Coarse Geometry via Grothendieck Topologies

By Alexander Schmidt

In the course of the last years several authors have studied index problems for open Riemannian manifolds. The abstract indices are elements in the K-theory of an associated C^*-algebra, which only depends on the "coarse" (or large scale) geometry of the underlying metric space. In order to make these indices computable J.Roe introduced a new cohomology theory, called coarse cohomology, which is sensitive only to this coarse geometry (see [5]). This theory takes values in $\mathbb{R}$-vector spaces and it is functorial on complete metric spaces and coarse maps. The coarse cohomology (which can be computed in many examples) is the source of a character map to the cyclic cohomology of a C^*-algebra associated to the metric space.

The coarse cohomology groups measure the behavior at infinity of the given metric space, i.e. they really depend on the metric, not just on the underlying topology. Roe defined his cohomology using a standard complex of locally bounded real valued functions satisfying a suitable support condition.

The object of this paper is to show that coarse geometry can be viewed as a special example of the general concept of a Grothendieck topology. In fact we will show that there is a natural Grothendieck topology on the category of metric spaces under which Roe's coarse cohomology is just the cohomology with compact support of the constant sheaf $\mathbb{R}$. From this point of view many properties of coarse cohomology are easy consequences of general principles. Several of the notions that we define in this article (like bornotopy) are taken from [5] and in order to give a self contained presentation we also give some corollaries, which are already in [5]. Our intention however is to put emphasis on the more functorial point of view and the greater freedom of working with arbitrary sheaves rather than constant systems.

The author wants to thank M.Puschnigg for his interest and for his comments on a draft version of this paper. Moreover we are grateful to the referee for his detailed and constructive criticism and for many valuable suggestions concerning the presentation.

1 Definition of the coarse topology

In the following let X be a metric space. Our concern is to explain how the technique of Grothendieck topologies can be applied in the investigation of "coarse" properties of metric spaces. The term "coarse" means that one is only interested in the behaviour at infinity of a given metric space. From the coarse point of view compact spaces are trivial (i.e. equivalent to a one-point space) and maps are considered only with respect to their asymptotic behaviour at infinity.

Recall (see [1] or [2] for the details) that a *Grothendieck topology* (or a *site*) $\mathcal{X}$ consists of a category $Cat(\mathcal{X})$ and a collection of coverings. This means that for every object B in $Cat(\mathcal{X})$ we have given a collection $Cov(B)$ of families $\{B_i \to B\}_{i\in I}$ of morphisms to B, such that the identity $B \stackrel{id}{\to} B$ is a covering and the collection of coverings is stable under composition and base change[1].

For example the usual topology of the metric space X consists of the category of open subsets of X (with inclusions as morphisms) and a covering is a family of open subsets such that every point of X is contained in at least one of the opens. We will denote this classical topology on X by X_{top}.

It is well known that many non-equivalent metrics on X give rise to the same topology X_{top}, in particular we loose the information about bounded sets.

Definition: The coarse topology X_{co} on X is the Grothendieck topology associated to the following data

(i) $Cat(X_{co})$ = the category of (all) subsets of X with inclusions as morphisms.

(ii) A family $\{U_i \subset U\}_{i\in I}$ of morphisms in $Cat(X_{co})$ is called a covering if every bounded subset $V \subset U$ is contained in U_i for some $i \in I$.

Remark: We do not restrict to the subcategory of open subsets of X, since working with the category of all subsets of X we get functoriality under a larger class of morphisms.

Associated with X_{co} are the categories of presheaves (i.e. contravariant functors from $Cat(X_{co})$ to the category *Ab* of abelian groups) and of sheaves of abelian groups on X_{co}, i.e. those presheaves F for which the sequence

$$F(U) \longrightarrow \prod_{i\in I} F(U_i) \rightrightarrows \prod_{i,j\in I} F(U_i \times_U U_j)$$

is exact for all coverings $\{U_i \to U\}_{i\in I}$. We denote these categories by $Presh(X_{co})$ and $Sh(X_{co})$, respectively. The presheaves on X_{co} are just the usual presheaves on X, when X is endowed with the discrete topology. The property of being a (co)-sheaf measures the behavior at infinity. For example, if X is bounded, then every presheaf is a (co)-sheaf.

As is known from the general theory, the categories $Presh(X_{co})$ and $Sh(X_{co})$ are abelian categories with sufficiently many injective objects. Furthermore there exists a sheafifycation functor $a : Presh(X_{co}) \to Sh(X_{co})$, which is exact and left adjoint to the canonical inclusion $i : Sh(X_{co}) \hookrightarrow Presh(X_{co})$ (see [1], II (1.6)). In particular, an injective sheaf is also injective as a presheaf.

[1] In [2] these data define a *pre-topology*, while the associated topology is given by a class of *sieves*, which is defined using the coverings. The coarse topology below will be defined by a pre-topology and it therefore suffices to work on the level of coverings, avoiding the use of sieves.

