Generic Injectivity for Étale Cohomology and Pretheories

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Let k be a field. We call W a smooth semi-local k-scheme if there exists a smooth affine k-scheme Y and finitely many closed points $y_1, \dots, y_n$ on Y such that W is the inverse limit of all Zariski open neighbourhoods of $\{y_1, \dots, y_n\}$ in Y. The objective of this paper is to show the following

Theorem 1 *Let W be a connected smooth semi-local scheme over a field k and let η be its generic point. Let $X \to W$ be a proper smooth morphism, n an integer prime to char(k) and $\mathcal{K}^\bullet$ a complex of sheaves of $\mathbb{Z}/n\mathbb{Z}$-modules on X_{et} whose cohomology sheaves are locally constant constructible and bounded below. Then for every $q \in \mathbb{Z}$ the canonical map*

$$H^q_{et}(X, \mathcal{K}^\bullet) \longrightarrow H^q_{et}(X_\eta, \mathcal{K}^\bullet)$$

is a universal monomorphism.

This injectivity result for étale cohomology was known before by the work of J.-L. Colliot-Thélène, R. Hoobler and B. Kahn [CHK] in the following cases:

- dim $W = 1$ ([CHK], Corollary B.3.3)
- $K^\bullet = \mu_n^{\otimes i}$ (concentrated in degree zero) and $X = W \times T$ with T a smooth (not necessarily proper) variety over k ([CHK], Theorem 8.1.1).

The reader should also compare Theorem 1 with O. Gabber's injectivity result [Ga] for henselizations.

The word "universal" in the statement of the theorem has the following meaning: Let $\phi : M \hookrightarrow N$ be a monomorphism in an abelian category $\mathcal{A}$ which has the property that filtered direct limits exist and are exact. ϕ is called *universal monomorphism* if for every abelian category $\mathcal{B}$ in which filtered direct limits exist and are exact and for every additive functor $T : \mathcal{A} \to \mathcal{B}$ commuting with filtered direct limits, the homomorphism $T(M) \to T(N)$ is again monomorphic. A typical example of a universal monomorphism is a filtered direct limit of split injections.

The techniques used in the proof of theorem 1 are those of Voevodsky [Vo], §§4.3,4.4. Let X be a smooth scheme over k and let X-$Sm(k)$ be the category of X-schemes which are smooth over the field k. Let $F : X\text{-}Sm(k)^{op} \to Ab$ be a presheaf of abelian groups which can be endowed with the structure of a homotopy invariant pretheory over X (see section 2). As usual, we extend F to pro-objects by setting

$$F(\lim_{\longleftarrow} Y_i) = \lim_{\longrightarrow} F(Y_i).$$

If F satisfies the additional property of being extensible (see Definition 2.5), we show the

Theorem 2 *Let X be a smooth scheme over a field k and let F be an extensible homotopy invariant pretheory over the k-scheme X. Then for every smooth semi-local scheme W over X and each dense open subscheme $U \subset W$, the restriction homomorphism*

$$F(W) \longrightarrow F(U)$$

is a universal monomorphism. In particular, the natural homomorphism

$$F_{Zar} \longrightarrow \bigoplus_{x \in X^0} (i_x)_* F(k(x))$$

of sheaves on X_{Zar} is injective.

If F comes by base change from a homotopy invariant pretheory over k, Theorem 2 is a result of V. Voevodsky ([Vo], Cor.4.18) and, if k is perfect, the given injection of Zariski sheaves is the first arrow of the Gersten resolution ([Vo], Theorem 4.37) for F_{Zar}. The essential difficulty in generalizing Voevodsky's result to the relative case is ([Vo] Proposition 4.9) which says that any finite set of closed points on a smooth quasiprojective variety over a field has an open neighbourhood being part of a "standard triple". This does not remain true in the relative case. The key idea of the present paper is the observation that one can overcome this difficulty if the pretheory F is "extensible". Étale cohomology is naturally equipped with the structure of a pretheory and we show that this pretheory structure is extensible. Therefore Theorem 1 follows from Theorem 2 and from the smooth-proper base change theorem.

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1 Relative Curves and Standard Triples

In the present section we recall some definitions and facts of [Vo], §2. We tacitly assume that all occurring schemes are noetherian.

