On the degeneration of some spectral sequences

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The following is my compilation of arguments given in [CE], [Gr], [De] and [Ja]. Maybe it is also useful for other people. I do not claim any originality. This material will very likely be contained in the second edition of Neukirch/Schmidt/Wingberg: Cohomology of Number Fields.

1 Spectral sequences

Let $\mathcal{A}$ be an abelian category. Recall that a (decreasing) filtration F of an object A of $\mathcal{A}$ is a family $(F^nA)_{n\in\mathbb{Z}}$ of subobjects of A such that $F^mA \subset F^nA$ for $n \leq m$. By convention, we put $F^\infty A = 0$ and $F^{-\infty}A = A$. We say that the filtration is finite if there exist $n, m \in \mathbb{Z}$ with $F^mA = 0$ and $F^nA = A$. Let $(X^\bullet, d)$ be a cochain complex consisting of objects of $\mathcal{A}$ and let $F^\bullet X^\bullet$ be a filtration of $X^\bullet$ by subcomplexes, i.e. for each n, $F^nX^\bullet$ is a subcomplex of $X^\bullet$. We say that $F^\bullet X^\bullet$ is biregular, if, for each $n \in \mathbb{Z}$, the filtration $F^\bullet X^n$ is finite.

A biregular filtration induces a spectral sequence

$$E_1^{pq} \Rightarrow H^{p+q}(X^\bullet)$$

by defining for $r \in \mathbb{Z} \cup \{\infty\}$

$$Z_r^{pq} = \ker\left(F^pX^{p+q} \to F^pX^{p+q+1}/F^{p+r}X^{p+q+1}\right),$$

$$B_r^{pq} = d(F^{p-r}X^{p+q-1}) \cap F^pX^{p+q},$$

$$E_r^{pq} = Z_r^{pq}/B_{r-1}^{pq} + Z_{r-1}^{p+1,q-1}$$

and

$$F^pH^{p+q}(X^\bullet) = \operatorname{im}\left(F^pH^{p+q}(X^\bullet) \to H^{p+q}(X^\bullet)\right).$$

One easily verifies that this spectral sequence converges, i.e. for fixed $p, q \in \mathbb{Z}$ there is an r_0 such that

$$E_{r_0}^{pq} = E_{r_0+1}^{pq} = \cdots = E_\infty^{pq}$$

and

$$E_\infty^{pq} = \mathrm{gr}_p H^{p+q}(X^\bullet).$$

We say that a spectral sequence degenerates at E_{r_0} if the differentials d_r are zero for all $r \geq r_0$, i.e. $E_{r_0}^{pq} = E_\infty^{pq}$ for all p, q.

Proposition 1.1. *For the above spectral sequence, the following assertions are equivalent*

(i) *The spectral sequence degenerates at E_1.*

(ii) *For all n, p we have $F^p X^n \cap d(X^{n-1}) = d(F^p X^{n-1})$.*

(iii) *For all n, p the natural map $F^p H^{p+q}(X^\bullet) \to H^{p+q}(X^\bullet)$ is injective.*

If, moreover, the maps in (iii) *are split-injections, we obtain a splitting*

$$H^n(X^\bullet) \cong \bigoplus_{p+q=n} E_1^{pq}.$$

Proof. [De] Proposition 1.3.2. □

If $A^{\bullet\bullet}$ is a double complex, with total complex

$$X^\bullet = \mathrm{tot}(A^{\bullet\bullet})$$

then we consider the filtration

$$F^p X^n = \bigoplus_{\substack{i+j=n \\ i \geq p}} A^{ij}.$$

If there exists an $m \in \mathbb{Z}$ with $A^{ij} = 0$, for $i < m$ or $j < m$, this filtration is biregular and induces a spectral sequence converging to the cohomology of $X^\bullet$. We will refer to this this spectral sequence as the spectral sequence associated to the double complex $A^{\bullet\bullet}$.

2 Displacing

By a formal reindexing procedure, we can displace a spectral sequence in the following sense: Assume we are given a spectral sequence $E_r^{pq} \Rightarrow E^{p+q}$. Putting $\tilde{E}_r^{pq} = E_{r+1}^{2p+q,-p}$, we obtain a new spectral sequence converging to the same end terms, but with shifted indices. It is a remarkable fact that, if the spectral sequence E arises from a biregular filtered cochain complex as in the last section, then the spectral sequence $\tilde{E}$ arises from another filtration of the same complex. This will be useful in showing that a spectral sequence degenerates at E_2, just by showing that the displaced spectral sequence $\tilde{E}$ satisfies the conditions of Proposition 1.1.

Let $F^\bullet X^\bullet$ be a biregular filtered cochain complex. Consider the "displaced filtration"[1]

$$\mathrm{Dis}(F)^p X^n = Z_1^{p+n,-p},$$

where the term on the right hand side has been formed with respect to the filtration F. We denote the complex $X^\bullet$ together with the filtration $\mathrm{Dis}(F)$ by $\mathrm{Dis}(X^\bullet)$. We have the

Proposition 2.1. *There are natural isomorphism for all $r \geq 1$ commuting with the corresponding differentials*

$$E_r^{pq}(Dis(X^\bullet)) \xrightarrow{\sim} E_{r+1}^{2p+q,-p}(X^\bullet).$$

Proof. [De] Proposition 1.3.4. □

Now we consider a special example. Let $C^\bullet$ and $K^\bullet$ be bounded below complexes of abelian groups and put $A^{\bullet\bullet} = C^\bullet \otimes K^\bullet$.