Lemma 1.1 *A sequence of sheaves $F \to G \to H$ is exact in $Sh(X_{co})$ if and only if the associated sequence $F(B) \to G(B) \to H(B)$ is exact for every bounded subset $B \subset X$.*

Proof: It follows from the general theory that with $F \to G \to H$ also the restricted sequence $F_B \to G_B \to H_B$ of sheaves in $Sh(B_{co})$ is exact. If B is bounded, then $Sh(B_{co}) = Presh(B_{co})$, hence the sequence of sheaves on B is exact if and only if the sequence $F(A) \to G(A) \to H(A)$ is exact for every $A \subset B$. This shows the only if part in the statement.
Now assume that the sequence $F(B) \to G(B) \to H(B)$ is exact for all bounded subsets $B \subset X$. In order to show the exactness of the sequence of sheaves $F \to G \to H$, we have to show that:

> *If $U \subset X$ is arbitrary and $t \in ker(G(U) \to H(U))$, then there exists a covering $\{U_i \to U\}_{i \in I}$ and a family $\{s_i \in F(U_i)\}_{i \in I}$ such that s_i maps to $res^U_{U_i}(t) \in G(U_i)$.*

But this is easy: Choose a covering $\{U_i \to U\}_{i \in I}$ with bounded sets U_i. Then $res^U_{U_i}(t) \in ker(G(U_i) \to H(U_i)) = im(F(U_i) \to G(U_i))$ for every $i \in I$. This completes the proof. □

Remark: Every bounded subset $B \subset X$ defines a *point* of X_{co}, i.e. $F \mapsto F(B)$ defines a functor from the category of set-valued sheaves on X to the category of sets which is right exact and commutes with direct limits (see [2] IV, 6.1.). Hence lemma 1.1 says that sequences of sheaves of abelian groups are exact if and only if they are stalk-wise exact.

For an abelian group A we will denote by $\underline{A}$ the constant sheaf to A, i.e. the presheaf $U \mapsto A$ (for all U), which is already a coarse sheaf: Indeed, assume that $\{U_i \subset U\}_{i \in I}$ is a coarse covering and suppose that $U_i, U_j \neq \emptyset$. Then there exists a bounded subset $B \subset X$ with $B \cap U_i, B \cap U_j \neq \emptyset$ and since $B \subset U_k$ for some index $k \in I$, we have that $U_k \cap U_i, U_k \cap U_j \neq \emptyset$. In particular we see that constant sheaves are flabby[2] and it is not difficult to prove that the usual sheaf cohomology (i.e. the right derived functor of the global sections functor) is trivial for constant sheaves. Therefore we use the natural analogue of the "sections with compact support"-functor in order to define coarse cohomology.

Definition: Let $F \in Sh(X_{co})$ be a sheaf and let $s \in \Gamma(X_{co}, F)$. For every real number $R > 0$ the we define the R-support of the section s by

$$supp_R(s) = \bigcup_{\substack{s_{|U} \neq 0 \\ d(U) < R}} U,$$

where $d(U) := \sup_{x,y \in U} d(x,y)$ denotes the diameter of the subset $U \subset X$.

Definition: The group of **sections with compact support** is defined by

$$\Gamma_c(X_{co}, F) = \{s \in \Gamma(X_{co}, F) \,|\, supp_R(s) \text{ is bounded } \forall R > 0\}.$$

[2] A sheaf is flabby if the restriction map $res^U_V : F(U) \to F(V)$ is surjective for every pair $V \subset U$.

One easily observes that $\Gamma_c(X_{co}, -)$ is an additive, left exact functor from $Sh(X_{co})$ to the category of abelian groups. By R^iT we will denote the i^{th} right derived functor of an additive (usually left exact) functor T.

Definition: The groups

$$H_c^i(X_{co}, F) := R^i\Gamma_c(X_{co}, -)(F)$$

are called the **coarse cohomology groups** of X with values in the sheaf $F \in Sh(X_{co})$. For an abelian group A we will denote $H_c^i(X_{co}, \underline{A})$ also by $H_{co}^i(X, A)$.

Remark: It is easily seen that Γ_c is an exact functor, when X is bounded. Therefore bounded metric spaces have the cohomology of a point.

2 Change of space

As one expects from the above discussion, the class of maps under which coarse cohomology is functorial is not the class of (top)-continuous maps.

Definition: A map $f : X \longrightarrow Y$ between metric spaces is **bornologous** if the images of bounded subsets in X are bounded subsets in Y. We say that f is **uniformly bornologous** (cf. [5],(2.1)) if for every $R > 0$ there exists $S > 0$ such that

$$\forall x, x' \in X, d(x, x') < R \Rightarrow d(f(x), f(x')) < S.$$

We call f **proper** if preimages of bounded sets are bounded. We say that f is **uniformly proper** if for every $R > 0$ there exists $S > 0$ such that

$$\forall x, x' \in X, d(f(x), f(x')) < R \Rightarrow d(x, x') < S.$$

Remark: Assume that X and Y are proper metric spaces, i.e. that bounded, closed subsets of X (resp. Y) are compact. Then every continuous map $f : X \to Y$ is bornologous and if f is proper in the sense of proper, continuous maps between locally compact Hausdorff spaces, then it is proper in our sense.

A bornologous map induces a continuous morphism of Grothendieck topologies $f : X_{co} \to Y_{co}$, i.e. the assignment $U \subset Y \mapsto f^{-1}(U) \subset X$ defines a functor $f^{-1} : Cat(Y_{co}) \to Cat(X_{co})$ which sends coverings to coverings. Without further mentioning we will assume all occurring maps to be bornologous. Associated to every (bornologous) map f are the usual functors f_* and f^* between the categories of coarse sheaves: The functor $f_* : Sh(X_{co}) \to Sh(Y_{co})$ is defined by $f_*F(U) = F(f^{-1}(U))$, $U \subset Y, F \in Sh(X_{co})$, and is left exact. By the general theory (see [2] III.2.3) f_* admits the exact left adjoint functor $f^* : Sh(Y_{co}) \to Sh(X_{co})$, in particular f_* sends injective sheaves to injective sheaves.