To begin with, let S be a regular connected scheme. Let $p : X \to S$ be a curve, i.e. a dominant morphism of finite type such that all nonempty fibers are of dimension 1. By $c_{equi}(X/S, 0)$ we denote the free abelian group generated by the set of closed integral subschemes $Z \subset X$ such that the projection $Z \to S$

is finite and surjective. As was shown in [SV1], for any morphism of connected regular schemes $f : S' \to S$ one can define a homomorphism

$$cycl(f) : c_{equi}(X/S, 0) \to c_{equi}(X \times_S S'/S', 0)$$

If f is dominant and Z is a closed integral subscheme in the group $c_{equi}(X/S, 0)$, then

$$cycl(f)(Z) = Cycl_{X \times_S S'}(Z \times_S S'),$$

where $Cycl_{X \times_S S'}(Z \times_S S')$ is the cycle of the closed subscheme $Z \times_S S'$ in $X \times_S S'$. Let $g : X_1 \to X_2$ be a morphism of curves over S. Then the direct image homomorphism

$$g_* : c_{equi}(X_1/S, 0) \to c_{equi}(X_2/S, 0)$$

is defined by setting $g_*(Z) = n(Z, g)g(Z)$, where Z is an integral closed subscheme in X_1 which belongs to $c_{equi}(X_1/S, 0)$ and $n(Z, g)$ is the degree of the finite extension of function fields $k(Z)|k(g(Z))$. By ([SV1], 3.6.2), the homomorphisms g_* and $cycl(f)$ respect each other, i.e. for a morphism $g : X_1 \to X_2$ of curves over S and a morphism $f : S' \to S$ the diagram

$$\begin{array}{ccc} c_{equi}(X_1/S, 0) & \xrightarrow{cycl(f)} & c_{equi}(X_1 \times_S S'/S', 0) \\ \downarrow g_* & & \downarrow g_* \\ c_{equi}(X_2/S, 0) & \xrightarrow{cycl(f)} & c_{equi}(X_2 \times_S S'/S', 0) \end{array}$$

commutes. Two elements $\mathcal{Z}_0$, $\mathcal{Z}_1$ of $c_{equi}(X/S, 0)$ are equivalent (homologous) if there is an element $\mathcal{Z}$ in the group $c_{equi}(\mathbb{A}^1_X/\mathbb{A}^1_S, 0)$ such that $cycl(i_0)(\mathcal{Z}) = \mathcal{Z}_0$ and $cycl(i_1)(\mathcal{Z}) = \mathcal{Z}_1$, where i_0, $i_1 : S \to \mathbb{A}^1_S$ are the closed embeddings corresponding to the points 0 and 1, respectively. The group of equivalence classes of elements of $c_{equi}(X/S, 0)$ with respect to this equivalence relation will be denoted by $h_0(X/S)$. It follows from the definition that the homomorphisms $cycl(f)$ induce homomorphisms $h_0(X/S) \to h_0(X \times_S S'/S')$ and that the groups $h_0(X/S)$ are covariantly functorial with respect to morphisms $X_1 \to X_2$ of curves over S.

According to ([SV2], §2), a good compactification of a smooth curve X/S is a pair $(\bar{p} : \bar{X} \to S, j : X \to \bar{X})$ where $\bar{X}$ is a normal proper curve over S and j is an open embedding over S such that the closed subset $X_\infty = \bar{X} - X$ in $\bar{X}$ has an open neighbourhood which is affine over S.

Suppose that $X \to S$ is quasi-affine and that we are given a line bundle $\mathcal{L}$ on $\bar{X}$ which is trivial over an open neighbourhood U of X_∞ in $\bar{X}$. Any trivialization of $\mathcal{L}$ over U, considered as a rational section of $\mathcal{L}$ over $\bar{X}$ defines a divisor on X whose class is in $h_0(X/S)$. We will use the following result ([Vo], 2.6).

Proposition 1.1 *Let $p : X \to S$ be a smooth quasi-affine curve over a regular scheme S and let $(\bar{p} : \bar{X} \to S, j : X \to \bar{X})$ be a good compactification of X. Let X_∞ be the reduced subscheme $\bar{X} - X$, $\mathcal{L}$ a line bundle on $\bar{X}$ and $s : \mathcal{O}_{X_\infty} \to \mathcal{L}|_{X_\infty}$ a trivialization of $\mathcal{L}$ over X_∞. Then the following statements hold.*

(i) *For any two extension* $\tilde{s}_1$, $\tilde{s}_2$ *of the trivialization* s *to an open neighbourhood* U *of* X_∞ *the cycles* $Cycl_X(D(\mathcal{L}, U, \tilde{s}_1))$ *and* $Cycl_X(D(\mathcal{L}, U, \tilde{s}_2))$ *give the same element in* $h_0(X/S)$ *(here* $D(\mathcal{L}, U, \tilde{s}_i)$ *is the associated divisor).*

(ii) *If* S *is affine, there exists an affine open neighbourhood* U *of* X_∞ *in* $\bar{X}$ *and an extension* $\tilde{s} : \mathcal{O}_U \to \mathcal{L}|_U$ *of* s *to a trivialization of* $\mathcal{L}$ *on* U*.*

Following ([Vo], 4.1), we recall the notion of a standard triple.

