On the étale site of marked schemes

Alexander Schmidt

When considering the étale site of a scheme it is often of interest to consider a variant which forces a given set of points to split in at least one member of a covering. Examples are the étale site of a marked curve used in [Scm], where a finite set of closed points is considered and the Nisnevich site [Nis], where all points are required to split. In this note we develop this approach in greater generality. Furthermore, we close a small gap in the literature by showing that any Nisnevich covering of a quasi-compact scheme has a finite subcovering.

1 Definition of the marked site

Let X be a scheme and let T be a set of points of X. We will loosely write $T \subset X$ and call the pair (X,T) a *marked scheme*. A morphism $f:(Y,S) \to (X,T)$ of marked schemes is a scheme morphism $f: Y \to X$ with $f(S) \subset T$.

Definition 1.1. Let (X,T) be a marked scheme. The *marked étale site* $(X,T)_{\mathrm{et}}$ consists of the following data: The category $\mathrm{Cat}(X,T)_{\mathrm{et}}$ is the category of morphisms $f:(U,S) \to (X,T)$ such that

a) $(f: U \to X)$ is étale, and
b) $S = p^{-1}(T)$.

A family $(p_i : (U_i, S_i) \to (U,S))_{i\in I}$ of morphisms in $\mathrm{Cat}(X,T)_{\mathrm{et}}$ is a covering if it is surjective and any point $s \in S$ *splits*, i.e., there exists an index i and a point $u_i \in S_i$ mapping to s such that the induced field homomorphism $k(s) \to k(u_i)$ is an isomorphism.

Example 1.2. For $T = \varnothing$, we obtain the small étale site of X, for $T = X$ the Nisnevich site [Nis].

A morphism of marked schemes induces a morphism of the associated marked étale sites in the obvious way.

We consider the following "geometric points" of $(X,T)_{\mathrm{et}}$: we fix a separable closure $k(x)^s$ of $k(x)$ for every scheme-theoretic point $x \in X$ and consider the following morphisms of marked schemes

1.) for $x \notin T$, the natural morphism $(\mathit{Spec}\, k(x)^s, \varnothing) \to (X,T)$.
2.) for $x \in T$, the natural morphisms $(\mathit{Spec}\, \kappa, \mathit{Spec}\, \kappa) \to (X,T)$ for every subextension $\kappa/k(x)$ of $k(x)^s/k(x)$.

If $f: P \to X$ is any of the morphisms described in 1.) and 2.), the assignment $F \mapsto \Gamma(P, f^*F)$ is a topos-theoretic point of $(X,T)_{\mathrm{et}}$ and one easily verifies that this family of points is conservative. In particular, exactness of sequences of abelian sheaves can be checked stalkwise.

We denote the cohomology of a sheaf $F \in \mathit{Sh}_{\mathrm{et}}(X,T)$ of abelian groups on $(X,T)_{\mathrm{et}}$ by $H^*_{\mathrm{et}}(X,T,F)$.

2 Excision

Let (X,T) be a marked scheme, $i : Z \hookrightarrow X$ a closed immersion and $U = X \smallsetminus Z$ the open complement. The right derivatives of the left exact functor "sections with support in Z"

$$F \mapsto \ker(F(X,T) \to F(U, T \cap U))$$

are called the *cohomology groups with support in* Z. Notation: $H^*_Z(X,T,F)$.

The proof of the next proposition is standard (cf. [Art, III, (2.11)] for the étale case without marking).