Lemma 2.1 *Assume that $f : X \longrightarrow Y$ is uniformly bornologous and proper. Then*

(i) *f_* is exact.*

(ii) *There are functorial morphisms:* $H^i(f^*): H^i_c(Y_{co}, G) \longrightarrow H^i_c(X_{co}, f^*G)$ *for all* i *and every* $G \in Sh(Y_{co})$*. In particular we have associated morphisms*

$$H^i(f^*): H^i_{co}(Y, A) \longrightarrow H^i_{co}(X, A)$$

for all i *and every abelian group* A*.*

Proof: (i) The exactness of f_* is easily verified on stalks because the preimages of bounded sets are bounded.
(ii) If $f: X \to Y$ is uniformly bornologous and $G \in Sh(Y_{co})$, then for every $R > 0$ there exists an $S > 0$ such that for every $s \in \Gamma_c(Y, G)$

$$supp_R(f^*(s)) \subset f^{-1}(supp_S(s)).$$

Therefore, if $f: X \longrightarrow Y$ is uniformly bornologous and proper, then there exists a functor morphism $\Gamma_c(Y_{co}, -) \longrightarrow \Gamma_c(X_{co}, -) \circ f^*$, which extends to a corresponding morphism for the derived functors

$$R^+\Gamma_c(Y_{co}, -) \longrightarrow R^+(\Gamma_c(X_{co}, -) \circ f^*) \longrightarrow R^+(\Gamma_c(X_{co}, -)) \circ f^*,$$

where the last morphism exists because f^* is exact. Therefore we have functorial morphisms: $H^i_c(Y_{co}, G) \longrightarrow H^i_c(X_{co}, f^*G)$ for all i and every $G \in Sh(Y_{co})$. Moreover $f^*(\underline{A}) = \underline{A}$, which shows the last claim. □

Theorem 1 *Assume that* $f: X \to Y$ *is uniformly bornologous, uniformly proper and surjective. Then for every sheaf* $G \in Sh(Y_{co})$ *the canonical homomorphism*

$$f^*: H^i_c(Y_{co}, G) \to H^i_c(X_{co}, f^*G)$$

is an isomorphism for all i*.*

Proof: Since f is surjective and proper, one verifies on stalks that the canonical adjunction homomorphism $G \longrightarrow f_*f^*G$ is an isomorphism.
Now let $F \in Sh(X_{co})$ be any sheaf. Then, since f is uniformly bornologous, for every $R > 0$ there exists an $S > 0$ such that for every $s \in \Gamma(X_{co}, F)$

$$supp_R(s) \subset f^{-1}(supp_S(f_*s)).$$

Because f is proper, we therefore have a canonical injective morphism of functors $\Gamma_c(Y_{co}, -) \circ f_* \hookrightarrow \Gamma_c(X_{co}, -)$. Since f is uniformly proper, for every $R > 0$ there is an $S > 0$ such that for every $s \in \Gamma_c(X_{co}, F)$ we have

$$supp_R(f_*s) \subset f(supp_S(s)).$$

Hence the canonical injective functor morphism $\Gamma_c(Y_{co}, -) \circ f_* \longrightarrow \Gamma_c(X_{co}, -)$ is an isomorphism in our situation. Since f_* is exact an sends injectives to injectives, we obtain canonical isomorphisms $H^i_c(Y_{co}, f_*F) \xrightarrow{\sim} H^i_c(X_{co}, F)$. For $F = f^*G$ we therefore get isomorphisms

$$H^i_c(Y_{co}, G) \xrightarrow{\sim} H^i_c(Y_{co}, f_*f^*G) \xrightarrow{\sim} H^i_c(X_{co}, f^*G).$$

□

Following [5],(2.5) we define the notion of bornotopy:

Definition: Two maps $f, g : X \to Y$ (assumed to be uniformly bornologous and proper throughout) are called to be **bornotopic** if there is a uniformly bornologous and proper map:

$$F : \{0,1\} \times X \longrightarrow Y$$

with $f = F(0,-)$, $g = F(1,-)$. Here we choose *any* metric on the two point space $\{0,1\}$.

We also could have worked with $[0,1]$ instead (cf. the remarks after (2.5) in [5]).

Corollary 2.2 *(Bornotopy invariance of coarse cohomology)*
Assume that $f, g : X \longrightarrow Y$ are bornotopic maps. Then for every abelian group A the induced homomorphisms

$$H^i(f^*), H^i(g^*) : H^i_{co}(Y, A) \longrightarrow H^i_{co}(X, A)$$

are the same.

Proof: Since the projection: $p : \{0,1\} \times X \longrightarrow X$ is uniformly bornologous, uniformly proper and surjective, it induces isomorphisms

$$H^i_{co}(X, A) \xrightarrow{\sim} H^i_{co}(\{0,1\} \times X, A).$$

Therefore the two obvious sections i_0 and i_1 of p both induce the same (namely the inverse to $H^i(p^*)$) map $H^i_{co}(\{0,1\} \times X, A) \xrightarrow{\sim} H^i_{co}(X, A)$. From this the statement of the corollary follows because $f^* = i_0^* \circ F^*$ and $g^* = i_1^* \circ F^*$ □

Definition: A subspace of $Z \subset X$ is called **uniformly dense** in X if there is an $R > 0$ such that for every $x \in X$ there exists an $z \in Z$ with $d(x, z) < R$.