Definition 1.2 *A standard triple* $(\bar{p} : \bar{X} \to S, X_\infty, Z)$ *over a regular scheme* S *is a proper normal curve* $\bar{p} : \bar{X} \to S$ *together with a pair of reduced closed subschemes* Z, X_∞ *in* $\bar{X}$ *such that the closed subset* $Z \cup X_\infty$ *has an open neighbourhood in* $\bar{X}$ *which is affine over* S, $Z \cap X_\infty = \varnothing$ *and the scheme* $X = \bar{X} - X_\infty$ *is quasi-affine and smooth over* S*.*

If $(\bar{p} : \bar{X} \to S, X_\infty, Z)$ is a standard triple, then $\bar{p} : \bar{X} \to S$ represents a good compactification of $X = \bar{X} - X_\infty$ and of $X - Z$. In the case of smooth schemes over a field, the existence of standard triples is provided by the following fact ([Wa], 4.13 or [Vo], 4.9).

Proposition 1.3 *Let* W *be a smooth quasi-projective variety over a field* k*. Let* $N \subset W$ *be a closed reduced subscheme in* W *such that* $N \neq W$ *and let* $\{x_1, \ldots, x_n\}$ *be a finite set of closed points of* N*. Then there exists an affine open neighbourhood* V *of* $\{x_1, \ldots, x_n\}$ *in* W *and a standard triple* $(\bar{p} : \bar{X} \to S, X_\infty, Z)$ *over a smooth affine variety* S *such that the pair* $(V, V \cap N)$ *is isomorphic to the pair* $(X = \bar{X} - X_\infty, Z)$*.*

For an open subscheme $U \subset X$ we denote by $\Delta_{X,U}$ the image of the canonical morphism $U \to X \times_S U$. Let $\pi : Y \to X \times_S U$ be a finite étale morphism having a section $\delta : \Delta_{X,U} \to Y$ over the closed subscheme $\Delta_{X,U}$. The image $\delta(\Delta_{X,U})$ is a divisor on Y (isomorphic to U) and we denote the associated line bundle by $\mathcal{L}_\delta$. The following definition generalizes ([Vo],4.4).

Definition 1.4 *A standard triple* $T = (\bar{p} : \bar{X} \to S, X_\infty, Z)$ *splits over* $(U, \pi : Y \to X \times_S U)$ *if the restriction of* $\mathcal{L}_\delta$ *to* $\pi^{-1}(Z \times_S U)$ *is trivial. A trivialization of* $\mathcal{L}_\delta|_{\pi^{-1}(Z \times_S U)}$ *is called a splitting of* T *over* $(U, \pi : Y \to X \times_S U)$*.*

Lemma 1.5 *Let* $T = (\bar{p} : \bar{X} \to S, X_\infty, Z)$ *be a standard triple which splits over* $(U, \pi : Y \to X \times_S U, \delta : \Delta_{X,U} \to Y)$*, where* U *is affine. Then there exists an element in* $h_0(\pi^{-1}((X - Z) \times_S U)/_{pr_2 \circ \pi} U)$ *whose direct image in* $h_0(Y/_{pr_2 \circ \pi} U)$ *coincides with the class of the* U*-point* $\delta_U : U \to Y$ *which is given as the composite of the canonical* U*-point* $U \to X \times_S U$ *and* $\delta : \Delta_{X,U} \to Y$*.*

Proof: Consider the standard triple $T_U = (pr_2 : \bar{X} \times_S U \to U, X_\infty \times_S U, Z \times_S U)$ which is given by base change of the standard triple T along $U \to S$. Let $\bar{Y}$ be the normalization of $\bar{X} \times_S U$ in the function field of Y. The projection $\pi : Y \to X \times_S U$ extends to a finite morphism $\bar{\pi} : \bar{Y} \to \bar{X} \times_S U$ and $pr_2 \circ \bar{\pi} : \bar{Y} \to U$ is a good compactification of $pr_2 \circ \pi : Y \to U$ and also of $pr_2 \circ \pi :$

