J. Ramanujan Math. Soc. **30,** No.4 (2015) 397–402

A Galois side analogue of a theorem of Bernstein

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Communicated by: Dipendra Prasad

Received: April 18, 2015

Abstract. Let G be a connected reductive group defined over a non archimedean local field k. A theorem of Bernstein states that for any compact subgroup K of $G(k)$, there are, upto unramified twists, only finitely many K-spherical supercuspidal representations of $G(k)$. We prove an analogous result on the Galois side of the Langlands correspondence.

1. Introduction

Let G be a connected reductive group defined over a non-archimedean local field k. Let $X_{\mathrm{nr}}(G(k))$ be the group of *unramified characters* of $G(k)$ (Definition 12). For a smooth representation π of $G(k)$, the various representations $\pi \otimes \chi$, $\chi \in X_{\mathrm{nr}}(G(k))$ are called the unramified twists of π. A Theorem of Bernstein [Roc09, Theorem 1.4.2.1] states that

Theorem 1 (Bernstein). *For each compact open subgroup K of $G(k)$, the number of isomorphism classes, up to unramified twists, of irreducible cuspidal representations of $G(k)$ having non-zero K-fixed vectors is finite.*

On the other hand, local Langlands conjectures predict that "packets" of irreducible admissible representations of $G(k)$ should be parametrized by *Langlands parameters*, which are *admissible* elements of $\mathrm{H}^1(W_k', \hat{G})$ (Definition 4), where W_k' is the Weil-Deligne group and $\hat{G}$ is the complex dual of G (see [Bor79]). Under this conjectural correspondence, supercuspidal representations are expected to correspond to *discrete Langlands parameters* (Definition 9). By Langlands philosophy, one should expect a result analogous to Theorem 1 on the parameter side.

Let W_k be the Weil group of k and I_k its inertia subgroup. Let Fr be a Frobenius element in W_k. Let J be an open subgroup of I_k which is normal in W_k. Call two Langlands parameters to be equivalent if they are in the same $\mathrm{H}^1(W_k/I_k, (Z(\hat{G})^{I_k})^\circ)$ orbit (see (5.1)), where $Z(\hat{G})$ is the center of $\hat{G}$. In Theorem 10, we show that upto this equivalence, there are only finitely many discrete Langlands parameters which are trivial on J. In Section 7, using Kottwitz homomorphism, we obeserve that $\mathrm{H}^1(W_k/I_k, (Z(\hat{G})^{I_k})^\circ)$ is isomorphic to the group $X_{\mathrm{nr}}(G(k))$ of unramified characters of $G(k)$.

These statements are thus consistent with the conjectures in [Bor79, Section 10.3 (2)].

2. Notations

Let k be a non-archimedean local field and fix an algebraic closure $\bar{k}$ of k. Let W_k denote the Weil group of k and I_k denote its inertia subgroup. We fix a Frobenius element Fr in W_k. For any algebraic group $\mathcal{G}$, we will denote by $Z(\mathcal{G})$ the center of $\mathcal{G}$. For any subgroup $\mathcal{H}$ of $\mathcal{G}$, we will denote by $Z_{\mathcal{G}}\mathcal{H}$, the centralizer of $\mathcal{H}$ in $\mathcal{G}$. The identity component of $\mathcal{G}$ will be denoted by $\mathcal{G}^\circ$.

3. Representations of the Weil group

Let J be an open subgroup of I_k which is normal in W_k.

Definition 2. *A representation of W_k is called unramified, if it is trivial on I_k.*

Lemma 3. *Upto unramified twists, there exist only finitely many irreducible representations of W_k which are trivial on J.*

Proof. Let (ρ, V) be an irreducible representation of W_k such that $J \subset \ker \rho$. Let $\mathrm{Fr} \in W_k$ be a Frobenius element in W_k. It acts by conjugation on the finite group I_k/J, so some power Fr^d, $d \geq 1$ acts trivially. Thus $\rho(\mathrm{Fr}^d)$ commutes with $\rho(I_k)$ and since ρ is irreducible, $\rho(\mathrm{Fr}^d)$ must be scalar by Schur's lemma. Let χ be an unramified character of W_k such that $\chi(\mathrm{Fr}^d) = \rho(\mathrm{Fr}^d)$. Thus ρ is of the form $\chi \otimes \tau$ where τ is an irreducible representation of the finite group $W_k/\langle \mathrm{Fr}^d, J\rangle$. Thus there are upto unramified twists, only finitely many irreducible representations of W_k which are trivial on J. □

4. Langlands parameters

Let G be a connected, reductive group over k and let k_0 be the splitting field in $\bar{k}$ of the quasi-split inner form of G. Let ${}^L G = \hat{G} \rtimes \mathrm{Gal}(k_0/k)$, where $\hat{G}$ is the complex dual of G. The center ${}^L Z$ of ${}^L G$ is the group of $\mathrm{Gal}(k_0/k)$-fixed points in the center of $\hat{G}$.