Proposition 2.1. *There is a long exact sequence*

$$0 \to H^0_Z(X,T,F) \to H^0_{\mathrm{et}}(X,T,F) \to H^0_{\mathrm{et}}(U,T\cap U,F) \to \\ H^1_Z(X,T,F) \to H^1_{\mathrm{et}}(X,T,F) \to H^1_{\mathrm{et}}(U,T\cap U,F) \to \dots$$

Proposition 2.2 (Excision)**.** *Let $\pi : (X',T') \to (X,T)$ be a morphism of marked schemes, $Z \hookrightarrow X$, $Z' \hookrightarrow X'$ closed immersions and $U = X \smallsetminus Z$, $U' = X' \smallsetminus Z'$ the open complements. Assume that*

- *$\pi : X' \to X$ is étale,*
- *$T' = \pi^{-1}(T)$,*
- *π induces an isomorphism $Z'_{\mathrm{red}} \xrightarrow{\sim} Z_{\mathrm{red}}$,*
- *$\pi(U') \subset U$.*

Then the induced homomorphism

$$H^p_Z(X,T,F) \xrightarrow{\sim} H^p_{Z'}(X',T',\pi^*F)$$

is an isomorphism for every sheaf $F \in Sh_{\mathrm{et}}(X,T)$ and all $p \geq 0$.

Proof. The standard proof for étale topology applies: By the general theory, π^* is exact. Since π belongs to $\mathrm{Cat}(X,T)_{\mathrm{et}}$, π^* has the exact left adjoint "extension by zero", hence π^* sends injectives to injectives. Therefore it suffices to deal with the case $p = 0$. Without changing the statement, we can replace all occurring schemes by their reductions. By assumption,

$$(X',T') \coprod (U, T \cap U) \longrightarrow (X,T)$$

is a covering. For $\alpha \in H^0_Z(X,T,F)$ mapping to zero in $H^0_{Z'}(X',T',\pi^*F)$ we therefore obtain $\alpha = 0$.

Now let $\alpha' \in H^0_{Z'}(X',T',\pi^*F)$ be given. We show that α' and $0 \in H^0(U,T \cap U,F)$ glue to an element in $H^0_Z(X,T,F)$. The only nontrivial compatibility on intersections is $p_1^*(\alpha') = p_2^*(\alpha')$ for $p_1,p_2 : (X' \times_X X', T' \times_T T') \to (X',T')$. This can be checked on stalks noting that $Z' \xrightarrow{\sim} Z$ implies that the two projections $Z' \times_Z Z' \to Z'$ are the same. □

3 Continuity

Proposition 3.1. *Let X be a quasi-compact scheme and let $T \subset X$ be a closed subscheme. Then every étale covering of (X,T) admits a finite subcovering.*

Proof. Since X has a finite affine Zariski-open covering, we may assume that X is affine, in particular X is quasi-separated. Then also T is quasi-compact and quasi-separated. Let

$$\coprod_{i \in I} (U_i, S_i) \to (X,T)$$

be an étale covering. Let, for $i \in I$, $K_i \subset T$ be the set of points in T which split in $U_i \to X$. By [Src, Lemma 13.3], K_i is ind-constructible, i.e., open in the constructible topology of T, which is compact by [EGA4, 1.9.15 (iii)]. Since $T = \bigcup_i K_i$ by assumption, we find a finite subset $J \subset I$ with $T = \bigcup_{i\in J} K_i$. Furthermore, since X is quasi-compact and étale morphisms are open, we find a finite subset $J' \subset I$ such that $\coprod_{i\in J'} U_i \to X$ is an étale covering. We conclude that $\coprod_{i\in J\cup J'}(U_i, S_i) \to (X,T)$ is a finite subcovering of $\coprod_{i\in I}(U_i, S_i) \to (X,T)$. □

As in [SGA4, VII, 3.2] for the unmarked étale site, we define the *restricted marked étale site*

$$(X,T)_{\mathrm{et}}^{\mathrm{res}}$$

as the restriction of $(X,T)_{\mathrm{et}}$ to the subcategory of all $(U,S) \in (X,T)_{\mathrm{et}}$ where $U \to X$ is of finite presentation. Assume that X is quasi-compact and quasi-separated. Then the same is true for any such U and Proposition 3.1 shows that the restricted site is noetherian. Moreover, the categories of sheaves on $(X,T)_{\mathrm{et}}$ and $(X,T)_{\mathrm{et}}^{\mathrm{res}}$ are naturally equivalent. Hence the same argument as in the unmarked étale case [SGA4, VII, Prop. 3.3] shows

Theorem 3.2. *Let X be a quasi-compact and quasi-separated scheme and let $T \subset X$ be a closed subscheme. Let (F_i) be a filtered direct system of abelian sheaves on $(X,T)_{\mathrm{et}}$. Then*

$$\operatorname*{colim}_i H^p_{\mathrm{et}}(X,T,F_i) \cong H^p_{\mathrm{et}}(X,T,\operatorname*{colim}_i F_i)$$

for all $p \ge 0$.

