_n} for the image of u under the projection $\mathcal{U}'(K'_\infty) \to \mathcal{O}^1_{L_n}$ to the principal units of $\mathcal{O}_{L_n}$. Then, we define the p-adic regulator of χ and u by*

$$R_p(u,\chi) = \sum_{g \in \Gamma_n \times H_L} \chi(g^{-1}) \cdot \log(g(u_{L_n}))$$

and note that this is equal to $\sum_{g\in\Gamma_n\times H_L} \chi(g^{-1}) \cdot g\big(\log(u_{L_n})\big)$ since every Galois automorphism $g \in \Gamma_n \times H_L$ is continuous.

4.3.4 The values of $\mathcal{E}_{u'}$ at Artin characters

Let $\chi : \mathcal{G}' = H \times \Gamma \longrightarrow \mathbb{C}_p^\times$ be an Artin character. Moreover, let $u' \in \mathcal{U}'(K'_\infty)$ be an element as in assumption 4.3, which exists by (4.13). We now want to determine the value of $\mathcal{E}_{u'}$ at χ, which, by definition, is given by

$$\int_{\mathcal{G}'} \chi \; d\mathcal{E}_{u'} := \Big(\int_{\mathcal{G}'} \chi \; d(\mathcal{E}_{u'}^{-1}) \Big)^{-1},$$

for which we recall that $\mathcal{E}_{u'}^{-1} \in \mathcal{O}_{K'}[[\mathcal{G}']] \cap (\tilde{\Lambda}'_{\tilde{\mathcal{S}}'})^\times$ so that $\mathcal{E}_{u'}$ is of the form $\frac{1}{\mathcal{E}_{u'}^{-1}} \in (\tilde{\Lambda}'_{\tilde{\mathcal{S}}'})^\times$. We interpret $\mathcal{E}_{u'}^{-1}$ as an $\mathcal{O}_{K'}$-integral measure on $\mathcal{G}'$. Our main result in this section is the following theorem.

Theorem 4.10 (Interpolation property of $\mathcal{E}_{u'}$) *The value of $\mathcal{E}_{u'}$ at an Artin character χ is given by*

$$\int_{\mathcal{G}'} \chi \; d\mathcal{E}_{u'} \overset{\text{def}}{=} \Big(\int_{\mathcal{G}'} \chi \; d(\mathcal{E}_{u'}^{-1}) \Big)^{-1} = -\frac{\varepsilon_p(\chi, \psi_{\epsilon^{-1}}, dx)}{R_p(u',\chi)}$$

whenever $R_p(u',\chi) \neq 0$. In fact, we always have

$$\Big(\int_{\mathcal{G}'} \chi \; d(\mathcal{E}_{u'}^{-1}) \Big) \cdot \varepsilon_p(\chi, \psi_{\epsilon^{-1}}, dx) = -R_p(u',\chi),$$

regardless of whether $R_p(u',\chi) \neq 0$.

Before we give the proof of this theorem let us adopt some notational conventions and make a remark. As in the previous subsection 4.3.3, for a fixed Artin character $\chi : \mathcal{G}' = H \times \Gamma \longrightarrow \mathbb{C}_p^\times$, L will denote the fixed field of the kernel of the restriction $\chi_{|H}$ of χ to H and χ restricted to Γ factors through Γ_n, but not through Γ_{n-1}, where $\Gamma_m = G(\mathbb{Q}_p(\zeta_{p^m})/\mathbb{Q}_p)$. Hence, χ factors through $\Gamma_n \times H_L$, $H_L = G(L/\mathbb{Q}_p)$. Moreover, let us make the convention that elements of H_L will be denoted by h or h' and elements of Γ_n will be denoted by γ or γ'. We identify h with $(h,1)$ in $H_L \times \Gamma_n$ and likewise we identify γ with $(1,\gamma)$. As before, we will also write u_{L_∞} and u_{L_n} for the images of any $u \in \mathcal{U}'(K'_\infty)$ under the maps $\mathcal{U}'(K'_\infty) \longrightarrow \mathcal{U}'(L_\infty)$ and $\mathcal{U}'(K'_\infty) \longrightarrow \mathcal{U}'(L_n) = \varprojlim_m \mathcal{O}_{L_n}^\times / (\mathcal{O}_{L_n}^\times)^{p^m} \cong \hat{\mathcal{O}}_{L_n}^\times$, respectively.

Remark 4.11. (i) while we work under the assumption that $K'/\mathbb{Q}_p$ is of infinite degree, we note that this is not necessary. The proof of theorem 4.10 below shows that the same interpolation property holds for finite unramified extensions $L'/\mathbb{Q}_p$, the corresponding extension $L'_\infty/\mathbb{Q}_p$ and elements $\mathcal{E}^{-1}_{u,L'} \in \mathcal{O}_{L'}[[\Gamma \times H_{L'}]]$ defined by

$$-\mathcal{L}_{L',\epsilon^{-1}}(u \otimes 1) = \mathcal{E}^{-1}_{u,L'}((1 \otimes \epsilon) \otimes 1), \quad u \in \mathcal{U}'(L'_\infty). \tag{4.19}$$

In fact, the value of $\mathcal{E}_{u'}$ at χ only depends on $\mathcal{E}^{-1}_{u'_{L_\infty}} \in \mathcal{O}_L[[\Gamma \times H_L]]$, the image of $\mathcal{E}^{-1}_{u'}$ under the projection $\mathcal{O}_{K'}[[\mathcal{G}']] \to \mathcal{O}_{K'}[[\Gamma \times H_L]]$, compare remark 4.6.

(ii) the interested reader can compare the interpolation property given in theorem 4.10 with the formula in ([21], §3.6.1) for the extension $\mathbb{Q}_p(\zeta_{p^\infty})/\mathbb{Q}_p$ which is stated without proof.

Proof (of theorem 4.10). We know from remark 4.6 that $\mathcal{E}^{-1}_{u'}$ belongs to $\mathcal{O}_{K'}[[\mathcal{G}']]$ and that it is given by the compatible system $\mathcal{E}^{-1}_{u'} = (\mathcal{E}^{-1}_{u'_{L'_\infty}})_{L'}$, where $\mathcal{E}^{-1}_{u'_{L'_\infty}}$ was defined in the same remark. Recall that under the canonical projection

$$\mathcal{O}_{K'}[[\mathcal{G}']] \to \mathcal{O}_{K'}[\Gamma_n \times H_L], \quad \mathcal{E}^{-1}_{u'} \longmapsto -\sum_{i=0}^{d_L-1} \phi^{-i}\big(\mathrm{Col}_{\epsilon^{-1},L}(u'_{L_\infty})_n\big)\bar{\varphi}_p^i \ \in\ \mathcal{O}_L[\Gamma_n \times H_L],$$

where $\mathrm{Col}_{\epsilon^{-1},L}(u'_{L_\infty})_n$ is the image of $\mathrm{Col}_{\epsilon^{-1},L}(u'_{L_\infty})$ under the projection $\mathcal{O}_L[[\Gamma]] \to \mathcal{O}_L[\Gamma_n]$. Elements of $\mathcal{O}_L[\Gamma_n \times H_L]$ have the form $\sum_{(\gamma,i)} \alpha[\gamma,i](\gamma,\bar{\varphi}_p^i)$ with $\alpha[\gamma,i] \in \mathcal{O}_L$, $(\gamma,\bar{\varphi}_p^i) \in \Gamma_n \times H_L$ and where we sum over $\Gamma_n \times \{0,\dots,d_L-1\}$. Using this notation, and an analogous notation for elements $\sum_{\gamma\in\Gamma_n} \beta[\gamma]\gamma$ in $\mathcal{O}_L[\Gamma_n]$, we can write

$$-\sum_{i=0}^{d_L-1} \phi^{-i}\big(\mathrm{Col}_{\epsilon^{-1},L}(u'_{L_\infty})_n\big)\bar{\varphi}_p^i = -\sum_{(\gamma,i)} \phi^{-i}\big(\mathrm{Col}_{\epsilon^{-1},L}(u'_{L_\infty})_n[\gamma]\big)(\gamma,\bar{\varphi}_p^i) \ \in\ \mathcal{O}_L[\Gamma_n \times H_L],$$

so the coefficient of $(\gamma,\bar{\varphi}_p^i)$ is the coefficient $\mathrm{Col}_{\epsilon^{-1},L}(u'_{L_\infty})_n[\gamma] \in \mathcal{O}_L$ acted upon by ϕ^{-i}. We can now start with the calculations. Since χ factors through $\Gamma_n \times H_L$, by definition of the integral, we get

$$\int_{\mathcal{G}'} \chi \ d(\mathcal{E}^{-1}_{u'}) = -\sum_{(\gamma',i)} \Big(\phi^{-i}\big(\mathrm{Col}_{\epsilon^{-1},L}(u'_{L_\infty})_n[\gamma']\big) \cdot \chi(\gamma',\bar{\varphi}_p^i)\Big). \tag{4.20}$$

Multiplying this expression with $\chi(\bar{\varphi}_p^n) \cdot \varepsilon_p(\chi,\psi_{\epsilon^{-1}},dx)$, in view of (4.18), we get

$$\begin{aligned}
&\Big(\int_{\mathcal{G}'} \chi \ d(\mathcal{E}^{-1}_{u'})\Big) \cdot \big(\chi(\bar{\varphi}_p^n) \cdot \varepsilon_p(\chi,\psi_{\epsilon^{-1}},dx)\big) \\
&= -\Big(\sum_{(\gamma',i)} \Big(\phi^{-i}\big(\mathrm{Col}_{\epsilon^{-1},L}(u'_{L_\infty})_n[\gamma']\big) \cdot \chi(\gamma',\bar{\varphi}_p^i)\Big)\Big) \cdot \Big(\sum_{\gamma\in\Gamma_n} \chi(\gamma)^{-1}\gamma(\zeta_{p^n}^{-1})\Big) && \text{(by definition)} \\
&= -\sum_{(\gamma',i)} \sum_{\gamma\in\Gamma_n} \Big(\phi^{-i}\big(\mathrm{Col}_{\epsilon^{-1},L}(u'_{L_\infty})_n[\gamma']\big) \cdot \chi(\gamma'\gamma^{-1},\bar{\varphi}_p^i) \cdot \gamma(\zeta_{p^n}^{-1})\Big) && \text{(multiplying)} \\
&= -\sum_{(\gamma',i)} \sum_{\gamma\in\Gamma_n} \Big(\phi^{-i}\big(\mathrm{Col}_{\epsilon^{-1},L}(u'_{L_\infty})_n[\gamma']\big) \cdot \chi(\gamma^{-1},\bar{\varphi}_p^i) \cdot \gamma\gamma'(\zeta_{p^n}^{-1})\Big) && (\gamma \overset{\text{subst.}}{\longmapsto} \gamma\gamma') \\
&= -\sum_i \sum_\gamma \sum_{\gamma'} \Big(\chi(\gamma^{-1},\bar{\varphi}_p^i) \cdot \gamma\gamma'(\zeta_{p^n}^{-1}) \cdot \phi^{-i}\big(\mathrm{Col}_{\epsilon^{-1},L}(u'_{L_\infty})_n[\gamma']\big)\Big) && \text{(rearrange).}
\end{aligned} \tag{4.21}$$

Let us consider the summands of the last sum; ϕ^{-i} acts on $\mathrm{Col}_{\epsilon^{-1},L}(u'_{L_\infty})_n[\gamma'] \in \mathcal{O}_L$ and γ acts on $\gamma'(\zeta_{p^n}^{-1})$. Since $\bar{\varphi}_p$ is the restriction of ϕ to L, we may also write $\phi^{-i}\big(\mathrm{Col}_{\epsilon^{-1},L}(u'_{L_\infty})_n[\gamma']\big) = \bar{\varphi}_p^{-i}\big(\mathrm{Col}_{\epsilon^{-1},L}(u'_{L_\infty})_n[\gamma']\big)$. Since $(\gamma,\bar{\varphi}_p^{-i}) \in G(L(\zeta_{p^n})/\mathbb{Q}_p)$ acts on L through $\bar{\varphi}_p^{-i}$ and on $\mathbb{Q}_p(\zeta_{p^n})$ through γ, we have

$$\gamma\gamma'(\zeta_{p^n}^{-1}) \cdot \phi^{-i}\big(\mathrm{Col}_{\epsilon^{-1},L}(u'_{L_\infty})_n[\gamma']\big) = (\gamma,\bar{\varphi}_p^{-i})\Big(\gamma'(\zeta_{p^n}^{-1}) \cdot \mathrm{Col}_{\epsilon^{-1},L}(u'_{L_\infty})_n[\gamma']\Big). \tag{4.22}$$

Therefore, the last sum in (4.21) is equal to

$$\begin{aligned} &-\sum_i\sum_\gamma\sum_{\gamma'}\Big(\chi(\gamma^{-1},\bar{\varphi}_p^{i}) \cdot (\gamma,\bar{\varphi}_p^{-i})\big(\gamma'(\zeta_{p^n}^{-1}) \cdot \mathrm{Col}_{\epsilon^{-1},L}(u'_{L_\infty})_n[\gamma']\big)\Big) && \text{(use (4.22))}\\ =&-\sum_i\sum_\gamma\Big[\chi(\gamma^{-1},\bar{\varphi}_p^{i}) \cdot (\gamma,\bar{\varphi}_p^{-i})\Big(\sum_{\gamma'}\gamma'(\zeta_{p^n}^{-1}) \cdot \mathrm{Col}_{\epsilon^{-1},L}(u'_{L_\infty})_n[\gamma']\Big)\Big] && \text{(factor out).} \end{aligned} \tag{4.23}$$

The element $\sum_{\gamma'}\gamma'(\zeta_{p^n}^{-1}) \cdot \mathrm{Col}_{\epsilon^{-1},L}(u'_{L_\infty})_n[\gamma']$ from the last sum looks familiar. Note that the function $\Gamma \to \mu_{p^n}$ defined by $\gamma \mapsto \gamma(\zeta_{p^n}^{-1})$ is locally constant modulo $G(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p(\mu_{p^n}))$. Therefore, considering $\mathrm{Col}_{\epsilon^{-1},L}(u'_{L_\infty}) \in \mathcal{O}_L[[\Gamma]]$ as a measure on Γ, we get

$$\begin{aligned} &\sum_{\gamma'}\gamma'(\zeta_{p^n}^{-1}) \cdot \mathrm{Col}_{\epsilon^{-1},L}(u'_{L_\infty})_n[\gamma'] \\ =& \int_\Gamma \gamma'(\zeta_{p^n}^{-1})\ d\big(\mathrm{Col}_{\epsilon^{-1},L}(u'_{L_\infty})\big)(\gamma') \\ =& \int_\Gamma \zeta_{p^n}^{-\kappa(\gamma')}\ d\big(\mathrm{Col}_{\epsilon^{-1},L}(u'_{L_\infty})\big)(\gamma') && \text{(def. of } \kappa) \\ =& \int_{\mathbb{Z}_p^\times} \zeta_{p^n}^{-x}\ d\big(\kappa_*\mathrm{Col}_{\epsilon^{-1},L}(u'_{L_\infty})\big)(x) && \text{(use } \mathcal{O}_L[[\Gamma]] \xrightarrow{\kappa_*} \mathcal{O}_L[[\mathbb{Z}_p^\times]]) \\ =& \int_{\mathbb{Z}_p} \zeta_{p^n}^{-x}\ d\big(\iota\kappa_*\mathrm{Col}_{\epsilon^{-1},L}(u'_{L_\infty})\big)(x) && \text{(extend measure by 0 to } p\mathbb{Z}_p), \end{aligned} \tag{4.24}$$

where $\iota : \mathcal{O}_L[[\mathbb{Z}_p^\times]] \hookrightarrow \mathcal{O}_L[[\mathbb{Z}_p]]$ is defined as in (4.10). By definition of the Coleman map $\mathrm{Col}_{\epsilon^{-1},L}$, see (4.11), the measure appearing in the last term of (4.24) is given by

$$\iota\kappa_*\mathrm{Col}_{\epsilon^{-1},L}(u'_{L_\infty}) = \mathcal{M}^{-1}\Big(\frac{1}{p}\log\Big(\frac{f^p_{u'_{L_\infty}}}{\varphi(f_{u'_{L_\infty}})}\Big)\Big),$$

where $f_{u'_{L_\infty}} \in \mathcal{O}_L[[T]]^\times$ is the Coleman power series attached to $u'_{L_\infty} \in \mathcal{U}'(L_\infty)$ (and with respect to the generator ϵ^{-1} of $\mathbb{Z}_p$). Using lemma 4.13 that we state after this theorem, we see that the last term of (4.24) is equal to

$$\frac{1}{p}\log\Big(\frac{f^p_{u'_{L_\infty}}(\zeta_{p^n}^{-1}-1)}{\varphi(f_{u'_{L_\infty}})(\zeta_{p^n}^{-1}-1)}\Big) = \log\big(f_{u'_{L_\infty}}(\zeta_{p^n}^{-1}-1)\big)\ -\ \frac{1}{p}\log\big(f^\phi_{u'_{L_\infty}}(\zeta_{p^{n-1}}^{-1}-1)\big), \tag{4.25}$$

where, we recall that for $f(T) \in \mathcal{O}_L[[T]]$ we defined $\varphi(f)(T) = f^\phi((1+T)^p-1)$. Note also that the evaluation map $ev_{\zeta_{p^n}^{-1}} : \mathcal{O}_L[[T]] \longrightarrow \mathcal{O}_{L_n},\ T \longmapsto \zeta_{p^n}^{-1}-1$, from lemma 4.13 is continuous. Hence, for a power series $f(T) \in \mathcal{O}_L[[T]]$ congruent to 1 modulo $(\mathfrak{m}_L, T)$, where $\mathfrak{m}_L$ is the maximal ideal of $\mathcal{O}_L$, evaluating $\log(f(T))$ at $\zeta_{p^n}^{-1}-1$ yields the same as evaluating log at $f(\zeta_{p^n}^{-1}-1)$.