Corollary 2.3 *Assume that $Z \subset X$ is uniformly dense. Then for every abelian group A the canonical restriction homomorphism*

$$H^i_{co}(X, A) \xrightarrow{\sim} H^i_{co}(Z, A)$$

is an isomorphism for all i.

Proof: We denote the canonical inclusion by $i : Z \hookrightarrow X$. Choose any map $p : X \to Z$ sending an $x \in X$ to a $z \in Z$ with distance $< R$. Then p and i are bornotopy inverse to each other, i.e. $p \circ i$ is bornotopic to id_Z and $i \circ p$ is bornotopic to id_X. □

3 An auxiliary topology

In order to compare the coarse cohomology as defined above with Roe's cohomology and with the usual cohomology with compact support it is useful to introduce an auxiliary topology X_{CO} on X.

Definition: We define the CO-topology on X by the following data:

(i) $Cat(X_{CO})$ = the category of (top-) open subsets of X together with inclusions.

(ii) A family $\{U_i \subset U\}_{i\in I}$ of morphisms in $Cat(X_{CO})$ is called a covering if every bounded (open) subset $V \subset U$ is contained in U_i for some $i \in I$.

The *co*-topology is a refinement of the CO-topology (having more "open" sets) and the obvious forgetful functor will be denoted by

$$v : X_{co} \longrightarrow X_{CO}.$$

However also the classical topology X_{top} is a refinement of X_{CO} (having more coverings) and we will denote this forgetful functor by

$$h : X_{top} \longrightarrow X_{CO}.$$

The presheaves on X_{CO} are the usual presheaves on X_{top} and the functor $h_* : Sh(X_{top}) \to Sh(X_{CO})$ reads as "looking at a top-sheaf as CO-sheaf", while h^* is the functor of "sheafifying a CO-sheaf to a top-sheaf". The CO-sheaves are the presheaves which are called ω-sheaves in [5],(3.19). We define the functor Γ_c on $Sh(X_{CO})$ in the same way as for $Sh(X_{co})$ and we define the cohomology $H^i_c(X_{CO}, -)$ as the i^{th} right derived functor of the functor Γ_c on $Sh(X_{CO})$. The next lemma assures that we do not loose cohomological information, when we change from X_{co} to X_{CO}.

Lemma 3.1 *The functor $v_* : Sh(X_{co}) \to Sh(X_{CO})$ is exact and the canonical homomorphism: $G \to v_*v^*G$ is an isomorphism for every $G \in Sh(X_{CO})$. The canonical homomorphisms*

$$H^i(v_*) : H^i_c(X_{CO}, v_*F) \xrightarrow{\sim} H^i_c(X_{co}, F) \text{ and } H^i(v^*) : H^i_c(X_{CO}, G) \xrightarrow{\sim} H^i_c(X_{co}, v^*G)$$

are isomorphisms for all i and all sheaves $G \in Sh(X_{CO})$, $F \in Sh(X_{co})$.

Proof: The exactness of v_* as well as the isomorphism $G \xrightarrow{\sim} v_*v^*G$ immediately follow from the definitions. For the cohomological statements note that $\Gamma_c(X_{co}, -) = \Gamma_c(X_{CO}, -) \circ v_*$ and that v_* is exact and sends injectives to injectives. This implies the first statement, while the second follows from the first because

$$H^i_c(X_{CO}, G) \xrightarrow{\sim} H^i_c(X_{CO}, v_*v^*G) \xrightarrow{\sim} H^i_c(X_{co}, v^*G).$$

□

Lemma 3.2 *Assume that X is a proper metric space. Then for every $G \in Sh(X_{CO})$ and all i there are canonical homomorphisms*

$$H^i(h^*) : H^i_c(X_{CO}, G) \to H^i_c(X_{top}, h^*G).$$

In particular we have homomorphisms: $H^i_{co}(X, A) \to H^i_c(X_{top}, A)$ for all i and every abelian group A.

Proof: Without difficulties one observes that for any section $s \in \Gamma_c(X_{CO}, G)$ the support (in the (top)-sense) of $h^*(s)$ is bounded. Therefore, if X is proper, we have a natural homomorphism

$$\Gamma_c(X_{CO}, G) \longrightarrow \Gamma_c(X_{top}, h^*G),$$

where $\Gamma_c(X_{top}, -)$ denotes the usual sections with compact support functor. Since h^* is exact (by general reasons) we obtain associated homomorphisms $H^i_c(X_{CO}, G) \to H^i_c(X_{top}, h^*G)$. In particular if $G = \underline{A}$ is constant, we obtain maps $H^i_{co}(X, A) \to H^i_c(X_{top}, A)$. □

Remark: 1. For $A = \mathbb{R}$ the homomorphism above coincides with the character map c defined in [5],(2.11).
2. Using corollary 4.2 below and standard properties of Čech cohomology for "good" covers one gets sufficient conditions under which this character map is an isomorphism, e.g. for $X = \mathbb{R}^n$, see corollary 4.3 (see also [5] 3.4. for variants).

