Tensor Categories and Representation Theory

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Seminar at the HU Berlin, Summer 2017

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Date and Venue:	Friday, 15-17 h, Room 1.115 / RUD 25 At some Fridays we may start earlier and have a double session from 13-17 h, to make up for cancellations due to BMS talks on other Fridays
Prerequisites:	Basic knowledge about groups, rings and modules
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Many problems in algebra, geometry, topology and physics lead to so-called[1] tensor categories: Categories with a product similar to the tensor product of vector spaces. Some simple examples are

- $\mathscr{C} = \mathrm{Vec}_k$: Vector spaces over a field k,
- $\mathscr{C} = \mathbb{Z} - \mathrm{Vec}_k$: Graded vector spaces $V = \oplus_{i\in\mathbb{Z}} V_i$ over k,
- $\mathscr{C} = \mathrm{MHS}_k$: Mixed k-Hodge structures over $k = \mathbb{Q}$ or $k = \mathbb{R}$,
- $\mathscr{C} = \mathrm{Loc}_k(X)$: Locally constant k-sheaves on a topological space X,
- $\mathscr{C} = \mathrm{Rep}_k(G)$: Finite-dimensional k-representations of a group G, algebraic representations of an algebraic group, or continuous representations of a topological group, etc.

Tensor categories, and more generally the so-called monoidal categories, provide a common framework for the study of Galois groups, Hodge theory, fundamental groups and motives. We will begin the seminar with a brief introduction to monoidal categories and then specialize to the case of tensor categories. As a fundamental example we then consider the category of representations of an algebraic group. We will see that the group is essentially determined by this category, which naturally leads to the question how to see whether a given tensor category $\mathscr{C}$ is equivalent to a category of representations. The crucial ingredient here is the existence of a fiber functor $\omega : \mathscr{C} \longrightarrow \mathrm{Vec}_k$ which sends a representation to the underlying vector space. This leads to the abstract notion of a Tannakian category; at the end of the seminar we will discuss a general criterion of Deligne which gives an intrinsic characterization of such categories among all tensor categories. If time permits, we may conclude with some applications from algebraic geometry or Hodge theory.

[1]The notion of a tensor category is not used consistently in the literature. For us it will mean a rigid symmetric monoidal abelian category with $\mathrm{End}(\mathbf{1}) \simeq k$ as in [4], see section 2.2 below.

1. Basic notions of category theory

The seminar does not assume any previous knowledge of category theory, the relevant notions can be developped in a first talk if needed:

1.1. **Categories and functors.** This talk consists of two parts: The first part should introduce the notion of categories, functors, natural transformations and equivalences as in [8, §1]; one might also consult [2, §§1,7,8] but this is much less concise. As a matter of convention, we only consider covariant functors and view contravariant functors to a category $\mathscr{C}$ as functors to the opposite category $\mathscr{C}^{op}$. Some easy examples should fill the abstract concepts with life. Representable functors and the Yoneda lemma should be explained. The second part of the talk should introduce in some detail the notions of additive and k-linear categories and functors [7, §1.2], abelian categories and exact functors [7, §§1.3 – 1.3.5, §1.6.1]. A good reference for this is [10, chapt. VIII]. Valentin Steinforth, 5.+12. Mai

2. Tensor categories

The next three talks will give a general introduction to the language of tensor categories, embedded in the larger context of monoidal categories (= categories with an associative product $\otimes$ and a unit object $\mathbf{1}$):

2.1. **Monoidal categories.** This talk introduces monoidal categories in the sense of [7, def. 2.2.8]; the simpler definition 2.1.1 and its equivalence with 2.2.8 can be omitted. As examples one should cover [7, §§2.3.1 - 2.3.4] and then introduce monoidal functors [7, §§2.4 - 2.5.2]. We then proceed to MacLane's coherence theorem that clarifies the role of the isomorphisms a, l, r in a monoidal category; while these cannot be ignored naïvely by choosing one object in every isomorphism class [10, §VII.1], we will see that any monoidal category is equivalent to one which is strict in the sense that a, l, r are the identity [7, §2.8]. As a result any diagram formed by compositions, inverses and $\otimes$ of a, l, r is commutative [7, §2.9]. Filip Gärber, 19 May

2.2. **Rigid and symmetric monoidal categories.** This talk adds two further notions to a monoidal category: The property of rigidity, which should be explained in detail [7, §2.10], and the datum of symmetry isomorphisms [7, Def. 8.1.12]. Not every symmetric monoidal category is equivalent to a strict one, since usually the symmetry isomorphism $c : A \otimes A \longrightarrow A \otimes A$ is not the identity. But there is still a coherence theorem for symmetric monoidal categories which may be quoted without proof from [11, th. 5.1]. After some examples [7, §8.2], one should introduce the notion of a tensor category as a rigid symmetric monoidal abelian category with $\mathrm{End}(\mathbf{1}) \simeq k$, and observe that such categories are automatically k-linear and their tensor product $\otimes$ is exact and k-linear in each variable [4, §§2.1 - 2.5]. Yingying Wang, 26 May

if needed, extra slot to replace 2 June

3. Affine group schemes and representations

In the next talks we will introduce the notion of an affine group scheme and its representations. We will see that the category of these representations, together with its fiber functor, uniquely determines the group scheme. This motivates the abstract notion of a neutral Tannakian category, and we will show the main result of Tannaka duality that any such is equivalent to a category of representations.