Since u'_{L_∞} is a norm-coherent series of principal units, the Coleman power series $f_{u'_{L_\infty}}(T)$ and also $f^{\phi}_{u'_{L_\infty}}(T)$ are congruent to 1 modulo $(\mathfrak{m}_L, T)$, see ([14], I, §3.3, p. 18). Let us write $\mathfrak{m}_{L_n}$ for the maximal ideal of $\mathcal{O}_{L_n}$, $L_n = L(\zeta_{p^n})$. Then, since $\zeta_{p^n}^{-1} - 1$ and $\zeta_{p^{n-1}}^{-1} - 1$ are uniformizers for $\mathbb{Q}_p(\zeta_{p^n})$ and $\mathbb{Q}_p(\zeta_{p^{n-1}})$, respectively, they also belong to $\mathfrak{m}_{L_n}$ and $\mathfrak{m}_{L_{n-1}}$, respectively, and we conclude that $f_{u'_{L_\infty}}(\zeta_{p^n}^{-1} - 1)$ and $f^{\phi}_{u'_{L_\infty}}(\zeta_{p^{n-1}}^{-1} - 1)$ are principal units in $\mathcal{O}^\times_{L_n}$ and $\mathcal{O}^\times_{L_{n-1}}$, respectively. Hence, their logarithms are given by the usual formula. Since $(1, \phi_{|L}) = (1, \bar{\varphi}_p) \in \Gamma_n \times H_L$ is a continuous automorphism of L_n (and acts trivially on $\mathbb{Q}_p(\zeta_{p^n})$ and the coefficients of log), we have

$$\begin{aligned}
&\log\big(f_{u'_{L_\infty}}(\zeta_{p^n}^{-1} - 1)\big) \; - \; \frac{1}{p}\log\big(f^{\phi}_{u'_{L_\infty}}(\zeta_{p^{n-1}}^{-1} - 1)\big) \\
&= (1, \bar{\varphi}_p)^n\Big(\log\big(f^{\phi^{-n}}_{u'_{L_\infty}}(\zeta_{p^n}^{-1} - 1)\big) \; - \; \frac{1}{p}\log\big(f^{\phi^{-(n-1)}}_{u'_{L_\infty}}(\zeta_{p^{n-1}}^{-1} - 1)\big)\Big) \\
&= (1, \bar{\varphi}_p)^n\big(\log(u'_{L_n}) - \frac{1}{p}\log(u'_{L_{n-1}})\big),
\end{aligned} \tag{4.26}$$

by definition of the Coleman power series (note we considered the maps constructed with respect to ϵ^{-1}). From (4.24) until now we have shown that

$$\sum_{\gamma'} \gamma'(\zeta_{p^n}^{-1}) \cdot \mathrm{Col}_{\epsilon^{-1},L}(u'_{L_\infty})_n[\gamma'] = (1, \bar{\varphi}_p)^n\big(\log(u'_{L_n}) - \frac{1}{p}\log(u'_{L_{n-1}})\big).$$

We get that the last term of (4.23) is equal to

$$\begin{aligned}
&-\sum_i \sum_\gamma \Big[\chi(\gamma^{-1}, \bar{\varphi}_p^i) \cdot (\gamma, \bar{\varphi}_p^{-i})\Big((1, \bar{\varphi}_p)^n\big(\log(u'_{L_n}) - \frac{1}{p}\log(u'_{L_{n-1}})\big)\Big)\Big] \\
&= -\sum_i \sum_\gamma \Big[\chi(\gamma^{-1}, \bar{\varphi}_p^i) \cdot (\gamma, \bar{\varphi}_p^{-i+n})\Big(\log(u'_{L_n}) - \frac{1}{p}\log(u'_{L_{n-1}})\Big)\Big] && \text{(associativity)} \\
&= -\sum_i \sum_\gamma \Big[\chi(\gamma^{-1}, \bar{\varphi}_p^{i+n}) \cdot (\gamma, \bar{\varphi}_p^{-i})\Big(\log(u'_{L_n}) - \frac{1}{p}\log(u'_{L_{n-1}})\Big)\Big] && (i \overset{\text{subst.}}{\longmapsto} i+n) \\
&= -\chi(\bar{\varphi}_p^n)\sum_i \sum_\gamma \Big[\chi(\gamma^{-1}, \bar{\varphi}_p^i) \cdot (\gamma, \bar{\varphi}_p^{-i})\Big(\log(u'_{L_n}) - \frac{1}{p}\log(u'_{L_{n-1}})\Big)\Big] && \text{(factor out)},
\end{aligned} \tag{4.27}$$

where we note that we may make the substitution $i \mapsto i + n$ since this just permutes H_L through multiplication by $\bar{\varphi}_p^n$. Since $-\frac{1}{p}\log(u'_{L_{n-1}})$ belongs to L_{n-1} ($u'_{L_{n-1}}$ is a principal unit in $\mathcal{O}^\times_{L_{n-1}}$), it follows from lemma 4.12, which we prove after this theorem, that the last term of (4.27) is equal to

$$-\chi(\bar{\varphi}_p^n)\sum_i \sum_\gamma \Big[\chi(\gamma^{-1}, \bar{\varphi}_p^i) \cdot (\gamma, \bar{\varphi}_p^{-i})\big(\log(u'_{L_n})\big)\Big] = -\chi(\bar{\varphi}_p^n) R_p(u', \chi).$$

So we have shown that

$$\Big(\int_{\mathcal{G}'} \chi \; d(\mathcal{E}_{u'}^{-1})\Big) \cdot \big(\chi(\bar{\varphi}_p^n) \cdot \varepsilon_p(\chi, \psi_{\epsilon^{-1}}, dx)\big) = -\chi(\bar{\varphi}_p^n) R_p(u', \chi),$$

which concludes the proof upon cancelling out the factor $\chi(\bar{\varphi}_p^n) \neq 0$.

Let us state the two lemmata that we referred to in the proof of theorem 4.10.

Lemma 4.12. *Let L be a finite unramified extension of $\mathbb{Q}_p$ as before and let $\tilde{\chi} : H_L \times \Gamma_n \longrightarrow \mathbb{C}_p^\times$ be a character such that $\tilde{\chi}_{|H_L}$ is injective and $\tilde{\chi}_{|\Gamma_n}$ does not factor through Γ_{n-1} for n, $n \geq 1$. If $n = 1$, then assume that $p > 2$. Then, for any $x \in L(\zeta_{p^{n-1}})$ the identity*

$$\sum_{(\sigma,g)\in H_L\times\Gamma_n} \left(\tilde{\chi}(\sigma,g)^{-1}(\sigma,g)(x)\right) = 0$$

holds.

Proof. By assumption we have $\tilde{\chi}\big(G\big(\mathbb{Q}_p(\zeta_{p^n})/\mathbb{Q}_p(\zeta_{p^{n-1}})\big)\big) \neq 1$. In case $n \geq 2$, the Galois group $G\big(\mathbb{Q}_p(\zeta_{p^n})/\mathbb{Q}_p(\zeta_{p^{n-1}})\big)$ is cyclic of order p and in case $n = 1$ it is cyclic of order $p-1$. In any case fix a generator g_0 of $G\big(\mathbb{Q}_p(\zeta_{p^n})/\mathbb{Q}_p(\zeta_{p^{n-1}})\big)$. We then have $\tilde{\chi}(g_0) \neq 1$. In $G(L_n/\mathbb{Q}_p) \cong H_L \times \Gamma_n$ the element g_0 corresponds to (id_{H_L}, g_0), hence it acts trivially on L and $\mathbb{Q}_p(\zeta_{p^{n-1}})$, i.e., on L_{n-1}. Write $\tilde{p}$ for the order of $G\big(\mathbb{Q}_p(\zeta_{p^n})/\mathbb{Q}_p(\zeta_{p^{n-1}})\big)$, so $\tilde{p} = p$ if $n \geq 2$ and $\tilde{p} = p-1$ if $n = 1$.

Now, for any $\overline{\gamma} \in \Gamma_{n-1}$ fix an element $\gamma \in \Gamma_n$ mapping to $\overline{\gamma}$ under the projection $\Gamma_n \longrightarrow \Gamma_{n-1}$. Then $\{\gamma, \gamma g_0, \gamma g_0^2, \dots, \gamma g_0^{\tilde{p}-1}\}$ is the preimage of $\{\overline{\gamma}\}$ under the projection and we get

$$\begin{aligned}
&\sum_{(\sigma,\gamma')\in H_L\times\Gamma_n} \tilde{\chi}(\sigma,\gamma')^{-1}(\sigma,\gamma')(x) \\
&= \sum_{\sigma\in H_L}\sum_{\overline{\gamma}\in\Gamma_{n-1}}\sum_{i=0}^{\tilde{p}-1} \tilde{\chi}(\sigma,\gamma g_0^i)^{-1}(\sigma,\gamma g_0^i)(x) \\
&= \sum_{\sigma\in H_L}\sum_{\overline{\gamma}\in\Gamma_{n-1}}\sum_{i=0}^{\tilde{p}-1} \tilde{\chi}(\sigma,\gamma g_0^i)^{-1}(\sigma,\gamma)(x) && (g_{0|L_{n-1}} = \mathrm{id}_{L_{n-1}}) \\
&= \sum_{\sigma\in H_L}\sum_{\overline{\gamma}\in\Gamma_{n-1}} \left[\tilde{\chi}(\sigma,\gamma)(\sigma,\gamma)(x)\Big(\sum_{i=0}^{\tilde{p}-1}\tilde{\chi}(g_0^i)^{-1}\Big)\right] && \text{(factor out)} \\
&= 0
\end{aligned}$$

where the last equality follows because $\tilde{\chi}(g_0^{-1})$ is a $\tilde{p}$-th root of unity and not equal to 1 and hence it is a root of the polynomial

$$(X^{\tilde{p}} - 1)\big/(X-1) = X^{\tilde{p}-1} + X^{\tilde{p}-2} + \cdots + 1,$$

showing that $\sum_{i=0}^{\tilde{p}-1} \tilde{\chi}(g_0^{-1})^i = 0$.

For the next lemma, let L be a finite unramified extension of $\mathbb{Q}_p$ and consider $L_n = L(\zeta_{p^n})$, where ζ_{p^n} is our (or any) fixed primitive p^n-th root of unity. The element $\zeta_{p^n}^{-1} - 1$ belongs to the maximal ideal $\mathfrak{m}_{\mathcal{O}_{L_n}}$ of $\mathcal{O}_{L_n}$, which is complete with respect to the $\mathfrak{m}_{\mathcal{O}_{L_n}}$-adic topology. By the universal property of the ring of power series we get a map

$$ev_{\zeta_{p^n}^{-1}} : \mathcal{O}_L[[T]] \longrightarrow \mathcal{O}_{L_n}, \quad T \longmapsto \zeta_{p^n}^{-1} - 1,$$

which is a continuous $\mathcal{O}_L$-algebra homomorphism. Recall that we have a topological algebra isomorphism $\mathcal{M} : \mathcal{O}_L[[\mathbb{Z}_p]] \xrightarrow{\sim} \mathcal{O}_L[[T]]$ given by $1_{\mathbb{Z}_p} \longmapsto 1 + T$.

Lemma 4.13. *For every $n \geq 1$ and every $\mu \in \mathcal{O}_L[[\mathbb{Z}_p]]$ we have the equality*

$$\int_{\mathbb{Z}_p} (\zeta_{p^n}^{-1})^x d\mu(x) = ev_{\zeta_{p^n}^{-1}} \circ \mathcal{M}(\mu).$$

Proof. The result follows from the universal property of the completed group ring, see ([36], lemma 3.3.9).

5 Selmer Groups and the Sequence of Poitou-Tate

In this section we will define the Selmer groups of p-adic Galois representations that we will use and briefly explain that they appear in a modified version of the sequence of Poitou-Tate.

5.1 Definition of Selmer groups

Let p be an odd prime and let us consider a number field F, a finite set of places Σ of F containing Σ_p, the primes of F above p, and Σ_∞, the archimedean places of F. Let T be a finitely generated free $\mathbb{Z}_p$-module endowed with a continuous action of G_F which is unramified outside Σ. We note that such T give rise to compact p-adic Lie extensions. In fact, if F_∞ denotes the fixed field of the kernel of the representation and r denotes the $\mathbb{Z}_p$-rank of T, then $G(F_\infty/F)$ is isomorphic to a closed subgroup of the compact p-adic Lie Group $GL_r(\mathbb{Z}_p)$ and therefore a compact p-adic Lie group itself, compare ([32], section 4.2, p. 561). Note that by general topology the image of $G(F_\infty/F)$ in $GL_r(\mathbb{Z}_p)$ is compact (since the representation is continuous) and since $GL_r(\mathbb{Z}_p)$ is Hausdorff it is then also closed, see ([7], I, 7.5. Theorem). Likewise, if $\bar{\nu}$ is a non-archimedean prime of F_∞ above a prime ν of F, then the decomposition group of $\bar{\nu}$ in $G(F_\infty/F)$ is closed in $G(F_\infty/F)$ and therefore a compact p-adic Lie group.

Before we define a Selmer group for the Galois representation T, we recall that the finite part $H^1_f(F_\nu, T \otimes \mathbb{Q}_p)$ of local cohomology is defined as follows, compare ([24], 14.1, p. 235). For a finitely generated $\mathbb{Q}_p$-vector space V with a continuous action of G_{F_ν} for some $\nu \in \Sigma$ we define $H^1_f(F_\nu, V)$ by

$$H^1_f(F_\nu, V) = \begin{cases} \ker\big(H^1(F_\nu, V) \longrightarrow H^1(F_\nu^{\mathrm{ur}}, V)\big) & \text{if } \nu \in \Sigma_f \setminus \Sigma_p, \\ \ker\big(H^1(F_\nu, V) \longrightarrow H^1(F_\nu, B_{\mathrm{crys}} \otimes_{\mathbb{Q}_p} V)\big) & \text{if } \nu \in \Sigma_p, \\ 0 & \text{if } \nu \in \Sigma_\infty, \end{cases}$$

where B_{crys} is the ring defined by Fontaine (and Messing) in [19], see also [17], [18] or the more recent [20]. We remark that in the literature the space $H^1_f(F_\nu, V)$ for $\nu \in \Sigma_f \setminus \Sigma_p$ is often denoted $H^1_{ur}(F_\nu, V)$ and called the subgroup of $H^1(F_\nu, V)$ of unramified cohomology classes.

We will also need the notion of finite parts of cohomology groups for finitely generated free $\mathbb{Z}_p$-modules T and discrete modules $T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p$ with an action of G_{F_ν} and define these to be the inverse image and image of $H^1_f(F_\nu, T \otimes \mathbb{Q}_p)$, respectively, under the natural maps

$$H^1(F_\nu, T) \xrightarrow{\iota_*} H^1(F_\nu, T \otimes \mathbb{Q}_p) \xrightarrow{pr_*} H^1(F_\nu, T \otimes \mathbb{Q}_p/\mathbb{Z}_p),$$

i.e., for any place $\nu \in \Sigma$ we define

$$H^1_f(F_\nu, T) := \iota_*^{-1}\big(H^1_f(F_\nu, T \otimes \mathbb{Q}_p)\big)$$

and

$$H^1_f(F_\nu, T\otimes \mathbb{Q}_p/\mathbb{Z}_p) := pr_*\big(H^1_f(F_\nu, T\otimes \mathbb{Q}_p)\big).$$

We note that, p being odd, the first cohomology groups $H^1(F_\nu, V)$ vanish for archimedean places $\nu \in \Sigma_\infty$. Accordingly, we have $H^1_f(F_\nu, T) = H^1(F_\nu, T)$ and $H^1_f(F_\nu, T\otimes \mathbb{Q}_p/\mathbb{Z}_p) = 0$ for $\nu \in \Sigma_\infty$. We make the following definition, similar to ([24], 14.1, p. 234).

We write F_Σ for the maximal extension of F unramified outside the primes of Σ and $G_\Sigma(F)$ for $\mathrm{Gal}(F_\Sigma/F)$. Note that $F_\infty \subset F_\Sigma$ since T is unramified outside Σ, by assumption. For finite subextensions $\dots F_n \subset F_{n+1}\cdots \subset F_\infty$ of F_∞/F such that $F_\infty = \cup_n F_n$, we also write $G_{n,\Sigma} = G_{\Sigma_n}(F_n) = \mathrm{Gal}(F_{n,\Sigma_n}/F_n) = G(F_\Sigma/F_n)$, where Σ_n is the set of places of F_n above those in Σ, $n \in \mathbb{N}\cup\{\infty\}$. We also write $T^* = \mathrm{Hom}_{\mathbb{Z}_p}(T,\mathbb{Z}_p)$ for the $\mathbb{Z}_p$-dual representation of T.

Definition 5.1 (Selmer group). *(i) for a number field F, a finite set of places Σ containing $\Sigma_p\cup\Sigma_\infty$, and a finitely generated free $\mathbb{Z}_p$-module T endowed with a continuous action of G_F that is unramified outside Σ we define*

$$\mathrm{Sel}(F,T) = \ker\Big(H^1\big(G_\Sigma(F), T\otimes\mathbb{Q}_p/\mathbb{Z}_p\big) \longrightarrow \bigoplus_{\nu\in\Sigma} H^1(F_\nu, T\otimes\mathbb{Q}_p/\mathbb{Z}_p)/H^1_f(F_\nu, T\otimes\mathbb{Q}_p/\mathbb{Z}_p)\Big),$$

where ν runs through all places of F in Σ.