$Y - \pi^{-1}(Z \times_S U) \to U$. The divisor $\delta(\Delta_{X,U}) \subset Y$ is closed in $\bar{Y}$ and we denote by $\mathcal{L}_\delta$ the associated line bundle. Put $Y_\infty = (\bar{Y} - Y) \cup \pi^{-1}(Z \times_S U)$. By assumption, the restriction of $\mathcal{L}_\delta$ to $\pi^{-1}(Z \times_S U)$ is trivial. Since $(\bar{Y} - Y) \cap \pi^{-1}(Z \times_S U) = \varnothing$, we can choose a trivialization of $\mathcal{L}_\delta$ over Y_∞ whose restriction to $\bar{Y} - Y$ coincides with the canonical trivialization. By proposition 1.1 (ii) applied to the compactification $\bar{Y}$ of $Y - \pi^{-1}(Z \times_S U)$, we obtain an extension of the chosen trivialization to an open neighbourhood of Y_∞ and therefore we get an element in $h_0(\pi^{-1}((X - Z) \times_S U)/_{pr_2 \circ \pi} U)$. By proposition 1.1 (i), the image of this element in $h_0(Y/_{pr_2 \circ \pi} U)$ has the required property. □

Lemma 1.6 *Let $T = (\bar{p} : \bar{X} \to S, X_\infty, Z)$ be a standard triple, $U \subset X$ a nonempty open affine subscheme and $\pi : Y \to X \times_S U$ a finite étale morphism having a splitting $\delta : \Delta_{X,U} \to Y$ over $\Delta_{X,U}$. Let $\{x_1, \dots, x_n\}$ be a finite set of closed points of U. Then there exists an affine open neighbourhood U' of $\{x_1, \dots, x_n\}$ in U such that the triple T splits over $(U', \pi : \pi^{-1}(X \times_S U') \to X \times_S U')$.*

Proof: Since Z is proper over S and has an open neighbourhood in $\bar{X}$ which is affine over S, the projection $Z \to S$ is finite. Consider the semi-local scheme $\mathcal{U} = \mathrm{Spec}\mathcal{O}_{U,\{x_1,\dots,x_n\}}$. The projection $Z \times_S \mathcal{U} \xrightarrow{pr_2} \mathcal{U}$ is finite and therefore also the composite $\pi^{-1}(Z \times_S \mathcal{U}) \xrightarrow{\pi} Z \times_S \mathcal{U} \xrightarrow{pr_2} \mathcal{U}$ is finite. Thus $\pi^{-1}(Z \times_S \mathcal{U})$ is a semi-local affine scheme. Since every line bundle on a semi-local affine scheme is trivial, the restriction of $\mathcal{L}_\delta$ to $\pi^{-1}(Z \times_S \mathcal{U})$ is trivial. Consequently we find an affine open neighbourhood U' of $\{x_1, \dots, x_n\}$ in U such that the restriction of $\mathcal{L}_\delta$ to $\pi^{-1}(Z \times_S U')$ is trivial. □

2 Extensible Pretheories

In this section we translate the definition of a homotopy invariant pretheory and some of its properties given in ([Vo], §3) to the relative case. We introduce the notion of an extensible pretheory.

From this point on we fix a connected smooth scheme B over a field k and we consider the category B-$Sm(k)$ of B-schemes which are smooth over k. Similar to ([Vo], 3.1.) we define

Definition 2.1 *A pretheory (F, ϕ) over B is the following collection of data:*

1. *A presheaf of abelian groups $F : B\text{-}Sm(k)^{op} \longrightarrow Ab$.*
2. *For any object $S \in B\text{-}Sm(k)$ and any smooth curve $p : X \to S$ a homomorphism of abelian groups*

$$\phi_{X/S} : c_{equi}(X/S, 0) \to Hom(F(X), F(S)).$$

These data should satisfy the following conditions:

1. *For any object $S \in B$-$Sm(k)$, any smooth curve $p : X \to S$ and any S-point $i : S \to X$ of X one has $\phi_{X/S}(i(S)) = F(i)$.*

2. *Let $f : S_1 \to S_2$ be a morphism in the category B-$Sm(k)$ and let $p : X_2 \to S_2$ be a smooth curve over S_2. Consider the Cartesian square*

$$\begin{array}{ccc} X_1 & \xrightarrow{g} & X_2 \\ \downarrow & & \downarrow p \\ S_1 & \xrightarrow{f} & S_2. \end{array}$$

   Then for any $\mathcal{Z}$ in $c_{equi}(X_2/S_2, 0)$ one has

$$F(f) \circ \phi_{X_2/S_2}(\mathcal{Z}) = \phi_{X_1/S_1}(cycl(f)(\mathcal{Z})) \circ F(g).$$

3. *For any pair X, Y of objects in B-$Sm(k)$ the canonical morphism*

$$F(X \amalg Y) \to F(X) \oplus F(Y)$$

   is an isomorphism.

Definition 2.2 *A pretheory (F, ϕ) be over B is called homotopy invariant if for any object X of B-$Sm(k)$ the projection $\mathbb{A}^1_X \to X$ induces an isomorphism $F(X) \to F(\mathbb{A}^1_X)$.*

The following properties of homotopy invariant pretheories ([Vo], 3.11, 3.12) remain true with the same proofs.

