Some consequences of Wiesend's higher dimensional class field theory

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G. Wiesend [W1] established a class field theory for arithmetic schemes, solely based on data attached to closed points and curves on the given scheme. Our goal is to deduce from his result the relation between the integral singular homology in degree zero and the abelianized tame fundamental group of a regular, connected scheme of finite type over $\mathrm{Spec}(\mathbb{Z})$.

1 Singular homology of schemes

In [S1] we extended the definition of Suslin's singular homology groups of varieties, see [SV], to schemes of finite type over a regular, connected and excellent base scheme S. We start by recalling this definition in the case $S = \mathrm{Spec}(\mathbb{Z})$. See [S1] for more details and motivation.

Let $\mathrm{Sch}(\mathbb{Z})$ denote the category of separated schemes of finite type over $\mathrm{Spec}(\mathbb{Z})$. Let $\Delta^\bullet$ be the standard cosimplicial object in $\mathrm{Sch}(\mathbb{Z})$, i.e., Δ^n is given as a subscheme in $\mathbb{A}^{n+1}_{\mathbb{Z}} = \mathrm{Spec}(\mathbb{Z}[T_0, \ldots, T_n])$ by the equation $\Sigma\, T_i = 1$, and the simplicial structure is given by the usual face and degeneracy morphisms. For a scheme $X \in \mathrm{Sch}(\mathbb{Z})$ and an integer $n \geq 0$ we put

$C_n(X)$ = free abelian group on closed integral subschemes $Z \subset X \times \Delta^n$ such that the restriction of the projection $X \times \Delta^n \to \Delta^n$ to Z induces a finite morphism $Z \to T \subset \Delta^n$ onto a closed integral subscheme T of codimension 1 in Δ^n which intersects all faces $\Delta^m \subset \Delta^n$ properly.

If Z is as above, then for each face map $\delta^i\colon \Delta^{n-1} \to \Delta^n$, $i = 0, \ldots, n$, each component of $(\delta^i)^{-1}(Z) \subset X \times \Delta^{n-1}$ is finite and surjective over an integral subscheme of codimension 1 in Δ^{n-1} which intersects all faces properly. Therefore the cycle theoretic inverse image $(\delta^i)^*(Z)$ is well-defined and lies in $C_{n-1}(X)$. This yields face operators $\partial_i = (\delta^i)^*\colon C_n(X) \to C_{n-1}(X)$. The homology groups of the complex

$$\big(C_\bullet(X), d\big), \qquad d = \sum (-1)^i \partial_i$$

are called the (integral) **singular homology groups** of X and will be denoted by $H_*^{sing}(X, \mathbb{Z})$. Singular homology is covariantly functorial in the scheme X. If $X \in \mathrm{Sch}(\mathbb{Z})$ is a variety over a finite field, then the singular homology groups defined above coincide with those defined by Suslin [SV].

By definition, $C_0(X)$ is the group $Z_0(X)$ of zero-cycles on X, i.e., the free abelian group on the set of closed points of X. Furthermore, $H_0^{sing}(X,\mathbb{Z})$ is the quotient of $C_0(X)$ by the subgroup $d_1(C_1(X))$. This imposes an equivalence relation on the group of zero-cycles which is in general finer than rational equivalence.

We call integral schemes of dimension one in $\mathrm{Sch}(\mathbb{Z})$ *curves*. Let $C \in \mathrm{Sch}(\mathbb{Z})$ be a curve. Then to each rational function $f \neq 0$ on C we can attach the zero-cycle $div(f) \in Z_0(C)$ (see [Fu], Ch.I,1.2). Let $\tilde{C}$ be the normalization of C in its function field and let $P(\tilde{C})$ be the regular compactification of $\tilde{C}$, i.e., $P(\tilde{C})$ is the uniquely determined regular curve which is proper over $\mathrm{Spec}(\mathbb{Z})$ and contains $\tilde{C}$ as an open subscheme. With this terminology, elements in the function field $k(C)$ of a curve C are in 1–1 correspondence to morphisms $P(\tilde{C}) \to \mathbb{P}^1_{\mathbb{Z}}$ which are not $\equiv \infty$. The following result of [S1] explicitly describes the singular homology groups in degree zero.