(ii) for an infinite algebraic extension F_∞/F with finite subextensions $\dots F_n \subset F_{n+1}\cdots\subset F_\infty$ such that $F_\infty = \cup_n F_n$, we define

$$\mathrm{Sel}(F_\infty, T^*(1)) := \varinjlim_n \mathrm{Sel}(F_n, T^*(1)),$$

which is a subgroup of $H^1(G_{\infty,\Sigma}, T^*(1)\otimes\mathbb{Q}_p/\mathbb{Z}_p)$.

5.2 The Sequence of Poitou-Tate

In this subsection we briefly explain that the Selmer groups defined in the previous subsection fit into a modified version of the exact sequence of Poitou-Tate, for which we need the notion of finite parts of global cohomology groups. Let us define $H^1_f(G_\Sigma(F), T)$ to be the preimage of $\bigoplus_{\nu\in\Sigma} H^1_f(F_\nu, T)$ under the map

$$H^1(G_\Sigma(F), T) \longrightarrow \bigoplus_{\nu\in\Sigma} H^1(F_\nu, T).$$

We will write $H^i_{/f}(\dots)$ for the quotient $H^i(\dots)/H^i_f(\dots)$ both in local and in global settings. Moreover, we denote by $M^\vee$ the Pontryagin dual of a module M. Using the duality

$$H^1_f(F_\nu, T) \cong H^1_{/f}(F_\nu, T^*(1)\otimes\mathbb{Q}/\mathbb{Z})^\vee, \qquad \nu\in\Sigma,$$

which follows from ([4], Proposition 3.8) and ([35], chapter 1, propositions 1.4.2 and 1.4.3), one derives the following proposition.

Proposition 5.2. *Let T be a finitely generated free $\mathbb{Z}_p$-module endowed with a $\mathbb{Z}_p$-linear continuous action of G_F that is unramified outside Σ. For $\nu\in\Sigma_p$ assume that $T\otimes\mathbb{Q}_p$ is* *de Rham as a representation of G_{F_ν}. The Poitou-Tate sequence induces a six term exact sequence*

$$0 \to H^1_{/f}(G_\Sigma(F),T) \to \bigoplus_{\Sigma_f} H^1_{/f}(F_\nu,T) \longrightarrow \mathrm{Sel}(F,T^*(1))^\vee$$
$$\to H^2(G_\Sigma(F),T) \to \prod_{\nu\in\Sigma_f} H^2(F_\nu,T) \to H^0(G_\Sigma(F),T^\vee(1))^\vee \to 0. \tag{5.1}$$

Proof. See ([36], section 4.2).

5.3 Tate module of an elliptic curve with CM

Next, consider the following setting.

Setting 5.3. *We fix a prime number p, $p \neq 2,3$, algebraic closures $\overline{\mathbb{Q}}$, $\overline{\mathbb{Q}}_p$ and an embedding $\overline{\mathbb{Q}} \subset \overline{\mathbb{Q}}_p$. All algebraic extensions of $\mathbb{Q}$ are considered inside $\overline{\mathbb{Q}}$. Let $E/\mathbb{Q}$ be an elliptic curve with complex multiplication by $\mathcal{O}_K$, K quadratic imaginary, and good ordinary reduction at p such that p splits in $\mathcal{O}_K$ into distinct primes $\mathfrak{p} = (\pi)$ and $\bar{\mathfrak{p}} = (\bar{\pi})$, $\bar{\mathfrak{p}} \neq \mathfrak{p}$, with generators $\pi := \psi(\mathfrak{p})$ and $\bar{\pi} = \overline{\psi(\mathfrak{p})}$, respectively, where ψ is the Größencharacter attached to E/K. Consider the extension $K_\infty = \bigcup_n K_n$, where $K_n = K(E[p^n])$. Let $\mathfrak{p}$ (resp. ν) be the prime of K (resp. K_∞) above p that is determined by the embedding $\overline{\mathbb{Q}} \subset \overline{\mathbb{Q}}_p$. Moreover, we set $\Sigma_{\mathbb{Q},f} = \{p\} \cup \Sigma_{\mathbb{Q},\mathrm{bad}}$, where $\Sigma_{\mathbb{Q},\mathrm{bad}}$ is the finite set of primes where $E/\mathbb{Q}$ has bad reduction, and define $\Sigma_{\mathbb{Q}} := \{\nu_\infty\} \cup \Sigma_{\mathbb{Q},f}$, where ν_∞ is the archimedean prime of $\mathbb{Q}$. We write*

$$\mathcal{G} = \mathrm{Gal}(K_\infty/\mathbb{Q}), \quad G = \mathrm{Gal}(K_\infty/K), \quad \mathcal{G}' = \mathrm{Gal}(K_{\infty,\nu}/\mathbb{Q}_p), \quad \text{and} \quad \mathcal{G}_{\nu_q} = \mathrm{Gal}(K_{\infty,\nu_q}/\mathbb{Q}_q),$$

where for every $q \in \Sigma_{\mathbb{Q},f}$ we fix a prime ν_q of K_∞ above q. By the splitting assumption we have $G(K_{\infty,\nu}/\mathbb{Q}_p) = G(K_{\infty,\nu}/K_\mathfrak{p})$ and we also write G' for $\mathcal{G}'$. We also define subgroups

$$\mathcal{H} = G(K_\infty/\mathbb{Q}^{\mathrm{cyc}}), \quad H = G(K_\infty/K^{\mathrm{cyc}}) \quad \text{and} \quad \mathcal{H}' = G(K_{\infty,\nu}/\mathbb{Q}_p^{\mathrm{cyc}})$$

of $\mathcal{G}$, G and $\mathcal{G}'$, respectively, and write

$$\mathcal{S} \subset \mathcal{S}^* \subset \Lambda(\mathcal{G}), \quad S \subset S^* \subset \Lambda(G) \quad \text{and} \quad \mathcal{S}' \subset \mathcal{S}'^* \subset \Lambda(\mathcal{G}').$$

for the corresponding Ore sets as defined in (2.1) and (2.2). Let us write Λ' for $\Lambda(\mathcal{G}')$.

We also write $T = T_pE$ and $T^0 = T_pE \cap M_p^0(p)$. The $\mathbb{Q}_p$-vector space $M_p^0(p)$ is defined as $M_p^0(p) = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T_p(\hat{E})$, where $T_p(\hat{E})$ is the Tate-module of the formal group $\hat{E}$ associated to a fixed Weierstraß equation for E.

Let us make a remark about Tate modules.

Remark 5.4 (Conventions for Tate modules). (i) one certainly defines the p-primary Tate module T_pE via the multpilication by p maps $E[p^{n+1}] \to E[p^n]$, $x \mapsto px$. For the π- and $\bar{\pi}$-primary Tate modules some authors adopt different conventions. Either one defines $T_\pi E$ via the maps $E[\pi^{n+1}] \to E[\pi^n]$, $x \mapsto \pi x$, which will be our definition. Or, alternatively, recall that $p = \pi \cdot \bar{\pi}$, one defines a π-primary Tate module via the maps $E[\pi^{n+1}] \to E[\pi^n]$, $x \mapsto px$, which also gives a well-defined module which we will denote by $T_{\pi,\cdot p}E$ (similarly we define $T_{\bar{\pi},\cdot p}E$). The modules $T_\pi E$ and $T_{\pi,\cdot p}E$ are isomorphic for which we recall that $E[\pi^n] \cong \mathcal{O}_K/(\mathfrak{p}^n)$ as $\mathcal{O}_K$-modules. π^n and $\bar{\pi}^n$ are coprime and hence multiplication with $\bar{\pi}^n$ induces an isomorphism $\cdot\bar{\pi}^n : E[\pi^n] \xrightarrow{\sim} E[\pi^n]$. Passing to the projective limit we get an isomorphism

$$(\cdot\bar{\pi}^n)_n : T_{\pi,\cdot p}E \xrightarrow{\sim} T_\pi E, \quad (x_n)_n \longmapsto (\bar{\pi}^n x_n)_n.$$

Note that $T_{\pi,\cdot p}E$ naturally embedds into T_pE since the maps of the corresponding projective systems are compatible. $T_\pi E$ embedds into T_pE via the above isomorphism composed with $T_{\pi,\cdot p}E \hookrightarrow T_pE$. An analogous statement holds for the Tate modules $T_{\bar{\pi}}E$ and $T_{\bar{\pi},\cdot p}E$, where the latter is defined in a fashion entirely similar to $T_{\pi,\cdot p}E$.

(ii) there is a direct sum decomposition of G_K-modules

$$T_pE = T_{\pi,\cdot p}E \oplus T_{\bar{\pi},\cdot p}E, \tag{5.2}$$

see ([36], subsection A.6.4), and an exact sequence of $\mathcal{G}'$-modules

$$0 \to T_p\hat{E} \xrightarrow{\iota_{\hat{E}}} T_pE \xrightarrow{red_{\mathfrak{p}}} T_p\tilde{E} \to 0, \tag{5.3}$$

such that $\iota_{\hat{E}}$ maps $T_p\hat{E}$ isomorphically to $T_{\pi,\cdot p}E$ and $red_{\mathfrak{p}}$ maps $T_{\bar{\pi},\cdot p}E$ isomorphically to $T_p\tilde{E}$, where $T_p\tilde{E}$ is the Tate module of the reduction $\tilde{E}$ of E modulo $\mathfrak{p}$ and $red_{\mathfrak{p}}$ is induced by the reduction modulo $\mathfrak{p}$ map $E(\overline{\mathbb{Q}}_p) \longrightarrow \tilde{E}(\overline{\mathbb{F}}_p)$, see (loc. cit. proposition A.6.13). In the sequel, we will use the isomorphisms from this remark as identifications.

Remark 5.5. Note that the extension $K_{\infty,\nu}/\mathbb{Q}_p$ meets all requirements of the setting that we studied in subsection 4.3. In fact, $G(K_{\infty,\nu}/\mathbb{Q}_p) \subset G$ is abelian, $\bigcup_n K(E[\bar{\pi}^n])$ contains a $\mathbb{Z}_p$-extension of K in which $\mathfrak{p}$ is unramified and $\mu_{p^\infty} \subset K_\infty$ by the Weil pairing so that the abelian extension $K_{\infty,\nu}/K_{\mathfrak{p}}$ is indeed of the form $K'(\mu_{p^\infty})/\mathbb{Q}_p$ for some infinite unramified abelian extension $K'/\mathbb{Q}_p$.

Let us briefly state some facts for $T = T_pE$:

(i) $\varprojlim_n H^1_f(G_{n,\Sigma_{\mathbb{Q}}}, T) = 0$, see ([36], subsection 4.3.5).

(ii) $H^1_{/f}(K_{n,\omega}, T) = 0$ for all places ω of K_n not above p, i.e., $\omega \nmid p$, $\forall n \geq 0$, which follows from Mattuck's theorem ([29], VI, 13., Theorem 7, p. 114) in combination with ([35], Corollary 1.3.3) and the duality $H^1_{/f}(K_{n,\omega}, T)^\vee \cong H^1_f(K_{n,\omega}, T^*(1) \otimes \mathbb{Q}_p/\mathbb{Z}_p)$ for which we refer to ([35], chapter 1, propositions 1.4.2 and 1.4.3).

(iii) writing $\Sigma_p(K_n)$ for the places of K_n above p and $K_{n,\nu}$ for the completion of K_n at the restriction of ν to K_n, we have

$$\begin{aligned}\varprojlim_n \Big(\bigoplus_{\omega \in \Sigma_p(K_n)} \big(H^1_{/f}(K_{n,\omega}, T) \big) &\cong \text{c-Ind}_{\mathcal{G}}^{\mathcal{G}'} \Big(\varprojlim_n H^1_{/f}(K_{n,\nu}, T) \Big) \\ &\cong \text{c-Ind}_{\mathcal{G}}^{\mathcal{G}'} \Big(\varprojlim_n H^1(K_{n,\nu}, T/T^0) \Big) \\ &\cong \text{Ind}_{\mathcal{G}}^{\mathcal{G}'} \Big(\varprojlim_n H^1(K_{n,\nu}, T/T^0) \Big),\end{aligned}$$

where, for the second isomorphism, we refer to ([36], proposition 4.3.5) building on a result of Fukaya and Kato. The last isomorphism holds since the completed tensor product in $\text{c-Ind}_{\mathcal{G}}^{\mathcal{G}'}\Big(\varprojlim_n H^1(K_{n,\nu}, T/T^0)\Big) = \Lambda(\mathcal{G})\hat{\otimes}_{\Lambda(\mathcal{G}')}\Big(\varprojlim_n H^1(K_{n,\nu}, T/T^0)\Big)$ coincides with the usual tensor product and since $\mathcal{G}'$ is of finite index in $\mathcal{G}$.

Passing to the projective limit of the general exact sequences (5.1) for the fields K_n, $n \geq 1$, with respect to the corestriction and dual restriction maps and using local Tate duality, we get an exact sequence

$$\begin{aligned} 0 \longrightarrow H^1(G_{\Sigma_{\mathbb{Q}}}(\mathbb{Q}), \Lambda(\mathcal{G})^{\#} \otimes T) &\longrightarrow \operatorname{Ind}_{\mathcal{G}}^{\mathcal{G}'} H^1(\mathbb{Q}_p, \Lambda(\mathcal{G}')^{\#} \otimes T/T^0) \\ \longrightarrow \operatorname{Sel}(K_\infty, T^*(1))^\vee &\longrightarrow H^2(G_{\Sigma_{\mathbb{Q}}}(\mathbb{Q}), \Lambda(\mathcal{G})^{\#} \otimes T) \\ \longrightarrow \bigoplus_{q \in \Sigma_{\mathbb{Q},f}} \text{c-Ind}_{\mathcal{G}}^{\mathcal{G}_{\nu_q}} T(-1) &\longrightarrow T(-1) \longrightarrow 0. \end{aligned} \tag{5.4}$$

6 *The element $\Omega_{p,u,u'}$*

Let $E/\mathbb{Q}$ be one of the representatives of elliptic curves with complex multiplication by $\mathcal{O}_K$, K quadratic imaginary, with minimal discriminant listed in Appendix A, §3 of Silverman's book [39]. Fix a prime p, $p \neq 2, 3$, as in setting 5.3 and consider the following assumption, where we use the notation from previous sections.

Assumption 6.1. *There exists $u \in \varprojlim_n \mathcal{O}_{K_n}^\times$ such that*

$$\Lambda(G)_{S^*} \longrightarrow (\mathcal{E}_\infty)_{S^*}, \quad 1 \longmapsto u$$

is an isomorphism of $\Lambda(G)_{S^*}$-modules, where $S^* \subset \Lambda(G)$ is the Ore set from (2.2) and $\mathcal{E}_\infty = \varprojlim_n (\mathcal{O}_{K_n}^\times \otimes_{\mathbb{Z}} \mathbb{Z}_p)$.

We will show in this section that a global unit u as in assumption 6.1 and a local unit u' as in assumption 4.3, under the assumption that $\operatorname{Sel}(K_\infty, T^*(1))^\vee$ is $\mathcal{S}^*$-torsion, each canonically determine a generator of $\big(\operatorname{Ind}_{\mathcal{G}}^{\mathcal{G}'}\big(T_pE/T^0(-1) \otimes_{\mathbb{Z}_p} \mathbb{H}^1_{\text{loc}}\big)\big)_{\mathcal{S}^*}$ which is a free $\Lambda(\mathcal{G})_{\mathcal{S}^*}$-module of rank one. The element $\Omega_{p,u,u'} \in \Lambda(\mathcal{G})_{\mathcal{S}^*}^\times$ is then defined to be the *base change* of these two bases. In (6.21) we will determine its image in $K_0(\mathfrak{M}_{\mathcal{H}}(\mathcal{G}))$ under the connecting homomorphism from K-theory, which will be an important ingredient in the proof of one of the main theorems in the next section.

6.1 Setting

Let $E/\mathbb{Q}$ be one of the elliptic curves with complex multiplication by $\mathcal{O}_K$ listed in Appendix A, §3 of [39] and fix $p \neq 2, 3$ as in the setting 5.3. Such a curve has bad reduction at precisely one prime of $\mathbb{Z}$. In the following we will write Σ for the set of places of K consisting of the places $\Sigma_p = \{\mathfrak{p}, \bar{\mathfrak{p}}\}$ above p, of the complex archimedean place $\Sigma_\infty = \{\nu_\infty\}$ and the places $\Sigma_{K,\text{bad}}$ where E/K has bad reduction, i.e.,

$$\Sigma = \Sigma_p \cup \Sigma_\infty \cup \Sigma_{K,\text{bad}}.$$

We write $\Sigma_{\mathbb{Q}}$ for the places of $\mathbb{Q}$ below those in Σ.

Remark 6.2. (i) we recall that the discriminant $-D$ of any quadratic imaginary field $K = \mathbb{Q}(\sqrt{-D})$ as above is divisible by one (positive) prime number in $\mathbb{Z}$ only, which we will denote by l_K, compare ([39], Appendix A, §3). l_K is the unique prime at which $E/\mathbb{Q}$ has bad reduction.

(ii) writing $\mathfrak{f} = \mathfrak{f}_\psi$ for the conductor of the Größencharacter ψ attached to E/K (considered as a curve over K), we deduce from a result of Stroeker ([40], (1.7) Main Theorem) that

$$\mathfrak{f} = \mathfrak{l}^r, \quad r \geq 1,$$

where $\mathfrak{l}$ is the unique prime ideal of K lying above $\mathbb{Z}l_K$ (it is unique since l_K ramifies in K), compare ([36], theorem A.6.8 and proposition A.6.9). We see that $\Sigma_{\mathrm{bad},K} = \{\mathfrak{l}\}$ so that $\Sigma_\mathbb{Q} = \{p\} \cup \{l_K\} \cup \{\nu_{\infty|_\mathbb{Q}}\}$. Thus, $\Sigma_\mathbb{Q}$ coincides with the set of places $\Sigma_\mathbb{Q}$ from setting 5.3. We conclude that Σ is precisely the set of primes of K above the primes of $\Sigma_\mathbb{Q}$ and, since $\Sigma_\mathbb{Q}$ contains the unique prime $\mathbb{Z}l_K$ that ramifies in $K/\mathbb{Q}$, that

$$\mathbb{Q}_{\Sigma_\mathbb{Q}} = K_\Sigma. \tag{6.1}$$

In particular, we have $G_\Sigma(K) \subset G_{\Sigma_\mathbb{Q}}(\mathbb{Q})$.