Next we consider inverse limits of marked schemes.

Theorem 3.3. *Let (X,T) be a marked scheme with T closed in X and let $X_i \to X$, $i \in I$, be an inverse system of X-schemes. Assume that all X_i are quasi-separated and quasi-compact and that all transition morphisms are affine. Let T_i be the preimage of T in X_i and put $X_\infty = \varprojlim X_i$, $T_\infty = \varprojlim T_i$.*

Then the restricted site $(X_\infty, T_\infty)_{\mathrm{et}}^{\mathrm{res}}$ is the limit site of the sites $(X_i, T_i)_{\mathrm{et}}^{\mathrm{res}}$.

Corollary 3.4. *With the notation and assumptions of Theorem 3.3, let F be a sheaf of abelian groups on $(X,T)_{\mathrm{et}}$. We denote its inverse image on $(X_i,T_i)_{\mathrm{et}}$ and $(X_\infty,T_\infty)_{\mathrm{et}}$ by F_i and F_∞. Then the natural map*

$$\operatorname*{colim}_i H^p_{\mathrm{et}}(X_i,T_i,F_i) \longrightarrow H^p_{\mathrm{et}}(X_\infty,T_\infty,F_\infty)$$

is an isomorphism for all $p \ge 0$.

Proof of Theorem 3.3. By [Art, III, Theorem 3.8], the site $(X_\infty)_{\mathrm{et}}^{\mathrm{res}}$ is naturally equivalent to the limit site of the $(X_i)_{\mathrm{et}}^{\mathrm{res}}$. In view of Proposition 3.1, it therefore suffices to show that for every quasi-compact étale surjection $U_i \to X_i$ with the property that every point of T_∞ splits in $U_\infty = U_i \times_{X_i} X_\infty \to X_\infty$ there exist $j \ge i$ such that every point of T_j splits in $U_j = U_i \times_{X_i} X_j \to X_j$. We follow the proof of [Src, Lemma 13.2] for Nisnevich coverings. By [Src, Lemma 13.3], the subset $S_j \subset T_j$ of points that split in $U_j \to X_j$ is ind-constructible for all $j \ge i$. Denoting the projection by $u_j : T_\infty \to T_j$, the assumption on $U_\infty \to X_\infty$ implies $T_\infty = \bigcup_j u_j^{-1}(S_j)$. Considering the $T_j \subset X_j$ as reduced, closed subschemes, we may apply [EGA4, Cor. 8.3.4] to obtain $S_j = T_j$ for some j. □

Remark 3.5. Let A be a ring and let $(A \to B_i)_{i\in I}$ be an affine Nisnevich covering. We write A as the union of its finitely generated subrings. Then, by Proposition 3.1 and Theorem 3.3, there exists a finite subset $J \subset I$, a subring $A' \subset A$ which is finitely generated over $\mathbb{Z}$ and a finite Nisnevich covering $(A' \to B'_j)_{j\in J}$ such that $B_j \cong A \otimes_{A'} B'_j$ for all $j \in J$.

Hence the refined definition of Nisnevich coverings for general rings introduced by Lurie in [DAG, XI, Definition 1.1 and Remark 1.15] coincides with the naive definition.