Theorem 2.2. *If $C^\bullet$ consists of flat (i.e. torsion-free) abelian groups, then the spectral sequence of the double complex $A^{\bullet\bullet}$ degenerates at E_2. Furthermore, we have a splitting*

$$H^n(X^\bullet) \cong \bigoplus_{p+q=n} E_2^{pq}.$$

Proof. Let F be the standard filtration on $X^\bullet = \mathrm{tot}(A^{\bullet\bullet})$ as in the last section, i.e. $F^p X^n = \bigoplus_{\substack{i+j=n \\ i \geq p}} A^{ij}$. Then, using the flatness of $C^\bullet$, one verifies that

$$\mathrm{Dis}^p(X^\bullet) = \mathrm{tot}(C^\bullet \otimes \tau_{\leq -p} K^\bullet),$$

[1] Deligne use the name "filtration décalée" in [De]

where $\tau_{\leq -p}$ is the canonical truncation functor. By Propositions 1.1 and 2.1, it therefore remains to show that for all n, m the natural homomorphism

$$H^n(C^\bullet \otimes \tau_{\leq m} K^\bullet) \to H^n(C^\bullet \otimes \tau_{\leq m+1} K^\bullet)$$

is a split-injection. The complex $\tau_{\leq m+1} K^\bullet$ is bounded in both directions and therefore we find a complex $Y^\bullet$ bounded in both directions and consisting of free $\mathbb{Z}$-modules together with a quasi-isomorphism $Y^\bullet \to \tau_{\leq m+1} K^\bullet$. The inclusion

$$\tau_{\leq m} Y^\bullet \to Y^\bullet$$

has a section and, by the flatness of $C^\bullet$, we obtain a compatible quasi-isomorphisms

$$C^\bullet \otimes \tau_{\leq m} Y^\bullet \to C^\bullet \otimes \tau_{\leq m} K^\bullet$$

and

$$C^\bullet \otimes Y^\bullet \to C^\bullet \otimes \tau_{\leq m+1} K^\bullet.$$

Finally, the inclusion $C^\bullet \otimes \tau_{\leq m} Y^\bullet \to C^\bullet \otimes Y^\bullet$ has a section, showing the result. □

3 The Hochschild-Serre spectral sequence

Let G be a profinite group and let $H \subset G$ be a closed normal subgroup. Let A be a G-module. To the standard resolution $0 \to A \to X^\bullet$ of the G-module A, we apply the functor $H^0(H, -)$, and get the complex

$$H^0(H, X^0) \to H^0(H, X^1) \to H^0(H, X^2) \to \cdots$$

of G/H-modules. For each $H^0(H, X^q)$, we consider the cochain complex

$$H^0(H, X^q)^{G/H} \to C^\bullet(G/H, H^0(H, X^q))$$

and obtain a double complex

$$C^{pq} = C^p(G/H, H^0(H, X^q)) = X^p(G/H, X^q(G, A)^H)^{G/H}, \quad p, q \geq 0.$$

We define the Hochschild-Serre spectral sequence as the spectral sequence

$$E_2^{pq} \Longrightarrow E^n$$

associated with this double complex. One calculates

$$E_2^{pq} = H^p(G/H, H^q(H, A)).$$

The functor 'homogeneous cochain complex' is a 'resolving functor' in the sense of [Gr], §2.5, and, by loc.cit. Proposition 2.5.3, the Hochschild-Serre spectral sequence as defined above coincides with the spectral sequence for the composition of the derived functors of $H^0(H, -)$ and $H^0(G/H, -)$.

Theorem 3.1. *Let G and H be profinite groups, and let B be a discrete H-module, regarded as a $(G \times H)$-module via trivial action of the group G.*

Then the Hochschild-Serre spectral sequence

$$E_2^{pq} = H^p(G, H^q(H, B)) \Rightarrow H^n(G \times H, B)$$

degenerates at E_2. Furthermore, it splits in the sense that there is a decomposition

$$H^n(G \times H, B) \cong \bigoplus_{p+q=n} H^p(G, H^q(H, B)).$$

Lemma 3.2. *Let A be a trivial G-module. Then we have a natural isomorphism of complexes*

$$C^\bullet(G, A) \cong C^\bullet(G, \mathbb{Z}) \otimes A.$$

Proof. This is easily verified for a finite group G. The result for profinite G follows by a straightforward limit process. □

Proof of the theorem. By our construction of the Hochschild-Serre spectral sequence and by the last lemma, it is the spectral sequence associated to the double complex

$$C^\bullet(G, \mathbb{Z}) \otimes X^\bullet(G \times H, B)^H.$$

Our result follows from Theorem 2.2. □

References

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