Recall that for $m \geq 0$ we defined the global universal cohomology groups

$$\mathbb{H}^m_\Sigma = \varprojlim_n H^m(G_\Sigma(K_n), \mathbb{Z}_p(1)) \cong H^m(G_\Sigma(K), \Lambda(G)^\#(1)) \cong H^m(G_\Sigma(\mathbb{Q}), \Lambda(\mathcal{G})^\#(1)).$$

Note that, since $G_\Sigma(K) \subset G_{\Sigma_\mathbb{Q}}(\mathbb{Q})$, each module $H^m(G_\Sigma(K_n), \mathbb{Z}_p(1))$ naturally carries an action of $G(K_n/\mathbb{Q}) = G_{\Sigma_\mathbb{Q}}(\mathbb{Q})/G_\Sigma(K_n)$ and, by restriction, a $G(K_n/K)$-action. For $m \geq 0$ we also have the local universal cohomology groups

$$\mathbb{H}^m_{\mathrm{loc}} \cong \varprojlim_n H^m(K_{n,\nu}, \mathbb{Z}_p(1)) \cong H^m(\mathbb{Q}_p, \Lambda(\mathcal{G}')^\#(1)),$$

where we write ν also for the prime of K_n below the prime ν of K_∞, compare the setting 5.3 where ν was defined. For all practical purposes we use the identification $\mathcal{E}_\infty \cong \mathbb{H}^1_\Sigma$ from (3.1). For the principal semi-local units we will write

$$\mathcal{U}_\infty = \varprojlim_n \prod_{\omega|\mathfrak{p}} \mathcal{O}^1_{K_{n,\omega}} \cong \varprojlim_n \prod_{\omega|\mathfrak{p}} \hat{\mathcal{O}}^\times_{K_{n,\omega}},$$

where $K_{n,\omega}$ denotes the completion of K_n at the prime ω of K_n above $\mathfrak{p}$. Recall that since Leopoldt's conjecture holds for the fields K_n, we have a natural embedding $\mathcal{E}_\infty \hookrightarrow \mathcal{U}_\infty$. For the local principal units in the extension $K_{\infty,\nu}/\mathbb{Q}_p$ we write $\mathcal{U}'(K_{\infty,\nu}) = \varprojlim_n \hat{\mathcal{O}}^\times_{K_{n,\nu}}$. Note that $G' = \mathcal{G}'$ is of finite index in G and that we have an isomorphism $\mathrm{Ind}^{G'}_G \mathcal{U}'(K_{\infty,\nu}) \cong \mathcal{U}_\infty$ induced by the natural embedding $\mathcal{U}'(K_{\infty,\nu}) \hookrightarrow \mathcal{U}_\infty$. Since $K_{\infty,\nu}/\mathbb{Q}_p$ is of infinite residue degree, the Kummer sequence induces $\mathcal{U}'(K_{\infty,\nu}) \cong \mathbb{H}^1_{\mathrm{loc}}$.

6.2 Choices of u and u'

Regarding assumption 6.1, let us make the following remark.

Remark 6.3. Assumption 6.1 is satisfied under the assumption that $\mathcal{E}_\infty/\mathcal{C}_\infty$ is S^*-torsion. In fact, in section 3 we considered compatible systems of elliptic units $e(\mathfrak{a}) \in \varprojlim_n \mathcal{O}_{K_n}^\times$ for an integral ideal $\mathfrak{a}$ of K prime to $6p\mathfrak{f}$, see definition 3.6, and wrote $u(\mathfrak{a})$ for the image of $e(\mathfrak{a})$ in $\mathcal{E}_\infty \subset \mathcal{U}_\infty$. For the commutative main theorem 3.9 we then restricted to a prime ideal $\mathfrak{q}$ of K that, in addition to being prime to $6p\mathfrak{f}$ has norm $N\mathfrak{q}$ congruent to 1 modulo (p). We have explained in remark 3.10 that $\mathcal{E}_\infty/\Lambda(G)u(\mathfrak{q}) \cong \mathbb{H}^1_\Sigma/\Lambda(G)u(\mathfrak{q})$ is S^*-torsion under the assumption that $\mathcal{E}_\infty/\mathcal{C}_\infty$ is S^*-torsion. Moreover, by ([36], proposition 2.4.28), λ is not a zero divisor in $\Lambda(G, \widehat{\mathbb{Z}}_p^{ur})$. Since neither $x_\mathfrak{q} = \sigma_\mathfrak{q} - N(\mathfrak{q})$ nor $12 \in \mathbb{Z}_p^\times$ are zero divisors, it follows that $12x_\mathfrak{q}\lambda = \mathbb{L}(u(\mathfrak{q}))$ is not a zero divisor in $\Lambda(G, \widehat{\mathbb{Z}}_p^{ur})$. In particular, $\mathrm{ann}_{\Lambda(G)}u(\mathfrak{q}) = 0$. In conclusion, if $\mathcal{E}_\infty/\mathcal{C}_\infty$ is S^*-torsion, then $e(\mathfrak{q})$ satisfies assumption 6.1.

The local assumption 4.3 about the existence of an element $u' \in \mathcal{U}'(K_{\infty,\nu})$ such that

$$\Lambda'_{\mathcal{S}'} \longrightarrow \mathcal{U}'(K_{\infty,\nu})_{\mathcal{S}'} \cong (\mathbb{H}^1_{\mathrm{loc}})_{\mathcal{S}'}, \quad 1 \longmapsto u'$$

is an isomorphism of $\Lambda'_{\mathcal{S}'}$-modules, was seen to be satisfied in (4.13).

6.3 Global contribution

Let u be an element as in assumption 6.1. Since S^* contains no non-trivial zero divisors by ([9], Theorem 2.4) we have $\mathrm{ann}_{\Lambda(G)}u = 0$. Hence, we get an exact sequence

$$0 \to \Lambda(G) \xrightarrow{1 \mapsto u} \mathrm{Res}^{\mathcal{G}}_G \mathbb{H}^1_\Sigma \to \mathbb{H}^1_\Sigma/\Lambda(G)u \to 0,$$

where we write $\mathrm{Res}^{\mathcal{G}}_G \mathbb{H}^1_\Sigma$ for $\mathbb{H}^1_\Sigma$ to emphasize that the $\Lambda(G)$ action on $\mathbb{H}^1_\Sigma$ is the restriction of a $\Lambda(\mathcal{G})$-action. We twist this sequence with $T_{\bar{\pi}}(-1) = T_{\bar{\pi}}(E)(-1)$ and get

$$0 \to T_{\bar{\pi}}(-1) \otimes_{\mathbb{Z}_p} \Lambda(G) \to T_{\bar{\pi}}(-1) \otimes_{\mathbb{Z}_p} \mathrm{Res}^{\mathcal{G}}_G \mathbb{H}^1_\Sigma \to T_{\bar{\pi}}(-1) \otimes_{\mathbb{Z}_p} \left(\mathbb{H}^1_\Sigma/\Lambda(G)u\right) \to 0, \tag{6.2}$$

where each modules is equipped with the $\Lambda(G)$-action induced by the diagonal G-action. We now fix a basis $t_{\bar{\pi}}$ of $T_{\bar{\pi}} \cong \mathbb{Z}_p$ and, as before, a basis ϵ of $\mathbb{Z}_p(1)$, which determines a basis of $\mathbb{Z}_p(-1)$. Together, $t_{\bar{\pi}}$ and ϵ determine a basis $t = t_{\bar{\pi},\epsilon}$ of $T_{\bar{\pi}}(-1)$, which, in turn, canonically determines an isomorphism $\tilde{\phi}_t : T_{\bar{\pi}}(-1) \cong \mathbb{Z}_p, t \mapsto 1$. Using $\tilde{\phi}_t$, we can define an isomorphism

$$\phi_t = \phi_{t_{\bar{\pi},\epsilon}} : T_{\bar{\pi}}(-1) \otimes_{\mathbb{Z}_p} \Lambda(G) \cong \Lambda(G) \tag{6.3}$$

of left $\Lambda(G)$-modules induced by $\tilde{t} \otimes g \mapsto \tilde{\phi}_t(g^{-1}.\tilde{t})g$, where $\tilde{t} \in T_{\bar{\pi}}(-1)$ and $g \in G$, compare ([43], Lemma 7.2). Using this isomorphism and applying $\mathrm{Ind}^G_{\mathcal{G}}$ to (6.2) we get an exact sequence

$$0 \to \Lambda(\mathcal{G}) \xrightarrow{\phi_{t,u}} T_pE(-1) \otimes_{\mathbb{Z}_p} \mathbb{H}^1_\Sigma \to \mathrm{Ind}^G_{\mathcal{G}}\Big(T_{\bar{\pi}}(-1) \otimes_{\mathbb{Z}_p} \left(\mathbb{H}^1_\Sigma/\Lambda(G)u\right)\Big) \to 0, \tag{6.4}$$

where we used the isomorphism $\mathrm{Ind}^G_{\mathcal{G}}\big(T_{\bar{\pi}}(-1) \otimes_{\mathbb{Z}_p} \mathrm{Res}^{\mathcal{G}}_G \mathbb{H}^1_\Sigma\big) \cong T_pE(-1) \otimes_{\mathbb{Z}_p} \mathbb{H}^1_\Sigma$. Using that $\mathbb{H}^1_\Sigma/\Lambda(G)u$ is S^*-torsion, one can show that $\mathrm{Ind}^G_{\mathcal{G}}\Big(T_{\bar{\pi}}(-1) \otimes_{\mathbb{Z}_p} \left(\mathbb{H}^1_\Sigma/\Lambda(G)u\right)\Big)$ is $\mathcal{S}^*$-torsion, which is explained in ([36], subsection 5.3.1). We conclude that the map

$$\phi_{t,u} : \Lambda(\mathcal{G})_{\mathcal{S}^*} \xrightarrow{\sim} \left(T_pE(-1) \otimes_{\mathbb{Z}_p} \mathbb{H}^1_\Sigma\right)_{\mathcal{S}^*}, \quad 1 \longmapsto \phi_{t,u}(1) = t \otimes u, \tag{6.5}$$

induced by (6.4) is an isomorphism of $\Lambda(\mathcal{G})_{\mathcal{S}^*}$-modules. It also follows from (6.4) that in $K_0(\mathfrak{M}_{\mathcal{H}}(\mathcal{G}))$ we have

$$\Big[\left(T_pE(-1) \otimes_{\mathbb{Z}_p} \mathbb{H}^1_\Sigma\right)/\Lambda(\mathcal{G})\phi_{t,u}(1)\Big] = \Big[\mathrm{Ind}^G_{\mathcal{G}}\Big(T_{\bar{\pi}}(-1) \otimes_{\mathbb{Z}_p} \left(\mathbb{H}^1_\Sigma/\Lambda(G)u\right)\Big)\Big]. \tag{6.6}$$

6.4 Local contribution

In the local situation we proceed similarly. Let $u' \in \mathcal{U}'(K_{\infty,\nu})$ be an element as in assumption 4.3. Since $\mathcal{S}'$ contains no non-trivial zero divisors $\mathrm{ann}_{\Lambda'}(u') = 0$, see ([9], Theorem 2.4), and we have an exact sequence

$$0 \to \Lambda' \xrightarrow{1 \mapsto u'} \mathbb{H}^1_{\mathrm{loc}} \to \mathbb{H}^1_{\mathrm{loc}}/\Lambda' u' \to 0.$$

Recall that $T^0 = T_pE \cap (\mathbb{Q}_p \otimes_{\mathbb{Z}_p} T_p(\hat{E}))$ and let us write T for T_pE considered as a $\mathcal{G}'$-module. We twist the above exact sequence with $T' := (T/T^0)(-1)$, which is free as a $\mathbb{Z}_p$-module of rank one, and get

$$0 \to T' \otimes_{\mathbb{Z}_p} \Lambda' \to T' \otimes_{\mathbb{Z}_p} \mathbb{H}^1_{\mathrm{loc}} \to T' \otimes_{\mathbb{Z}_p} \left(\mathbb{H}^1_{\mathrm{loc}}/\Lambda' u'\right) \to 0. \tag{6.7}$$

Fixing a basis t_0 of T/T^0 and a basis ϵ of $\mathbb{Z}_p(1)$ as above (the one induced by the global choice and the fixed embedding $\overline{\mathbb{Q}} \subset \overline{\mathbb{Q}}_p$) we get a basis $t' = t'_{0,\epsilon}$ of T'. This choice determines an isomorphism

$$\phi_{t'} : T' \otimes_{\mathbb{Z}_p} \Lambda' \cong \Lambda' \tag{6.8}$$

of left Λ'-modules similarly to the global case (6.3) above. Using this isomorphism and applying $\mathrm{Ind}_{\mathcal{G}}^{\mathcal{G}'}$ to (6.7) we get an exact sequence

$$0 \to \Lambda(\mathcal{G}) \xrightarrow{\phi_{t',u'}} \mathrm{Ind}_{\mathcal{G}}^{\mathcal{G}'}\big(T' \otimes_{\mathbb{Z}_p} \mathbb{H}^1_{\mathrm{loc}}\big) \to \mathrm{Ind}_{\mathcal{G}}^{\mathcal{G}'}\Big(T' \otimes_{\mathbb{Z}_p} \big(\mathbb{H}^1_{\mathrm{loc}}/\Lambda' u'\big)\Big) \to 0. \tag{6.9}$$

Similar to the global case, one can now show that $\mathrm{Ind}_{\mathcal{G}}^{\mathcal{G}'}\Big(T' \otimes_{\mathbb{Z}_p} \big(\mathbb{H}^1_{\mathrm{loc}}/\Lambda' u'\big)\Big)$ is $\mathcal{S}^*$-torsion. Let us conclude that the map

$$\phi_{t',u'} : \Lambda(\mathcal{G})_{\mathcal{S}^*} \xrightarrow{\sim} \big(\mathrm{Ind}_{\mathcal{G}}^{\mathcal{G}'}\big(T' \otimes_{\mathbb{Z}_p} \mathbb{H}^1_{\mathrm{loc}}\big)\big)_{\mathcal{S}^*}, \quad 1 \longmapsto \phi_{t',u'}(1) = 1 \otimes t' \otimes u', \tag{6.10}$$

induced by (6.9) is an isomorphism of $\Lambda(\mathcal{G})_{\mathcal{S}^*}$-modules. Moreover, it follows from (6.9) that in $K_0(\mathfrak{M}_{\mathcal{H}}(\mathcal{G}))$ we have

$$\Big[\big(\mathrm{Ind}_{\mathcal{G}}^{\mathcal{G}'}\big(T' \otimes_{\mathbb{Z}_p} \mathbb{H}^1_{\mathrm{loc}}\big)\big)/\Lambda(\mathcal{G})\phi_{t',u'}(1)\Big] = \Big[\mathrm{Ind}_{\mathcal{G}}^{\mathcal{G}'}\Big(T' \otimes_{\mathbb{Z}_p} \big(\mathbb{H}^1_{\mathrm{loc}}/\Lambda' u'\big)\Big)\Big]. \tag{6.11}$$

6.5 Definition

Using the isomorphisms

$$H^1(G_{\Sigma_{\mathbb{Q}}}(\mathbb{Q}), \Lambda(\mathcal{G})^{\#} \otimes T) \cong T(-1) \otimes_{\mathbb{Z}_p} \mathbb{H}^1_{\Sigma} \tag{6.12}$$

and

$$H^1(\mathbb{Q}_p, \Lambda(\mathcal{G}')^{\#} \otimes T/T^0) \cong T/T^0(-1) \otimes_{\mathbb{Z}_p} \mathbb{H}^1_{\mathrm{loc}}, \tag{6.13}$$

which can be deduced from a result of Fukaya and Kato, see ([36], corollary A.3.10), we can extract from the Poitou-Tate sequence (5.4) an exact sequence

$$0 \to T(-1) \otimes_{\mathbb{Z}_p} \mathbb{H}^1_{\Sigma} \xrightarrow{\mathrm{loc}} \mathrm{Ind}_{\mathcal{G}}^{\mathcal{G}'}\big(T/T^0(-1) \otimes_{\mathbb{Z}_p} \mathbb{H}^1_{\mathrm{loc}}\big) \to \mathrm{Sel}(K_\infty, T^*(1))^\vee, \tag{6.14}$$

where $T = T_pE$ and $T^0 = T_pE \cap (\mathbb{Q}_p \otimes_{\mathbb{Z}_p} T_p(\hat{E}))$. From now on we assume the following fundamental conjecture.

Conjecture 6.4 (Torsion Conjecture). *The dual* $\mathrm{Sel}(K_\infty, T^*(1))^\vee$ *of the Selmer group is* $\mathcal{S}^*$*-torsion.*

This torsion assumption implies that the first two terms of (6.14) become isomorphic after extending scalars to $\Lambda(\mathcal{G})_{\mathcal{S}^*}$. In particular, using the isomorphisms (6.5) and (6.10), we see that

$$\phi_{t',u'}(1) \quad \text{and} \quad \mathrm{loc}(\phi_{t,u}(1))$$

both each constitute a basis of the rank one $\Lambda(\mathcal{G})_{\mathcal{S}^*}$-module $\left(\mathrm{Ind}_{\mathcal{G}}^{\mathcal{G}'}\left(T/T^0(-1) \otimes_{\mathbb{Z}_p} \mathbb{H}^1_{\mathrm{loc}}\right)\right)_{\mathcal{S}^*}$.