Proposition 2.3 *A pretheory (F, ϕ) is homotopy invariant if and only if for any object $S \in B$-$Sm(k)$ and any smooth curve $X \to S$ the morphism*

$$\phi_{X/S} : c_{equi}(X/S, 0) \to Hom(F(X), F(S))$$

factors through the natural projection $c_{equi}(X/S, 0) \to h_0(X/S)$.

Proposition 2.4 *Let (F, ϕ) be a homotopy invariant pretheory over B, S an object of B-$Sm(k)$ and $j : U \to X$ an open embedding of smooth curves over S. Then for any element $a \in c_{equi}(U/S, 0)$ one has*

$$\phi_{X/S}(j_*(a)) = \phi_{U/S}(a) \circ F(j).$$

Now we introduce the notion of an extensible pretheory. Let $X \to B$ be smooth and let (F, ϕ) be a pretheory over X. Consider the two projections pr_1, pr_2 : $X \times_B X \to X$. The pull-back by means of pr_1 and pr_2 gives two different presheaves $pr_1^* F$, $pr_2^* F$ on the category $(X \times_B X)$-$Sm(k)$.

Definition 2.5 *We say that a pretheory (F, ϕ) over X is an extensible pretheory over the B-scheme X if for any finite set $\{x_1, \ldots, x_n\}$ of closed points of X there exist an open neighbourhood $U \overset{i}{\hookrightarrow} X$ of $\{x_1, \ldots, x_n\}$ in X and*

1. *A connected scheme Y together with a finite étale morphism $\pi : Y \to X \times_B U$ having a section $\delta : \Delta_{X,U} \to Y$ over $\Delta_{X,U}$.*

2. *A homomorphism of presheaves on Y-$Sm(k)$*

$$\Psi : p_1^* F \to p_2^* F,$$

where $p_1 = pr_1 \circ \pi$ and $p_2 = i \circ pr_2 \circ \pi$, such that the restriction of Ψ to the U-point $\delta_U : U \xrightarrow{can} \Delta_{X,U} \xrightarrow{\delta} Y$

$$\Psi_{\delta_U} : \delta_U^*(p_1^* F) \longrightarrow \delta_U^*(p_2^* F)$$

is the canonical isomorphism $id : i^* F \xrightarrow{\sim} i^* F$ over U.

If (F, ϕ) comes by base change from a pretheory over B, then the presheaves $p_1^* F$ and $p_2^* F$ on the category $(X \times_B X)$-$Sm(k)$ are canonically isomorphic, and hence (F, ϕ) is extensible over B. More general, we have the following lemma whose proof is straightforward.

Lemma 2.6 *Assume we are given a commutative diagram of smooth morphisms*

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ B' & \longrightarrow & B. \end{array}$$

If (F, ϕ) is an extensible pretheory over the B-scheme X, then the presheaf $f^* F$ has a canonical structure of an extensible pretheory over the B'-scheme X'.

3 Proof of Theorem 2

Theorem 2 of the introduction is a straightforward consequence of the following

Theorem 3.1 *Let W be a smooth quasi-projective variety over a field k. Let (F, ϕ) be an extensible homotopy invariant pretheory over the k-scheme W. Let $\{x_1, \ldots, x_n\}$ be a finite set of points of W. Then for any nonempty open subset V of W there exists an open neighbourhood U of $\{x_1, \ldots, x_n\}$ and a homomorphism $F(V) \to F(U)$ such that the following diagram commutes:*

$$\begin{array}{ccc} F(W) & \longrightarrow & F(V) \\ \downarrow & \swarrow & \\ F(U). & & \end{array}$$

Proof: Let N be the reduced closed subscheme $W - V$ of W. By Proposition 1.3 there exists an open affine neighbourhood W' of $\{x_1, \ldots, x_n\}$ and a standard triple $T = (\bar{p} : \bar{X} \to S, X_\infty, Z)$ over a smooth affine variety S such that the pair $(W', W' \cap N)$ is isomorphic to the pair $(X = \bar{X} - X_\infty, Z)$. Note that $X - Z = W' \cap V$. Applying Lemma 2.6 to the diagram

$$\begin{array}{ccc} X & \xrightarrow{\text{incl}} & W \\ \downarrow & & \downarrow \\ S & \longrightarrow & \mathrm{Spec}(k), \end{array}$$

the pull-back $(\text{incl})^* F$ has a canonical structure of an extensible pretheory over the S-scheme X. We denote it again by (F, ϕ).
By Definition 2.5, there exist an affine open neighbourhood U of $\{x_1, \ldots, x_n\}$ in X and a finite étale morphism $\pi : Y \to X \times_S U$ together with a section $\delta : \Delta_{X,U} \to Y$ over $\Delta_{X,U}$ and a homomorphism of presheaves $\Psi : p_1^* F \to p_2^* F$ over Y such that the restriction of Ψ to the U-point $\delta_U : U \xrightarrow{can} \Delta_{X,U} \xrightarrow{\delta} Y$