Corollary 3.6. *Let (X,T) be a marked scheme with T closed in X and $Z = \{z_1,\dots,z_n\}$ a finite set of closed points of X. Put $X^h_{z_i} = Spec(\mathcal{O}^h_{X,z_i})$. Then, for every sheaf F of abelian groups on $(X,T)_{\text{et}}$ and all $p \ge 0$*

$$H^p_Z(X,T,F) \cong \bigoplus_{i=1}^n H^p_{\{z_i\}}(X^h_{z_i}, T \cap X^h_{z_i}, F).$$

Proof. Since $H^p_Z(X,T,F) \cong \bigoplus_{i=1}^n H^p_{\{z_i\}}(X,T,F)$, we may assume that $Z = \{z\}$ consists of a single closed point. Excision shows that

$$H^p_{\{z\}}(X,T,F) = H^p_{\{z\}}(U, T\cap U, F)$$

for every affine étale open neighbourhood U of z. Since X^h_z is the limit over all these U, the long exact sequences of Proposition 2.1 together with Corollary 3.4 show the result. □

Using Corollary 3.4, it is easy to calculate the stalks of the higher direct images of the site morphism $(X,T)_{\text{et}} \to (X,X)_{\text{et}} = X_{\text{Nis}}$. The Leray spectral sequence together with the fact that the Nisnevich cohomological dimension of noetherian schemes is bounded by the Krull dimension [Nis, Theorem 1.32] yields:

Corollary 3.7. *Let X be a noetherian scheme of finite Krull dimension d, $T \subset X$ closed and assume that there exists a nonnegative integer N such that*

$$cd(k(x)) \le N$$

for all points $x \in X \smallsetminus T$. Then for every abelian torsion sheaf F on $(X,T)_{\text{et}}$ we have

$$H^p_{\text{et}}(X,T,F) = 0 \quad \text{for } p > N + d.$$

4 Galois covers

Definition 4.1. A *Galois cover* of X with finite Galois group G in the site $(X,T)_{\text{et}}$ is a morphism $(Y,S) \to (X,T)$ in $(X,T)_{\text{et}}$ together with a right action of G on Y over X such that the following holds:

1. $(Y,S) \to (X,T)$ is a covering for the site $(X,T)_{\text{et}}$.
2. $Y \to X$ is an étale Galois cover, i.e.,

   $$Y \times G \to Y \times_X Y, \quad (y,g) \mapsto (y, yg)$$

   is an isomorphism.

Since G acts transitively on the set of points in Y over a given point $x \in X$, we see that every $t \in T$ splits *completely* in Y/X.

Proposition 4.2 (Hochschild-Serre spectral sequence)**.** *Let $(Y,S) \to (X,T)$ be a Galois cover with finite group G und $F \in Sh_{\text{et}}(X,T)$. Then there is a natural spectral sequence*

$$E_2^{pq} = H^p(G, H^q_{\text{et}}(Y,S,F)) \Longrightarrow H^{p+q}_{\text{et}}(X,T,F).$$

Proof. The proof is word-by-word the same as for the étale cohomology, see [Mi, Theorem 2.20]. □

Remark 4.3. Assume that X is quasi-compact and quasi-separated and $T \subset X$ closed. Let

$$(Y_i, S_i) \to (X, T)$$

be a directed inverse system of Galois covers with finite Galois groups G_i and $(Y, S) = \lim(Y_i, S_i)$. Then $(Y, S) \to (X, T)$ is a pro-Galois cover with profinite Galois group $G = \lim G_i$. By Theorem 3.3, for $F \in Sh_{\mathrm{et}}(X, T)$, the groups $H^q_{\mathrm{et}}(Y, S, F) = \operatorname{colim} H^q_{\mathrm{et}}(Y_i, S_i, F)$ are discrete G-modules and we obtain the profinite Hochschild-Serre sequence

$$E_2^{pq} = H^p(G, H^q_{\mathrm{et}}(Y, S, F)) \Longrightarrow H^{p+q}_{\mathrm{et}}(X, T, F),$$

where $H^*(G, -)$ is the continuous cohomology of the profinite group G with values in a discrete G-module (see [NSW, I, §2]).

5 Fundamental group

We recall some facts from Artin-Mazur [AM]. Let $\mathscr{C}$ be a pointed site and $\mathrm{HR}(\mathscr{C})$ the category of pointed hypercovers of $\mathscr{C}$ [AM, §8]. If $\mathscr{C}$ is locally connected, then the "connected component functor" π defines an object

$$\Pi\mathscr{C} = \{\pi(K_\bullet)\}_{K_\bullet \in \mathrm{HR}(\mathscr{C})}$$

in the pro-category of the homotopy category of pointed simplicial sets. By definition, the fundamental group of $\mathscr{C}$ is the pro-group $\pi_1(\Pi(\mathscr{C}))$.