Definition 6.5 (of $\boldsymbol{\Omega_{p,u,u'}}$). *We define the element* $\Omega_{p,u,t,u',t'} \in \Lambda(\mathcal{G})_{\mathcal{S}^*}$ *by the equation*

$$\Omega_{p,u,t,u',t'} \cdot \phi_{t',u'}(1) = \mathrm{loc}(\phi_{t,u}(1))$$

and note that $\Omega_{p,u,t,u',t'}$ *actually belongs to* $\Lambda(\mathcal{G})^\times_{\mathcal{S}^*}$ *since* $\phi_{t',u'}(1)$ *and* $\mathrm{loc}(\phi_{t,u}(1))$ *are both each a basis of* $\left(\mathrm{Ind}_{\mathcal{G}}^{\mathcal{G}'}\left(T/T^0(-1) \otimes_{\mathbb{Z}_p} \mathbb{H}^1_{\mathrm{loc}}\right)\right)_{\mathcal{S}^*}$. *We will also write* $\Omega_{p,u,u'} = \Omega_{p,u,t,u',t'}$ *when making the canonical choice explained in the following remark.*

Remark 6.6 (Independence of $\boldsymbol{t, t'}$). In the local setting we fixed a generator t_0 of $T_pE/T_p\hat{E}$ and in the global setting we fixed a generator $t_{\bar{\pi}}$ of $T_{\bar{\pi}}E$. Under our CM and good ordinary reduction assumptions, we have a canonical $\mathcal{G}'$-isomorphism

$$\iota : T_{\bar{\pi}}E \cong T_pE/T_p\hat{E}, \tag{6.15}$$

see remark 5.4 or ([36], proposition A.6.13 and equation (A.6.14)). But this means that a global choice, i.e., a generator of $T_{\bar{\pi}}E$, canonically determines a local choice, i.e., a generator of $T_pE/T_p\hat{E}$, and vice versa. Let us choose a generator $t_{\bar{\pi}}$ of $T_{\bar{\pi}}E$ and write t_0 for the image of $t_{\bar{\pi}}$ under ι. For a fixed generator ϵ of $\varprojlim_n \mu_{p^n}(\bar{\mathbb{Q}})$ let us also write ϵ for the generator of $\varprojlim_n \mu_{p^n}(\bar{\mathbb{Q}}_p)$ induced by $\bar{\mathbb{Q}} \subset \bar{\mathbb{Q}}_p$. With these choices ι induces

$$\iota_1 : T_{\bar{\pi}}E(-1) \cong (T_pE/T_p\hat{E})(-1), \quad t \longmapsto t'.$$

Any other choice of generator of $T_{\bar{\pi}}E(-1)$ is of the form at for $a \in \mathbb{Z}_p^\times$ and at determines at' via ι_1. Since $\phi_{at',u'}(1) = 1 \otimes at' \otimes u'$ and $\phi_{at,u}(1) = at \otimes u$ for any $a \in \mathbb{Z}_p^\times$ we have

$$\mathrm{loc}(\phi_{at,u}(1)) = a \cdot \mathrm{loc}(\phi_{t,u}(1)) \qquad \text{and} \qquad \phi_{at',u'}(1) = a\phi_{t',u'}(1)$$

and, hence,

$$\Omega_{p,u,t,u',t'} = \Omega_{p,u,at,u',at'}$$

for any $a \in \mathbb{Z}_p^\times$. We conclude that $\Omega_{p,u,t,u',t'}$ is independent of t and t' as long as we let t determine t' canonically via ι_1. From now on we let t determine t' and simply write $\Omega_{p,u,u'}$ for $\Omega_{p,u,t,u',t'}$.

Next we derive the relation for which we introduced the element $\Omega_{p,u,u'}$. Let us write

$$\begin{aligned}
&\frac{\lambda_\Omega}{s_\Omega} = \Omega_{p,u,u'} \qquad \text{with } \lambda_\Omega \in \Lambda(\mathcal{G}),\ s_\Omega \in \mathcal{S}^* \\
&x = \phi_{t',u'}(1) \\
&y = \mathrm{loc}(\phi_{t,u}(1)) \\
&M = \mathrm{Ind}_{\mathcal{G}}^{\mathcal{G}'}\big((T/T^0)(-1) \otimes_{\mathbb{Z}_p} \mathbb{H}^1_{\mathrm{loc}}\big) \\
&\Lambda = \Lambda(\mathcal{G}).
\end{aligned}$$

Let us explain in the following remark that we have

$$\lambda_\Omega \cdot x = s_\Omega \cdot y$$

in M.

Remark 6.7. Let us briefly explain how we deduce from $\lambda_\Omega \cdot x = s_\Omega \cdot y$ in $M_{\mathcal{S}^*}$ (which holds by definition of $\Omega_{p,u,u'}$) that the same equation already holds in M. We will show that we have an embedding of $\Lambda(\mathcal{G})$-modules $\varphi : M \hookrightarrow \Lambda(\mathcal{G})$ and then the statement follows from the commutative diagram

$$\begin{array}{ccc}
M & \overset{\varphi}{\hookrightarrow} & \Lambda(\mathcal{G}) \\
\downarrow & & \downarrow \\
M_{\mathcal{S}^*} & \overset{\varphi}{\hookrightarrow} & \Lambda(\mathcal{G})_{\mathcal{S}^*},
\end{array} \tag{6.16}$$

where the vertical map on the right is injective since $\mathcal{S}^*$ does not contain any zero-divisors, see ([9], Theorem 2.4). In fact, only note now that the vertical map on the left must be injective, too. Let us show that $\varphi : M \hookrightarrow \Lambda(\mathcal{G})$ exists. First, recall that for a fixed generator ϵ of $\mathbb{Z}_p(1) = \varprojlim_n \mu_{p^n}$ we have the exact sequence from (4.12)

$$0 \longrightarrow \mathcal{U}'(K_{\infty,\nu}) \overset{-\mathcal{L}_{\epsilon^{-1}}}{\longrightarrow} \mathbb{T}_{\mathrm{un}}(K_{\infty,\nu}) \otimes_{\Lambda'} \Lambda'_{\varphi_p} \longrightarrow \mathbb{Z}_p(1) \longrightarrow 0 \tag{6.17}$$

induced by the Coleman maps, where we write $\Lambda' = \Lambda(\mathcal{G}')$. The module $\mathbb{T}_{\mathrm{un}}(K_{\infty,\nu}) \otimes_{\Lambda'} \Lambda'_{\varphi_p}$ in the middle of (6.17) is free of rank one as a Λ'-module, see ([44], Proposition 2.1). Fixing a Λ'-isomorphism $\mathbb{T}_{\mathrm{un}}(K_{\infty,\nu}) \otimes_{\Lambda'} \Lambda'_{\varphi_p} \cong \Lambda'$ and using $\mathcal{U}'(K_{\infty,\nu}) \cong \mathbb{H}^1_{\mathrm{loc}}$, we see that (6.17) induces an injection

$$\mathbb{H}^1_{\mathrm{loc}} \hookrightarrow \Lambda' \tag{6.18}$$

Tensoring (6.18) with the free $\mathbb{Z}_p$-module $(T/T^0)(-1)$ of rank one, we get an injection of Λ'-modules

$$(T/T^0)(-1) \otimes_{\mathbb{Z}_p} \mathbb{H}^1_{\mathrm{loc}} \hookrightarrow \Big((T/T^0)(-1) \otimes_{\mathbb{Z}_p} \Lambda'\Big) \cong \Lambda', \tag{6.19}$$

where the isomorphism on the right is as in (6.8). Applying $\mathrm{Ind}_{\mathcal{G}}^{\mathcal{G}'}$ to (6.19), we get an injection of $\Lambda(\mathcal{G})$-modules

$$\mathrm{Ind}_{\mathcal{G}}^{\mathcal{G}'}\Big((T/T^0)(-1) \otimes_{\mathbb{Z}_p} \mathbb{H}^1_{\mathrm{loc}}\Big) \hookrightarrow \Lambda(\mathcal{G}), \tag{6.20}$$

which is what we wanted to show. This concludes the remark.

Now, consider the two exact sequences

$$0 \to \Lambda x/\Lambda(\lambda_\Omega \cdot x) \to M/\Lambda(\lambda_\Omega \cdot x) \to M/\Lambda x \to 0$$

and

$$0 \to \Lambda y/\Lambda(s_\Omega \cdot y) \to M/\Lambda(s_\Omega \cdot y) \to M/\Lambda y \to 0,$$

which are both exact sequences in $\mathfrak{M}_{\mathcal{H}}(\mathcal{G})$, just note that $M/\Lambda x$, $M/\Lambda y$ and $\Lambda y/\Lambda(s_\Omega \cdot y)$ are certainly $\mathcal{S}^*$-torsion and $\Lambda x/\Lambda(\lambda_\Omega \cdot x)$ is $\mathcal{S}^*$-torsion, because $\frac{\lambda_\Omega}{s_\Omega}$, and hence λ_Ω, belongs to $\Lambda(\mathcal{G})^\times_{\mathcal{S}^*}$. It follows by exactness that the middle terms are also $\mathcal{S}^*$-torsion.

Since x and y are generators of $M_{\mathcal{S}^*} \cong \Lambda_{\mathcal{S}^*}$ and since $\mathcal{S}^*$ does not contain any non-trivial zero divisors we have $\mathrm{ann}_\Lambda(x) = 0 = \mathrm{ann}_\Lambda(y)$. It follows that

$$\Lambda x/\Lambda(\lambda_\Omega \cdot x) \cong \Lambda/\Lambda\lambda_\Omega \quad \text{and} \quad \Lambda y/\Lambda(s_\Omega \cdot y) \cong \Lambda/\Lambda s_\Omega.$$

In $K_0(\mathfrak{M}_{\mathcal{H}}(\mathcal{G}))$ we get

$$\begin{aligned} [M/\Lambda x] &= [M/\Lambda(\lambda_\Omega \cdot x)] - [\Lambda x/\Lambda(\lambda_\Omega \cdot x)] \\ &= [M/\Lambda(s_\Omega \cdot y)] - [\Lambda/\Lambda\lambda_\Omega] \\ &= [M/\Lambda y] + [\Lambda y/\Lambda(s_\Omega \cdot y)] - [\Lambda/\Lambda\lambda_\Omega] \\ &= [M/\Lambda y] + [\Lambda/\Lambda s_\Omega] - [\Lambda/\Lambda\lambda_\Omega] \end{aligned}$$

Rewriting this reads

$$\begin{aligned} &[\Lambda/\Lambda\lambda_\Omega] - [\Lambda/\Lambda s_\Omega] + \left[\mathrm{Ind}_{\mathcal{G}}^{\mathcal{G}'}\left((T_pE/T^0)(-1) \otimes_{\mathbb{Z}_p} \mathbb{H}^1_{\mathrm{loc}}\right)/\Lambda\phi_{t',u'}(1)\right] \\ &= \left[\mathrm{Ind}_{\mathcal{G}}^{\mathcal{G}'}\left((T_pE/T^0)(-1) \otimes_{\mathbb{Z}_p} \mathbb{H}^1_{\mathrm{loc}}\right)/\Lambda\mathrm{loc}(\phi_{t,u}(1))\right] \end{aligned} \tag{6.21}$$

and we note that $\partial([\Omega_{p,u,u'}]) = [\Lambda/\Lambda\lambda_\Omega] - [\Lambda/\Lambda s_\Omega]$ where $\partial : K_1(\Lambda(\mathcal{G})_{\mathcal{S}^*}) \to K_0(\mathfrak{M}_{\mathcal{H}}(\mathcal{G}))$ is the connecting homomorphism from K-theory and $[\Omega_{p,u,u'}]$ denotes the image of $\Omega_{p,u,u'}$ in $K_1(\Lambda(\mathcal{G})_{\mathcal{S}^*})$.

7 *Twist conjecture for Elliptic Curves $E/\mathbb{Q}$ with CM*

In this section we study the third and last of Kato's conjectures which were mentioned in the introduction. This last conjecture connects the work from the previous sections. While the study of $L_{p,u} \in K_1(\mathbb{Z}_p[[G]]_{S^*})$ and $\mathcal{E}_{p,u'} \in K_1(\widehat{\mathbb{Z}}_p^{ur}[[\mathcal{G}']]_{\tilde{\mathcal{S}}'^*})$ is certainly interesting in its own right, it is strongly motivated by the following question: Is it possible to express p-adic L-functions of motives M (satisfying, at the minimum, conditions (C1) and (C2) from [21]), up to elements of the form $\Omega_{p,u,u'}$, as twists of universal elements such as $L_{p,u}$ and $\mathcal{E}_{p,u'}$ by p-adic Galois representations associated to M? Moreover, are such twist elements characteristic elements of the Pontryagin dual of the respective Selmer groups (or Selmer complexes as in loc. cit.)?

Since Kato did not give a precise interpolation formula for the element starring in his third conjecture, we do not aim to state the conjecture in the greatest generality and simply restrict to the setting in which, under a torsion assumption, we can prove it. Therefore, let us consider one of the elliptic curves $E/\mathbb{Q}$ from the setting described in the next subsection with complex multiplication by $\mathcal{O}_K$ and prime power conductor over K. In this setting, under the assumption that $\mathrm{Sel}(K_\infty, T_pE^*(1))^\vee$ is $\mathcal{S}^*$-torsion, we will define an element $\mathcal{L}_{p,u,E} \in$

$K_1(\hat{\mathbb{Z}}_p^{\mathrm{ur}}[[\mathcal{G}]]_{\tilde{\mathcal{S}}^*})$ in terms of twists of the elements $L_{p,u}$ and $\mathcal{E}_{p,u'}$ studied in sections 3 and 4, respectively, and in terms of $\Omega_{p,u,u'}$ defined in subsection 6.5. We will then prove that, up to an Euler factor, $\mathcal{L}_{p,u,E}$ is a characteristic element of the dual Selmer group, see theorem 7.5. Moreover, we show in corollary 7.8 that $\mathcal{L}_{p,u,E}$ coincides with $\tau_{\psi^{-1}}(\lambda)$, the twist of de Shalit's element λ from definition 3.6 by the inverse ψ^{-1} of the Größencharacter ψ attached to E/K which gives the action of $G = G(K([p^\infty])/K)$ on $T_\pi E$. For $\tau_{\psi^{-1}}(\lambda)$ an interpolation property is immediately derived from ([14], Theorem 4.14, p. 80) and ([5], Lemma 2.10, p. 394).

7.1 Setting

Let $E/\mathbb{Q}$ be one of the elliptic curves listed in Appendix A, §3 of Silverman's book [39] with complex multiplication by $\mathcal{O}_K$, K quadratic imaginary, which are representatives of their $\bar{\mathbb{Q}}$-isomorphism classes with minimal discriminant. We fix a prime number $p \in \mathbb{Z}$, $p \geq 5$, at which E has good ordinary reduction so that p splits in $\mathcal{O}_K$ into distinct primes $\mathfrak{p}$ and $\bar{\mathfrak{p}}$, $\bar{\mathfrak{p}} \neq \mathfrak{p}$. We fix an embedding $\overline{\mathbb{Q}} \subset \overline{\mathbb{Q}}_p$ as in setting 5.3 and adopt the same notation. Moreover, we assume that the torsion conjecture 6.4 holds, i.e., that $\mathrm{Sel}(K_\infty, T_pE^*(1))^\vee$ is $\mathcal{S}^*$-torsion.

Lemma 7.1. *Assume that* $\mathrm{Sel}(K_\infty, T_pE^*(1))^\vee$ *is* $\mathcal{S}^*$*-torsion. Then,* $\mathbb{H}^2_\Sigma$ *and* $\mathcal{A}_\infty$ *are* S^**-torsion, i.e., the torsion assumption from the commutative main theorem 3.9 is satisfied.*

Proof. Recall from remark 3.10 that $\mathbb{H}^2_\Sigma$ is S^*-torsion if and only if $\mathcal{A}_\infty$ is S^*-torsion. It follows from the Poitou-Tate sequence (5.4) that $H^2(G_\Sigma(\mathbb{Q}), \Lambda(\mathcal{G})^\# \otimes T_pE) \cong T_pE(-1) \otimes \mathbb{H}^2_\Sigma$ is $\mathcal{S}^*$-torsion. Since G is of finite index in $\mathcal{G}$ it follows that $T_pE(-1) \otimes \mathbb{H}^2_\Sigma$ is S^*-torsion. But as G-modules we have an injection $T_\pi E(-1) \otimes \mathbb{H}^2_\Sigma \hookrightarrow T_pE(-1) \otimes \mathbb{H}^2_\Sigma$, which shows that $T_\pi E(-1) \otimes \mathbb{H}^2_\Sigma$ is S^*-torsion. Since $T_\pi E(-1)$ is of $\mathbb{Z}_p$-rank one we can twist this module by the inverse of the character giving the action of G on $T_\pi E(-1)$ and, hence, conclude that $\mathbb{H}^2_\Sigma$ must be S^*-torsion.

Let us fix a complex period Ω such that for the period lattice L of E we have $L = \mathcal{O}_K\Omega$. We also fix a generator ϵ of $\mathbb{Z}_p(1)$ as in [14] and let these choices determine the period Ω_p as in (loc. cit., p. 67f), which determines an isomorphism $\theta = \theta_{\Omega_p} : \mathbb{G}_m \cong \hat{E}$ of formal groups. We write ψ for the Größencharacter attached to E/K and set $\pi = \psi(\mathfrak{p})$ and $\bar{\pi} = \overline{\psi(\mathfrak{p})} = \psi(\bar{\mathfrak{p}})$ (where the last equation holds by ([26], p. 559)) for the generators of $\mathfrak{p}$ and $\bar{\mathfrak{p}}$, respectively. For the Ore sets from (2.2) in the Iwasawa algebras with $\hat{\mathbb{Z}}_p^{ur}$-coefficients $\hat{\mathbb{Z}}_p^{\mathrm{ur}}[[G]]$, $\hat{\mathbb{Z}}_p^{\mathrm{ur}}[[\mathcal{G}]]$ and $\hat{\mathbb{Z}}_p^{\mathrm{ur}}[[\mathcal{G}']]$ we will write

$$\tilde{S}^* \subset \hat{\mathbb{Z}}_p^{\mathrm{ur}}[[G]], \quad \tilde{\mathcal{S}}^* \subset \hat{\mathbb{Z}}_p^{\mathrm{ur}}[[\mathcal{G}]] \quad \text{and} \quad \tilde{\mathcal{S}'}^* \subset \hat{\mathbb{Z}}_p^{\mathrm{ur}}[[\mathcal{G}']].$$

Let us make a remark linking the setting considered in this chapter to some of the previous chapters. We write $K_n = K(E[p^n])$, $n \geq 1$, and $\Lambda' = \Lambda(\mathcal{G}') = \Lambda(G')$.