4 Coarse Čech cohomology

In this section we want to develop compactly supported Čech cohomology for the coarse topology. We do this for proper metric spaces using the notion of an anti-Čech system (cf. [5],(3.13)):

Definition: Let $\mathfrak{U}$ be an open covering (in the (top)-sense) of X. We say that $\mathfrak{U}$ is locally finite if every bounded set only meets finitely many members of $\mathfrak{U}$. A sequence $\{\mathfrak{U}_n\}_{n\in\mathbb{N}}$ of locally finite open coverings of X is called an **anti-Čech system** if there is a sequence of real numbers $R_n \to \infty$ such that for all n

(i) Each set $U \in \mathfrak{U}_n$ has diameter less than R_n.

(ii) The covering $\mathfrak{U}_{n+1}$ has a Lebesgue number greater than or equal to R_n, i.e. every subset of diameter less than R_n is contained in at least one member of $\mathfrak{U}_{n+1}$.

In particular $\mathfrak{U}_n$ is a refinement of $\mathfrak{U}_{n+1}$ for all n. Anti-Čech systems exist for every proper metric space (see [5] lemma 3.15).

Now we recall the definition of the compactly supported Čech complex associated to a locally finite open covering $\mathfrak{U} = \{U_i\}_{i\in I}$ and a presheaf $F \in Presh(X_{CO})(= Presh(X_{top}))$. It is defined by

$C^p_c(\mathfrak{U}, F) :=$

$$\{s \in \bigoplus_{(i_0,\ldots,i_p)\in I^{p+1}} F(U_{i_0} \cap \ldots \cap U_{i_p}) \,|\, s_{\sigma(i_0),\ldots,\sigma(i_p)} = sgn(\sigma) s_{(i_0,\ldots,i_p)} \text{ for all } \sigma \in \mathfrak{S}_{p+1}\}$$

together with the usual alternating differential: $d : C^p \to C^{p+1}$, where $\mathfrak{S}_{p+1}$ is the symmetric group over $p+1$ elements. (The compact support comes in via the $\oplus$ instead of $\prod$.)

Defining $\check{H}^i_c(\mathfrak{U},F) := H^i(C^*_c(\mathfrak{U},F))$, it is well known (cf. [1], I.(3.1)) that

$$\check{H}^i_c(\mathfrak{U},F) = R^i\check{H}^0_c(\mathfrak{U},F).$$

Choosing an anti-Čech system $\mathfrak{U}_n$, the refinement maps define inverse systems $\{\check{H}^i_c(\mathfrak{U}_n,F)\}_{n\in\mathbb{N}}$. Denoting the category of inverse systems of abelian groups by $Ab^{\mathbb{N}}$ and the canonical inclusion of $Sh(X_{CO})$ to $Presh(X_{CO})$ by i, we obtain the following series of functors

$$Sh(X_{CO}) \xrightarrow{\ i\ } Presh(X_{CO}) \xrightarrow{\check{H}^0_c(\mathfrak{U}_*,-)} Ab^{\mathbb{N}} \xrightarrow{\varprojlim} Ab.$$

An easy calculation shows that we have:

$$\varprojlim_n \circ \check{H}^0_c(\mathfrak{U}_n,-)\circ i = \Gamma_c(X_{CO},-)$$

as functors: $Sh(X_{CO}) \longrightarrow Ab$.

Lemma 4.1 **(i)** *The functor $\check{H}^0_c(\mathfrak{U}_n,-)$ sends injective presheaves to $\varprojlim$-acyclic inverse systems.*

(ii) $\check{H}^p_c(\mathfrak{U}_n,R^qi) = 0$ *for all n, p and all $q>0$.*

Corollary 4.2 *For every sheaf $G\in Sh(X_{CO})$ and all p there is a short exact sequence*

$$0\longrightarrow \varprojlim_n{}^1 \check{H}^{p-1}_c(\mathfrak{U}_n,iG)\longrightarrow H^p_c(X_{CO},G)\longrightarrow \varprojlim_n \check{H}^p_c(\mathfrak{U}_n,iG)\longrightarrow 0.$$

Proof of the corollary: The functor i is left exact and sends injectives to injectives and as is well known, $\varprojlim^k : Ab^{\mathbb{N}}\longrightarrow Ab$ is trivial for all $k\geq 2$ (see [4] Sect.1 for the basic properties of the category $Ab^{\mathbb{N}}$ and of the functor $\varprojlim$). Therefore, composing the derived functors, the statement of the corollary follows from lemma 4.1. □

Proof of lemma 4.1: (i) It suffices to show that every presheaf G can be embedded into a presheaf G' having $\varprojlim$-acyclic $\{\check{H}^0_c(\mathfrak{U}_n,-)\}_{n\in\mathbb{N}}$ (G is a direct summand in G', if G is injective). We define G' (which is the presheaf analogue of the first step of the classical Godement resolution) by

$$G'(U) := \prod_{V\subset U} G(V),$$

together with the obvious restriction maps and the canonical inclusion $G\hookrightarrow G'$. If $\mathfrak{U}=\{U_i\}_{i\in I}$ is a locally finite open cover, then one obtains the following equality, in which we denote by $B_{\mathfrak{U}}(X)$ the family of bounded open subset $V\subset X$, such that $V\subset U_i$ for at least one $i\in I$):

$$\begin{aligned}\check{H}^0_c(\mathfrak{U},G') \;=\; &\{(s_V)\in \textstyle\prod_{V\in B_{\mathfrak{U}}(X)} G(V)\mid \text{ there exists a finite subset } J\subset I \\ &\text{such that } s_V=0 \text{ if } V\subset U_i \text{ for one } i\notin J\}\end{aligned}$$