$$\Psi_{\delta_U} : \delta_U^*(p_1^* F) \longrightarrow \delta_U^*(p_2^* F)$$

is the identity $id : i^* F \xrightarrow{\sim} i^* F$ over U. By Lemma 1.6 we may assume (making U smaller, if necessary) that the standard triple $T = (\bar{p} : \bar{X} \to S, X_\infty, S)$ splits over $(U, \pi : Y \to X \times_S U,\ \delta : \Delta_{X,U} \to Y)$.

Setting $Z' = \pi^{-1}(Z \times_S U)$, Lemma 1.5 yields an element τ in the group $h_0(Y - Z'/_{pr_2 \circ \pi} U)$ whose image in $h_0(Y/_{pr_2 \circ \pi} U)$ coincides with the class of the U-point δ_U. Since (F, ϕ) is homotopy invariant, τ defines (cf. Proposition 2.3) a homomorphism

$$\phi(\tau) : F(Y - Z'/_{p_2 \circ j'} X) \to F(U),$$

where the structure of an X-scheme on $Y - Z'$ is given by the composite

$$Y - Z' \overset{j'}{\hookrightarrow} Y \xrightarrow{p_2} X.$$

Proposition 2.4 applied to the open embedding $j' : (Y - Z') \to Y$, yields the commutative diagram

$$\begin{array}{ccc} F(Y/_{p_2} X) & \xrightarrow{F(j')} & F(Y - Z'/_{p_2 \circ j'} X) \\ & \searrow^{F(\delta_U) = \phi(j'_*(\tau))} & \downarrow \phi(\tau) \\ & & F(U). \end{array}$$

We therefore obtain the commutative diagram

$$\begin{array}{ccc}
F(W) & \stackrel{res}{\longrightarrow} & F(V) \\
\downarrow{\scriptstyle res} & & \downarrow{\scriptstyle res} \\
F(X) & \stackrel{res}{\longrightarrow} & F(X-Z) \\
\downarrow{\scriptstyle pr_1^*} & & \downarrow{\scriptstyle pr_1*} \\
F(X \times_S U/_{pr_1} X) & \stackrel{res}{\longrightarrow} & F((X-Z)\times_S U/_{pr_1} X) \\
\downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle \pi^*} \\
p_1^* F(Y) & \stackrel{res}{\longrightarrow} & p_1^* F(Y-Z') \\
\downarrow{\scriptstyle \Psi} & & \downarrow{\scriptstyle \Psi} \\
p_2^* F(Y) & \stackrel{res}{\longrightarrow} & p_2^* F(Y-Z') \\
\| & & \| \\
F(Y/_{p_2} X) & \stackrel{res}{\longrightarrow} & F(Y-Z'/_{p_2\circ j'} X) \\
\downarrow{\scriptstyle \delta_U^*} & & \downarrow{\scriptstyle \phi(\tau)} \\
F(U) & \stackrel{id}{\longrightarrow} & F(U).
\end{array}$$

By the defining property of Ψ, the composite of all vertical arrows in the left hand column coincides with $res : F(W) \to F(U)$. We obtain the required homomorphism $F(V) \to F(U)$ as the composite of all vertical arrows in the right hand column □

4 Application to Étale Cohomology

Let X be a smooth scheme over a field k, n an integer prime to char(k) and $\mathcal{K}^\bullet$ a bounded complex of locally constant constructible sheaves of $\mathbb{Z}/n\mathbb{Z}$-modules on X_{et}. For an integer q we consider the presheaf F on X-$Sm(k)$ which is given by

$$F(Y \xrightarrow{f} X) = H^q_{et}(Y, f^*\mathcal{K}^\bullet),$$

where the object on the right hand side is étale hypercohomology. We will show that F carries in a natural way the structure of an extensible homotopy invariant pretheory over X.