Remark 7.2. (i) if we consider the curve $E/\mathbb{Q}$ as a curve over K all the assumptions from the setting of section 3 are satisfied, including assumption 3.1, compare remark 6.2. Hence, up to the assumption that $\varprojlim_n (Cl(K_n)\{p\})$ is S^*-torsion, the commutative main theorem 3.9 holds.

(ii) when we consider the decomposition group $\mathcal{G}' \cong G(K_{\infty,\nu}/\mathbb{Q}_p)$ of a fixed prime ν of $K_\infty = K(E[p^\infty])$ above $\mathfrak{p}$, then we are in the setting of section 4.3, compare remark 5.5. In particular, the local main conjecture holds in this case.

(iii) since we assume that the torsion conjecture 6.4 holds (which was used to define $\Omega_{p,u,u'}$), our present setting satisfies all conditions of the setting considered in section 6.1 where we studied $\Omega_{p,u,u'}$, including (6.1), compare also remark 6.2.

(iv) recall from (2.3) that in order to define the twist operators on K_1-groups, we fixed a $\mathbb{Z}_p$-basis of the representation space. Throughout this chapter we fix the same basis $t_{\bar{\pi}}$ of $T_{\bar{\pi}}E$ as in section 6.5 where we defined $\Omega_{p,u,u'}$. Let us write $[-] : T_pE \longrightarrow T_pE/T_p\hat{E}$ for the canonical projection and recall from remark 6.6 that it restricts to an isomorphism $[-] : T_{\bar{\pi}}E \xrightarrow{\sim} T_pE/T_p\hat{E}$. Hence, $t_{\bar{\pi}}$ canonically determines a basis $t_0 = [t_{\bar{\pi}}]$ of $T_pE/T_p\hat{E}$. Together with our fixed generator ϵ of $\mathbb{Z}_p(1)$ which determines a basis ϵ_0 of $\mathbb{Z}_p(-1)$, these determine bases $t = t_{\bar{\pi}} \otimes \epsilon_0$ and $t' = [t_{\bar{\pi}}] \otimes \epsilon_0$ of $T_{\bar{\pi}}E(-1)$ and $(T_pE/T_p\hat{E})(-1)$, respectively, such that under the natural map $\iota_1 : T_pE(-1) \longrightarrow (T_pE/T_p\hat{E})(-1)$ induced by $[-]$ we have $t \mapsto t'$. We also write ι_1 for the restriction of ι_1 to $T_{\bar{\pi}}E(-1)$.

7.2 Definition of of $\mathcal{L}_{p,u,E}$

Let us fix $\mathfrak{q}$ as in subsection 3.3 and write $u = u(\mathfrak{q})$, where $u(\mathfrak{q})$ was defined in definition 3.6. Our first task is to define the element $\mathcal{L}_{p,u,E} = \mathcal{L}_{p,u,u',E} \in K_1(\hat{\mathbb{Z}}_p^{\mathrm{ur}}[[\mathcal{G}]]_{\tilde{\mathcal{S}}^*})$ in terms of twists of the elements $L_{p,u}$, and $\mathcal{E}_{p,u'}$ and in terms of $\Omega_{p,u,u'}$. The element $\mathcal{E}_{p,u'}$ from the local theory will be twisted by the $\mathcal{G}'$-representation $(T_pE/T_p\hat{E})(-1)$, which is of $\mathbb{Z}_p$-rank one. We will write $\tau_{E/\hat{E}(-1)}$ for the twist operator from corollary 2.2 on $K_1(\hat{\mathbb{Z}}_p^{\mathrm{ur}}[[\mathcal{G}']]_{\tilde{\mathcal{S}}'^*})$ induced by $(T_pE/T_p\hat{E})(-1)$. The element $L_{p,u} := [\frac{1}{x_{\mathfrak{q}}}]$ from the commutative global theorem 3.9 will be twisted by the G-representation $T_{\bar{\pi}}E(-1)$, which is also of $\mathbb{Z}_p$-rank one. We will write $\tau_{E_{\bar{\pi}}(-1)}$ for the twist operator from corollary 2.2 on $K_1(\Lambda(G)_{S^*})$ induced by $T_{\bar{\pi}}E(-1)$. We want to consider the elements

$$\tau_{E_{\bar{\pi}}(-1)}(L_{p,u}) \in K_1(\Lambda(G)_{S^*}), \quad \tau_{E/\hat{E}(-1)}(\mathcal{E}_{p,u'}) \in K_1(\hat{\mathbb{Z}}_p^{\mathrm{ur}}[[\mathcal{G}']]_{\tilde{\mathcal{S}}'^*}), \quad \Omega_{p,u,u'} \in K_1(\Lambda(\mathcal{G})_{\mathcal{S}^*}) \tag{7.1}$$

in one common K_1-group. We have inclusions

$$S^* \subset \mathcal{S}^* \subset \tilde{\mathcal{S}}^* \quad \text{and} \quad \tilde{\mathcal{S}}'^* \subset \tilde{\mathcal{S}}^*,$$

compare ([36], corollary A.8.15 and lemma A.8.19), and, hence, get natural maps

$$\begin{array}{ccccc} K_1(\Lambda(G)_{S^*}) & \longrightarrow & K_1(\Lambda(\mathcal{G})_{\mathcal{S}^*}) & \longrightarrow & K_1(\hat{\mathbb{Z}}_p^{\mathrm{ur}}[[\mathcal{G}]]_{\tilde{\mathcal{S}}^*}) \\ & & & & \uparrow \\ & & & & K_1(\hat{\mathbb{Z}}_p^{\mathrm{ur}}[[\mathcal{G}']]_{\tilde{\mathcal{S}}'^*}). \end{array} \tag{7.2}$$

It is through these maps that we consider the images of the elements from (7.1) in the group $K_1(\hat{\mathbb{Z}}_p^{\mathrm{ur}}[[\mathcal{G}]]_{\tilde{\mathcal{S}}^*})$.

Definition 7.3 (of $\mathcal{L}_{p,u,E}$). *We define $\mathcal{L}_{p,u,E} \in K_1(\hat{\mathbb{Z}}_p^{\mathrm{ur}}[[\mathcal{G}]]_{\tilde{\mathcal{S}}^*})$ by*

$$\mathcal{L}_{p,u,E} = \frac{\tau_{E_{\bar{\pi}}(-1)}(L_{p,u})}{\tau_{E/\hat{E}(-1)}(\mathcal{E}_{p,u'})} \cdot \Omega_{p,u,u'} \cdot \frac{1}{12},$$

which is independent of u' as we show in theorem 7.7 and note that $12 \in \mathbb{Z}_p^\times$ since $p \geq 5$ by assumption.

Remark 7.4. We could have also defined $\mathcal{L}_{p,u,E}$ as an element in the intersection of Iwasawa algebras

$$\left(\hat{\mathbb{Z}}_p^{\mathrm{ur}}[[\mathcal{G}]]_{\tilde{\mathcal{S}}^*}\right)^\times \bigcap \hat{\mathbb{Z}}_p^{\mathrm{ur}}[[\mathcal{G}]]_{\mathcal{S}^*}$$

since the representations by which we twist have $\mathbb{Z}_p$-rank one. In fact, $\tau_{E_{\bar{\pi}}(-1)}(L_{p,u})$ comes from $\tau_{E_{\bar{\pi}}(-1)}(1/x_{\mathfrak{q}}) \in \Lambda(G)_S^\times$ with $x_{\mathfrak{q}} = \mathrm{Frob}_{\mathfrak{q}} - N\mathfrak{q} \in \Lambda(G)_S^\times$, where we also write $\tau_{E_{\bar{\pi}}(-1)}$ for the ring homomorphism $\Lambda(G)_{S^*} \to \Lambda(G)_{S^*}$ from proposition 2.1 corresponding to $T_{\bar{\pi}}E(-1)$. Likewise, recall from (4.16) that $\mathcal{E}_{p,u'}$ was originally defined as an element in $\left(\hat{\mathbb{Z}}_p^{\mathrm{ur}}[[\mathcal{G}']]_{\tilde{\mathcal{S}}'^*}\right)^\times$ and $\mathcal{E}_{p,u'}^{-1} \in \hat{\mathbb{Z}}_p^{\mathrm{ur}}[[\mathcal{G}']]$, so that we can interpret $\tau_{E/\hat{E}(-1)}(\mathcal{E}_{p,u'})$ as an element of $\left(\hat{\mathbb{Z}}_p^{\mathrm{ur}}[[\mathcal{G}']]_{\tilde{\mathcal{S}}'^*}\right)^\times$, where, as above, we consider $\tau_{E/\hat{E}(-1)}$ also as the ring homomorphism $\hat{\mathbb{Z}}_p^{\mathrm{ur}}[[\mathcal{G}']]_{\tilde{\mathcal{S}}'^*} \to \hat{\mathbb{Z}}_p^{\mathrm{ur}}[[\mathcal{G}']]_{\tilde{\mathcal{S}}'^*}$ from proposition 2.1 induced by $(T_pE/T_p\hat{E})(-1)$. Lastly, $\Omega_{p,u,u'}$ is, by definition 6.5, an element in $\Lambda(\mathcal{G})_{\mathcal{S}^*}^\times$.

7.3 Main theorems

Before stating our first theorem let us recall from K-theory the following maps

$$\begin{aligned}
&K_0(\mathfrak{M}_H(G)) \longrightarrow K_0(\mathfrak{M}_{\mathcal{H}}(\mathcal{G})), && [M] \longmapsto [\Lambda(\mathcal{G}) \otimes_{\Lambda(G)} M] = [\mathrm{Ind}_{\mathcal{G}}^{G} M]\\
&K_0(\mathfrak{M}_{\mathcal{H}}(\mathcal{G})) \longrightarrow K_0(\mathfrak{M}_{\hat{\mathbb{Z}}_p^{\mathrm{ur}},\mathcal{H}}(\mathcal{G})), && [M] \longmapsto [\hat{\mathbb{Z}}_p^{\mathrm{ur}} \hat{\otimes}_{\mathbb{Z}_p} M] = [\hat{\mathbb{Z}}_p^{\mathrm{ur}}[[\mathcal{G}]] \otimes_{\Lambda(\mathcal{G})} M]\\
&K_0(\mathfrak{M}_{\hat{\mathbb{Z}}_p^{\mathrm{ur}},\mathcal{H}'}(\mathcal{G}')) \longrightarrow K_0(\mathfrak{M}_{\hat{\mathbb{Z}}_p^{\mathrm{ur}},\mathcal{H}}(\mathcal{G})), && [M] \longmapsto [\hat{\mathbb{Z}}_p^{\mathrm{ur}}[[\mathcal{G}]] \otimes_{\hat{\mathbb{Z}}_p^{\mathrm{ur}}[[\mathcal{G}']]} M] = [\mathrm{Ind}_{\mathcal{G}}^{\mathcal{G}'} M],
\end{aligned}$$

compare ([36], corollary A.8.16 and lemma A.8.20) for the fact that these maps are well-defined. Let us also recall the following exact sequence from K-theory

$$K_1(\hat{\mathbb{Z}}_p^{\mathrm{ur}}[[\mathcal{G}]]) \longrightarrow K_1(\hat{\mathbb{Z}}_p^{\mathrm{ur}}[[\mathcal{G}]]_{\tilde{\mathcal{S}}^*}) \xrightarrow{\partial} K_0(\mathfrak{M}_{\hat{\mathbb{Z}}_p^{\mathrm{ur}},\mathcal{H}}(\mathcal{G})) \longrightarrow K_0(\hat{\mathbb{Z}}_p^{\mathrm{ur}}[[\mathcal{G}]]) \twoheadrightarrow K_0(\hat{\mathbb{Z}}_p^{\mathrm{ur}}[[\mathcal{G}]]_{\tilde{\mathcal{S}}^*}).$$

Next, we come to one of the main theorems of this paper.

Theorem 7.5 *Let the setting be as above. Assume that* $\mathrm{Sel}(K_\infty, T^*(1))^\vee$ *is* $\mathcal{S}^*$*-torsion. Then, up to a twisted Euler factor,* $\mathcal{L}_{p,u,E}$ *is a characteristic element of* $\hat{\mathbb{Z}}_p^{\mathrm{ur}} \hat{\otimes}_{\mathbb{Z}_p} \mathrm{Sel}(K_\infty, T_pE^*(1))^\vee$*. More concretely, we have*

$$\partial(\mathcal{L}_{p,u,E}) = [\mathrm{Sel}(K_\infty, T_pE^*(1))^\vee]_{\tilde{\Lambda}} + [\text{c–}\mathrm{Ind}_{\mathcal{G}}^{\mathcal{G}_{\nu_l}} T_pE(-1)]_{\tilde{\Lambda}},$$

where l *is the unique prime at which* $E/\mathbb{Q}$ *has bad reduction,* $\mathcal{G}_{\nu_l}$ *is the decomposition group of some place of* K_∞ *above* l *and the notation* $[-]_{\tilde{\Lambda}}$ *is explained in (7.3) below.*

Proof. Let us write $\tilde{\Lambda} = \hat{\mathbb{Z}}_p^{\mathrm{ur}}[[\mathcal{G}]]$. For any finitely generated $\Lambda(\mathcal{G})$-module M that is $\mathcal{S}^*$-torsion let us write

$$[M]_{\tilde{\Lambda}} := [\hat{\mathbb{Z}}_p^{\mathrm{ur}}[[\mathcal{G}]] \otimes_{\Lambda(\mathcal{G})} M] = [\hat{\mathbb{Z}}_p^{\mathrm{ur}} \hat{\otimes}_{\mathbb{Z}_p} M] \tag{7.3}$$

for its class in $K_0(\mathfrak{M}_{\hat{\mathbb{Z}}_p^{\mathrm{ur}},\mathcal{H}}(\mathcal{G}))$. We will adopt a similar notation

$$[M]_{\tilde{\Lambda}} := [\hat{\mathbb{Z}}_p^{\mathrm{ur}}[[\mathcal{G}]] \otimes_{\hat{\mathbb{Z}}_p^{\mathrm{ur}}[[\mathcal{G}']]} M]$$

for a finitely generated $\hat{\mathbb{Z}}_p^{\mathrm{ur}}[[\mathcal{G}']]$-module M that is $\tilde{\mathcal{S}}'^*$-torsion. We will use repeatedly the diagram (2.4) which states that twisting on K_1-groups corresponds to taking the tensor product with the respective representation on K_0-groups. The contribution of $\tau_{E_{\bar{\pi}}(-1)}(L_{p,u})$, by the defining property (3.7) of $L_{p,u} = \frac{1}{x_{\mathfrak{q}}}$, is given by

$$\begin{aligned}\partial(\tau_{E_{\bar{\pi}}(-1)}(L_{p,u})) &= \left[\mathrm{Ind}_{\mathcal{G}}^{G}(T_{\bar{\pi}}E(-1) \otimes_{\mathbb{Z}_p} \mathrm{Res}_{G}^{\mathcal{G}}\mathbb{H}^2_{\Sigma})\right]_{\tilde{\Lambda}} - \left[\mathrm{Ind}_{\mathcal{G}}^{G}(T_{\bar{\pi}}E(-1) \otimes_{\mathbb{Z}_p} \mathbb{H}^1_{\Sigma}/\Lambda(G)u)\right]_{\tilde{\Lambda}} \\ &= \left[T_pE(-1) \otimes_{\mathbb{Z}_p} \mathbb{H}^2_{\Sigma})\right]_{\tilde{\Lambda}} - \left[\left(T_pE(-1) \otimes_{\mathbb{Z}_p} \mathbb{H}^1_{\Sigma}\right)/\Lambda(\mathcal{G})\phi_{t,u}(1)\right]_{\tilde{\Lambda}} \qquad (7.4)\end{aligned}$$

where the second equality follows from (6.6).