Theorem 1.1. *The group $H_0^{sing}(X,\mathbb{Z})$ is the quotient of $Z_0(X)$ by the subgroup generated by elements of the form $div(f)$, where*

- *$C \subset X$ is a closed curve and*
- *f is a rational function on C which, considered as a morphism $P(\tilde{C}) \to \mathbb{P}^1_{\mathbb{Z}}$, is $\equiv 1$ on $P(\tilde{C}) \smallsetminus \tilde{C}$.*

Proof. See [S1], Theorem 5.1. □

Corollary 1.2. *All relations on $C_0(X) = Z_0(X)$ defining $H_0^{sing}(X,\mathbb{Z})$ come from curves. More precisely, the subgroup $d_1(C_1(X)) \subset C_0(X)$ is generated by elements of the form $f_*(x)$, where $C \in \mathrm{Sch}(\mathbb{Z})$ is a regular curve, $f: C \to X$ is a finite morphism and $x \in d_1(C_1(C))$.*

2 The reciprocity homomorphism

Assume that $X \in \mathrm{Sch}(\mathbb{Z})$ is regular and connected. Then either X is a smooth variety over a finite field or the structural morphism $X \to \mathrm{Spec}(\mathbb{Z})$ is flat. We will refer to these cases as the geometric and the arithmetic one, respectively. Besides the abelianized étale fundamental group $\pi_1^{et}(X)^{ab}$, we consider the group $\pi_1^t(X)^{ab}$, which classifies finite, abelian, étale coverings of X with at most tame ramification along the boundary of a compactification, see [S2], [W2]. We also consider the modified group $\tilde{\pi}_1^t(X)^{ab}$, the quotient of $\pi_1^t(X)^{ab}$ which classifies those coverings in which, in addition, every $\mathbb{R}$-valued point splits completely (this gives nothing new in the geometric case, as well in the arithmetic case if $X(\mathbb{R}) = \varnothing$). For each closed point $x \in X$, the field $k(x)$ is finite. Therefore the étale fundamental group $\pi_1^{et}(\{x\})$ is isomorphic to $\hat{\mathbb{Z}}$ with the Frobenius automorphism *Frob* as a canonical generator. We denote by $Frob_x$ the image of *Frob* under the natural homomorphisms $\pi_1^{et}(\{x\}) \to \pi_1^{et}(X)^{ab}$, and we consider the homomorphism

$$r_X\colon Z_0(X) \longrightarrow \pi_1^{et}(X)^{ab}, \quad x \longmapsto Frob_x.$$

By the density theorem of Čebotarev-Lang [La], the image of r_X is a dense subgroup of the profinite group $\pi_1^{et}(X)^{ab}$.

Proposition 2.1. *Let $X \in \mathrm{Sch}(\mathbb{Z})$ be regular and connected. Then the composite map*

$$Z_0(X) \xrightarrow{r_X} \pi_1^{et}(X)^{ab} \xrightarrow{p_X} \tilde{\pi}_1^t(X)^{ab},$$

where $p_X : \pi_1^{et}(X)^{ab} \to \tilde{\pi}_1^t(X)^{ab}$ denotes the canonical projection, factors through $H_0^{sing}(X, \mathbb{Z})$, thus defining a reciprocity homomorphism

$$rec_X : H_0^{sing}(X, \mathbb{Z}) \longrightarrow \tilde{\pi}_1^t(X)^{ab}.$$

Proof. The case $\dim X = 0$ (i.e., X is the spectrum of a finite field) is trivial. Let us consider the case $\dim X = 1$ first. Then the function field $K = k(X)$ is a global field. By Theorem 1.1, we obtain a natural isomorphism between $H_0^{sing}(X, \mathbb{Z})$ and the ray class group $C_{\mathfrak{m}}(K)$, where $\mathfrak{m}$ is the (square-free) modulus obtained by multiplying all places of K with center outside X (including the archimedean ones). The statement of the proposition follows easily from classical (one-dimensional) global class field theory.

Now we come to the general case. By Corollary 1.2, it suffices to show that for any finite morphism $f : C \to X$ from a regular curve C to X and for any $x \in d_1(C_1(C))$, we have $p_X \circ r_X(f_*(x)) = 0$. This follows from the corresponding result in dimension 1 and from the commutative diagram

$$\begin{array}{ccccccc} C_1(C) & \xrightarrow{d_1} & C_0(C) & \xrightarrow{r_C} & \pi_1^{et}(C)^{ab} & \xrightarrow{p_C} & \tilde{\pi}_1^t(C)^{ab} \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\ C_1(X) & \xrightarrow{d_1} & C_0(X) & \xrightarrow{r_X} & \pi_1^{et}(X)^{ab} & \xrightarrow{p_X} & \tilde{\pi}_1^t(X)^{ab}. \end{array}$$

□

3 Tame class field theory

Now we use the power of Wiesend's higher dimensional class field theory to establish the properties of the tame reciprocity morphism rec_X of the last section.