Let X be a locally noetherian scheme. Then (cf. [AM, §9]) the site X_{et}, and hence also $(X, T)_{\mathrm{et}}$ is locally connected. Pointing $(X, T)_{\mathrm{et}}$ by choosing any "geometric" point $\bar{x}$ described at the end of Section 1, we obtain the étale fundamental group $\pi_1^{\mathrm{et}}(X, T, \bar{x})$. It is independent of the choice of $\bar{x}$ up to isomorphism, which is canonical up to inner automorphisms. By [AM, Cor. 10.7], for any group G, the set $\mathrm{Hom}(\pi_1^{\mathrm{et}}(X, T, \bar{x}), G)$ is in bijection with the set of isomorphism classes of pointed (over $\bar{x}$) G-torsors in $(X, T)_{\mathrm{et}}$. In particular, $\pi_1^{\mathrm{et}}(X, \varnothing, \bar{x})$ is the enlarged étale fundamental group of [SGA3, X, §6] and its profinite completion is the usual étale fundamental group of X defined in [SGA1]. If $\bar{x}$ is a geometric point of X, then $\pi_1^{\mathrm{et}}(X, T, \bar{x})$ is a factor group of $\pi_1^{\mathrm{et}}(X, \varnothing, \bar{x})$, which is profinite for normal X by [AM, Thm. 11.1]. Hence we obtain the following result.

Proposition 5.1. *Let X be a noetherian, normal, connected scheme and $T \subset X$. Then (for any choice of base point) $\pi_1^{\mathrm{et}}(X, T)$ is a profinite group. Its finite quotients are in bijection with the isomorphism classes of finite connected pointed étale Galois covers of X in which every point $t \in T$ splits completely.*

Example 5.2. For general (X, T), the fundamental group need not be profinite. For example, let k be a field and $N = \mathbb{P}^1_k/(0 \sim 1)$ the node over k. Then

$$\pi_1^{\mathrm{et}}(N, T) \cong \begin{cases} \mathbb{Z} \times \mathrm{Gal}_k, & T = \varnothing \\ \mathbb{Z}, & T = X. \end{cases}$$

We will use the notation $\hat{\pi}_1^{\mathrm{et}}(X, T)$ for the profinite completion of $\pi_1^{\mathrm{et}}(X, T)$, hence we have a completion map $\pi_1^{\mathrm{et}}(X, T) \to \hat{\pi}_1^{\mathrm{et}}(X, T)$ which is an isomorphism by Proposition 5.1 if X is a noetherian, normal and connected scheme.

We end this section with the following observation concerning products.

Proposition 5.3. *Let k be a field, X and Y geometrically connected schemes of finite type over k and $S \subset X(k)$, $T \subset Y(k)$ nonempty sets of k-rational points. Let a and b be geometric points of $X \smallsetminus S$ and $Y \smallsetminus T$ with values in a common separably closed extension field of k. Assume that at least one of the schemes X and Y is proper over k. Then the natural map*

$$\hat{\pi}_1^{\mathrm{et}}(X \times_k Y, S \times T, (a,b)) \longrightarrow \hat{\pi}_1^{\mathrm{et}}(X, S, a) \times \hat{\pi}_1^{\mathrm{et}}(Y, T, b)$$

is an isomorphism of profinite groups.

Proof. We omit the base points from notation. For a connected scheme X, let $\widetilde{X}$ denote the profinite universal cover. For a subset $S \subset X$, the kernel of $\hat{\pi}_1^{\mathrm{et}}(X) \to \hat{\pi}_1^{\mathrm{et}}(X,S)$ is the (closed) normal subgroup of $\hat{\pi}_1^{\mathrm{et}}(X) = \mathrm{Gal}(\widetilde{X}|X)$ generated by the decomposition groups of the points in S, i.e., it is the (closed) subgroup of $\mathrm{Gal}(\widetilde{X}|X)$ generated by all automorphisms which fix a point $\tilde{s} \in \widetilde{X}$ lying over some $s \in S$. We denote this group by $K(X,S)$.