First of all, F is homotopy invariant (see, e.g., [Mi] VI, 4.20). Next we will construct natural trace maps which will endow F with the structure of a pretheory over X. This is well known, but knowing no good reference, we give a construction of the trace maps below:

Let $g : Y \to S$ be a smooth curve in X-$Sm(k)$. Since the base scheme X does not play any further role, we denote the pull-back of $\mathcal{K}^\bullet$ to S by the same letter. Using the isomorphism $Rg^!\mathcal{K}^\bullet \cong g^*\mathcal{K}^\bullet(1)[2]$ ([SGA4], XVIII, thm.3.2.5), we obtain

$$\mathrm{Hom}(g^*\mathcal{K}^\bullet, g^*\mathcal{K}^\bullet) \quad \cong \quad \mathrm{Hom}(Rg_!g^*\mathcal{K}^\bullet(1)[2], \mathcal{K}^\bullet),$$

and thus the identity of $g^*\mathcal{K}^\bullet$ induces a natural homomorphism

$$\mathrm{Tr}_g : H^{q+2}_{c,et}(Y, g^*\mathcal{K}^\bullet(1)) \longrightarrow H^q_{et}(S, \mathcal{K}^\bullet).$$

Here cohomology with compact support of Y is meant with respect to its S-scheme structure (and not as a variety over k). Let $i : Z \hookrightarrow Y$ be an integral subscheme of codimension one in Y such that $f = g \circ i : Z \to S$ is quasifinite and let $cl(Z) \in H^2_Z(Y, \mathbb{Z}/n\mathbb{Z}(1))$ be its fundamental class (see [SGA4.5], (`cycle` 2.3)). The composition of the cup product ([SGA4.5], (`cycle` 1.2.2.2))

$$cl(Z) \cup ? : \ H^q_{c,et}(Z, f^*\mathcal{K}^\bullet) \to H^{q+2}_{c,et}(Y, g^*\mathcal{K}^\bullet(1))$$

with Tr_g induces trace maps

$$\mathrm{Tr}_f : H^q_{c,et}(Z, f^*\mathcal{K}^\bullet) \longrightarrow H^q_{et}(S, \mathcal{K}^\bullet).$$

Finally, if $f : Z \to S$ is finite and surjective, then we get the required trace map $\phi_{Y/S}(Z) : F(Y) \to F(S)$ as the composite map

$$H^q_{et}(Y, g^*\mathcal{K}^\bullet) \xrightarrow{i^*} H^q_{et}(Z, f^*\mathcal{K}^\bullet) \xrightarrow{\mathrm{Tr}_f} H^q_{et}(S, \mathcal{K}^\bullet).$$

All necessary compatibilities follow from the properties of the fundamental class $cl(Z)$ proven in [SGA4.5], (`cycle`, 2.3).

Remark: A more “advanced” way to construct the trace maps would consist of the following steps
1. Since $\mathcal{K}^i$ is represented by an étale X-scheme for all i, we can extend $\mathcal{K}^\bullet$ to a complex of *qfh*-sheaves ([SV2], appendix) on the category X-*Nor*(k) of X-schemes which are normal k-schemes of finite type.
2. Since *qfh*-sheaves admit transfer maps ([SV2],§5), the presheaf of abelian groups $G(Y) = H^q_{qfh}(Y, \mathcal{K}^\bullet)$ on X-*Nor*(k) admits natural transfer maps.
3. The isomorphism $H^q_{qfh}(Y, \mathcal{K}^\bullet) \cong H^q_{et}(Y, \mathcal{K}^\bullet)$ ([SV2], thm.10.3) shows that the restriction of G to X-$Sm(k)$ coincides with F.

Finally we have to show that F is extensible over k. Let $\tilde{X}$ be a finite Galois covering with Galois group G such that the pull-back of $\mathcal{K}^\bullet$ to $\tilde{X}$ is a complex of constant sheaves. Consider the étale covering $\tilde{X} \times_k \tilde{X} \to X \times_k X$ with Galois group $G \times G$. Let $\widetilde{X \times_k X} = (\tilde{X} \times_k \tilde{X})_G$ be the unique intermediate covering associated with the diagonal subgroup $G = (g, g) \in G \times G$. The diagonal map $\tilde{X} \to \tilde{X} \times_k \tilde{X}$ induces a map $\delta : X \to \widetilde{X \times_k X}$ which is a section to the projection $\widetilde{X \times_k X} \to X \times_k X$ over the diagonal $X \cong \Delta_X \subset X \times_k X$. Let Y be the connected component of $X = \mathrm{im}(\delta)$ in $\widetilde{X \times_k X}$. Then Y is a connected Galois covering of the connected component of the diagonal $\Delta_X \subset X \times_k X$ having a section over Δ_X. The projections pr_1 and pr_2 on the first and second factor induce two structures of an X-scheme on Y and we use the notation Y_1 and Y_2 for Y in order to indicate the X-scheme structure on Y.

We will now extend the identity $id : \delta^* pr_1^* F \xrightarrow{\sim} \delta^* pr_2^* F$ over $X = \mathrm{im}(\delta) \subset Y$ to the full scheme Y, thus verifying the condition of definition 2.5 with $U = X$.