3.1. **Affine group schemes.** After a short motivating reminder of the notion of affine varieties over a field [9, §I.1], we define the category of affine schemes over a commutative ring R to be the opposite of the category of commutative R-algebras. An affine group scheme over R is then defined as a group object in this category; one should mention that by Yoneda this is the same as a representable functor from commutative R-algebras to groups, and explain the description in terms of Hopf algebras [16, §6.1] [17]. As examples one should consider $\mathbb{G}_m$, $\mathbb{G}_a$, Gl_n, Sl_n and μ_n. Finally one should introduce homomorphisms and representations of affine group schemes over a field k and describe representations via comodules [6, prop. 2.2]. Uwe Möhlenbruch, 9 June

? 16 June

3.2. **Reconstruction of a group from its representation category.** Here one should first note that any comodule is a union of sub-comodules of finite dimension, and give some corollaries as in [6, §§2.3 – 2.7]. The main goal of the talk then is to reconstruct an affine group scheme from its category of representations with its fiber functor [6, prop. 2.8, cor. 2.9]. For the proof note that [3, prop. 3.1(b)] remains valid over a field of positive characteristic.

? 23 June

3.3. **Neutral Tannakian categories.** Motivated by the above, we now define the abstract notion of a neutral Tannakian category [6, def. 2.19]. Such categories are ubiquitous in mathematics, and the main goal of the talk is the proof of Tannaka duality: Any such category is equivalent to the category of representations of an affine group scheme [6, th. 2.11].

4. An intrinsic criterion by Deligne

How do you see whether a given tensor category admits a fiber functor and hence is equivalent to the representation category of an affine group scheme? For the rest of the seminar we assume $char(k) = 0$. The following talks discuss a simple intrinsic criterion for the existence of fiber functors due to Deligne, provided k is algebraically closed or we allow the passage to a finite extension field $K \supset k$.

Yingying Wang ?

4.1. **Deligne's criterion I.** For applications the notion of a neutral Tannakian category is sometimes too strong, like for motives over $\mathbb{F}_{q^2}$ [1, rem. 6.2.6.1]. So we define a – not necessarily neutral – Tannakian category to be a tensor category which has a k-linear exact faithful tensor functor to finite dimensional vector spaces over some extension field $K \supset k$, see [4, §2.8]. One should note without proof that for finitely generated tensor categories one can always take K/k finite. The goal of the talk is then to explain the formulation of Deligne's criterion [13, th. 3.1]; for this one should first introduce the dimension, symmetric and exterior powers of objects in a symmetric monoidal category as in [13, §1.5].

Fiorela Rossi ?

4.2. **Deligne's criterion II.** We now prove Deligne's criterion: A tensor category admits a fiber functor over some unspecified extension field $K \supset k$ iff the dimension of every non-zero object in it is a natural number, which holds iff every object is annihilated by a sufficiently high exterior power. For this we need to introduce algebras and modules in a symmetric monoidal category and study the splitting of objects and morphisms [13, §§1-2]. One should remark however that not every tensor category fulfils Deligne's criterion — see talks 5.1 & 5.2.

5. Further topics

Depending on the time and motivation of the participants, the seminar may conclude with a variety of topics from representation theory, algebraic geometry or Hodge theory such as the following:

5.1. **Super representations.** An interesting generalization of the above notions is obtained if one replaces vector spaces and algebras by their $\mathbb{Z}/2\mathbb{Z}$-graded cousins, which leads to the notion of affine super groups and super representations. Note that for graded objects, the symmetry constraint $c : A \otimes B \longrightarrow B \otimes A$ involves an extra sign given by the Koszul rule; as a consequence, the dimension of objects may become negative. This talk should explain the basic definitions and some examples of super groups and then formulate – presumably without proof – the criterion by Deligne in the super context [12] [5]. Yoshua Kesting ?

5.2. **Representations of non-integral dimension.** Whereas the dimension of the objects in the previous talk was at least an integer, this talk will construct tensor categories with objects whose dimension is any complex number: Deligne's categories $\mathrm{Rep}(\mathfrak{S}_t)$ or $\mathrm{Rep}(Gl_t)$ for $t \notin \mathbb{N}$, see [7, §9.12] and the references which are given therein. In particular, these categories are not equivalent to the category of super representations of any affine super group! André Knispel ?

5.3. **Mumford-Tate groups.** The Tannakian formalism allows to control Hodge structures in terms of representation theory; the arising algebraic groups are known as Mumford-Tate groups and play an important role in algebraic geometry. This talk could begin with a general introduction to Hodge structures and their geometric applications, and then explain the link to Tannaka duality as in [14, §3.4].

5.4. **Differential Galois theory.** Another nice application of Tannaka duality lies in the Galois theory of differential equations. This talk could give an elementary introduction to this classical subject following [16, §6.6].

References

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