We write $\tilde{\Lambda}' = \hat{\mathbb{Z}}_p^{\mathrm{ur}}[[\mathcal{G}']]$ and $T' = (T_pE/T_p\hat{E})(-1)$ as before. The element $\mathcal{E}_{p,u'}$ from the local main theorem, by the defining property (4.5) of $\mathcal{E}_{p,u'}$, maps to

$$\begin{aligned}\partial(\tau_{E/\hat{E}(-1)}(\mathcal{E}_{p,u'})) &= \left[T' \otimes_{\mathbb{Z}_p} \left(\tilde{\Lambda}' \otimes_{\Lambda'} \mathbb{H}^2_{\mathrm{loc}}\right)\right]_{\tilde{\Lambda}} - \left[T' \otimes_{\mathbb{Z}_p} \left(\tilde{\Lambda}' \otimes_{\Lambda'} \left(\mathbb{H}^1_{\mathrm{loc}}/\Lambda' u'\right)\right)\right]_{\tilde{\Lambda}} \\ &= \left[\tilde{\Lambda}' \otimes_{\Lambda'} \left(T' \otimes_{\mathbb{Z}_p} \mathbb{H}^2_{\mathrm{loc}}\right)\right]_{\tilde{\Lambda}} - \left[\tilde{\Lambda}' \otimes_{\Lambda'} \left(T' \otimes_{\mathbb{Z}_p} \left(\mathbb{H}^1_{\mathrm{loc}}/\Lambda' u'\right)\right)\right]_{\tilde{\Lambda}} \\ &= \left[\tilde{\Lambda} \otimes_{\Lambda'} \left(T' \otimes_{\mathbb{Z}_p} \mathbb{H}^2_{\mathrm{loc}}\right)\right] - \left[\tilde{\Lambda} \otimes_{\Lambda'} \left(T' \otimes_{\mathbb{Z}_p} \left(\mathbb{H}^1_{\mathrm{loc}}/\Lambda' u'\right)\right)\right] \\ &= \left[\mathrm{Ind}_{\mathcal{G}}^{\mathcal{G}'}\left(T' \otimes_{\mathbb{Z}_p} \mathbb{H}^2_{\mathrm{loc}}\right)\right]_{\tilde{\Lambda}} - \left[\mathrm{Ind}_{\mathcal{G}}^{\mathcal{G}'}\left(T' \otimes_{\mathbb{Z}_p} \left(\mathbb{H}^1_{\mathrm{loc}}/\Lambda' u'\right)\right)\right]_{\tilde{\Lambda}} \\ &= \left[\mathrm{Ind}_{\mathcal{G}}^{\mathcal{G}'}\left(T' \otimes_{\mathbb{Z}_p} \mathbb{H}^2_{\mathrm{loc}}\right)\right]_{\tilde{\Lambda}} - \left[\left(\mathrm{Ind}_{\mathcal{G}}^{\mathcal{G}'}\left(T' \otimes_{\mathbb{Z}_p} \mathbb{H}^1_{\mathrm{loc}}\right)\right)/\Lambda(\mathcal{G})\phi_{t',u'}(1)\right]_{\tilde{\Lambda}} \qquad (7.5)\end{aligned}$$

where the second equality follows from ([36], corollary 1.1.7) (twisting commutes with extension of scalars to $\hat{\mathbb{Z}}_p^{\mathrm{ur}}$), the last equality from (6.11) and in the third and fourth equation we use the isomorphisms

$$\tilde{\Lambda} \otimes_{\tilde{\Lambda}'} \tilde{\Lambda}' \otimes_{\Lambda'} M \cong \tilde{\Lambda} \otimes_{\Lambda'} M \cong \tilde{\Lambda} \otimes_{\Lambda(\mathcal{G})} \Lambda(\mathcal{G}) \otimes_{\Lambda'} M$$

for a finitely generated Λ'-module M.

Let us write $\Lambda = \Lambda(\mathcal{G})$. The element $\Omega_{p,u,u'}$, by (6.21), maps to

$$\begin{aligned}\partial(\Omega_{p,u,u'}) = &\quad [\Lambda/\Lambda\lambda_{\Omega}]_{\tilde{\Lambda}} - [\Lambda/\Lambda s_{\Omega}]_{\tilde{\Lambda}} \\ = &\quad \left[\mathrm{Ind}_{\mathcal{G}}^{\mathcal{G}'}\left(T' \otimes_{\mathbb{Z}_p} \mathbb{H}^1_{\mathrm{loc}}\right)/\Lambda\mathrm{loc}(\phi_{t,u}(1))\right]_{\tilde{\Lambda}} \\ &\quad - \left[\mathrm{Ind}_{\mathcal{G}}^{\mathcal{G}'}\left(T' \otimes_{\mathbb{Z}_p} \mathbb{H}^1_{\mathrm{loc}}\right)/\Lambda\phi_{t',u'}(1)\right]_{\tilde{\Lambda}}. \qquad (7.6)\end{aligned}$$

We conclude from (7.4), (7.5) and (7.6) that the element $\mathcal{L}_{p,u,E}$ (note that $\partial([1/12]) = 0$) maps to

$$\begin{aligned}\partial(\mathcal{L}_{p,u,E}) = &\quad \partial(\tau_{E_{\bar{\pi}}(-1)}(L_{p,u})) - \partial(\tau_{E/\hat{E}(-1)}(\mathcal{E}_{p,u'})) + \partial(\Omega_{p,u,u'}) \\ = &\quad \left[T_pE(-1) \otimes_{\mathbb{Z}_p} \mathbb{H}^2_{\Sigma})\right]_{\tilde{\Lambda}} - \left[\left(T_pE(-1) \otimes_{\mathbb{Z}_p} \mathbb{H}^1_{\Sigma}\right)/\Lambda(\mathcal{G})\phi_{t,u}(1)\right]_{\tilde{\Lambda}} \\ &- \left[\mathrm{Ind}_{\mathcal{G}}^{\mathcal{G}'}\left(T' \otimes_{\mathbb{Z}_p} \mathbb{H}^2_{\mathrm{loc}}\right)\right]_{\tilde{\Lambda}} + \left[\mathrm{Ind}_{\mathcal{G}}^{\mathcal{G}'}\left(T' \otimes_{\mathbb{Z}_p} \mathbb{H}^1_{\mathrm{loc}}\right)/\Lambda\mathrm{loc}(\phi_{t,u}(1))\right]_{\tilde{\Lambda}}. \qquad (7.7)\end{aligned}$$

Using the isomorphisms from (6.12) and (6.13) and

$$T_pE(-1) \otimes_{\mathbb{Z}_p} \mathbb{H}^2_{\Sigma} \cong H^2(G_{\Sigma_{\mathbb{Q}}}(\mathbb{Q}), \Lambda^{\#} \otimes T),$$

compare ([36], corollary A.3.10), the Poitou-Tate sequence from (5.4) induces the exact sequence

$$
\begin{array}{l}
0 \longrightarrow \big(T_pE(-1)\otimes_{\mathbb{Z}_p}\mathbb{H}^1_{\Sigma}\big)/\Lambda\phi_{t,u}(1) \xrightarrow{\text{loc}} \mathrm{Ind}_{\mathcal{G}}^{\mathcal{G}'}\big(T'\otimes_{\mathbb{Z}_p}\mathbb{H}^1_{\text{loc}}\big)/\Lambda\mathrm{loc}(\phi_{t,u}(1)) \\
\longrightarrow \mathrm{Sel}(K_\infty, T_pE^*(1))^\vee \longrightarrow T_pE(-1)\otimes_{\mathbb{Z}_p}\mathbb{H}^2_{\Sigma} \\
\longrightarrow \bigoplus_{q\in\Sigma_{\mathbb{Q},f}} \text{c–}\mathrm{Ind}_{\mathcal{G}}^{\mathcal{G}_{\nu_q}} T_pE(-1) \longrightarrow T_pE(-1) \longrightarrow 0.
\end{array}
\tag{7.8}
$$

where we recall that $\Sigma_{\mathbb{Q},f} = \{p, l\}$, l being the unique prime at which $E/\mathbb{Q}$ has bad reduction, and where we used that the kernel of the map to the dual Selmer group from (6.14) contains $\Lambda\mathrm{loc}(\phi_{t,u}(1))$ by exactness of (6.14). By assumption $\mathrm{Sel}(K_\infty, T_pE^*(1))^\vee$ is $\mathcal{S}^*$-torsion so that (7.8) is an exact sequence in $\mathfrak{M}_{\mathcal{H}}(\mathcal{G})$. We note that for any $q \in \Sigma_{\mathbb{Q},f}$

$$\text{c–}\mathrm{Ind}_{\mathcal{G}}^{\mathcal{G}_{\nu_q}} T_pE(-1) \overset{\text{def}}{=} \Lambda(\mathcal{G})\hat{\otimes}_{\Lambda(\mathcal{G}_{\nu_q})} T_pE(-1) \cong \Lambda(\mathcal{G}) \otimes_{\Lambda(\mathcal{G}_{\nu_q})} T_pE(-1),$$

where the isomorphism on the right holds since $\Lambda(\mathcal{G})$ is a pseudo-compact $(\Lambda(\mathcal{G}), \Lambda(\mathcal{G}_{\nu_q}))$-bimodule and since $T_pE(-1)$ is finitely presented as a $\Lambda(\mathcal{G}_{\nu_q})$-module (note that $\Lambda(\mathcal{G}_{\nu_q})$ is Noetherian since $\mathcal{G}_{\nu_q}$, as a closed subgroup of $\mathcal{G}$, is a compact p-adic Lie group) and for such modules the completed tensor product coincides with the usual tensor product, see ([47], proposition 1.14). For $q = p$, $\mathcal{G}_{\nu_q} = \mathcal{G}'$ is of finite index in $\mathcal{G}$ and therefore

$$\Lambda(\mathcal{G}) \otimes_{\Lambda(\mathcal{G}')} T_pE(-1) \cong \mathrm{Ind}_{\mathcal{G}}^{\mathcal{G}'} T_pE(-1),$$

so that compact induction and usual induction from $\mathcal{G}_{\nu_q}$ to $\mathcal{G}$ coincide for $q = p$.

Since $T_pE(-1)$, $T_\pi E$ and $T'\otimes_{\mathbb{Z}_p}\mathbb{H}^2_{\text{loc}} \cong T'$ (recall that $\mathbb{H}^2_{\text{loc}} \cong \mathbb{Z}_p$) are finitely generated as $\mathbb{Z}_p$-modules, we have

$$[\mathrm{Ind}_{\mathcal{G}}^{\mathcal{G}'} T_pE(-1)] = 0, \quad [T_pE(-1)] = [\mathrm{Ind}_{\mathcal{G}}^{G} T_\pi E] = 0, \quad [\mathrm{Ind}_{\mathcal{G}}^{\mathcal{G}'}\big(T'\otimes_{\mathbb{Z}_p}\mathbb{H}^2_{\text{loc}}\big)] = 0$$

in $K_0(\mathfrak{M}_{\mathcal{H}}(\mathcal{G}))$, see ([36], remark 6.1.3 (ii) and corollary 1.2.4). It follows that also their images in $K_0(\mathfrak{M}_{\hat{\mathbb{Z}}_p^{\text{ur}},\mathcal{H}}(\mathcal{G}))$ vanish. We conclude from (7.7) and the above Poitou-Tate sequence that

$$\partial(\mathcal{L}_{p,u,u'}) = [\mathrm{Sel}(K_\infty, T_pE^*(1))^\vee]_{\tilde{\Lambda}} + [\text{c–}\mathrm{Ind}_{\mathcal{G}}^{\mathcal{G}_{\nu_l}} T_pE(-1)]_{\tilde{\Lambda}},$$

which finishes the proof.

Next, we want to study the interpolation property of $\mathcal{L}_{p,u,E}$. Let us make a preparatory remark.

Remark 7.6. (i) it is well-known that the Weil pairing induces an isomorphism $\bigwedge^2(T_pE) \cong \mathbb{Z}_p(1)$. Hence, the determinant of $\rho_E : G \longrightarrow \mathrm{Aut}_{\mathbb{Z}_p}(T_pE)$ is given by the cyclotomic character κ, i.e., we have

$$\psi\cdot\bar{\psi} = \det(\rho_E) = \kappa,$$

where $\bar{\psi}$ is the Größencharacter of infinity type $(0,1)$ attached to E. Since the action of G on $T_{\bar{\pi}}E$ is given by $\bar{\psi}$ we see that $T_{\bar{\pi}}E(-1)$ corresponds to $\bar{\psi}\cdot\kappa^{-1} = \psi^{-1}$. In particular, under the ring homomorphism $\tau_{E_{\bar{\pi}}(-1)} : \Lambda(G) \to \Lambda(G)$ we have for any $g \in G$

$$\tau_{E_{\bar{\pi}}(-1)}(g) \overset{\text{def}}{=} (\bar{\psi}\cdot\kappa^{-1})(g^{-1})g = \psi^{-1}(g^{-1})g = \psi(g)g.$$

(ii) recall from proposition 2.1 that the twist operators $\tau_{E/\hat{E}(-1)}$ and $\tau_{E_{\bar{\pi}}(-1)}$ are induced by ring homomorphisms between the respective localized Iwasawa algebras and note by the isomorphism from (6.15) that these fit into a commutative diagram

$$\begin{array}{ccccc}
\hat{\mathbb{Z}}_p^{\mathrm{ur}}[[\mathcal{G}']]_{\tilde{\mathcal{S}}'^*} & \xrightarrow{\tau_{E/\hat{E}(-1)}} & \hat{\mathbb{Z}}_p^{\mathrm{ur}}[[\mathcal{G}']]_{\tilde{\mathcal{S}}'^*} & \hookrightarrow & \hat{\mathbb{Z}}_p^{\mathrm{ur}}[[\mathcal{G}]]_{\tilde{\mathcal{S}}^*} \\
\big\downarrow & & \big\downarrow & \nearrow & \\
\hat{\mathbb{Z}}_p^{\mathrm{ur}}[[G]]_{\tilde{S}^*} & \xrightarrow{\tau_{E_{\bar{\pi}}(-1)}} & \hat{\mathbb{Z}}_p^{\mathrm{ur}}[[G]]_{\tilde{S}^*}, & &
\end{array} \tag{7.9}$$

where the vertical injections and the maps on the right are induced by the natural embeddings $\mathcal{G}' \subset G \subset \mathcal{G}$ and the inclusions $\tilde{\mathcal{S}}'^* \subset \tilde{S}^* \subset \tilde{\mathcal{S}}^*$ for which we refer to ([36], corollary A.8.15).

The second main theorem of this chapter, which we prove next, will enable us to derive as a corollary an expression of $\mathcal{L}_{p,u,E}$ for which an interpolation property is known. Let us recall from (3.6) the semi-local version $\mathbb{L} : \mathcal{U}_\infty \longrightarrow \hat{\mathbb{Z}}_p^{ur}[[G]]$ of the Coleman map associated to $\hat{E}$, where $\mathcal{U}_\infty = \varprojlim_n \prod_{\omega|\mathfrak{p}} \mathcal{O}^1_{K_{n,\omega}}$ denote the semi-local principal units for the tower $K_n = K(E[p^n])$, $n \geq 1$. We explain in the proof of theorem 7.7 below that the (local) Coleman map $-\mathcal{L}_{\epsilon^{-1}}$ for $\mathbb{G}_m$ from (6.17) may be viewed as a map $-\mathcal{L}_{\epsilon^{-1}} : \mathcal{U}'(K_{\infty,\nu}) \longrightarrow \hat{\mathbb{Z}}_p^{\mathrm{ur}}[[\mathcal{G}']]$ of $\Lambda(\mathcal{G}')$-modules, where $\mathcal{U}'(K_{\infty,\nu}) = \varprojlim_n \mathcal{O}^1_{K_{n,\nu}}$ are the local principal units for the extension $K_{\infty,\nu}/\mathbb{Q}_p$. Taking $\mathrm{Ind}_G^{\mathcal{G}'}$ and using the natural isomorphism $\mathcal{U}_\infty \cong \mathrm{Ind}_G^{\mathcal{G}'}\mathcal{U}'(K_{\infty,\nu})$ we obtain the map

$$\mathcal{L}_{semi-loc} :\ \mathcal{U}_\infty \cong \mathrm{Ind}_G^{\mathcal{G}'}\mathcal{U}'(K_{\infty,\nu}) \xrightarrow{\mathrm{Ind}_G^{\mathcal{G}'}(-\mathcal{L}_{\epsilon^{-1}})} \mathrm{Ind}_G^{\mathcal{G}'}\hat{\mathbb{Z}}_p^{\mathrm{ur}}[[\mathcal{G}']] \cong \hat{\mathbb{Z}}_p^{\mathrm{ur}}[[G]], \tag{7.10}$$

which is the semi-local version of the Coleman map for $\mathbb{G}_m$. Recall that via the natural (diagonal) map $\mathcal{E}_\infty = \varprojlim_n (\mathcal{O}^\times_{K_n} \otimes_{\mathbb{Z}} \mathbb{Z}_p)$ embeds into $\mathcal{U}_\infty$, which we use to evaluate $\mathcal{L}_{semi-loc}$ at global units. We also consider the natural map

$$loc_\nu : \mathcal{E}_\infty \longrightarrow \mathcal{U}'(K_{\infty,\nu})$$

from global to local principal units, which we also interpret as a map $loc_\nu : \mathbb{H}^1_\Sigma \to \mathbb{H}^1_{\mathrm{loc}}$ via the isomorphisms $\mathcal{E}_\infty \cong \mathbb{H}^1_\Sigma$ and $\mathcal{U}'(K_{\infty,\nu}) \cong \mathbb{H}^1_{\mathrm{loc}}$.

We will intepret the elements $\Omega_{p,u,u'}$ and $\tau_{E/\hat{E}(-1)}(\mathcal{E}_{p,u'})^{-1} = \tau_{E/\hat{E}(-1)}(\mathcal{E}^{-1}_{p,u'})$ as elements in $\left(\hat{\mathbb{Z}}_p^{\mathrm{ur}}[[\mathcal{G}]]\right)_{\mathcal{S}^*}$ which is possible as we have explained in remark 7.4. Here and in the following proof, when we localize with respect to $\mathcal{S}^*$ we always consider the modules in question as $\Lambda(\mathcal{G})$-modules.