We start with the geometric case. Let X be a smooth, connected variety over a finite field k. The structural morphism $X \to \mathrm{Spec}(k)$ induces degree maps:

$$\begin{array}{rcl} H_0^{sing}(X, \mathbb{Z}) & \longrightarrow & H_0^{sing}(\mathrm{Spec}(k), \mathbb{Z}) = \mathbb{Z} \\ \pi_1^t(X)^{ab} & \longrightarrow & \pi_1^t(\mathrm{Spec}(k))^{ab} = Gal(\bar{k}|k) \cong \hat{\mathbb{Z}}. \end{array}$$

The next theorem was proved in 1999 by M. Spieß and the author, see [SS]. We used deep results on motivic cohomology to deduce the assertion for surfaces from the unramified class field theory of Bloch-Kato-Saito [KS]. Then we used a version of Colliot-Thélène's hypersurface section argument to extend the result to arbitrary dimensions. For technical reasons, we also had to assume that X is quasi-projective and admits a smooth, projective compactification. Using Wiesend's result, the proof becomes much simpler now.

Theorem 3.1 (Schmidt/Spieß)**.** *Let X be a smooth, connected variety over a finite field k. Then the reciprocity homomorphism*

$$rec_X : H_0^{sing}(X, \mathbb{Z}) \longrightarrow \pi_1^t(X)^{ab}$$

is injective. The image of rec_X *consists of all elements whose degree in* $Gal(\bar{k}|k)$ *is an integral power of the Frobenius automorphism. In particular, the cokernel* $coker(rec_X) \cong \hat{\mathbb{Z}}/\mathbb{Z}$ *is uniquely divisible. The induced map on the degree-zero parts* $rec_X^0\colon H_0^{sing}(X,\mathbb{Z})^0 \xrightarrow{\sim} \pi_1^t(X)^{ab,\,0}$ *is an isomorphism of finite abelian groups.*

Proof. Let $\mathcal{C}_X^t$ be the tame idèle class group of [W1], Definition 4, i.e.

$$\mathcal{C}_X^t = \operatorname{coker}\Big(\bigoplus_{C\subset X} k(C)^\times \to Z_0(X) \oplus \bigoplus_{C\subset X}\bigoplus_{v\in C_\infty} k(C)_v^\times/U^1_{k(C)_v}\Big),$$

where $C_\infty = P(\tilde{C}) \smallsetminus \tilde{C}$ is the set of places of the global field $k(C)$ with center outside C and $k(C)_v$ is the completion of $k(C)$ with respect to v. Elementary approximation on curves shows that the obvious map

$$\phi\colon Z_0(X) \longrightarrow \mathcal{C}_X^t$$

is surjective. The kernel of ϕ is the subgroup in $Z_0(X)$ generated by elements of the form $div(f)$ where $C \subset X$ is a closed curve and f is an invertible rational function on C which is in $U^1_{k(C)_v}$ for all $v \in C_\infty$. By Theorem 1.1, we obtain $\ker(\phi) = d_1(C_1(X))$. Therefore ϕ induces an isomorphism

$$H_0^{sing}(X,\mathbb{Z}) \xrightarrow{\sim} \mathcal{C}_X^t.$$

By construction, this isomorphism is compatible with the respective reciprocity homomorphisms to $\pi_1^t(X)^{ab}$. Therefore the statement of the theorem follows from [W1], Theorem 1(b), which establishes the respective result for the reciprocity homomorphism $\mathcal{C}_X^t \longrightarrow \pi_1^t(X)^{ab}$. □

Next we consider the arithmetic case. We have shown in [S2] that $\tilde{\pi}_1^t(X)^{ab}$ is finite in this case. We conjectured that rec_X is an isomorphism of finite abelian groups and we proved in [S3] a version of this conjecture with $H_0^{sing}(X,\mathbb{Z})$ replaced by $\mathrm{CH}_0(\bar{X}, X)$, the relative Chow group of zero-cycles with respect to a regular compactification $\bar{X}$ of X. See [S4] for a discussion on the relation between this result and the (at that time) conjecture on H_0. Having Wiesend's class field theory at hand, the proof of the following theorem is quite simple.

Theorem 3.2. *Let* X *be a regular, connected scheme, flat and of finite type over* $\operatorname{Spec}(\mathbb{Z})$*. Then the reciprocity homomorphism*

$$rec_X\colon H_0^{sing}(X,\mathbb{Z}) \longrightarrow \tilde{\pi}_1^t(X)^{ab}$$

is an isomorphism of finite abelian groups.