Definition 4. *A homomorphism* $\varphi : W_k \times \mathrm{SL}(2, \mathbb{C}) \to {}^L G$ *is called admissible if*

(1) $\varphi : \mathrm{SL}(2, \mathbb{C}) \to \hat{G}$ *is a homomorphism of algebraic groups over* $\mathbb{C}$.
(2) φ *is continuous on* I_k *and* $\varphi(\mathrm{Fr})$ *is semisimple.*
(3) The composite $W_k \to^L G \to \mathrm{Gal}(k_0/k)$ *is the canonical surjection* $W_k \to \mathrm{Gal}(k_0/k)$.

Two admissible homomorphisms are equivalent if they are conjugate by $\hat{G}$. *A Langlands parameter is an equivalence class of admissible homomorphisms.*

The group $W'_k := W_k \times \mathrm{SL}(2, \mathbb{C})$ is called the Weil-Deligne group of k. It is sometimes more convenient to see a Langlands parameter as an element of $\mathrm{H}^1(W'_k, \hat{G})$.

Definition 5. *A Langlands parameter is unramified if it is trivial on* I_k *and* $\mathrm{SL}(2, \mathbb{C})$.

5. Finiteness result

Let the notations be as in Section 4. So G is as before a connected reductive group defined over k. Let $\hat{\mathfrak{g}}$ be the Lie algebra of the complex dual $\hat{G}$ of G. Let $W'_k := W_k \times \mathrm{SL}(2, \mathbb{C})$ denote the Weil-Deligne group of k. Let J be an open subgroup of I_k which is normal in W_k. Let $\Phi(G)$ denote the set of Langlands parameters of G.

We have a well defined action

$$\mathrm{H}^1(W_k, Z(\hat{G})) \times \Phi(G) \to \Phi(G), \qquad [\alpha] \cdot [\phi] \mapsto [\alpha \cdot \phi] \tag{5.1}$$

Definition 6. *Call two parameters* φ, φ' *to be equivalent if they are in the same* $\mathrm{H}^1(W_k/I_k, (Z(\hat{G})^I)^\circ)$ *orbit.*

Lemma 7. *Let* T *be a tori defined over* k *and let* $\hat{T}$ *be its comples dual. There are only finitely many equivalence classes of Langlands parameters for* T *which are trivial on* J.

Proof. We have a canonical decomposition $\mathrm{H}^1((\langle \mathrm{Fr} \rangle \ltimes I_k)/J, \hat{T}) = \mathrm{H}^1(\langle \mathrm{Fr} \rangle, (\hat{T}^{I_k})^\circ) \times \mathrm{H}(I_k/J, \hat{T})$. Let $d_J = |I_k/J|$. Then any element of $\mathrm{H}^1(I_k/J, \hat{T})$ is killed by d_J. Thus the image of these elements lies in the d_J-torsion points of $\hat{T}$ which is a finite set. Therefore $\mathrm{H}^1(I_k/J, \hat{T})$ is finite. □

Lemma 8. *Let* $\varphi : W'_k \to {}^L G$ *be an admissible homomorphism which is trivial on* J. *If* image(φ) *is not contained in any proper parabolic subgroup of* ${}^L G$, *then there exists a number* $n = n(J, G)$ *such that* $\varphi(\mathrm{Fr}^n) \in Z({}^L G)$.

Proof. Let d be a positive integer such that Fr^d acts trivially on I_k/J. Then $\varphi(\mathrm{Fr}^d) \in Z(\mathrm{image}(\varphi))$. Let $l = |\mathrm{Gal}(k_0/k)|$. Then $s := \varphi(\mathrm{Fr}^d)^l \in \hat{G}$. Let $H = Z_{{}^LG}(s)$. Then $\mathrm{image}(\varphi) \subset H$ and $s \in Z(H)$. The group $Z_{{}^LG}(Z(H)^\circ)$ is a Levi subgroup of ${}^LG$ containing H and therefore must be ${}^LG$ since $\mathrm{image}(\varphi)$ is not contained in any proper parabolic subgroup. Thus $Z(H)^\circ \subset Z({}^LG)$. From the structure theorem of the centralizers of semisimple elements, we know that there can be only finitely many possibilities for H [Kur83, Prop. 2.1]. Since $s \in Z(H)$, the fact that there are only finitely many possibilities for H allows us to choose a positive integer $a = a(G)$ independent of H such that $s^a \in Z(H)^\circ$. The Lemma follows. □

Definition 9. *A Langlands parameter is called discrete if its image is not contained in any parabolic subgroup of ${}^LG$.*

Theorem 10. *Let G be a connected reductive group over k. Then there exist only finitely many equivalence classes of discrete Langlands parameters for G which are trivial on J.*

Proof. Let $\varphi : W_k' \to {}^LG$ be a an admissible homomorphism. By Lemma 8, there exists an integer $n = n(J, G)$ such that the composite map $\bar{\varphi} : W_k' \to {}^LG \to \mathcal{G} := \hat{G}_{\mathrm{ad}} \rtimes \mathrm{Gal}(k_0/k)$ factors through $W_k/\langle \mathrm{Fr}^n, J\rangle \times \mathrm{SL}(2, \mathbb{C})$. By [Slo97, II.3, Theorem 1], there are only finitely many $\mathcal{G}$ conjugacy classes of homomorphisms $W_k/\langle \mathrm{Fr}^n, J\rangle \to \mathcal{G}$. It follows that there are only finitely many $\mathcal{G}$ conjugacy classes of homomorphisms $W_k' \to \mathcal{G}$ which are trivial on J.