If $\mathfrak{V} = \{V_{i'}\}_{i' \in I'}$ is a refinement of $\mathfrak{U}$, then it can be seen from the above identification that the canonical refinement map

$$\check{H}^0_c(\mathfrak{U}, G') \longrightarrow \check{H}^0_c(\mathfrak{V}, G')$$

is surjective. Indeed, we can lift an $s = (s_V)_{V \in B_{\mathfrak{V}}(X)} \in \check{H}^0_c(\mathfrak{V}, G')$ to an $s' = (s'_V)_{V \in B_{\mathfrak{U}}(X)} \in \check{H}^0_c(\mathfrak{U}, G')$ setting

$$s'_V = \begin{cases} s_V \text{ if } V \subset V_{i'} \text{ for an } i' \in I' \\ 0 \text{ otherwise.} \end{cases}$$

The construction is correct because $J' := \{j' \in I' \mid V_{j'} \cap U_j \neq \emptyset \text{ for an } j \in J\}$ is a finite subset of I' if J is a finite subset of I. As is well known, surjective systems are $\varprojlim$-acyclic, hence we proved (i).
(ii) Recall that the sheafification functor $a : Presh(X_{CO}) \to Sh(X_{CO})$ is exact and that $a \circ i = id_{Sh}$. Therefore, for every open $U \subset X$ and every $G \in Sh(X_{CO})$ we have $R^q iG(U) = R^i\Gamma(U_{CO}, G)$. In particular $R^q iG(U) = 0$ for $q > 0$ if U is bounded. This implies (ii). □

Following the terminology in [5],(3.28) we call a covering $\mathfrak{U}$ a **Leray covering** if every finite intersection $U_1 \cap \ldots \cap U_k$ of elements of $\mathfrak{U}$ is contractible. It is well known that for every locally finite open Leray covering $\mathfrak{U}$ of an locally compact Hausdorff space X the canonical homomorphism $\check{H}^i_c(\mathfrak{U}, A) \to H^i_c(X_{top}, A)$ is an isomorphism for all i and every abelian group A. The exact sequence of corollary 4.2 therefore implies:

Corollary 4.3 *Suppose that X is a proper metric space which admits an anti-Čech system $\mathfrak{U}_n$ consisting of Leray coverings. Then for every abelian group A and every i the canonical homomorphism*

$$H^i_{co}(X, A) \longrightarrow H^i_c(X_{top}, A)$$

is an isomorphism.

5 Comparison with Roe's cohomology

Since Roe used the notation HX^* for his cohomology we will use the letter M for the ground space in this chapter. Our aim is to prove the following theorem

Theorem 2 *Assume that M is a proper metric space. Then there are canonical isomorphisms for all i:*

$$HX^i(M) \xrightarrow{\sim} H^i_{co}(M, \mathbb{R})$$

between Roe's cohomology and the coarse cohomology of the constant sheaf $\mathbb{R}$.

Recall that Roe's coarse (or "eXotic") cohomology $HX^i(M)$ is defined as the i^{th} cohomology group of the complex $CX^*(M)$, which is given by

$$\begin{aligned} CX^q(M) \;=\; & \{\text{locally bounded Borel functions } \phi : M^{q+1} \to \mathbb{R} \text{ such that} \\ & supp(\phi) \cap Pen(\Delta_q, R) \text{ is precompact in } M^{q+1} \text{ for every } R > 0\}, \end{aligned}$$

where $Pen(\Delta_q, R) = \{(x_0, \ldots, x_q) \in M^{q+1} \,|\, \exists x \in M : \sup_i d(x, x_i) \leq R\}$. The coboundary map $d : CX^q(M) \to CX^{q+1}(M)$ is given by

$$d\phi(x_0, \ldots, x_{q+1}) = \sum_{i=0}^{q+1} (-1)^i \phi(x_0, \ldots, \hat{x}_i, \ldots, x_{q+1}).$$

One observes that $CX^*(M)$ is just the complex of sections with compact support of the complex $\mathbf{CX}^*_M$ of CO-sheaves, which is defined by

$$\mathbf{CX}^q_M(U) = \{\text{locally bounded Borel functions } \phi : U^{q+1} \to \mathbb{R}\}$$

together with the obvious boundary map. The complex $\mathbf{CX}^*_M$ is a resolution of the constant sheaf $\mathbb{R}$ in the category $Sh(M_{CO})$. In order to compare Roe's cohomology with the coarse cohomology of the present paper, it therefore suffices to show that the sheaves $\mathbf{CX}^q_M$ are Γ_c-acyclic. For this we recall the following definitions:

Definition: Let $F \in Presh(M_{CO}) (= Presh(M_{top}))$. We say that F is **fine** if associated to every locally finite open covering $\mathfrak{U}$ there exists a partition of the unity for F, i.e. there is a family $\{e_U\}_{U \in \mathfrak{U}}$ of presheaf endomorphisms such that

(i) if V is open and $V \cap \bar{U} = \emptyset$, then $e_{U|V} : Presh(V) \to Presh(V)$ is zero,

(ii) if V is precompact, then

$$\sum_{U \in \mathfrak{U}} e_{U|V} : Presh(V) \to Presh(V)$$

is the identity (only finitely many summands are non-trivial by (i)).