Proof. We make the notational convention $U^1(K) = K^\times$ for the archimedean local fields $K = \mathbb{R}, \mathbb{C}$. We consider the quotient $\mathcal{C}_X^t$ of $\mathcal{C}_X$ obtained by cutting out the 1-unit groups of all places not on X. More precisely,

$$\mathcal{C}_X^t := \operatorname{coker}\Big(\bigoplus_{C\subset X} k(C)^\times \to Z_0(X) \oplus \bigoplus_{C\subset X}\bigoplus_{v\in C_\infty} k(C)_v^\times/U^1_{k(C)_v}\Big).$$

$\mathcal{C}_X^t$ is a discrete quotient of $\mathcal{C}_X$. By [W1], Theorem 1, its subgroups classify those finite, étale, abelian coverings of X such that the base change to each regular curve C defines a finite, étale, abelian covering which is at most tamely ramified along the boundary of a regular compactification of C and in which every $\mathbb{R}$-valued point splits completely. Therefore (cf. [W2], Theorem 2) the reciprocity homomorphism $\rho_X\colon \mathcal{C}_X \to \pi_1^{et}(X)^{ab}$ of [W1] induces an isomorphism of finite abelian groups $\mathcal{C}_X^t \xrightarrow{\sim} \tilde{\pi}_1^t(X)^{ab}$. Now we proceed as in the proof of Theorem 3.1 and consider the obvious map

$$\phi\colon Z_0(X) \longrightarrow \mathcal{C}_X^t.$$

The same argument as in the proof of Theorem 3.1 shows that ϕ induces an isomorphism $H_0^{sing}(X,\mathbb{Z}) \xrightarrow{\sim} \mathcal{C}_X^t$ which is compatible with the respective reciprocity homomorphisms to $\tilde{\pi}_1^t(X)^{ab}$. This completes the proof of the theorem. □

Finally, assume that X is regular, flat and proper over $\mathrm{Spec}(\mathbb{Z})$, and let $D \subset X$ be a divisor. Then (cf. [S3], [S4]) the relative Chow group of zero cycles $\mathrm{CH}_0(X,D)$ is a quotient of $H_0^{sing}(X \smallsetminus D,\mathbb{Z})$ in a natural way. In [S3] we constructed, under a mild technical assumption, a reciprocity isomorphism $\mathrm{rec}'_X\colon \mathrm{CH}_0(X,D) \xrightarrow{\sim} \tilde{\pi}_1^t(X \smallsetminus D)^{ab}$. As rec is the composite of rec′ with the natural projection, we obtain the following corollary.

Corollary 3.3. *Let X be a regular, connected scheme, flat and proper over* $\mathrm{Spec}(\mathbb{Z})$*, such that its generic fibre $X \otimes_{\mathbb{Z}} \mathbb{Q}$ is projective over $\mathbb{Q}$. Let D be a divisor on X whose vertical irreducible components are normal schemes. Then the natural homomorphism*

$$H_0^{sing}(X \smallsetminus D,\mathbb{Z}) \longrightarrow \mathrm{CH}_0(X,D)$$

is an isomorphism of finite abelian groups.

References

[Fu] W. Fulton *Intersection Theory.* Ergebnisse der Mathematik und ihrer Grenzgebiete vol. 3, Springer 1984, 2nd ed. 1998.

[KS] K. Kato, S. Saito, *Unramified class field theory of arithmetical surfaces.* Ann. of Math. **118** (1983), 241–275.

[La] S. Lang *Sur le séries L d'une variété algébrique.* Bull. Soc. Math. France **84** (1956), 555–563.

[S1] A. Schmidt *Singular homology of arithmetic schemes.* Preprint 2000. `http://www.math.uiuc.edu/K-theory/0418/` .

[S2] A. Schmidt *Tame coverings of arithmetic schemes.* Math. Annalen **322** (2002), 1–18.

[S3] A. Schmidt *Tame class field theory for arithmetic schemes.* Invent. Math. **160** (2005), 527–565.

[S4] A. Schmidt *Higher dimensional class field theory from a topological point of view.* Algebraic number theory and related topics, Sūrikaisekikenkyūsho Kōkyūroku (RIMS Proceedings) **10** (2005), 168–178.

[SS] A. Schmidt, M. Spieß *Singular homology and class field theory of varieties over finite fields*. J. reine u. angew. Math. **527** (2000), 13–37.

[SV] A. Suslin, V. Voevodsky *Singular homology of abstract algebraic varieties.* Invent. Math. **123** (1996), 61–94.

[W1] G. Wiesend *Class field theory for arithmetic schemes.* Math. Zeitschrift, to appear.

[W2] G. Wiesend *Tamely ramified covers of varieties and arithmetic schemes.* Preprint 2006.

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