Now if $\varphi_1, \varphi_2 \in \mathrm{H}^1(W_k', \hat{G})$ are two Langlands parameters such that their images in $\mathrm{H}^1(W_k', \hat{G}_{\mathrm{ad}})$ are equal, then $\varphi_1 = \varphi_c \cdot \varphi_2$ where $\varphi_c \in \mathrm{H}^1(W_k, Z(\hat{G}))$. By Lemma 7, there are only finitely many such φ_c upto equivalence. The theorem follows. □

The Weil group W_k carries an upper numbering filteration $\{W_k^r\}_{r \geq 0}$. The depth of a parameter $\varphi : W_k' \to {}^LG$ is defined to be

$$\inf\{r \geq 0 : W_k^s \subset \ker(\varphi) \text{ for } s > r\}.$$

Corollary 11. *There exist only finitely many equivalence classes of Langlands parameters of a given depth.*

6. Unramified characters

Let $X_k(G) = \mathrm{Hom}(G, \mathbb{G}_m)$, the lattice of k-rational characters of G. Let

$$G(k)^1 = \{g \in G(k) : \mathrm{val}_k(\chi(g)) = 0, \forall \chi \in X_k(G)\}.$$

Then $G(k)^1$ is an open normal subgroup of $G(k)$ that contains each compact subgroup of $G(k)$. It also has the following properties:

(1) $G(k)^1$ has compact center;
(2) $G(k)/G(k)^1$ is a free abelian group of finite rank;
(3) The center $Z(G(k)^1)$ of $G(k)^1$ has finite rank in $G(k)$.

Definition 12. *The group $X_{\mathrm{nr}}(G(k))$ of unramified characters of $G(k)$ is defined by*

$$X_{\mathrm{nr}}(G(k)) = \mathrm{Hom}(G(k)/G(k)^1, \mathbb{C}^\times).$$

Definition 13. *For a smooth representation π of $G(k)$, the representations $\pi \otimes \chi$, $\chi \in X_{\mathrm{nr}}(G(k))$ are called the unramified twists of π.*

Theorem 14 (Bernstein). *For each compact open subgroup K of $G(k)$, the number of isomorphism classes, up to unramified twists, of irreducible cuspidal representations τ of $G(k)$ with $\tau^K \neq 0$ is finite.*

7. Langlands parameters for unramified characters

In [Kot97, Section 7], Kottwitz defined a surjective homomorphism

$$\kappa_G : G(k) \to X^*(Z(\hat{G}))^{\mathrm{Fr}}_{I_k}.$$

Let $\upsilon_G : G(k) \to X^*(Z(\hat{G}))^{\mathrm{Fr}}_{I_k}/\,\mathrm{torsion}$ be the homomorphism induced by the Kottwitz homomorphism. Then $\ker \upsilon_G = G(k)^1$ (see [HR08, Remark 10]). We therefore have:

$$\begin{aligned} X_{\mathrm{nr}}(G(k)) &\cong \mathrm{Hom}(X^*(Z(\hat{G}))^{\mathrm{Fr}}_{I_k}/\,\mathrm{torsion}, \mathbb{C}^\times) \\ &\cong \mathrm{Hom}(X^*((Z(\hat{G})^{I_k})^\circ{}_{\mathrm{Fr}}), \mathbb{C}^\times) \\ &\cong (Z(\hat{G})^{I_k})^\circ{}_{\mathrm{Fr}}. \end{aligned} \tag{7.1}$$

The last equality holds by Cartier duality.

We have

$$(Z(\hat{G})^{I_k})^\circ_{\mathrm{Fr}} \cong \mathrm{H}^1(W_k/I_k, (Z(\hat{G})^{I_k})^\circ) \hookrightarrow \mathrm{H}^1(W_k, Z(\hat{G})). \tag{7.2}$$

Combining equations (7.1) and (7.2), we get a map

$$X_{\mathrm{nr}}(G(k)) \hookrightarrow \mathrm{H}^1(W_k, Z(\hat{G})) \to \mathrm{H}^1(W'_k, \hat{G}). \tag{7.3}$$

One can thus associate to the unramified characters, Langlands parameters whose image lie in the center of $\hat{G}$.

Acknowledgement

The author is very thankful to Sandeep Varma for pointing out several mistakes in the earlier drafts of this article. He is also very thankful to Dipendra Prasad for pointing out a gap in the proof of Theorem 10 and for his careful proof reading. He is grateful to his host Rainer Weissauer for many helpful suggestions and to the Mathematics Center Heidelberg (MATCH) where this work was written. He would also like to thank Sudhanshu Shekhar for many helpful conversations.

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