We say that a presheaf F is **soft** if for every precompact open set V and every open $U \supset \bar{V}$ we have

$$im(F(U) \overset{res^U_V}{\longrightarrow} F(V)) = im(F(M) \overset{res^M_V}{\longrightarrow} F(V)) \subset F(V).$$

It is well known that a fine sheaf in $Sh(X_{top})$ is soft, but this is not true for CO-sheaves.

Lemma 5.1 *The CO-sheaves $\mathbf{CX}^q_M$ defined above are fine and soft presheaves.*

Proof: This follows easily using a partition of the unity on M. □

Summarizing, the theorem is proved if we have shown the following

Proposition 5.2 *Assume that $F \in Sh(X_{CO})$ is fine and soft. Then F is Γ_c-acyclic, i.e. we have*

$$H^q_c(X_{CO}, F) = 0$$

for all $q > 0$.

Proof: We choose an anti-Čech-system $\mathfrak{U}_n$ of locally finite open coverings, such that all refinement inclusions $U_n \subset U_{n+1}$ for $U_n \in \mathfrak{U}_n, U_{n+1} \in \mathfrak{U}_{n+1}$ have the property that $U_{n+1} \supset \bar{U}_n$. (This is possible by [5],(3.15).) Then we apply corollary 4.2. Since F is soft, all transition maps in the inverse systems $\{\check{H}^i_c(\mathfrak{U}_n, F)\}_{n\in\mathbb{N}}$ are zero for $i > 0$ (see [6], 6.8. Thm.4 for ordinary cohomology and [5],(3.10) for the necessary changes in the case of $\check{H}_c$). Further, since F is soft, the inverse system $\{\check{H}^0_c(\mathfrak{U}_n, F)\}_{n\in\mathbb{N}}$ is a Mittag-Leffler system, i.e. $\varprojlim$-acyclic. This completes the proof. □

Remark: The same arguments are also valid for all variants (continuous functions, antisymmetric functions,...) of the standard complex, as they are given in [5] 3.3 and Theorem 3.23 of [5] implicitly already contains a statement about the Grothendieck topology. However, theorem 3.23 of [5] should be modified, because the author applies it to sheaves which are not flabby (but soft, indeed).

6 Mayer-Vietoris sequence

In this last section we give a proof of a Mayer-Vietoris sequence for the coarse cohomology, similar to that which has been shown for closed pairs and Roe's cohomology in [3].

Recall that for two closed subsets M, N of X with $M \cup N = X$ and for every (top)-sheaf F on X we have the (top)-exact sequence

$$0 \to F \to i_{M_*} i^*_M F \oplus i_{N_*} i^*_N F \to i_{M\cap N_*} i^*_{M\cap N} F \to 0,$$

in which $i_\square$ denotes the embedding of a closed subset. The exactness is easily verified on stalks and this sequence is responsible for the topological Mayer-Vietoris sequence for a pair of closed subsets.

Unfortunately the above sequence is never exact in the coarse topology. A first trivial obstruction against the exactness is that no non-trivial pair M, N, $M \cup N = X$ fulfills the following property

"*a bounded subset U which intersects M and N also intersects $M \cap N$*".

Hence there is no chance for the validity of a Mayer-Vietoris sequence for arbitrary coarse sheaves.

Setting $Pen(Z, R) = \{x \in X \mid \exists z \in Z : d(x, z) \leq R\}$, every subset $Z \subset X$ is bornotopy equivalent to $Pen(Z, R)$ for every R. Therefore, by restricting to constant sheaves and using the bornotopy invariance of coarse cohomology (corollary 2.2), one can prove a Mayer-Vietoris sequence for constant coefficients and for pairs satisfying an approximate form of the above property. Following [3] we define coarsely excisive pairs:

Definition: We call a pair of subset $M, N \subset X$, $M \cup N = X$ **coarsely excisive** if for every $R > 0$ there exists an $S > 0$ such that

$$Pen(M, R) \cap Pen(N, R) \subset Pen(M \cap N, S).$$

Theorem 3 *Assume that the pair $M, N \subset X$, $M \cup N = X$ is coarsely excisive. Then for every abelian group A we have a long exact sequence*

$$\cdots \to H^i_{co}(X, A) \to H^i_{co}(M, A) \oplus H^i_{co}(N, A) \to H^i_{co}(M \cap N, A) \to \cdots.$$

Proof: For a subset $Z \subset X$ we introduce the following functor on $Sh(X_{co})$:

$$\Gamma^Z_c(X_{co}, F) := \{s \in \Gamma(X_{co}, F) \mid supp_R(s) \cap Z \text{ is bounded } \forall R > 0\}.$$

(Observe that $\Gamma^X_c(X_{co}, -) = \Gamma_c(X_{co}, -)$ and $\Gamma^Z_c(X_{co}, -) = \Gamma(X_{co}, -)$ if Z is bounded.) In order to proceed we need the following

Lemma 6.1 *If M, N is coarsely excisive, then the sequence of functors*

$$0 \to \Gamma_c(X_{co}, -) \to \Gamma^M_c(X_{co}, -) \oplus \Gamma^N_c(X_{co}, -) \to \Gamma^{M\cap N}_c(X_{co}, -) \to 0$$

is exact on injectives.