By definition, for $i = 1, 2$, $pr_i^* F$ is the presheaf on Y-$Sm(k)$ given by $(f : U \to Y) \mapsto H^q_{et}(U, (pr_i \circ f)^* \mathcal{K}^\bullet)$. It therefore suffices to construct compatible isomorphisms of sheaves on Y-$Sm(k)_{et}$

$$(pr_1 \circ f)^* \mathcal{K}^i \xrightarrow{\sim} (pr_2 \circ f)^* \mathcal{K}^i.$$

for all i. Let K^i be the finite étale X-scheme representing $\mathcal{K}^i$ on X_{et}. Then we have to construct a natural isomorphism

$$Y_1 \times_X K^i \xrightarrow{\sim} Y_2 \times_X K^i.$$

Since $\mathcal{K}^i$ becomes constant over $\tilde{X}$, we are reduced to show that there exists a natural G-invariant isomorphism of Y-schemes

$$Y_1 \times_X \tilde{X} \xrightarrow{\sim} Y_2 \times_X \tilde{X},$$

where G acts from the right on the second factors. We obtain this by restricting a natural G-invariant isomorphism of $\widetilde{X \times_k X}$

$$\Psi : (\widetilde{X \times_k X})_1 \times_X \tilde{X} \xrightarrow{\sim} (\widetilde{X \times_k X})_2 \times_X \tilde{X}.$$

to be constructed below to Y. Let us give the isomorphism Ψ on (geometric) points. A point on $(\widetilde{X \times_k X})_1 \times_X \tilde{X}$ is a pair $((a, b)G, c)$ where $(a, b)G$ is a G-orbit (diagonal action) of points in $\tilde{X} \times_k \tilde{X}$ and c is a point on $\tilde{X}$ such that a and c project to the same point in X. Let $g \in G$ be the unique element with $c = ag$. We define $\Psi(((a, b)G, c))$ as the pair $((a, b)G, bg)$, which is a point in $(\widetilde{X \times_k X})_2 \times_X \tilde{X}$. If $(a', b')G = (a, b)G$, then there exists an element $h \in G$ with $ah = a'$, $bh = b'$ and $c = a'h^{-1}g$, $bg = bhh^{-1}g = b'h^{-1}g$. Therefore Ψ is correctly defined. If one wants to obtain Ψ in a more formal way, one can give it as a G-invariant map of the $(G \times G) \times G$-sets associated with the schemes in question. Finally note that the diagram

$$\begin{array}{ccc} \mathrm{im}(\delta)_1 \times_X \tilde{X} & \xrightarrow{\Psi} & \mathrm{im}(\delta)_2 \times \tilde{X} \\ \uparrow \wr & & \uparrow \wr \\ X \times_X \tilde{X} & \xrightarrow{id} & X \times_X \tilde{X} \end{array}$$

commutes. Therefore Ψ respects the connected component of $\mathrm{im}(\delta)$ and induces an isomorphism

$$Y_1 \times_X \tilde{X} \xrightarrow{\sim} Y_2 \times_X \tilde{X}.$$

This shows that F carries in a natural way the structure of an extensible homotopy invariant pretheory over the k-scheme X.

Now it is easy to prove Theorem 1 of the introduction. Since étale cohomology commutes with inductive limits, we may suppose that the cohomology of $\mathcal{K}^\bullet$ is also bounded from above. Denoting the projection by $\pi : X \to W$, the complex $R\pi_*\mathcal{K}^\bullet$ has locally constant constructible cohomology sheaves bounded in both directions (see, e.g., [Mi], VI.4.2). By the main result of [PS], a bounded complex of sheaves on W_{et} with locally constant constructible cohomology sheaves is in the derived category isomorphic to a bounded complex of locally constant constructible sheaves. Therefore the assignment

$$(U \xrightarrow{f} W) \quad \longmapsto \quad H^q_{et}(U, f^* R\pi_* \mathcal{K}^\bullet),$$

defines an extensible homotopy invariant pretheory over the smooth semi-local k-scheme W. Applying Theorem 2, we obtain Theorem 1. □

Finally, we mention the following variant of Theorem 1, which can be deduced by a straightforward limit argument.

Theorem 4.1 *Let W be the spectrum of the henselization of the local ring of a closed point on a smooth k-scheme and let $\eta \in W$ be the generic point. Let $X \to W$ be a proper smooth morphism, n an integer prime to char(k) and $\mathcal{K}^\bullet$ a complex of sheaves of $\mathbb{Z}/n\mathbb{Z}$-modules on X_{et} whose cohomology sheaves are locally constant constructible and bounded below. Then the canonical map*

$$H^q_{et}(X, \mathcal{K}^\bullet) \longrightarrow H^q_{et}(X_\eta, \mathcal{K}^\bullet)$$

is a universal monomorphism for all $q \in \mathbb{Z}$.

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