Theorem 7.7 *The element* $\frac{\Omega_{p,u,u'}}{\tau_{E/\hat{E}(-1)}(\mathcal{E}_{p,u'})}$ *is equal to the* $\tau_{E_{\bar{\pi}}(-1)}$*-twist of the image of* u *under the semi-local version* $\mathcal{L}_{semi-loc}$ *of the Coleman map for* $\mathbb{G}_m$*, i.e.,*

$$\frac{\Omega_{p,u,u'}}{\tau_{E/\hat{E}(-1)}(\mathcal{E}_{p,u'})} = \tau_{E_{\bar{\pi}}(-1)}(\mathcal{L}_{semi-loc}(u)) = \tau_{E_{\bar{\pi}}(-1)}\Big(\sum_{\sigma\in G/\mathcal{G}'} \sigma\cdot\big(-\mathcal{L}_{\epsilon^{-1}}(loc_\nu(\sigma^{-1}u))\big)\Big)$$

in $\big(\hat{\mathbb{Z}}_p^{\mathrm{ur}}[[\mathcal{G}]]\big)_{\mathcal{S}^*}$*, which shows that the element on the left side does not depend on* u'*. In particular,* $\mathcal{L}_{p,u,E}$ *is independent of* u'*.*

Proof. Let us write $\tilde{\Lambda}' = \hat{\mathbb{Z}}_p^{\mathrm{ur}}[[\mathcal{G}']]$. By abuse of notation let us also write $-\mathcal{L}_{\epsilon^{-1}}$ for the composition of the map $-\mathcal{L}_{\epsilon^{-1}}$ from (6.17) with the isomorphism $\mathbb{T}_{\mathrm{un}}(K_{\infty,\nu})\otimes_{\Lambda'}\Lambda'_{\varphi_p}\cong\Lambda'_{\varphi_p}$ and the natural embedding $\Lambda'_{\varphi_p}\subset\tilde{\Lambda}'$, i.e., for the map

$$-\mathcal{L}_{\epsilon^{-1}}:\mathcal{U}'(K_{\infty,\nu})\longrightarrow\tilde{\Lambda}' \tag{7.11}$$

of Λ'-modules. By definition of $\mathcal{E}_{p,u'}$ from (4.16) we have $-\mathcal{L}_{\epsilon^{-1}}(u') = \mathcal{E}_{p,u'}^{-1}$. Let us write $T' = (T_pE/T_p\hat{E})(-1)$, which is free of $\mathbb{Z}_p$-rank one. Tensoring (7.11) with $T'\otimes_{\mathbb{Z}_p}-$, fixing an isomorphism $T'\otimes_{\mathbb{Z}_p}\tilde{\Lambda}'\cong\tilde{\Lambda}'$ similarly as in (6.8) (corresponding to our fixed basis t' of T'), applying $\mathrm{Ind}_{\mathcal{G}}^{\mathcal{G}'}$ and applying $(-)_{\mathcal{S}^*}$ (viewing all modules as $\Lambda(\mathcal{G})$-modules), we get the composite map

$$\big(\mathrm{Ind}_{\mathcal{G}}^{\mathcal{G}'}(T'\otimes_{\mathbb{Z}_p}\mathbb{H}^1_{\mathrm{loc}})\big)_{\mathcal{S}^*}\xrightarrow{(I)}\big(\mathrm{Ind}_{\mathcal{G}}^{\mathcal{G}'}(T'\otimes_{\mathbb{Z}_p}\tilde{\Lambda}')\big)_{\mathcal{S}^*}\xrightarrow{(II)}\big(\mathrm{Ind}_{\mathcal{G}}^{\mathcal{G}'}\tilde{\Lambda}'\big)_{\mathcal{S}^*}\xrightarrow{(III)}\big(\hat{\mathbb{Z}}_p^{\mathrm{ur}}[[\mathcal{G}]]\big)_{\mathcal{S}^*}. \tag{7.12}$$

We note that under $T'\otimes_{\mathbb{Z}_p}\tilde{\Lambda}'\cong\tilde{\Lambda}'$ the element $t'\otimes\lambda$, $\lambda\in\tilde{\Lambda}'$, maps to

$$t'\otimes\lambda\longmapsto\tau_{E/\hat{E}(-1)}(\lambda), \tag{7.13}$$

see ([36], lemma 1.1.11). By the defining property of $\Omega_{p,u,u'} = \frac{\lambda_\Omega}{s_\Omega}\in\Lambda(\mathcal{G})_{\mathcal{S}^*}$ we have an equality

$$\lambda_\Omega\cdot\phi_{t',u'}(1) = s_\Omega\cdot\mathrm{loc}(\phi_{t,u}(1))$$

in $\big(\mathrm{Ind}_{\mathcal{G}}^{\mathcal{G}'}(T'\otimes_{\mathbb{Z}_p}\mathbb{H}^1_{\mathrm{loc}})\big)_{\mathcal{S}^*}$. We want to compare the images of $\lambda_\Omega\cdot\phi_{t',u'}(1) = s_\Omega\cdot\mathrm{loc}(\phi_{t,u}(1))$ under the composite map from (7.12). Recall first that $\phi_{t',u'}(1) = 1\otimes t'\otimes u'$ and

$$\mathrm{loc}(\phi_{t,u}(1)) = \mathrm{loc}(t\otimes u) = \sum_{\sigma\in\mathcal{G}/\mathcal{G}'}\sigma\otimes\iota_1(\sigma^{-1}t)\otimes loc_\nu(\sigma^{-1}u),$$

where ι_1 is the natural projection $T_pE(-1)\longrightarrow(T_pE/T_p\hat{E})(-1)$ mentioned in remark 7.2 (iv). Now, we make the important observation that for $g\in\mathcal{G}\smallsetminus G$, i.e., for an element that can be written as $g = g_0c$ where c is complex conjugation and g_0 belongs to G, the element $g.t_{\bar{\pi}}$ belongs to the submodule $T_\pi E$ of T_pE, where $t_{\bar{\pi}}$ is our fixed basis of $T_{\bar{\pi}}E$. This is true since the action of complex conjugation turns $\bar{\pi}$-power divisions into π-power division points. We have $T_pE/T_p\hat{E} = T_pE/T_\pi E$ and hence $[g^{-1}.t_{\bar{\pi}}] = 0$ in $T_pE/T_p\hat{E}$ and $\iota_1(g^{-1}t) = 0$ in T' for all $g\in\mathcal{G}\smallsetminus G$. We conclude that

$$\sum_{\sigma\in\mathcal{G}/\mathcal{G}'}\sigma\otimes\iota_1(\sigma^{-1}t)\otimes loc_\nu(\sigma^{-1}u) = \sum_{\sigma\in G/\mathcal{G}'}\sigma\otimes\iota_1(\sigma^{-1}t)\otimes loc_\nu(\sigma^{-1}u),$$

where on the right we sum only over representatives of $G/\mathcal{G}'$. We have noted in remark 7.6 (i) that G acts on $t \in T_{\bar\pi}(-1)$ through $\bar\psi \cdot \kappa^{-1}$ and hence we can rewrite the last sum as

$$\begin{aligned}\sum_{\sigma\in G/\mathcal{G}'} \sigma \otimes \iota_1(\sigma^{-1}t) \otimes loc_\nu(\sigma^{-1}u) &= \sum_{\sigma\in G/\mathcal{G}'} \big((\bar\psi\cdot\kappa^{-1})(\sigma^{-1})\cdot\sigma\big) \otimes \iota_1(t) \otimes loc_\nu(\sigma^{-1}u) \\ &= \sum_{\sigma\in G/\mathcal{G}'} \tau_{E_{\bar\pi}(-1)}(\sigma) \otimes t' \otimes loc_\nu(\sigma^{-1}u), \end{aligned} \tag{7.14}$$

where the last equation holds by definition of $\tau_{E_{\bar\pi}(-1)}$ and t'.

Next, we compare the images of $\lambda_\Omega \cdot \phi_{t',u'}(1) = s_\Omega \cdot \mathrm{loc}(\phi_{t,u}(1))$ under the composite map from (7.12) using (7.13). On the one hand we have

$$\begin{aligned}\lambda_\Omega\cdot\phi_{t',u'}(1) &\overset{(I)}{\longmapsto} \lambda_\Omega\cdot\big(1\otimes t'\otimes -\mathcal{L}_{\epsilon^{-1}}(u')\big) && \text{in } \big(\mathrm{Ind}_{\mathcal{G}}^{\mathcal{G}'}(T'\otimes_{\mathbb{Z}_p}\tilde\Lambda')\big)_{\mathcal{S}^*} \\ &\overset{(II)}{\longmapsto} \lambda_\Omega\cdot\big(1\otimes\tau_{E/\hat E(-1)}(\mathcal{E}_{p,u'}^{-1})\big) && \text{in } \big(\mathrm{Ind}_{\mathcal{G}}^{\mathcal{G}'}\tilde\Lambda'\big)_{\mathcal{S}^*} \\ &\overset{(III)}{\longmapsto} \frac{\lambda_\Omega}{\tau_{E/\hat E(-1)}(\mathcal{E}_{p,u'})} && \text{in } \big(\hat{\mathbb{Z}}_p^{\mathrm{ur}}[[\mathcal{G}]]\big)_{\mathcal{S}^*}.\end{aligned} \tag{7.15}$$

On the other hand we have

$$\begin{aligned}s_\Omega\cdot\mathrm{loc}(\phi_{t,u}(1)) &\overset{(I)}{\longmapsto} s_\Omega\cdot\Big(\sum_{\sigma\in G/\mathcal{G}'}\tau_{E_{\bar\pi}(-1)}(\sigma)\otimes t'\otimes -\mathcal{L}_{\epsilon^{-1}}(loc_\nu(\sigma^{-1}u))\Big) \\ &\overset{(II)}{\longmapsto} s_\Omega\cdot\Big(\sum_{\sigma\in G/\mathcal{G}'}\tau_{E_{\bar\pi}(-1)}(\sigma)\otimes\tau_{E/\hat E(-1)}\big(-\mathcal{L}_{\epsilon^{-1}}(loc_\nu(\sigma^{-1}u))\big)\Big) \\ &\overset{(III)}{\longmapsto} s_\Omega\cdot\Big(\sum_{\sigma\in G/\mathcal{G}'}\tau_{E_{\bar\pi}(-1)}(\sigma)\cdot\tau_{E/\hat E(-1)}\big(-\mathcal{L}_{\epsilon^{-1}}(loc_\nu(\sigma^{-1}u))\big)\Big) \\ &= s_\Omega\cdot\tau_{E_{\bar\pi}(-1)}\Big(\sum_{\sigma\in G/\mathcal{G}'}\sigma\cdot\big(-\mathcal{L}_{\epsilon^{-1}}(loc_\nu(\sigma^{-1}u))\big)\Big),\end{aligned} \tag{7.16}$$

where the last equality follows from the commutativity of (7.9). We conclude so far that

$$\frac{\Omega_{p,u,u'}}{\tau_{E/\hat E(-1)}(\mathcal{E}_{p,u'})} = \tau_{E_{\bar\pi}(-1)}\Big(\sum_{\sigma\in G/\mathcal{G}'}\sigma\cdot\big(-\mathcal{L}_{\epsilon^{-1}}(loc_\nu(\sigma^{-1}u))\big)\Big) \quad \text{in } \big(\hat{\mathbb{Z}}_p^{\mathrm{ur}}[[\mathcal{G}]]\big)_{\mathcal{S}^*}, \tag{7.17}$$

which shows, in particular, that the left side does not depend on u'. Next, let us consider the composite map

$$\mathcal{E}_\infty \xrightarrow{\mathrm{Coind}_G^{\mathcal{G}'}(loc_\nu)} \mathrm{Coind}_G^{\mathcal{G}'}\mathcal{U}'(K_{\infty,\nu}) \cong \mathrm{Ind}_G^{\mathcal{G}'}\mathcal{U}'(K_{\infty,\nu}) \xrightarrow{\mathrm{Ind}_G^{\mathcal{G}'}(-\mathcal{L}_{\epsilon^{-1}})} \mathrm{Ind}_G^{\mathcal{G}'}\tilde\Lambda' \cong \hat{\mathbb{Z}}_p^{\mathrm{ur}}[[G]] \tag{7.18}$$

under which u maps to $\sum_{\sigma\in G/\mathcal{G}'}\sigma\cdot\big(-\mathcal{L}_{\epsilon^{-1}}(loc_\nu(\sigma^{-1}u))\big)$, the argument of $\tau_{E_{\bar\pi}(-1)}$ on the right side of (7.17). To conclude the proof, we only have to note that the composite map from (7.18) coincides with the composite of the embedding $\mathcal{E}_\infty \hookrightarrow \mathcal{U}_\infty$ with $\mathcal{L}_{semi-loc}$.

As before, compare remark 7.4, we consider $\Omega_{p,u,u'}$, $\tau_{E/\hat E(-1)}(\mathcal{E}_{p,u'})^{-1}$, $\tau_{E_{\bar\pi}(-1)}(L_{p,u})$ and $\mathcal{L}_{p,u,E}$ as elements in $\hat{\mathbb{Z}}_p^{\mathrm{ur}}[[\mathcal{G}]]_{\mathcal{S}^*}$ and get the following

Corollary 7.8. *We have an equality of elements in* $\hat{\mathbb{Z}}_p^{\mathrm{ur}}[[\mathcal{G}]]_{\mathcal{S}^*}$

$$\mathcal{L}_{p,u,E} = \tau_{\psi^{-1}}(\lambda),$$

where $\tau_{\psi^{-1}}(\lambda)$ *denotes the twist of de Shalit's element* $\lambda \in \Lambda(G)$ *(from definition 3.6) by the* G*-module* $(T_\pi E)^*$. *The action of* G *on* $(T_\pi E)^*$ *is given by* ψ^{-1}. *Moreover, for an Artin character* χ *of* $\mathcal{G}$ *we have*

$$\frac{1}{\Omega_p} \cdot \int_G \mathrm{Res}_G^{\mathcal{G}} \chi \; d\mathcal{L}_{p,u,E} \;=\; \frac{1}{\Omega} \cdot G\big(\psi \cdot \mathrm{Res}\chi\big) \cdot \Big(1 - \frac{(\psi \cdot \mathrm{Res}\chi)(\mathfrak{p})}{p}\Big) \cdot L_{\mathfrak{f}\bar{\mathfrak{p}}}((\psi \cdot \mathrm{Res}\chi)^{-1}, 0), \quad (7.19)$$

where we refer to ([14], p. 80) for the definition of $G\big(\psi \cdot \mathrm{Res}\chi\big)$ *which is related to a local constant and in the expression* $(\psi \cdot \mathrm{Res}\chi)(\mathfrak{p})$ *we consider* $\psi \cdot \mathrm{Res}\chi$ *as a map on ideals of* K *prime to* $\mathfrak{f}$. *The periods* Ω *and* Ω_p *are defined in subsection 7.1.*

Proof. By the previous theorem 7.7 we know that

$$\frac{\Omega_{p,u,u'}}{\tau_{E/\hat{E}(-1)}(\mathcal{E}_{p,u'})} \;=\; \tau_{E_{\bar{\pi}}(-1)}(\mathcal{L}_{semi\text{-}loc}(u)).$$

But for *large enough* (unramified) abelian extensions the Coleman maps (composed with an integral logarithm and an isomorphism to $\mathbb{G}_m$ and then extended to measures) induced by two formal groups (in our case $\hat{E}$ and $\mathbb{G}_m$) coincide, which is proven in ([14], Proposition 3.9, p. 23) for the maximal abelian extension. The arguments in loc. cit. also work in our case using the fact that for any $n \geq 1$ we have

$$K_{\mathfrak{p}}(E[\bar{\pi}^n], E[\pi^n]) = K_{\mathfrak{p}}(E[\bar{\pi}^n], \mu_{p^n}),$$

i.e., adjoining to the local field $K_{\mathfrak{p}}(E[\bar{\pi}^n])$ the p^n division points of $\hat{E}$ or $\mathbb{G}_m$ yields the same extension, which follows from the Weil pairing. Hence, we have an equality

$$\mathcal{L}_{semi\text{-}loc}(u) = \mathbb{L}(u), \quad (7.20)$$

where $\mathcal{L}_{semi\text{-}loc}$ is the semi-local Coleman map for $\mathbb{G}_m$ and $\mathbb{L}$ is the one for $\hat{E}$. By definition 3.6, $u = u(\mathfrak{q})$ is the the compatible system of elliptic units attached to $\mathfrak{q} \subset \mathcal{O}_K$ and $L_{p,u} = \frac{1}{(\mathrm{Frob}_{\mathfrak{q}} - N\mathfrak{q})}$. By definition of λ and since $\tau_{E_{\bar{\pi}}(-1)}(12) = 12$ we get

$$\begin{aligned}
\mathcal{L}_{p,u,E} &\overset{\text{thm 7.7}}{=} \tau_{E_{\bar{\pi}}(-1)}(L_{p,u}) \cdot \tau_{E_{\bar{\pi}}(-1)}(\mathcal{L}_{semi\text{-}loc}(u)) \cdot \frac{1}{12} \\
&\overset{(7.20)}{=} \tau_{E_{\bar{\pi}}(-1)}\Big(\frac{\mathbb{L}(u(\mathfrak{q}))}{12 \cdot (\mathrm{Frob}_{\mathfrak{q}} - N\mathfrak{q})}\Big) \\
&\overset{\text{def.}\lambda}{=} \tau_{E_{\bar{\pi}}(-1)}(\lambda).
\end{aligned}$$

Now, to obtain the first assertion of the corollary, we only have to note that $\psi^{-1} = \bar{\psi} \cdot \kappa^{-1}$ as was explained in remark 7.6 (i) and that $\bar{\psi} \cdot \kappa^{-1}$ gives the Galois action on $T_{\bar{\pi}}E(-1)$, i.e., $\tau_{E_{\bar{\pi}}(-1)} = \tau_{\psi^{-1}}$.

Regarding the interpolation property, we note that for an element $g \in G$, by definition of the twist operator, we have $\tau_{\psi^{-1}}(g) = \psi^{-1}(g^{-1})g = \psi(g)g$, which shows that for an Artin character δ of G and any measure μ we have

$$\int_G \delta \; d(\tau_{\psi^{-1}}(\mu)) = \int_G \delta \cdot \psi \; d\mu. \quad (7.21)$$

But $\delta \cdot \psi$ is a Größencharacter of type $(1,0)$ and for such Größencharacters de Shalit ([14], Theorem 4.14, p. 80) determines the interpolation property for λ. In combination with ([5], Lemma 2.10, p. 394), (7.21) concludes the proof.

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