Now assume we are in the situation of the proposition. By the topological invariance of the étale topology we may assume that k is perfect. Let $\bar{k}$ be an algebraic closure of k. We denote the base changes to $\bar{k}$ by $(\bar{X}, \bar{S})$ and $(\bar{Y}, \bar{T})$. By [SGA1, X, 1.7], we have a natural isomorphism

$$\hat{\pi}_1^{\mathrm{et}}(\bar{X} \times_{\bar{k}} \bar{Y}) \xrightarrow{\sim} \hat{\pi}_1^{\mathrm{et}}(\bar{X}) \times \hat{\pi}_1^{\mathrm{et}}(\bar{Y}).$$

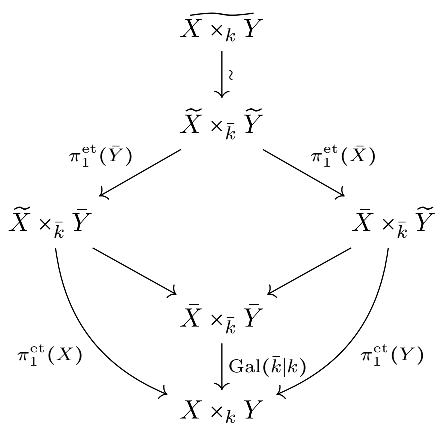
Moreover, by [SGA1, IX, 6.1], we have a natural exact sequence

$$1 \longrightarrow \hat{\pi}_1^{\mathrm{et}}(\bar{X}) \longrightarrow \hat{\pi}_1^{\mathrm{et}}(X) \longrightarrow \mathrm{Gal}(\bar{k}|k) \longrightarrow 1.$$

This and the similar sequence for Y shows the isomorphism

$$\hat{\pi}_1^{\mathrm{et}}(X \times_k Y) \xrightarrow{\sim} \hat{\pi}_1^{\mathrm{et}}(X) \times_{\mathrm{Gal}(\bar{k}|k)} \hat{\pi}_1^{\mathrm{et}}(Y), \qquad (*)$$

where the term on the right hand side is a fibre product in the category of profinite groups. We consider the corresponding diagram of étale Galois covers.



Let $(s,t) \in S \times T \subset X \times_k Y$ and let $(\tilde{s}, \tilde{t}) \in \widetilde{X \times_k Y} = \widetilde{X} \times_{\bar{k}} \widetilde{Y}$ be a point lying above (s,t). An element $\sigma \in \hat{\pi}_1^{\mathrm{et}}(X \times_k Y) = \mathrm{Gal}(\widetilde{X \times_k Y}|X \times_k Y)$ fixes $(\tilde{s}, \tilde{t})$ if and only if its image in $\hat{\pi}_1^{\mathrm{et}}(X) = \mathrm{Gal}(\widetilde{X} \times_{\bar{k}} \bar{Y}|X \times_k Y)$ fixes $\tilde{s} \in \widetilde{X}$ and its image in $\hat{\pi}_1^{\mathrm{et}}(Y) = \mathrm{Gal}(\bar{X} \times_{\bar{k}} \widetilde{Y}|X \times_k Y)$ fixes $\tilde{t} \in \widetilde{Y}$. Hence, the isomorphism $(*)$ induces an isomorphism of subgroups

$$K(X \times_k Y, S \times T) \xrightarrow{\sim} K(X,S) \times_{\mathrm{Gal}(\bar{k}|k)} K(Y,T). \qquad (**)$$