Proof of the lemma: One easily verifies that the above sequence is well defined and left exact. We proceed in a similar way as in the proof of lemma 4.1. In order to complete the proof it suffices to show, that every sheaf F can be embedded into a sheaf F' for which the right hand arrow is surjective (F is a direct summand in F' if F is injective). We define F' (which is the natural coarse analogue to the first step of the classical Godement resolution) by

$$F'(U) := \prod_{V \subset U \text{bounded}} F(V),$$

together with the obvious restriction maps and the canonical inclusion $F \hookrightarrow F'$. Then F' is a coarse sheaf. Denoting the set of bounded subsets of X by $B(X)$, let $F \in Sh(X_{co})$ and $s = \{s_V\}_{V \in B(X)} \in \Gamma^{M\cap N}_c(X_{co}, F')$. Then by definition we have

$$supp_R(s) = \bigcup \{U \mid d(U) < R, \exists V \subset U : s_V \neq 0\}.$$

Now choose a disjoint decomposition $B(X) = B_M(X) \sqcup B_N(X)$ with the property: $U \in B_M(X) \Rightarrow U \cap N \neq \emptyset$ and vice versa. Then we construct a pair $(s^M, s^N) \in \Gamma^M_c(X_{co}, F') \times \Gamma^N_c(X_{co}, F')$ with $s = s^M + s^N$. This is done by

$$s^M_V = \begin{cases} s_V & \text{for } V \in B_M(X) \\ 0 & \text{otherwise} \end{cases}$$

and symmetrically for s^N. Obviously $s = s^M + s^N$ and it remains to check that s^M and s^N satisfy the required support condition. By symmetry we restrict to s^M. However, if R and S are chosen as in the definition of coarsely excisive pairs, a straightforward verification shows

$$supp_R(s^M) \cap M \subset Pen(supp_{R+2S}(s) \cap M \cap N, S).$$

This completes the proof. $\square$

End of the proof of theorem 3: By the above lemma we have for every sheaf $F \in Sh(X_{co})$ a long exact sequence

$$\cdots \to H^i_c(X_{co}, F) \to R^i\Gamma^M_c(X_{co}, F) \oplus R^i\Gamma^N_c(X_{co}, F) \to R^i\Gamma^{M\cap N}_c(X_{co}, F) \to \cdots .$$

Therefore the next lemma finishes the proof of theorem 3. $\square$

Lemma 6.2 *For every subset $Z \subset X$ and every abelian group A we have canonical isomorphisms for all i*

$$R^i\Gamma^Z_c(X_{co}, A) \xrightarrow{\sim} H^i_{co}(Z, A).$$

Proof: Let $F \in Sh(X_{co})$ and $s \in \Gamma(X_{co}, F)$. For a pair $n \geq R > 0$ a straightforward verification shows

$$supp_R(s) \cap Z \subset supp_R(s_{|Pen(Z,n)}) \subset Pen(supp_{R+2n}(s) \cap Z, n + R).$$

Hence we have a canonical isomorphism of functors $\Gamma^Z_c(X, -) \cong \varprojlim_n \Gamma_c(Pen(Z, n), -)$. The fact that injective sheaves are sent to $\varprojlim$-acyclic systems can be verified on sheaves of the form F' (compare the prove of lemma 6.1). Since Z is uniformly dense in $Pen(Z, R)$ for every $R > 0$, corollary 2.3 shows that all transition maps $H^i_{co}(Pen(Z, n + 1), A) \to H^i_{co}(Pen(Z, n), A)$ are isomorphisms. This completes the proof. $\square$

Final remark: Perhaps it would be more conceptual to treat coarse and classical cohomology in a unified manner. Indeed, for every family $\mathfrak{V}$ of subsets of X we can find an appropriate Grothendieck topology such that the members of the family are points. This can be achieved by defining a covering to be a family $\mathfrak{U}$ of subsets, s.t. every $V \in \mathfrak{V}$ is contained in at least one $U \in \mathfrak{U}$. It is clear, which families give rise to the classical resp. to the coarse topology. In this way one can define many different topologies and associated cohomology theories, each of them adapted to a particular way of filtering information.

References

[1] M.Artin *Grothendieck topologies*
Lect. Notes Harvard Univ. Math. Dept. Cambridge Mass. 1962

[2] M.Artin, A.Grothendieck, J.L.Verdier *Thèorie des topos et cohomologie étale des schemas* (SGA 4) tome 1,2 Springer Lect. Not. Math. 269, 270

[3] H.Higson, J.Roe, G.Yu *A coarse Mayer-Vietoris sequence*
Math. Proc. Camb. Philos. Soc. **114** (1993), 85-97

[4] U.Jannsen *Continuous étale cohomology*
Math. Ann. **280** (1988), 207-245

[5] J.Roe *Coarse cohomology and index theory on complete Riemannian Manifolds* Mem. Amer. Math. Soc. No.**497** (1993)

[6] E.Spanier *Algebraic topology* McGraw-Hill 1966

Alexander Schmidt
Universität Heidelberg
Mathematisches Institut
Im Neuenheimer Feld 288
D-69120 Heidelberg
Germany
e-mail: schmidt@harmless.mathi.-heidelberg.de