The isomorphisms $(*)$ and $(**)$ together induce an isomorphism

$$\hat{\pi}_1^{\mathrm{et}}(X \times_k Y, S \times T) \xrightarrow{\sim} C$$

with

$$C = \mathrm{coker}\big(K(X,S) \times_{\mathrm{Gal}(\bar{k}|k)} K(Y,T) \longrightarrow \hat{\pi}_1^{\mathrm{et}}(X) \times_{\mathrm{Gal}(\bar{k}|k)} \hat{\pi}_1^{\mathrm{et}}(Y)\big).$$

The natural homomorphism $C \to \hat{\pi}_1^{\mathrm{et}}(X,S) \times \hat{\pi}_1^{\mathrm{et}}(Y,T)$ is injective. To conclude the proof of the proposition, it remains to show surjectivity, i.e., we have to show that every element in $\hat{\pi}_1^{\mathrm{et}}(X,S) \times \hat{\pi}_1^{\mathrm{et}}(Y,T)$ has a preimage in $\hat{\pi}_1^{\mathrm{et}}(X) \times_{\mathrm{Gal}(\bar{k}|k)} \hat{\pi}_1^{\mathrm{et}}(Y) \subset \hat{\pi}_1^{\mathrm{et}}(X) \times \hat{\pi}_1^{\mathrm{et}}(Y)$. For this it suffices to show that the composite map $K(X,S) \hookrightarrow \pi_1^{\mathrm{et}}(X) \to \mathrm{Gal}(\bar{k}|k)$ is surjective. This is true since $K(X,S)$ contains the decomposition group of a k-rational point. □

6 A modification

We consider a modification of the marked étale site which was used in [Scm] for one-dimensional, noetherian regular schemes.

Definition 6.1. The *strict marked étale site* $(X,T)_{\text{et-s}}$ consists of the following data: $\mathrm{Cat}(X,T)_{\text{et-s}}$ is the category of morphisms $f:(U,S) \to (X,T)$ such that

a) $(f : U \to X)$ is étale,
b) $S = p^{-1}(T)$, and
c) for every $u \in S$ mapping to $t \in T$ the induced field homomorphism $k(s) \to k(u)$ is an isomorphism.

Coverings are surjective families.

Proposition 6.2. (i) *If $T \subset X$ consists of a finite set of closed points, then the natural morphism of sites $\varphi : (X,T)_{\mathrm{et}} \to (X,T)_{\mathrm{et-s}}$ induces isomorphisms*

$$H^p_{\mathrm{et-s}}(X,T,F) \xrightarrow{\sim} H^p_{\mathrm{et}}(X,T,\varphi^* F),\ H^p_{\mathrm{et-s}}(X,T,\varphi_* G) \xrightarrow{\sim} H^p_{\mathrm{et}}(X,T,G)$$

for any $F \in Sh_{\mathrm{et-s}}(X,T)$, $G \in Sh_{\mathrm{et-s}}(X,T)$ and $p \geq 0$.

(ii) *For locally noetherian X (and any chosen base point), the natural map*

$$\pi_1^{\mathrm{et}}(X,T) \longrightarrow \pi_1^{\mathrm{et-s}}(X,T)$$

is an isomorphism.

Proof. Let $(U,S) \in \mathrm{Cat}(X,T)_{\text{et-s}}$ and assume that $(f_i : (U_i,S_i) \to (U,S))$ is a covering in $(X,T)_{\mathrm{et}}$. Removing for all i the finitely many points $s \in S_i$ such that $k(f(s_i)) \to k(s_i)$ is not an isomorphism from U_i, we obtain a strict covering $(f_i : (U_i', S_i') \to (U,S))$ which is a refinement of the original one. Hence $\varphi_*\varphi^* F = F$ and $R^q\varphi_* G = 0$ for $q \geq 1$. In view of the Leray spectral sequence, this shows (i). Assertion (ii) follows since both pro-groups represent the same functor: for any group G, a G-torsor in $(X,T)_{\mathrm{et}}$ is the same as a G-torsor in $(X,T)_{\text{et-s}}$. □

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Alexander Schmidt, Mathematisches Institut, Universität Heidelberg, Im Neuenheimer Feld 205, 69120 Heidelberg, Deutschland
email: `schmidt@mathi.uni-heidelberg.de`